SCHUR-HORN THEOREMS IN II_∞-FACTORS

MARTÍN ARGERAMI AND PEDRO MASSEY

Abstract. We describe majorization between selfadjoint operators in a σ-finite II_∞ factor $(\mathcal{M},\tau)$ in terms of simple spectral relations. For a diffuse abelian von Neumann subalgebra $\mathcal{A}\subset\mathcal{M}$ with trace-preserving conditional expectation $E_{\mathcal{A}}$, we characterize the closure in the measure topology of the image through $E_{\mathcal{A}}$ of the unitary orbit of a selfadjoint operator in $\mathcal{M}$ in terms of majorization (i.e., a Schur-Horn theorem). We also obtain similar results for the contractive orbit of positive operators in $\mathcal{M}$ and for the unitary and contractive orbits of τ-integrable operators in $\mathcal{M}$.

1. Introduction

Given two vectors $x,y\in\mathbb{R}^n$, we say that x is *majorized* by y ($x\prec y$) if

$$\sum_{j=1}^k x_j^\downarrow \le \sum_{j=1}^k y_j^\downarrow,\quad k=1,\dots,n-1;\quad \sum_{j=1}^n x_j=\sum_{j=1}^n y_j\,,$$

where $x^\downarrow\in\mathbb{R}^n$ denotes the vector obtained from x by re-arranging the entries in non-increasing order. The first systematic study of the notion of majorization is attributed to Hardy, Littlewood, and Polya [14]. We refer the reader to [8] for further references and properties of majorization. It is well known that (vector) majorization is intimately related with the theory of doubly stochastic matrices. Indeed, $x\prec y$ if and only if $x=Dy$ for some doubly stochastic matrix D; then, as a consequence of Birkhoff's characterization of the extreme points of the set of doubly stochastic matrices [9], one can conclude that

$$\{x\in\mathbb{R}^n:\ x\prec y\}=\operatorname{conv}\{y_\sigma:\ \sigma\in\mathbb{S}_n\}\,, \tag{1.1}$$

where $\operatorname{conv}\{y_\sigma:\ \sigma\in\mathbb{S}_n\}$ denotes the convex hull of the set of vectors y_σ that are obtained from y by re-arrangement of its components through permutations $\sigma\in\mathbb{S}_n$.

It turns out that majorization also characterizes the relation between the spectrum and the diagonal of a selfadjoint matrix. Let $M_n(\mathbb{C})$ denote the algebra of complex $n\times n$ matrices. For $A\in M_n(\mathbb{C})$, let $\operatorname{diag}(A)=(a_{11},a_{22},\dots,a_{nn})\in\mathbb{C}^n$, and let $\lambda(A)\in\mathbb{C}^n$ be the vector whose coordinates are the eigenvalues of A, counted with multiplicity. I. Schur [30] proved that for $A\in M_n(\mathbb{C})$ selfadjoint, $\operatorname{diag}(A)\prec\lambda(A)$; while A. Horn [18] proved the converse: given $x,y\in\mathbb{R}^n$ with $x\prec y$, there exists a selfadjoint matrix $A\in M_n(\mathbb{C})$, with $\operatorname{diag}(A)=x$, $\lambda(A)=y$. For $y\in\mathbb{C}^n$ let $M_y\in M_n(\mathbb{C})$ denote the diagonal matrix with main diagonal y and let $\mathcal{U}_n\subset M_n(\mathbb{C})$ denote the group of unitary matrices. The results from Schur and Horn can then be combined in the following assertion: given $y\in\mathbb{R}^n$,

$$\{x\in\mathbb{R}^n:\ x\prec y\}=\{\operatorname{diag}(U\,M_y\,U^*):\ U\in\mathcal{U}_n\}, \tag{1.2}$$

usually known as the Schur-Horn Theorem. The fact that majorization relations imply a family of entropic-like inequalities makes the Schur-Horn theorem an important tool in matrix analysis theory [8]. It has also been observed that the Schur-Horn theorem plays a crucial role in frame theory [1, 11, 25].

Majorization in the context of von Neumann algebras has been widely studied (see for instance [4, 15, 16, 17, 23, 24]). In [15] and [16] F. Hiai showed several characterizations of majorization in a semifinite von Neumann algebra, including a generalization of (1.1), i.e. a "Birkhoff" theorem. Nevertheless, the lack of the corresponding "Schur-Horn" theorems in the general context of von Neumann factors was only recently observed. Early work on this topic was developed by A. Neumann [26, 27] in relation with an extension to infinite dimensions of the linear Kostant convexity theorem in Lie theory.

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It was in [7] that W. Arveson and R.V. Kadison conjectured a Schur-Horn theorem in II_1 factors. Although this conjecture remains an open problem, there has been progress on related (but weaker) Schur-Horn theorems in this context [2, 3, 5]. There has also been significant improvements of Neumann's work on majorization between sequences in $c_0(\mathbb{R}^+)$ due to V. Kaftal and G. Weiss [21, 22] because of the relations between infinite dimensional versions of the Schur-Horn theorem (via majorization of bounded structured real sequences) and arithmetic mean ideals (see also [7] for improvements in the compact case in $B(H)$).

In this paper we prove versions of the Schur-Horn theorem (i.e. generalizations of (1.2)) in the case of a σ-finite II_∞-factor. These results extend those obtained in [2, 3, 26]. Our results are in the vein of Neumann's work, and they are related with a weak version of Arveson-Kadison's scheme for Schur-Horn theorems, but modeled in II_∞ factors. These extensions are formally analogous to the Schur-Horn theorems in [2, 3], but the techniques are more involved in the infinite case. We show that our results are optimal, in the sense that they can not be strengthened for a general selfadjoint operator in a II_∞ factor.

The paper is organized as follows. In section 2 we develop notation and some basic results on the measure topology and the τ-singular values in von Neumann algebras. Section 3 deals with majorization in $B(H)$, including some results complementing those in [26]. In Section 4 we consider a notion of majorization between selfadjoint operators in a II_∞ factor $(\mathcal{M},\tau)$ – in line with Neumann's idea [26] – together with several of its basic properties. Although majorization in II_∞ factors is not a new notion [15, 16], our approach is quite different from the previous presentations. In section 5 we state and prove the generalizations of the Schur-Horn theorem in II_∞ factors. Our strategy is to reduce the problem to a discrete version, where we can apply the Schur-Horn theorems developed in Section 3 for $B(H)$. We then proceed to show that Hiai's notion of majorization in terms of Choquet's theory of comparison of measures [16] coincides with ours. We finally consider similar results for the contractive orbit of a positive operator and for the unitary and contractive orbits of bounded τ-measurable operators.

2. Preliminaries

Let $(\mathcal{M},\tau)$ be a σ-finite, semi-finite, diffuse von Neumann algebra. The real subspace of selfadjoint elements in $\mathcal{M}$ is denoted by $\mathcal{M}^{\mathrm{sa}}$; the group of unitary operators by $\mathcal{U}_{\mathcal{M}}$; and the set of selfadjoint projections by $\mathcal{P}(\mathcal{M})$. Given $p\in\mathcal{P}(\mathcal{M})$, we use the notation $p^\perp = I-p$. For any $a\in\mathcal{M}^{\mathrm{sa}}$ and any Borel set $\Delta\subset\mathbb{R}$, $p^a(\Delta)\in\mathcal{P}(\mathcal{M})$ denotes the spectral projection of a corresponding to Δ.

In [12] T. Fack considered in $\mathcal{M}$ the ideals $\mathcal{F}(\mathcal{M})=\{x\in\mathcal{M}:\ \tau(\mathrm{supp}\ x^*)<\infty\}$ – the *τ-finite rank operators* – and $\mathcal{K}(\mathcal{M})=\overline{\mathcal{F}(\mathcal{M})}$, the ideal of *$\tau$-compact operators*. The quotient C*-algebra $\mathcal{M}/\mathcal{K}(\mathcal{M})$ is called the generalized Calkin algebra. The *essential spectrum* of x – denoted $\sigma_{\mathrm{e}}(x)$ – is the spectrum of $x+\mathcal{K}(\mathcal{M})$ as an element of $\mathcal{M}/\mathcal{K}(\mathcal{M})$. The complement of $\sigma_{\mathrm{e}}(x)$ within $\sigma(x)$ is the *discrete spectrum* $\sigma_{\mathrm{d}}(x)$ of x. As shown in [16], for $x\in\mathcal{M}^{\mathrm{sa}}$,

$$\sigma_{\mathrm{e}}(x)=\{t\in\sigma(x):\ \forall\varepsilon>0,\ \tau(p^x(t-\varepsilon,t+\varepsilon))=\infty\}.$$

It follows from the previous definitions that $x\in\mathcal{M}^{\mathrm{sa}}$ is τ-compact if and only if $\sigma_{\mathrm{e}}(x)=\{0\}$.

We consider in $\mathcal{M}$ the *measure topology* $\mathcal{T}$, which is the linear topology given by the neighborhoods of $0\in\mathcal{M}$,

$$V(\varepsilon,\delta)=\{r\in\mathcal{M}:\ \exists p\in\mathcal{P}(\mathcal{M}),\ \|rp\|<\varepsilon,\ \tau(p^\perp)<\delta\},$$

where $\varepsilon,\delta>0$. For a II_1 factor, $\mathcal{T}$ reduces to the σ-strong topology on bounded sets, while in a type I_∞ factor it reduces to the norm topology.

Definition 2.1. *The* upper spectral scale *of $b\in\mathcal{M}^{\mathrm{sa}}$ is the non-increasing right-continuous real function*

$$\lambda_t(b)=\min\{s\in\mathbb{R}:\ \tau(p^b(s,\infty))\le t\},\quad t\in[0,\infty).$$

The lower spectral scale *of b is the non-decreasing right-continuous function*

$$\mu_t(b)=-\lambda_t(-b)=\max\{s\in\mathbb{R}:\ \tau(p^b(-\infty,s))\le t\},\quad t\in[0,\infty).$$

A direct consequence of these definitions is that $\lambda_t(b),\,\mu_t(b)\in\sigma(b)$ for every $t\in\mathbb{R}^+$. The function $t\mapsto\lambda_t(b)$ is the analogue of the re-arrangement of the eigenvalues (in non-increasing order and counting multiplicities) of a self-adjoint matrix.

For $x \in \mathcal{M}$ we can consider the τ-*singular values* of x given by $\nu_t(x) = \lambda_t(|b|)$, $t \in [0, \infty)$. The spectral scale and τ-singular values have been extensively studied [12, 13, 17, 20, 29] in the broader context of τ-measurable operators affiliated to $(\mathcal{M}, \tau)$.

The elements of $\mathcal{K}(\mathcal{M})$ can be described in terms of τ-singular values. Indeed, $x \in \mathcal{M}$ is τ-compact if and only if $\lim_{t\to\infty} \nu_t(x) = 0$ [15]. We will make frequent use of the fact that (since $\mathcal{M}$ is diffuse) a given τ-compact $x \in \mathcal{M}^+$ admits a complete flag, i.e. an increasing assignment $\mathbb{R}^+ \ni t \mapsto e(t) \in \mathcal{P}(\mathcal{M})$ such that $\tau(e(t)) = t$, and

$$x = \int_0^\infty \lambda_t(x)\, de(t)\,. \tag{2.1}$$

As opposed to the finite case [2], the equality in (2.1) does not hold for arbitrary τ-compact selfadjoint operators in $\mathcal{M}$. This is possibly one of the reasons why majorization has been considered mainly between positive operators in the semi-finite algebras (see the remarks at the end of [15]). We shall overcome this issue by considering both the upper and lower spectral scale, as done in [26] in the case of separable I_∞ factors.

The following fact is used in [16] (in the context of possibly unbounded operators) but we do not know of an explicit proof in the literature. For $x \in \mathcal{M}$, we denote its usual one-norm or trace norm in $(\mathcal{M}, \tau)$ by $\|x\|_1 = \tau(|x|) \in [0, \infty]$.

Proposition 2.2. *Let $(\mathcal{M}, \tau)$ be a semifinite von Neumann algebra. For $s > 0$ let $\|\cdot\|_{(s)}$ be the norm given by*

$$\|x\|_{(s)} = \inf\{\|x_1\|_1 + s\,\|x_2\| :\ x = x_1 + x_2,\ x_1, x_2 \in \mathcal{M}\}, \quad x \in \mathcal{M}.$$

Then $\|x\|_{(s)} = \int_0^s \nu_t(x)\, dt$, and the topology induced by $\|\cdot\|_{(s)}$ agrees with the measure topology on bounded sets.

Proof. The equality $\|x\|_{(s)} = \int_0^s \nu_t(x)\, dt$ is proven in [13] in the argument after Theorem 4.4. We now show that the topology induced by $\|\cdot\|_{(s)}$ and the measure topology agree on bounded sets. Indeed, if $0 < s \le r$ then there exists $k \in \mathbb{N}$ such that $r \le k\,s$ and therefore $\|x\|_{(s)} \le \|x\|_{(r)} \le k\,\|x\|_{(s)}$, since $t \mapsto \nu_t(x)$ is a non-increasing function. This shows that the norms $\|\cdot\|_{(s)}$, for $s > 0$, are all equivalent and induce the same topology. Hence we can assume without loss of generality that $s = 1$.

If $\|x\|_{(1)} < d$, then $\int_0^1 \nu_t(x)\, dt < d$. Using that $\nu_t(x)$ is non-increasing, there exists t_0 with $0 < t_0 < \sqrt{d}$ such that $\nu_{t_0}(x) < \sqrt{d}$. By [13, Proposition 2.2],

$$\nu_{t_0}(x) = \inf\{\|xq\| :\ \tau(q^\perp) \le t_0\}, \tag{2.2}$$

so there is a projection $q \in \mathcal{P}(\mathcal{M})$ such that $\|xq\| < \sqrt{d}$ and $\tau(q^\perp) < \sqrt{d}$; that is, $x \in V(\sqrt{d}, \sqrt{d})$. Thus every ball in the $\|\cdot\|_{(1)}$-topology lies inside a neighborhood of 0 in the measure topology.

Conversely, if $x \in V(\varepsilon, \delta)$ and $\|x\| \le k$, there exists a projection $q \in \mathcal{P}(\mathcal{M})$ such that $\|xq\| < \varepsilon$, $\tau(q^\perp) < \delta$. Since $x = xq^\perp + xq$,

$$\|x\|_{(1)} \le \|xq^\perp\|_1 + \|xq\| \le k\delta + \varepsilon;$$

that is, $V(\varepsilon, \delta) \cap \{x \in \mathcal{M} :\ \|x\| \le k\} \subset \{x \in \mathcal{M} :\ \|x\|_{(1)} \le k\delta + \varepsilon\}$. □

Corollary 2.3. *Let $\mathcal{N}$ be a II_1-factor with trace $\tau_\mathcal{N}$, and let $\{x_j\}$ be a bounded net. Then $x_j \xrightarrow{\|\cdot\|_1} x$ if and only if $x_j \xrightarrow{\mathcal{T}} x$.*

Proof. For any $x \in \mathcal{N}^{\mathrm{sa}}$ we have $\|x\|_1 = \tau_\mathcal{N}(|x|) = \int_0^1 \nu_t(x)\, ds$. Then $\|\cdot\|_1 = \|\cdot\|_{(1)}$ and Proposition 2.2 yields the result. □

We will often and without mention make use of the following properties of the measure topology.

Corollary 2.4. *Let $\mathcal{A} \subset \mathcal{M}$ be a von Neumann subalgebra and let $E_\mathcal{A}$ be the trace preserving conditional expectation onto $\mathcal{A}$. Let $\{x_j\} \subset \mathcal{M}^{\mathrm{sa}}$, $\alpha, \beta \in \mathbb{R}$ with $\alpha\, I \le x_j \le \beta\, I$ for every j and such that $x_j \xrightarrow{\mathcal{T}} x$. Then*

(i) *$x \in \mathcal{M}^{\mathrm{sa}}$, and $\alpha \le x \le \beta$.*

(ii) $E_{\mathcal{A}}(x_j) \xrightarrow{\mathcal{T}} E_{\mathcal{A}}(x)$.

Proof. In order to prove (i) first notice that if $x_j \xrightarrow{\mathcal{T}} x$ with $x_j \geq 0$ for every j then $x \in \mathcal{M}^{\mathrm{sa}}$; indeed, this follows from the facts that the operation of taking adjoint is continuous in the measure topology and that this topology is Hausdorff. If $x \notin \mathcal{M}^+$, there exists a nonzero projection $q \in \mathcal{M}$ and $k \in \mathbb{R}^+$ such that $q\,x\,q \leq (-k)\,q$. By replacing q by a smaller projection if necessary, we may assume that $\tau(q) < \infty$. We have $q\,x_j\,q \xrightarrow{\mathcal{T}} q\,x\,q$, so for j big enough there exists a projection p such that $\|(q\,x\,q - q\,x_j\,q)p\| < k/3$ and $\tau(p^\perp) < \tau(q)/2$. Then $pqp \neq 0$, since

$$\tau(p\,q\,p) = \tau(p\,q) = \tau(q) - \tau(p^\perp q) \geq \tau(q) - \tau(q)/2 = \tau(q)/2 > 0.$$

We also get from above that $\tau(q) \leq 2\tau(p\,q\,p)$. But then $\tau(p\,q\,(x_j - x)\,q\,p) = \tau(q\,[q\,(x_j - x)\,q\,p]) \leq \frac{k}{3}\tau(q)$, so

$$\begin{aligned} 0 \leq \tau(p\,q\,x_j\,q\,p) &= \tau(p\,q\,x\,q\,p) + \tau(p\,q\,(x_j - x)\,q\,p) \leq (-k)\tau(p\,q\,p) + \frac{k}{3}\,\tau(q) \\ &\leq (-k)\tau(p\,q\,p) + \frac{2k}{3}\tau(p\,q\,p) = -\frac{k}{3}\,\tau(p\,q\,p) < 0, \end{aligned}$$

a contradiction. This shows that $x \geq 0$. By linearity we get that if $x_j \xrightarrow{\mathcal{T}} x$ and $\alpha \leq x_j \leq \beta$ then $\alpha \leq x \leq \beta$.

Item (ii) follows from the fact that $E_{\mathcal{A}}$ is contractive with respect to $\|\cdot\|_{(1)}$ together with Proposition 2.2. Indeed, it is well known that $\|E_{\mathcal{A}}(x)\| \leq \|x\|$ for $x \in \mathcal{M}$. Using that $\tau(E_{\mathcal{A}}(x)\,y) = \tau(x\,E_{\mathcal{A}}(y)) \leq \|E_{\mathcal{A}}(y)\|\,\tau(|x|)$ we get

$$\|E_{\mathcal{A}}(x)\|_1 = \sup\{|\tau(E_{\mathcal{A}}(x)\,y)| : \ y \in \mathcal{M},\ \|y\| \leq 1\} \leq \|x\|_1.$$

For any decomposition $x = y + z$, since $E_{\mathcal{A}}(x) = E_{\mathcal{A}}(y) + E_{\mathcal{A}}(z)$,

$$\|E_{\mathcal{A}}(x)\|_{(1)} \leq \|E_{\mathcal{A}}(y)\|_1 + \|E_{\mathcal{A}}(z)\| \leq \|y\|_1 + \|z\|.$$

So, by Proposition 2.2, $\|E_{\mathcal{A}}(x)\|_{(1)} \leq \|x\|_{(1)}$ for all $x \in \mathcal{M}$, and so $E_{\mathcal{A}}$ is $\mathcal{T}$-continuous. $\square$

3. Majorization in $\ell^\infty(\mathbb{N})$ and $B(H)$ revisited

Let H be a complex separable Hilbert space. In this section we revise and complement A. Neumann's [26] theory on majorization between self-adjoint operators in $B(H)$. These results will play a key role in our proof of the Schur-Horn theorem in II_∞-factors (Theorem 5.5). For conceptual and notational convenience, we shall follow the exposition in [1] (see also [20]).

In $B(H)$ we consider the canonical trace Tr. We write $\mathcal{U}(H)$ for the group of unitary operators in H, and $\mathcal{C}(H)$ for the semigroup of contractive operators in $B(H)$, i.e.

$$\mathcal{C}(H) = \{v \in B(H) : \ v^*v \leq I\}.$$

For $k \in \mathbb{N}$, let $\mathcal{P}_k$ be the set of orthogonal projections $p \in B(H)$ such that $\mathrm{Tr}\,(p) = k$. For $b \in B(H)^{\mathrm{sa}}$, $k \in \mathbb{N}$, we consider

$$U_k(b) = \sup_{p \in \mathcal{P}_k} \mathrm{Tr}\,(b\,p), \quad \text{and} \quad L_k(b) = \inf_{p \in \mathcal{P}_k} \mathrm{Tr}\,(b\,p). \tag{3.1}$$

For each $k \in \mathbb{N}$, both $b \mapsto U_k(b)$ and $b \mapsto L_k(b)$ are norm-continuous in $B(H)$, with $L_k(b) = -U_k(-b)$. Moreover, $U_k(u^*\,b\,u) = U_k(b)$ for every $b \in B(H)^{\mathrm{sa}}$, $u \in \mathcal{U}(H)$.

Following [26] (but with a different notation) we define, for $f \in \ell^\infty(\mathbb{N})$ and $k \in \mathbb{N}$,

$$U_k(f) = \sup\{\sum_{j \in K} f_j : \ |K| = k\}, \quad L_k(f) = \inf\{\sum_{j \in K} f_j : \ |K| = k\}. \tag{3.2}$$

Again, for each $k \in \mathbb{N}$, $L_k(f) = -U_k(-f)$. The similarity of the notations in (3.1) and (3.2) is justified by the following fact: if $b \in B(\mathcal{H})$ is selfadjoint and such that there exists an orthonormal basis $\{e_i\}_{i \in \mathbb{N}}$ of H and $f = (f_i)_{i \in \mathbb{N}} \in \ell^\infty_{\mathbb{R}}(\mathbb{N})$ such that $be_i = f_i\,e_i$, $i \in \mathbb{N}$ (i.e. if b is diagonal), then by [1, Proposition 3.3]

$$U_k(b) = U_k(f), \quad L_k(b) = L_k(f), \quad k \in \mathbb{N}\,. \tag{3.3}$$

Definition 3.1 (Operator majorization in $B(H)$ [1]). *Let a, $b \in B(H)^{\rm sa}$. We say that:*

(i) *a is* submajorized *by b, denoted $a \prec_w b$, if $U_k(a) \le U_k(b)$ for every $k \in \mathbb{N}$;*
(ii) *a is* majorized *by b, denoted $a \prec b$, if $a \prec_w b$ and $L_k(a) \ge L_k(b)$ for every $k \in \mathbb{N}$.*

We will also use the notion of vector majorization in $\ell^\infty_{\mathbb{R}}(\mathbb{N})$ (used implicitly in [26]) as follows:

Definition 3.2 (Vector majorization in $\ell^\infty_{\mathbb{R}}(\mathbb{N})$). *Let f, $g \in \ell^\infty_{\mathbb{R}}(\mathbb{N})$. We say that:*

(i) *f is* submajorized *by g, denoted $f \prec_w g$ if $U_k(f) \le U_k(g)$ for every $k \in \mathbb{N}$;*
(ii) *f is* majorized *by g, denoted $f \prec g$, if $f \prec_w g$ and $L_k(f) \ge L_k(g)$ for every $k \in \mathbb{N}$.*

We fix an orthonormal basis $\mathcal{B} = \{e_i\}_{i\in\mathbb{N}}$ on H, with associated system of matrix units $\{e_{ij}\}_{i,j\in\mathbb{N}}$ in $B(H)$. For each $f \in \ell^\infty(\mathbb{N})$ we denote by $M_f \in B(H)$ the induced diagonal operator with respect to $\mathcal{B}$, i.e. $M_f = \sum_{i\in\mathbb{N}} f_i\, e_{ii}$. By (3.3), it is immediate that for all $f, g \in \ell^\infty_{\mathbb{R}}(\mathbb{N})$,

$$M_f \prec M_g \iff f \prec g, \quad M_f \prec_w M_g \iff f \prec_w g. \tag{3.4}$$

We denote by $P_D : B(H) \to B(H)$ the trace preserving conditional expectation onto the (discrete) diagonal masa with respect to the fixed orthonormal basis. Explicitly, for each $x \in B(H)$,

$$P_D(x) = \sum_i e_{ii}\, x\, e_{ii} = \sum_i f_i\ e_{ii} = M_f\,, \quad \text{where } f_i = \langle x\, e_i, e_i\rangle,\ i \in \mathbb{N}.$$

We will use the following result from Neumann, which is a combination of [26, Theorem 2.18] and [26, Theorem 3.13]. Although Neumann's result is phrased in terms of vectors in $\ell^\infty_{\mathbb{R}}(\mathbb{N})$, we phrase the result in terms of operators in $B(H)$, as in [1, Theorem 3.10].

Theorem 3.3 (A Schur-Horn theorem for $B(H)$). *Let H be a separable complex Hilbert space and let P_D denote the unique trace preserving conditional expectation onto the discrete masa of diagonal operators with respect to the orthonormal basis $\mathcal{B}$ of H. Then, for $b \in B(H)^{\rm sa}$,*

$$\overline{\{P_D(u\, b\, u^*) :\ u \in \mathcal{U}(H)\}}^{\|\,\|} = \{M_f :\ f \in \ell^\infty_{\mathbb{R}}(\mathbb{N}),\, M_f \prec b\}\,.$$

As a consequence of Theorem 3.3 and (3.4) we recover Neumann's result for majorization in $\ell^\infty_{\mathbb{R}}(\mathbb{N})$ which states that, for f, $g \in \ell^\infty_{\mathbb{R}}(\mathbb{N})$,

$$M_f \in \overline{\{P_D(u\, M_g\, u^*) :\ u \in \mathcal{U}(H)\}}^{\|\,\|} \ \text{ if and only if } \ f \prec g\,. \tag{3.5}$$

In the rest of this section we will develop a contractive version of Theorem 3.3 for positive operators of $B(H)$ (Theorem 3.7). We will need a few preliminary results.

A proof of the following elementary inequality can be found in [20, Lemma 24].

Lemma 3.4. *Let $y_1 \ge y_2 \ge \cdots$ be positive real numbers and $\alpha_1, \alpha_2, \ldots \in [0, 1]$ with $\sum_{j=1}^\infty \alpha_j \le k$. Then*

$$\sum_{j=1}^\infty \alpha_j\ y_j \le \sum_{j=1}^k y_j. \tag{3.6}$$

Lemma 3.5. *For any $g \in \ell^\infty(\mathbb{N})^+$, $k \in \mathbb{N}$ we have*

$$U_k(g) = \sup\{\operatorname{Tr}(M_g\, x) :\ x \in \mathcal{C}(H)^+,\ \operatorname{Tr}(x) \le k\}\,.$$

Proof. The inequality "$\le$" is clear by (3.1) and (3.3). To prove the reverse inequality, fix $k \in \mathbb{N}$, let $\varepsilon > 0$, and fix $x \in \mathcal{C}(H)^+$ with $\operatorname{Tr}(x) \le k$. As x is a compact and positive contraction, $x = \sum_j \gamma_j h_j$, where $\{h_j\}_j$ is a pairwise-orthogonal family of rank-one projections, $0 \le \gamma_j \le 1$ for all j, and $\sum_j \gamma_j \le k$. We also have that $M_g = \sum_i g_i e_{ii}$, where $\{e_{ii}\}_i$ is the pairwise-orthogonal family of rank-one projections associated with the canonical basis $\mathcal{B}$. Let $\beta = \limsup_n g_n = \max \sigma_{\rm e}(M_g)$ and define $g' \in \ell^\infty(\mathbb{N})$ by

$$g'_i = \begin{cases} g_i & \text{if } g_i \ge \beta + \varepsilon \\ \beta & \text{otherwise} \end{cases}$$

Using [26, Lemma 2.17] it is readily seen that $|U_k(g') - U_k(g)| < k\varepsilon$. Since the entries in g that are strictly greater than β can only appear a finite number of times, we have that the set $D = \{i : \ g'_i > \beta\}$ is finite. So there is a unitary $u \in \mathcal{U}(H)$ (induced by an appropriate permutation) such that g'' given by $M_{g''} = uM_{g'}u^*$ satisfies $g''_1 \geq g''_2 \geq \cdots \geq g''_m$, where $m = |D|$, and $g''_i = \beta$ if $i > m$. For each $j \in \mathbb{N}$, let $h'_j = u^* h_j u$; then $\{h'_j\}_j$ is another family of pairwise orthogonal rank-one projections with sum I. We have

$$\sum_i \left(\sum_j \gamma_j \operatorname{Tr}(e_{ii}\, h'_j) \right) = \sum_j \gamma_j \operatorname{Tr}(h'_j) = \sum_j \gamma_j \leq k$$

and

$$0 \leq \sum_j \gamma_j \operatorname{Tr}(e_{ii}\, h'_j) \leq \sum_j \operatorname{Tr}(e_{ii}\, h'_j) = \operatorname{Tr}(e_{ii}) = 1.$$

Since $x \geq 0$ and $g \leq g'$,

$$\operatorname{Tr}(M_g x) \leq \operatorname{Tr}(M_{g'}\, x) = \operatorname{Tr}(M_{g''}\, u^*\, x\, u) = \sum_i g''_i \left(\sum_j \gamma_j \operatorname{Tr}(e_{ii}\, h'_j) \right) \tag{3.7}$$

Now, starting from (3.7) and applying the inequality (3.6) to the numbers $g''_1 \geq g''_2 \geq \cdots \geq 0$ and $\{\sum_j \gamma_j \operatorname{Tr}(e_{ii}\, h_j)\}_i$, we get

$$\begin{aligned} \operatorname{Tr}(M_g\, x) &\leq \sum_i g''_i \left(\sum_j \gamma_j \operatorname{Tr}(e_{ii}\, h'_j) \right) \leq \sum_{i=1}^k g''_i \\ &= U_k(g'') = U_k(g') < U_k(g) + \varepsilon k. \end{aligned}$$

As ε and x were arbitrary, we have proven the reverse inequality. □

Remark 3.6. Two operators $a, b \in B(H)$ are said to be *approximately unitarily equivalent* if there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{U}(H)$ such that

$$\lim_{n\to\infty} \|a - u_n\, b\, u_n^*\| = 0\,.$$

This equivalence is well-known to operator theorists and operator algebraists. As a consequence of the Weyl-von Neumann theorem, it follows [10, II.4.4] that $a, b \in B(H)^{\mathrm{sa}}$ are approximately unitarily equivalent if and only if their essential spectrums (with respect to the classical Calkin algebra) coincide and $\dim\ker(a - \lambda I) = \dim\ker(b - \lambda I)$ for every λ that is not in the essential spectrum of these operators. From this it can be deduced (see the proof of [10, II.4.4]) that for every $b \in B(H)^+$ and every orthonormal basis $\mathcal{B}$ of H, there exists $M_g \in B(H)^+$ – diagonal with respect to $\mathcal{B}$ – that is approximately unitarily equivalent to b.

The following is the main result of this section.

Theorem 3.7 (A contractive Schur-Horn theorem for $B(H)$)**.** *Let H be a separable complex Hilbert space and let P_D denote the unique trace preserving conditional expectation onto the discrete masa of diagonal operators with respect to the orthonormal basis $\mathcal{B}$ of H. Then, for $b \in B(H)^+$,*

$$\overline{\{P_D(v\, b\, v^*) : \ v \in \mathcal{C}(H)\}}^{\|\,\|} = \{M_f : \ f \in \ell^\infty(\mathbb{N})^+,\ M_f \prec_w b\}\,.$$

Proof. We first consider a reduction to the case where b is diagonalizable with respect to the orthonormal basis $\mathcal{B}$. Indeed, by Remark 3.6 there exists $g \in \ell^\infty(\mathbb{N})^+$ such that b and M_g are approximately unitarily equivalent. It is then straightforward to see that

$$\overline{\{v\, b\, v^* : \ v \in \mathcal{C}(H)\}}^{\|\,\|} = \overline{\{v\, M_g\, v^* : \ v \in \mathcal{C}(H)\}}^{\|\,\|}\,,$$

and that

$$\overline{\{P_D(v^*\, b\, v) : \ v \in \mathcal{C}(H)\}}^{\|\,\|} = \overline{\{P_D(v^*\, M_g\, v) : \ v \in \mathcal{C}(H)\}}^{\|\,\|}\,. \tag{3.8}$$

By (3.3), $U_k(b) = U_k(M_g)$ and $L_k(b) = L_k(M_g)$ for all $k \in \mathbb{N}$. These identities, together with (3.8), imply that – without loss of generality – we can assume that $b = M_g$ for some $g \in \ell^\infty(\mathbb{N})^+$.

Let $v \in \mathcal{C}(H)$ and let $p \in B(H)$ be a projection with $\mathrm{Tr}\,(p) = k$. Since $vv^* \le I$ and $0 \le P_D(p) \le I$ we have $v^* P_D(p)\, v \in \mathcal{C}(H)^+$ and $\mathrm{Tr}\,(v^* P_D(p)\, v) = \mathrm{Tr}\,(P_D(p)^{1/2}\, vv^*\, P_D(p)^{1/2}) \le \mathrm{Tr}\,(P_D(p)) = k$. Put $M_f = P_D(v\, M_g\, v^*)$. Then

$$\begin{aligned}
U_k(M_f) &= \sup\{\mathrm{Tr}\,(P_D(vM_gv^*)\, p) : \ \mathrm{Tr}\,(p) = k\} \\
&= \sup\{\mathrm{Tr}\,((vM_gv^*)\, P_D(p)) : \ \mathrm{Tr}\,(p) = k\} \\
&= \sup\{\mathrm{Tr}\,(M_g\,(v^* P_D(p)\, v)) : \ \mathrm{Tr}\,(p) = k\} \le U_k(M_g),
\end{aligned}$$

where in the last inequality we are using Lemma 3.5 and the fact that $v^* P_D(p)\, v \in \mathcal{C}(H)^+$. Thus, $M_f \prec_w M_g$ and, as $U_k(\cdot)$ is norm-continuous for every $k \in \mathbb{N}$, we get the inclusion "$\subset$".

For the reverse inclusion, assume that $M_f \prec_w M_g$ (i.e., $f \prec_w g$) and let $\varepsilon > 0$. We follow the idea of the proof of [8, Theorem II.2.8]. Consider $f', g' \in \ell^\infty(\mathbb{N}) \oplus \ell^\infty(\mathbb{N})$, given by

$$f' = (f + \varepsilon\, e) \oplus \varepsilon\, e, \quad g' = (g + \varepsilon\, e) \oplus 0\,.$$

where $e \in \ell^\infty(\mathbb{N})$ is the identity. Note that $\|f \oplus 0 - f'\|_\infty$, $\|g \oplus 0 - g'\|_\infty < \varepsilon$. Since $f, g \ge 0$, we have $U_k(f') = U_k(f) + k\varepsilon$, $U_k(g') = U_k(g) + k\varepsilon$, $L_k(f') = k\varepsilon$, $L_k(g') = 0$, for all $k \in \mathbb{N}$. Hence we have $f' \prec g'$. By Theorem 3.3, there exists a unitary operator $u \in B(H \oplus H)$ such that

$$\|M_{f'} - P_{D\oplus D}(u\, M_{g'}\, u^*)\| < \varepsilon. \tag{3.9}$$

We have

$$\|M_{g\oplus 0} - M_{g'}\| < \varepsilon, \quad \|M_{f\oplus 0} - M_{f'}\| < \varepsilon. \tag{3.10}$$

Now let $q = I \oplus 0 \in B(H \oplus H)$, and let $c = quq$ (clearly a contraction), seen as an operator in $B(H)$. Then, as $q\, P_{D\oplus D} = P_D \oplus 0$ and $q\, M_{f\oplus 0} = q\, M_{f\oplus 0}\, q = M_{f\oplus 0}$, we can use (3.9) and (3.10) to get

$$\begin{aligned}
\|M_f - P_D(c\, M_g\, c^*)\| &= \|q(M_{f\oplus 0} - P_{D\oplus D}(u\, M_{g\oplus 0}\, u^*))q\| \\
&\le \|M_{f\oplus 0} - P_{D\oplus D}(u\, M_{g\oplus 0}\, u^*)\| \\
&< 2\varepsilon + \|M_{f'} - P_{D\oplus D}(u\, M_{g'}\, u^*)\| < 3\varepsilon.
\end{aligned}$$

As ε was arbitrary, we conclude that $M_f \in \overline{\{P_D(v^* M_g\, v) : \ v \in \mathcal{C}(H)\}}^{\|\,\|}$. ☐

Remark 3.8. The positivity assumption in Theorem 3.7 is not just a technicality: even in dimension one we have $-1 \prec_w 0$, and $\{v\, 0\, v^* : |v| \le 1\} = \{0\}$.

As a consequence of Theorem 3.7 we get that, for $f,\, g \in \ell^\infty(\mathbb{N})^+$,

$$M_f \in \overline{\{P_D(v\, M_g\, v^*) : \ v \in \mathcal{C}(H)\}}^{\|\,\|} \ \text{ if and only if } \ f \prec_w g\,. \tag{3.11}$$

4. Majorization in II_∞-factors

Recall that $(\mathcal{M}, \tau)$ denotes a σ-finite and semi-finite diffuse von Neumann algebra. Given $a \in \mathcal{M}^{\mathrm{sa}}$, we consider the functions

$$U_t(a) = \int_0^t \lambda_s(a)\; ds \quad \text{and} \quad L_t(a) = \int_0^t \mu_s(a)\; ds\,, \quad t \in \mathbb{R}^+\,,$$

where $t \mapsto \lambda_t(a)$ and $t \mapsto \mu_t(a)$ denote the upper and lower spectral scales (Definition 2.1).

Our next goal is to describe the maps $b \mapsto U_t(b)$ and $b \mapsto L_t(b)$ by means of [13, Lemma 4.1]. We will make use of the following relation between spectral scales and singular values:

$$\lambda_t(a) = \nu_t(a + \gamma I) - \gamma\,, \quad \mu_t(a) = \rho - \nu_t(-a + \rho I), \quad a \in \mathcal{M}^{\mathrm{sa}}, \tag{4.1}$$

for any $\gamma,\, \rho \in \mathbb{R}$ such that $a + \gamma I$, $-a + \rho I \in \mathcal{M}^+$. We will denote by $\mathcal{P}_t(\mathcal{M})$ the set of all projections in $\mathcal{M}$ of trace t, i.e.

$$\mathcal{P}_t(\mathcal{M}) = \{p \in \mathcal{P}(\mathcal{M}) : \tau(p) = t\}.$$

Since $(\mathcal{M}, \tau)$ is diffuse and semifinite, $\mathcal{P}_t(\mathcal{M}) \ne \emptyset$ for every $t \ge 0$.

Lemma 4.1. *For any $a \in \mathcal{M}^{\rm sa}$,*

$$U_t(a) = \sup\{\tau(a\,p) : \ p \in \mathcal{P}_t(\mathcal{M})\}, \quad L_t(a) = \inf\{\tau(a\,p) : \ p \in \mathcal{P}_t(\mathcal{M})\}, \quad t \in \mathbb{R}^+.$$

Proof. The equalities are an immediate consequence of the identities (4.1) together with [13, Lemma 4.1] and the fact that, for every $t \in \mathbb{R}^+$,

$$\sup\{\tau(ap) : \ p \in \mathcal{P}_t(\mathcal{M})\} = \sup\{\tau((a+\gamma\, I)\,p) : \ p \in \mathcal{P}_t(\mathcal{M})\} - \gamma t\,. \qquad \square$$

Remark 4.2. If $a \in \mathcal{K}(\mathcal{M})^+$, then $\mu_t(a^+) = 0$ for $t \in \mathbb{R}^+$. Let $\{e(t)\}_{t\in\mathbb{R}^+} \subset \mathcal{M}$ be a complete flag for a such that $a = \int_0^\infty \lambda_t(a)\ de(t)$ (which exists by the assumptions on $\mathcal{M}$). Then, using [13, Proposition 2.7] and (4.1), we have

$$U_t(a) = \int_0^t \lambda_s(a)\ ds = \tau(a\,e(t)) \quad \text{and} \quad L_t(a) = 0, \quad t \in \mathbb{R}^+.$$

Thus, for a positive τ-compact operator a the supremum in Lemma 4.1 is attained explicitly by means of the projection $e(t)$ in $\mathcal{P}_t(\mathcal{M}) \cap \{a\}'$.

Lemma 4.3. *Let $b \in \mathcal{M}^{\rm sa}$. Then, for each $t \in \mathbb{R}^+$, the functions $b \mapsto U_t(b)$, $b \mapsto L_t(b)$ are $\|\cdot\|_1$-continuous, and they are also $\mathcal{T}$-continuous on bounded sets of $\mathcal{M}^{\rm sa}$.*

Proof. It is enough to prove the statement for $U_t(\cdot)$, since $L_t(b) = -U_t(-b)$. Given $\varepsilon > 0$, by Lemma 4.1 there exists $p \in \mathcal{P}_t(\mathcal{M})$ with $U_t(x) \le \tau(xp) + \varepsilon$. Then

$$U_t(x) - U_t(y) \le \tau(xp) + \varepsilon - \tau(yp) \le \|x-y\|_{(t)} + \varepsilon \le \|x-y\|_1 + \varepsilon\,,$$

where we used the inequality $\tau((x-y)p) \le \tau(|x-y|p) \le \|x-y\|_{(t)}$ that follows from Lemma 4.1. By letting $\varepsilon \to 0$ and reversing the roles of x and y we conclude the $\mathcal{T}$ and $\|\cdot\|_1$ continuity of $b \mapsto U_t(b)$ on bounded sets, by Proposition 2.2. $\square$

From now on we will specialize $(\mathcal{M}, \tau)$ to be a σ-finite II_∞-factor with faithful normal semifinite tracial weight τ.

We begin by describing the notion of majorization between selfadjoint operators in the II_∞-factor $\mathcal{M}$. In the setting of non-finite von Neumann algebras, this concept was developed for selfadjoint operators in [16]. Our presentation, inspired by Neumann's work [26], is fairly different (see Remark 4.5 below).

Definition 4.4. *Let $a, b \in \mathcal{M}^{\rm sa}$.*

(i) *We say that a is submajorized by b (denoted $a \prec_w b$) if*

$$U_t(a) \le U_t(b), \quad \textit{for every } t \in \mathbb{R}^+.$$

(ii) *We say that a is majorized by b, denoted $a \prec b$, if $a \prec_w b$ and*

$$L_t(a) \ge L_t(b), \quad \textit{for every } t \in \mathbb{R}^+.$$

Remark 4.5. If $b \in \mathcal{K}(\mathcal{M})^+$, then $\mu_t(b) = 0$ for all $t \in \mathbb{R}^+$ and therefore $L_t(b) = 0$ for all $t \in \mathbb{R}^+$. Thus, if $a \in \mathcal{M}^+$ and $a \prec_w b$, then $a \prec b$.

For a, $b \in \mathcal{M}^+$, our notion of majorization is strictly stronger than the one considered in [15]. As we have already mentioned, our notion of majorization does coincide with that of [16] for selfadjoint operators in a II_∞-factor (see Corollary 5.7). It is worth pointing out that in [16] majorization is described (for normal operators) in terms of Choquet's theory on comparison of measures, rather than in the simple terms used above: Lemma 4.1 shows that the notion of majorization in a II_∞-factor from definition 4.4 is an analogue of the notion of operator majorization in $B(H)$ as described in Definition 3.1.

For a fixed $b \in \mathcal{M}^{\rm sa}$, we write $\Omega_{\mathcal{M}}(b)$ for the set of all elements in $\mathcal{M}^{\rm sa}$ that are majorized by b, i.e.

$$\Omega_{\mathcal{M}}(b) = \{a \in \mathcal{M}^{\rm sa} : \ a \prec b\}.$$

Proposition 4.6. *Let $b \in \mathcal{M}^{\rm sa}$. Then $\Omega_{\mathcal{M}}(b)$ is a bounded $\mathcal{T}$-closed convex set that contains the unitary orbit $\mathcal{U}_{\mathcal{M}}(b)$.*

Proof. For any $x \in \mathcal{M}^{\mathrm{sa}}$, the definition of $U_t(x)$ and $L_t(x)$, together with the right-continuity of $\lambda_t(x)$ and $\mu_t(x)$, imply that

$$\lim_{t\to 0^+} \frac{U_t(x)}{t} = \lambda_t(0) = \max \sigma(x) \quad \text{and} \quad \lim_{t\to 0^+} \frac{L_t(x)}{t} = \mu_t(0) = \min \sigma(x).$$

Hence, $a \prec b$ implies $\sigma(a) \subset [\min \sigma(b), \max \sigma(b)]$; in particular $\|a\| \le \|b\|$, so $\Omega_{\mathcal{M}}(b)$ is a bounded set. Lemma 4.3 immediately implies that it is closed in the measure topology. Moreover, if $u \in \mathcal{U}_{\mathcal{M}}$, it is easy to see that $\lambda_t(ubu^*) = \lambda_t(b)$. So $U_t(ubu^*) = U_t(b)$ and, similarly, $L_t(ubu^*) = L_t(b)$. Thus $ubu^* \prec b$, and $\mathcal{U}_{\mathcal{M}}(b) \subset \Omega_{\mathcal{M}}(b)$.

Let a_1, $a_2 \in \mathcal{M}^{\mathrm{sa}}$, $\gamma \in [0,1]$, with $a_1 \prec b$, $a_2 \prec b$. Using Lemma 4.1,

$$\begin{aligned} U_t(\gamma\, a_1 + (1-\gamma)\, a_2) &= \sup\{\tau(p\,(\gamma\, a_1 + (1-\gamma)\, a_2)) :\ \tau(p) = t\} \\ &= \sup\{\gamma\, \tau(p\, a_1) + (1-\gamma)\, \tau(p\, a_2) :\ \tau(p) = t\} \\ &\le \gamma\, U_t(a_1) + (1-\gamma)\, U_t(a_2) \le U_t(b)\,. \end{aligned}$$

Similarly,

$$L_t(\gamma\, a_1 + (1-\gamma)\, a_2) \ge \gamma\, L_t(a_1) + (1-\gamma)\, L_t(a_2) \ge L_t(b)\,,$$

so $\gamma\, a_1 + (1-\gamma)\, a_2 \prec b$, and $\Omega_{\mathcal{M}}(b)$ is convex. □

Remark 4.7. Let $b \in \mathcal{M}^{\mathrm{sa}}$. The function $t \mapsto \lambda_t(b)$ is non-increasing and bounded; therefore the numbers $\lambda^{\mathrm{e}}_{\max}(b) = \lim_{t\to\infty} \lambda_t(b)$ and $\lambda^{\mathrm{e}}_{\min}(b) = \lim_{t\to\infty} \mu_t(b)$ exist. Indeed, we have

$$\lambda^{\mathrm{e}}_{\max}(b) = \max\ \sigma_{\mathrm{e}}(b) = \lim_{t\to\infty} \frac{U_t(b)}{t}\ \ ,\ \ \lambda^{\mathrm{e}}_{\min}(b) = \min\ \sigma_{\mathrm{e}}(b) = \lim_{t\to\infty} \frac{L_t(b)}{t}\,. \tag{4.2}$$

Consider the operators $\bar{b}$, $\underline{b} \in \mathcal{M}^+$ given by

$$\bar{b} = (b - \lambda^{\mathrm{e}}_{\max}(b)\, I)^+ \quad \text{and} \quad \underline{b} = (\lambda^{\mathrm{e}}_{\min}(b)\, I - b)^+\,. \tag{4.3}$$

Both $\bar{b}$, $\underline{b}$ are positive τ-compact operators with orthogonal support. It is easy to check that, for all $t \ge 0$, $U_t(b) = U_t(\bar{b}) + t\ \lambda^{\mathrm{e}}_{\max}(b)$, $L_t(b) = -U_t(\underline{b}) + t\ \lambda^{\mathrm{e}}_{\min}(b)$, and $L_t(\underline{b}) = L_t(\bar{b}) = 0$. If $a \prec b$ then, by (4.2),

$$\lambda^{\mathrm{e}}_{\min}(b) \le \lambda^{\mathrm{e}}_{\min}(a) \le \lambda^{\mathrm{e}}_{\max}(a) \le \lambda^{\mathrm{e}}_{\max}(b).$$

We finish the section with three lemmas on perturbations that will be used in Section 5.

Lemma 4.8. *Let $x \in \mathcal{K}(\mathcal{M})^+$, $z \in \mathcal{P}(\mathcal{M})$ infinite with $zx = 0$ and $\varepsilon > 0$. Then there exists $x' \in \mathcal{K}(\mathcal{M})^+$ such that:*

(i) *the support of x' contains z;*
(ii) *$\|x' - x\| < \varepsilon$;*
(iii) *$\lambda_t(x') = \lambda_t(x) + \varepsilon/(6+t)$, $t \in [0,\infty)$.*

Proof. Since x is τ-compact, there exists $s_0 > 0$ such that $\lambda_{s_0}(x) < \varepsilon/6$. Let $p_1 = p^x(\lambda_{s_0}(x), \infty)$. The τ-compactness of x guarantees that $\tau(p_1) < \infty$.

As x is τ-compact and positive, there exists a complete flag $e_x(t)$ with $x = \int_0^\infty \lambda_t(x)\, de_x(t)$. Note that $p_1 = e_x(s_0)$. Let $e_1(t)$ be a complete flag over z, and define

$$x' = \int_0^{s_0} \left(\lambda_t(x) + \frac{\varepsilon}{6+t}\right) de_x(t) + \int_0^\infty \left(\lambda_{t+s_0}(x) + \frac{\varepsilon}{6+t+s_0}\right) de_1(t)$$

The second term above equals $x' p_1^\perp = x' z$ and its norm is less than $\varepsilon/3$; so

$$\|x - x'\| \le \left\| \int_0^{s_0} \frac{\varepsilon}{6+t}\, de_x(t) \right\| + \|x p_1^\perp\| + \|x' p_1^\perp\| < \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{3} < \varepsilon$$

It is clear by construction (since $e_x(t) e_1(s) = 0$ for all t, s) that

$$\lambda_t(x') = \lambda_t(x) + \frac{\varepsilon}{6+t}, \quad t \in [0,\infty),$$

and this implies $x' \in \mathcal{K}(\mathcal{M})$. □

Lemma 4.9. *Let $\mathcal{A} \subset \mathcal{M}$ be a diffuse von Neumann subalgebra. Let $a \in \mathcal{A}^{\text{sa}}$, $b \in \mathcal{M}^{\text{sa}}$ with $a \prec b$, and fix $\varepsilon > 0$. Then there exist $a' \in \mathcal{A}^{\text{sa}}$, $b' \in \mathcal{M}^{\text{sa}}$ such that*

(i) $\|a - a'\| < \varepsilon$, $\|b - b'\| < \varepsilon$;
(ii) $a' \prec b'$;
(iii) $\overline{a'}$, $\underline{a'}$, $\overline{b'}$, $\underline{b'}$ *(as defined in Remark 4.7) have infinite support.*

Proof. We first consider a partition of the identity

$$s_1 = p^b[\lambda^{\text{e}}_{\max}(b) + \frac{\varepsilon}{8}, \infty)\,, \quad s_2 = p^b(\lambda^{\text{e}}_{\min}(b) - \frac{\varepsilon}{8}, \lambda^{\text{e}}_{\max}(b) + \frac{\varepsilon}{8})\,, \quad s_3 = p^b(-\infty, \lambda^{\text{e}}_{\min}(b) - \frac{\varepsilon}{8}].$$

The projection s_2 is infinite, while the others may or may not be infinite. We consider a decomposition $s_2 = z_1 + z_2 + z_3$ into three mutually orthogonal infinite projections, such that

$$z_1 \le p^b(\lambda^{\text{e}}_{\max}(b) - \frac{\varepsilon}{8}, \lambda^{\text{e}}_{\max}(b) + \frac{\varepsilon}{8}), \quad z_3 \le p^b(\lambda^{\text{e}}_{\min}(b) - \frac{\varepsilon}{8}, \lambda^{\text{e}}_{\min}(b) + \frac{\varepsilon}{8}).$$

Let $\underline{a}$, $\bar{a} \in \mathcal{K}(\mathcal{A})^+$ and $\underline{b}$, $\bar{b} \in \mathcal{K}(\mathcal{M})^+$ as in (4.3). Apply Lemma 4.8 to $\bar{b}s_1$ with the projection z_1 and to $\underline{b}\, s_3$ with z_3, to obtain $(\bar{b})'$, $(\underline{b})' \in \mathcal{K}(\mathcal{M})^+$, both with infinite support and such that $\|(\bar{b})' - \bar{b}\, s_1\| < \varepsilon/4$, $\|(\underline{b})' - \underline{b}\, s_3\| < \varepsilon/4$. Define

$$b' = ((\bar{b})' + \lambda^{\text{e}}_{\max}(b)(s_1 + z_1)) + (s_2 - z_1 - z_3)b - ((\underline{b})' - \lambda^{\text{e}}_{\min}(b)(s_3 + z_3)).$$

As $b = (\bar{b}\, s_1 + \lambda^{\text{e}}_{\max}(b)\, s_1) + bs_2 - (\underline{b}\, s_3 - \lambda^{\text{e}}_{\min}(b)s_3)$, we get

$$\begin{aligned}\|b' - b\| &\le \|(\bar{b})' - \bar{b}\, s_1\| + \|\lambda^{\text{e}}_{\max}(b)\, z_1 - b\, z_1\| + \|\lambda^{\text{e}}_{\min}(b)\, z_3 - b\, z_3\| + \|(\underline{b})' - \underline{b}\, s_3\| \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon.\end{aligned}$$

Note that $\lambda^{\text{e}}_{\max}(b') = \lambda^{\text{e}}_{\max}(b)$; then $\overline{b'} = (\bar{b})'$, $\underline{b'} = (\underline{b})'$ have infinite support,

$$\begin{aligned}\lambda_t(b') &= \lambda_t(\overline{b'}) + \lambda^{\text{e}}_{\max}(b') = \lambda_t((\bar{b})') + \lambda^{\text{e}}_{\max}(b) \\ &= \lambda_t(\bar{b}) + \frac{\varepsilon}{6+t} + \lambda^{\text{e}}_{\max}(b) = \lambda_t(b) + \frac{\varepsilon}{6+t}\end{aligned} \tag{4.4}$$

and similarly $\mu_t(b') = \mu_t(b) - \frac{\varepsilon}{6+t}$.

Proceeding with a in the same way we did for b, we obtain $a' \in \mathcal{A}^{\text{sa}}$ with $\|a - a'\| < \varepsilon$, with $\overline{a'}$ and $\underline{a'}$ having infinite support, and such that

$$\lambda_t(a') = \lambda_t(a) + \frac{\varepsilon}{6+t}, \quad \mu_t(a') = \mu_t(a) - \frac{\varepsilon}{6+t}, \quad t \in [0, \infty). \tag{4.5}$$

From (4.4), (4.5), and the fact that $a \prec b$, we deduce that $a' \prec b'$. □

Let $\mathcal{N}$ be a semifinite diffuse von Neumann algebra with fns trace τ. We consider the set $L^1(\mathcal{N}) \cap \mathcal{N}$, which consists of those $x \in \mathcal{N}$ with $\|x\|_1 < \infty$. The elements in $L^1(\mathcal{N}) \cap \mathcal{N}$ are necessarily compact, since $\int_0^\infty \lambda_t(|x|)\, dt < \infty$ forces $\nu_t(x) = \lambda_t(|x|) \xrightarrow[t\to\infty]{} 0$.

Lemma 4.10. *Let $\mathcal{N}$ be a semifinite diffuse von Neumann algebra with fns trace τ, and let $x \in L^1(\mathcal{N})^{\text{sa}}$, $\varepsilon > 0$. Then there exists $x' \in L^1(\mathcal{N})^{\text{sa}}$ such that*

(i) $\|x' - x\|_1 < \varepsilon$;
(ii) $\lambda_t(x') = \lambda_t(x) + \varepsilon/(10 + 4t^2)$;
(iii) $\mu_t(x') = \mu_t(x) - \varepsilon/(10 + 4t^2)$;
(iv) $\tau(p^{x'}(0, \infty)) = \infty$, $\tau(p^{x'}(-\infty, 0)) = \infty$;
(v) $p^{x'}(-\infty, 0) + p^{x'}(0, \infty) = I$.

Proof. Since x is τ-compact, its essential spectrum contains zero. Then $\lambda_t(x) \ge 0$, $\mu_t(x) \le 0$ for all t. With that in mind, the proof runs as the proof of Lemma 4.8, using the L^1 property instead of compactness to choose p_1 and considering the positive and negative parts of x separately. □

5. Schur-Horn theorems in II_∞-factors

In this section we prove versions of the Schur-Horn theorem in the σ-finite II_∞-factor $(\mathcal{M},\tau)$ (Theorems 5.5 and 5.8), in the spirit of Neumann's work [26]. We also consider versions of these results for τ-integrable operators (Theorems 5.10 and 5.12).

We begin with the following result, which comprises the main technical part of the proof of Theorem 5.5 (by allowing us to reduce the argument to a discrete case). Recall that $V(\varepsilon,\delta)$ denotes the canonical basis of neighborhoods of 0 in the measure topology, indexed by ε, $\delta>0$.

Proposition 5.1. *Let $\mathcal{A}\subset\mathcal{M}$ be a diffuse von Neumann subalgebra. Let $a\in\mathcal{A}^{\rm sa}$, $b\in\mathcal{M}^{\rm sa}$ be such that $a\prec b$ and fix $m\in\mathbb{N}$. Then there exist $\{p_n\}_{n\ge1}\subset\mathcal{P}(\mathcal{A})$, $\{q_n\}_{n\ge1}\subset\mathcal{P}(\mathcal{M})$ such that*

(i) $p_i\,p_j=q_i\,q_j=0$ *for* $i\neq j$;
(ii) $\tau(p_n)=\tau(q_n)=\tau(p_1)$ *for all* $n\in\mathbb{N}$;
(iii) $\tau(1-\sum_{n\ge1}p_n)=\tau(1-\sum_{n\ge1}q_n)<\frac{1}{m}$;
(iv) *there exist* $f,g\in\ell^\infty_{\mathbb{R}}(\mathbb{N})$ *such that:*
  (a) $f\prec g$;
  (b)
$$(a-\sum_{n\ge1}f(n)\,p_n),\ (b-\sum_{n\ge1}g(n)\,q_n)\in V(\frac{1}{m},\frac{1}{m}).$$

Proof. By Lemma 4.9 there exist $a'\in\mathcal{A}^{\rm sa}$, $b'\in\mathcal{M}^{\rm sa}$ with $\|a-a'\|<1/2m$, $\|b-b'\|<1/2m$, $a'\prec b'$, and such that $\bar a$, $\underline{a}$, $\bar b$, $\underline{b}$ (as defined in Remark 4.7) have infinite support. So, at the cost of replacing $1/m$ with $2/m$ in (ivb) above, we can assume without loss of generality that $\tau(r_1)=\tau(s_1)=\tau(r_3)=\tau(s_3)=\infty$, where $r_1,s_1,r_3,s_3\in\mathcal{P}(\mathcal{M})$ are as in the proof of Lemma 4.9.

Since $\mathcal{A}$ is diffuse, there exist complete flags $\{e_{\bar a}(t)\}_{t\in[0,\infty)}$, $\{e_{\underline{a}}(t)\}_{t\in[0,\infty)}$ in $\mathcal{A}$ over r_1 and r_3 respectively such that $\tau(e_{\bar a}(t))=\tau(e_{\underline{a}}(t))=t$ for $t\ge0$ and
$$\bar a=\int_0^\infty\lambda_s(\bar a)\ de_{\bar a}(s)\,,\ \underline{a}=\int_0^\infty\lambda_s(\underline{a})\ de_{\underline{a}}(s).$$
Similarly, there exist complete flags $\{e_{\bar b}(t)\}_{t\in[0,\infty)}$, $\{e_{\underline{b}}(t)\}_{t\in[0,\infty)}$ over s_1 and s_3 respectively such that $\tau(e_{\bar b}(t))=\tau(e_{\underline{b}}(t))=t$ for $t\ge0$ and
$$\bar b=\int_0^\infty\lambda_s(\bar b)\ de_{\bar b}(s)\,,\ \underline{b}=\int_0^\infty\lambda_s(\underline{b})\ de_{\underline{b}}(s).$$
Let $q_t=I-(e_{\bar b}(t)+e_{\underline{b}}(t))$, $p_t=I-(e_{\bar a}(t)+e_{\underline{a}}(t))$. Then $\{q_t\},\{p_t\}$ are decreasing nets of projections that converge strongly to s_2, r_2 respectively. For the rest of the proof, we will fix $t>0$ big enough so that the following three properties hold (all guaranteed by the fact that $\lambda_t(x)\to0$ as $t\to\infty$ if $x\in\mathcal{K}(\mathcal{M})$):
$$\left(\lambda^{\rm e}_{\min}(b)-\frac{1}{m}\right)q_t\ \le\ b\,q_t\ \le\ \left(\lambda^{\rm e}_{\max}(b)+\frac{1}{m}\right)q_t\,. \tag{5.1}$$
$$\left(\lambda^{\rm e}_{\min}(b)-\frac{1}{m}\right)p_t\ \le\ a\,p_t\ \le\ \left(\lambda^{\rm e}_{\max}(b)+\frac{1}{m}\right)p_t\,. \tag{5.2}$$
$$\max\{\lambda_t(\bar a),\lambda_t(\bar b),\lambda_t(\underline{a}),\lambda_t(\underline{b})\}\ <\ \frac{1}{m}. \tag{5.3}$$

Now apply [2, Lemma 3.2.] and Corollary 2.3 to $a\,e_{\bar a}(t)$ in the II_1 factor $e_{\bar a}(t)\mathcal{M}e_{\bar a}(t)$ and to $a\,e_{\underline{a}}(t)$ in the II_1-factor $e_{\underline{a}}(t)\mathcal{M}e_{\underline{a}}(t)$. This way we get $N\in\mathbb{N}$ with $N\ge t\cdot3\,m\cdot(2\,\|b\|\,m+3)$, partitions $\{p_j\}_{j=1}^N$ and $\{p'_j\}_{j=1}^N$ of $e_{\bar a}(t)$ and $e_{\underline{a}}(t)$ respectively given by
$$p_j=e_{\bar a}\left(\frac{j\,t}{N}\right)-e_{\bar a}\left(\frac{(j-1)\,t}{N}\right)\,,\ p'_j=e_{\underline{a}}\left(\frac{j\,t}{N}\right)-e_{\underline{a}}\left(\frac{(j-1)\,t}{N}\right)\,,\ 1\le j\le N,$$

and coefficients $\alpha_1' \geq \alpha_2' \geq \cdots \geq \alpha_N'$, $\alpha_1'' \geq \alpha_2'' \geq \cdots \geq \alpha_N''$ given by

$$\alpha_j' = \frac{N}{t}\int_{(j-1)t/N}^{jt/N} \lambda_s(ae_{\bar{a}}(t))\,ds = \frac{N}{t}\tau(ap_j), \quad \alpha_j'' = \frac{N}{t}\tau(ap_j'),$$

such that

$$(a\,e_{\bar{a}}(t) - \sum_{j=1}^N \alpha_j'\,p_j)\,,\ (a\,e_{\underline{a}}(t) - \sum_{j=1}^N \alpha_j''\,p_j')\ \in V(\frac{1}{m},\frac{1}{2\,m}) \tag{5.4}$$

(recall that $\|x\|_{(1)} \leq \|x\|_1$ and that if $\|x\|_{(1)} < 1/4m^2$, then $x \in V(1/2m, 1/2m)$; see the proof of Proposition 2.2). Similarly, we obtain for b partitions $\{q_j\}_{j=1}^N$ and $\{q_j'\}_{j=1}^N$ of $e_{\bar{b}}(t)$ and $e_{\underline{b}}(t)$ respectively such that

$$q_j = e_{\bar{b}}\left(\frac{j\,t}{N}\right) - e_{\bar{b}}\left(\frac{(j-1)\,t}{N}\right)\,,\ q_j' = e_{\underline{b}}\left(\frac{j\,t}{N}\right) - e_{\underline{b}}\left(\frac{(j-1)\,t}{N}\right)\,,\ 1 \leq j \leq N,$$

and coefficients $\beta_1' \geq \beta_2' \geq \cdots \geq \beta_N'$, $\beta_1'' \geq \beta_2'' \geq \cdots \geq \beta_N''$ given by

$$\beta_j' = \frac{N}{t}\tau(bq_j), \quad \beta_j'' = \frac{N}{t}\tau(bq_j')$$

with

$$(b\,e_{\bar{b}}(t) - \sum_{j=1}^N \beta_j'\,q_j)\,,\ (b\,e_{\underline{b}}(t) - \sum_{j=1}^N \beta_j''\,q_j')\ \in V(\frac{1}{m},\frac{1}{2\,m})\,. \tag{5.5}$$

Consider now a partition $\{I_j\}_{j=1}^L$ of $[\lambda_{\min}^{\mathrm{e}}(b) - \frac{1}{m}, \lambda_{\max}^{\mathrm{e}}(b) + \frac{1}{m}]$ into L consecutive disjoint sub-intervals with $2 \leq L \leq 2\,\|b\|\,m+3$, with $I_1 = [\lambda_{\min}^{\mathrm{e}}(b) - \frac{1}{m}, \lambda_{\min}^{\mathrm{e}}(b))$, $I_L = (\lambda_{\max}^{\mathrm{e}}(b), \lambda_{\max}^{\mathrm{e}}(b) + \frac{1}{m}]$, and such that the length of each I_j is no greater than $\frac{1}{m}$. Define

$$a_e = p_t\,a, \quad b_e = q_t\,b.$$

Let $\gamma_1 = \lambda_{\min}^{\mathrm{e}}(b)$, $\gamma_L = \lambda_{\max}^{\mathrm{e}}(b)$, and choose $\gamma_j \in I_j$ for $2 \leq j \leq L-1$. The choice of the γ_j, together with (5.1) and (5.2), imply that

$$\|\,a_e - \sum_{j=1}^L \gamma_j\,p^{a_e}(I_j)\,\| < \frac{1}{m}\,, \quad \|\,b_e - \sum_{j=1}^L \gamma_j\,p^{b_e}(I_j)\,\| < \frac{1}{m}\,. \tag{5.6}$$

For $j \in \{1, \ldots, L\}$ let

$$t_j^a = \begin{cases} \lfloor \frac{\tau(p^{a_e}(I_j))\,N}{t} \rfloor & \text{if } \tau(p^{a_e}(I_j)) < \infty \\ \infty & \text{if } \tau(p^{a_e}(I_j)) = \infty\,, \end{cases}$$

where $\lfloor x \rfloor$ denotes the integer part of $x \in \mathbb{R}$. We construct $\{t_j^b\}_{j=1}^L$ in the same way. For each j, if $t_j^a = \infty$ we consider a partition $\{p_i^{(j)}\}_{i\in\mathbb{N}} \subset \mathcal{P}(\mathcal{A})$ of $p^{a_e}(I_j)$ with $\tau(p_i^{(j)}) = \frac{t}{N}$ for all $i \in \mathbb{N}$; otherwise, if $t_j^a < \infty$, we consider a partition $\{p_i^{(j)}\}_{i=1}^{t_j^a+1} \subset \mathcal{P}(\mathcal{A})$ with $\tau(p_i^{(j)}) = \frac{t}{N}$ for $1 \leq i \leq t_j^a$, and $\tau(p_{t_j^a+1}^{(j)}) < \frac{t}{N}$.

Analogously, we consider partitions $\{q_i^{(j)}\}_i \subset \mathcal{P}(\mathcal{M})$ of $p^{b_e}(I_j)$ for $1 \leq j \leq L$. Since $\bar{b}$ and $\underline{b}$ have infinite support,

$$t_1^b = t_L^b = \infty\,, \quad \lambda_{\min}^{\mathrm{e}}(b) \leq \min_{1\leq j\leq L} \gamma_j \leq \max_{1\leq j\leq L} \gamma_j \leq \lambda_{\max}^{\mathrm{e}}(b) \tag{5.7}$$

and there exists $i_0 \in \{1, \ldots, L\}$ with $t_{i_0}^a = \infty$. As $L \leq 2\|b\|m+3$, $N \geq t \cdot 3\,m \cdot (2\,\|b\|\,m+3)$,

$$\sum_{j:\ t_j^a<\infty} \tau(p_{t_j^a+1}^{(j)}) \leq \sum_{i=1}^L \frac{t}{N} \leq \frac{1}{3\,m}\,, \quad \sum_{j:\ t_j^b<\infty} \tau(q_{t_j^b+1}^{(j)}) \leq \frac{1}{3\,m}. \tag{5.8}$$

We can assume that the two projections $\sum_{j:\ t_j^a<\infty} p_{t_j^a+1}^{(j)}$, $\sum_{j:\ t_j^b<\infty} q_{t_j^b+1}^{(j)}$ have equal trace; indeed we can take the necessary mass (which will be certainly less than $1/2m$) from one of the projections $p^{a_e}(I_{i_0})$,

$p^{b_e}(I_L)$ respectively (since each of them is an infinite projection) before considering the partitions of these projections (this, at the cost of replacing the "$\|\cdot\| < 1/m$" in (5.6) by "$\in V(1/m, 1/2m)$"). From (5.6) and (5.8),

$$(a_e - \sum_{j=1}^{L} \gamma_j \sum_{i=1}^{t_j^a} p_i^{(j)}) , \quad (b_e - \sum_{j=1}^{L} \gamma_j \sum_{i=1}^{t_j^b} q_i^{(j)}) \in V(\frac{1}{m}, \frac{1}{m}) . \tag{5.9}$$

Let $\{(\alpha_i, p_i)\}_{i\geq 1}$ be an enumeration of the countable set

$$\{(\alpha'_j, p_j) : 1 \leq j \leq N\} \cup \{(\alpha''_j, p'_j) : 1 \leq j \leq N\} \\ \cup \{(\gamma_j, p_i^{(j)}) : 1 \leq j \leq L \,,\, 1 \leq i \leq t_j^a\}$$

and let $\{(\beta_i, q_i)\}_{i\geq 1}$ be an enumeration of the countable set

$$\{(\beta'_j, q_j) : 1 \leq j \leq N\} \cup \{(\beta''_j, q'_j) : 1 \leq j \leq N\} \\ \cup \{(\gamma_j, q_i^{(j)}) : 1 \leq j \leq L \,,\, 1 \leq i \leq t_j^b\}.$$

By construction, $\{p_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$. It also follows that (i), (ii), and (iii) in the statement of the Theorem hold. Moreover, from (5.4), (5.5) and (5.9) we get part b) of (iv) (with $f = \{\alpha_n\}_{n\geq 1}$, $g = \{\beta_n\}_{n\geq 1}$). It remains to show that $f \prec g$ in the sense of Definition 3.1. We will only prove that $U_k(f) \leq U_k(g)$ for $k \geq 1$, since the L_k inequalities follow in a similar way. We have

$$U_k(g) = \begin{cases} \sum_{i=1}^k \beta'_j & \text{if } 1 \leq k \leq N\,, \\ \sum_{i=1}^N \beta'_j + (k-N)\lambda^{\mathrm{e}}_{\max}(b) & \text{if } N < k \end{cases}$$

(recall that $\gamma_L = \lambda^{\mathrm{e}}_{\max}(b)$ and that there is an infinity of γ_L in the list $\{\beta_n\}$)). For $U_k(f)$ we get

$$U_k(f) = \begin{cases} \sum_{i=1}^k \alpha'_j & \text{if } 1 \leq k \leq N\,, \\ \sum_{i=1}^N \alpha'_j + \sum_{i=N+1}^k \gamma_{\sigma(i)} & \text{if } N < k \end{cases}$$

for appropriate choices $\sigma(i) \in \{1, \dots, L\}$. If $1 \leq k \leq N$, then

$$\begin{aligned} U_k(g) &= \sum_{i=1}^k \beta'_i = \frac{N}{t} \int_0^{\frac{kt}{N}} \lambda_s(b)\, ds = \frac{N}{t}\, U_{kt/N}(b) \\ &\geq \frac{N}{t}\, U_{kt/N}(a) = \frac{N}{t} \int_0^{\frac{kt}{N}} \lambda_s(a)\, ds = \sum_{i=1}^k \alpha'_i = U_k(f). \end{aligned}$$

If $N < k$,

$$\begin{aligned} U_k(g) &= \frac{N}{t} \int_0^t \lambda_s(b)\, ds + (k-N)\lambda^{\mathrm{e}}_{\max}(b) \\ &\geq \frac{N}{t} \int_0^t \lambda_s(a)\, ds + \sum_{i=N+1}^k \gamma_{\sigma(i)} = U_k(f) \end{aligned}$$

since, by (5.7), $\gamma_{\sigma(i)} \leq \lambda^{\mathrm{e}}_{\max}(b)$ for all i. □

Remark 5.2. Let $\mathcal{A} \subset \mathcal{M}$ be a diffuse von Neumann subalgebra. Fix $a \in \mathcal{A}^+$, $b \in \mathcal{M}^+$ such that $a \prec_w b$ and let $m \in \mathbb{N}$. Then a slightly modified version of the proof of Proposition 5.1 (with $r_3 = s_3 = 0$, $\lambda^{\mathrm{e}}_{\min}(b) = \lambda^{\mathrm{e}}_{\min}(a) = 0$) shows that there exist $\{p_n\}_{n\geq 1} \subset \mathcal{P}(\mathcal{A})$, $\{q_n\}_{n\geq 1} \subset \mathcal{P}(\mathcal{M})$ and $f, g \in \ell^\infty(\mathbb{N})^+$ such that conditions (i)-(iii) and (ivb) hold, and such that $f \prec_w g$. We will use these facts for the proof of the contractive Schur-Horn theorem (Theorem 5.8).

Lemma 5.3. *Let $\mathcal{N} \subset \mathcal{M}$ be a von Neumann subalgebra, with $E_{\mathcal{N}}$ the unique trace-preserving conditional expectation onto $\mathcal{N}$. Let $\{p_j\}_{j\in\mathbb{N}} \subset \mathcal{Z}(\mathcal{N})$ be a family of mutually orthogonal projections, pairwise* *equivalent in $\mathcal{M}$. Let $\{e_{ij}\}$ be a system of matrix units in $B(H)$. Then there exists a (possibly non-unital) normal *-monomorphism $\pi : B(H) \to \mathcal{M}$ such that*

$$\pi(e_{jj}) = p_j, \quad j \in \mathbb{N}, \tag{5.10}$$

$$E_{\mathcal{N}}(\pi(x)) = \pi(P_D(x)), \quad x \in B(H). \tag{5.11}$$

Proof. Let $p = \sum_j p_j$. Since the projections $\{p_j\}_j$ are mutually orthogonal and equivalent, they can be extended to a system of matrix units $\{p_{ij}\}_{i,j\in\mathbb{N}}$ in $p\mathcal{M}p$, with $p_{ij}^* = p_{ji}$, $p_{ij}p_{kh} = \delta_{jk}\, p_{ih}$, $p_{jj} = p_j$ for all $i, j, k, h \in \mathbb{N}$. Also, it is easy to check (using that $p_j \in \mathcal{Z}(\mathcal{N})$ for all j) that $E_{\mathcal{N}}(p_{ij}) = \delta_{ij}\, p_j$.

Since p is an infinite projection, $p\mathcal{M}p$ is a II_∞-factor. It is standard that $p\mathcal{M}p \simeq B(H) \otimes p_{11}\mathcal{M}p_{11}$ via the (normal) *-isomorphism

$$\eta : y \mapsto \sum_{i,j} e_{ij} \otimes p_{1i}\; y\; p_{j1}.$$

In particular, $\eta(p_{ij}) = e_{ij} \otimes p_{11}$. Now let $\pi : B(H) \to \mathcal{M}$ be given by $\pi(x) = \eta^{-1}(x \otimes p_{11})$. So $\pi(e_{ij}) = p_{ij}$ for all i, j. For any $x \in B(H)$, we have $x = \sum_{i,j} x_{ij}\, e_{ij}$ for coefficients $x_{ij} \in \mathbb{C}$. If $\pi(x) = 0$, then for all i, j we have

$$0 = p_{ij}\, \pi(x)\, p_{ij} = \pi(e_{ij})\, \pi(x)\, \pi(e_{ij}) = \pi(e_{ij}\, x\, e_{ij}) = \pi(x_{ij}\, e_{ij}) = x_{ij}\, p_{ij}.$$

So $x_{ij} = 0$ for all i, j and this shows that π is a monomorphism. Finally, using the normality of π and $E_{\mathcal{N}}$,

$$\begin{aligned} \pi(P_D(x)) &= \pi\left(\sum_j x_{jj}\; e_{jj}\right) = \sum_j x_{jj}\; \pi(e_{jj}) = \sum_j x_{jj}\; p_{jj} \\ &= \sum_{i,j} x_{ij}\; E_{\mathcal{N}}(p_{ij}) = E_{\mathcal{N}}\left(\sum_{i,j}\; x_{ij}\; p_{ij}\right) = E_{\mathcal{N}}(\pi(x)). \end{aligned}$$ □

The characterization of U_t in Lemma 4.1 allows us to prove that conditional expectations are "contractive" from a majorization point of view:

Lemma 5.4. *Let $\mathcal{A} \subset \mathcal{M}$ be a diffuse abelian von Neumann subalgebra. Then, for every $b \in \mathcal{M}^{\mathrm{sa}}$, we have $E_{\mathcal{A}}(b) \prec b$.*

Proof. Fix $t > 0$ and let $\varepsilon > 0$. Then we can apply Lemma 4.1 in $\mathcal{A}$ to get a projection $q \in \mathcal{P}(\mathcal{A})$ with $\tau(q) = t$ and such that $U_t(E_{\mathcal{A}}(b)) \le \tau(E_{\mathcal{A}}(b)\, q) + \varepsilon$. Since $\tau(E_{\mathcal{A}}(b)\, q) = \tau(E_{\mathcal{A}}(b\, q)) = \tau(b\, q) \le U_t(b)$, we conclude that $U_t(E_{\mathcal{A}}(b)) \le U_t(b) + \varepsilon$ for all $\varepsilon > 0$; so, $U_t(E_{\mathcal{A}}(b)) \le U_t(b)$. Applying the same proof to $-b$, we get $L_t(E_{\mathcal{A}}(b)) = -U_t(E_{\mathcal{A}}(-b)) \ge -U_t(-b)) = L_t(b)$. As t was arbitrary, we get $E_{\mathcal{A}}(b) \prec b$. □

We are finally in position to state and prove our main theorem.

Theorem 5.5 (A Schur-Horn theorem for II_∞-factors)**.** *Let $\mathcal{A} \subset \mathcal{M}$ be a diffuse abelian von Neumann subalgebra. Then, for any $b \in \mathcal{M}^{\mathrm{sa}}$,*

$$\overline{E_{\mathcal{A}}(\mathcal{U}_{\mathcal{M}}(b))}^{\mathcal{T}} = \{a \in \mathcal{A}^{\mathrm{sa}} :\; a \prec b\}.$$

Proof. By Proposition 4.6 and Lemma 5.4, $\overline{E_{\mathcal{A}}(\mathcal{U}_{\mathcal{M}}(b))}^{\mathcal{T}} \subset \{a \in \mathcal{A} :\; a \prec b\}$. To show the reverse inclusion, fix $a \in \mathcal{A}^{\mathrm{sa}}$ with $a \prec b$ and fix $m \in \mathbb{N}$. Applying Proposition 5.1 to a, b we obtain sequences $f = \{\alpha_n\}$, $g = \{\beta_n\} \subset \ell^\infty_{\mathbb{R}}(\mathbb{N})$, $\{p_n\} \subset \mathcal{P}(\mathcal{A})$, $\{q_n\} \subset \mathcal{P}(\mathcal{M})$ with

$$p_i\; p_j = q_i\; q_j = 0 \text{ if } i \ne j;\quad \tau(p_1) = \tau(p_j) = \tau(q_j) \text{ for all } j; \tag{5.12}$$

$$\tau(1 - \sum_{n\ge1} p_n) = \tau(1 - \sum_{n\ge1} q_n) < \frac{1}{m}; \tag{5.13}$$

$$(a - \sum_{n\ge1} \alpha_n\, p_n),\ (b - \sum_{n\ge1} \beta_n\, q_n) \in V(\frac{1}{m}, \frac{1}{m}); \tag{5.14}$$

$$f \prec g.$$

By Theorem 3.3 there exists a unitary $v \in B(H)$ such that

$$\|M_f - P_D(v\, M_g\, v^*)\| < \frac{1}{m}.$$

The conditions on the projections in (5.12) and (5.13) guarantee that we can choose $w \in \mathcal{U}_\mathcal{M}$ with $w\, q_n\, w^* = p_n$ for all n. Let $p = \sum_n p_n$, $q = \sum_n q_n$; then by (5.13) there exists a partial isometry $z \in \mathcal{M}$ with $z^*z = p^\perp$, $zz^* = q^\perp$. Let u be the unitary $u = (\pi(v) + z)\, w$, where π is the *-monomorphism from Lemma 5.3 with respect to the projections $\{p_n\}_n$. From (5.14),

$$a - \pi(M_f) \in V(\frac{1}{m}, \frac{1}{m}), \quad w\, b\, w^* - \pi(M_g) \in V(\frac{1}{m}, \frac{1}{m}).$$

Note that by (5.13) we have $\tau(p^\perp) < 1/m$, $\tau(q^\perp) < 1/m$, so $z, z^* \in V(\varepsilon, 1/m)$ for any $\varepsilon > 0$. From this we conclude that

$$(\pi(v) + z)\ \pi(M_g)\ (\pi(v) + z)^* - \pi(vM_gv^*) \in V(\varepsilon, \frac{2}{m}), \quad \varepsilon > 0.$$

It follows that

$$u\, b\, u^* - \pi(v\, M_g\, v^*) \in V(\frac{2}{m}, \frac{3}{m}).$$

Letting m vary all along $\mathbb{N}$, we have constructed sequences of unitaries $\{u_m\}_m \subset \mathcal{M}$ and $\{v_m\}_m \subset \mathcal{U}(H)$, and sequences $\{f_m\}_m, \{g_m\}_m \subset \ell^\infty_\mathbb{R}(\mathbb{N})$ with

$$\pi(M_{f_m}) - a \xrightarrow[m\to\infty]{\mathcal{T}} 0, \quad M_{f_m} - P_D(v_m\, M_{g_m}\, v_m^*) \xrightarrow[m\to\infty]{\|\ \|} 0, \tag{5.15}$$

$$u_m\, b\, u_m^* - \pi(v_m\, M_{g_m}\, v_m^*) \xrightarrow[m\to\infty]{\mathcal{T}} 0.$$

Using that π is a *-monomorphism, the $\mathcal{T}$-continuity of $E_\mathcal{A}$ (Corollary 2.4) and the fact that $E_\mathcal{A} \circ \pi = \pi \circ P_D$ (Lemma 5.3) we get from (5.15) that

$$\pi(M_{f_m}) - \pi(P_D(v_m\, M_{g_m}\, v_m^*)) \xrightarrow[m\to\infty]{\|\ \|} 0, \tag{5.16}$$

$$E_\mathcal{A}(u_m\, b\, u_m^*) - \pi(P_D(v_m\, M_{g_m}\, v_m^*)) \xrightarrow[m\to\infty]{\mathcal{T}} 0. \tag{5.17}$$

From (5.15), (5.16), and (5.17),

$$E(u_m\, b\, u_m^*) - a \xrightarrow[m\to\infty]{\mathcal{T}} 0.$$

That is, $a \in \overline{E_\mathcal{A}(\mathcal{U}_\mathcal{M}(b))}^{\mathcal{T}}$. □

Remark 5.6. It is natural to ask whether one can remove the closure bar in the description of the set $\{a \in \mathcal{A}^{\mathrm{sa}} : a \prec b\}$ given in Theorem 5.5. Next we show an example in which

$$E_\mathcal{A}(\mathcal{U}_\mathcal{M}(b)) \subset E_\mathcal{A}(\overline{\mathcal{U}_\mathcal{M}(b)}^{\mathcal{T}}) \subsetneq \overline{E_\mathcal{A}(\mathcal{U}_\mathcal{M}(b))}^{\mathcal{T}}.$$

This implies that the characterization of $\{a \in \mathcal{A}^{\mathrm{sa}} : a \prec b\}$ given in Theorem 5.5 cannot be strengthened in the II_∞ case.

We consider $p \in \mathcal{P}(\mathcal{M})$ an infinite projection with $p^\perp$ also infinite. Then $U_t(p) = t$, $L_t(p) = 0$ for all t. Since $U_t(I) = t$, $L_t(I) = t$, we have $I \prec p$; then

$$I \in \overline{E_\mathcal{A}(\mathcal{U}_\mathcal{M}(p))}^{\mathcal{T}} \quad \text{but} \quad I \notin E_\mathcal{A}(\overline{\mathcal{U}_\mathcal{M}(p)}^{\mathcal{T}}). \tag{5.18}$$

Indeed, Theorem 5.5 guarantees the claim to the left in (5.18). On the other hand, assume that there exists $x \in \overline{\mathcal{U}_\mathcal{M}(p)}^{\mathcal{T}}$ with $I = E_\mathcal{A}(x)$. By Corollary 2.4, $0 \le x \le I$ and then

$$0 = \tau(I - E_\mathcal{A}(x)) = \tau(E_\mathcal{A}(I - x)) = \tau(I - x).$$

This last fact implies that $I = x \in \overline{\mathcal{U}_{\mathcal{M}}(p)}^{\mathcal{T}}$ by the faithfulness of τ. But as $\|\cdot\|_{(1)}$ is a unitarily invariant norm, for any $u \in \mathcal{U}_{\mathcal{M}}$ we get

$$\|I - u\,p\,u^*\|_{(1)} = \|u\,(I-p)\,u^*\|_{(1)} = \|I-p\|_{(1)} > 0$$

as $p \neq I$. Since $\|\cdot\|_{(1)}$ is $\mathcal{T}$-continuous (see Proposition 2.2), there is positive distance from I to the $\mathcal{T}$-closure of the unitary orbit of p, a contradiction.

It would be interesting to have a description of the set $E_{\mathcal{A}}(\overline{\mathcal{U}_{\mathcal{M}}(b)}^{\mathcal{T}})$ for an abelian diffuse von Neumann subalgebra of a general σ-finite semifinite factor $(\mathcal{M},\tau)$. But even in the I_∞ factor case this problem is known to be hard (see [19, Thm 15], [6, 7] for further discussion). In the II_1-factor case Arveson and Kadison [7] conjectured that

$$E_{\mathcal{A}}\left(\overline{\mathcal{U}_{\mathcal{M}}(b)}^{\mathcal{T}}\right) = \{a \in \mathcal{A}^{\mathrm{sa}} : a \prec b\}\,, \tag{5.19}$$

which is still an open problem (see [2, 3, 5] for a detailed discussion). □

The following result shows that the notion of majorization in $\mathcal{M}^{\mathrm{sa}}$ from Definition 4.4 coincides with the majorization introduced by Hiai in [16]. Thus, several other characterizations of majorization can be obtained from Hiai's work. Following Hiai, we say that a map is *doubly stochastic* if it is unital, positive and preserves the trace.

Corollary 5.7. *Let $\mathcal{A} \subset \mathcal{M}$ be a diffuse abelian von Neumann subalgebra and let $a, b \in \mathcal{M}^{\mathrm{sa}}$. Then the following statements are equivalent:*

(i) *$a \prec b$;*
(ii) *$a \in \overline{E_{\mathcal{A}}(\mathcal{U}_{\mathcal{M}}(b))}^{\mathcal{T}}$;*
(iii) *$a \in \overline{\mathrm{conv}\,\{\mathcal{U}_{\mathcal{M}}(b)\}}^{\mathcal{T}}$;*
(iv) *there exists a doubly stochastic map F on $\mathcal{M}$ with $a = F(b)$;*
(v) *there exists a completely positive doubly stochastic map F on $\mathcal{M}$ with $a = F(b)$;*
(vi) *$\tau(f(a)) \le \tau(f(b))$ for every convex function $f : I \to [0,\infty)$ with $\sigma(a), \sigma(b) \subset I$.*
(vii) *a is spectrally majorized by b in the sense of* [16].

Proof. By Theorem 5.5, (i) and (ii) are equivalent. The statements (iii)-(vii) are mutually equivalent by [16, Theorem 2.2]. Also, (iii) implies (i) by Proposition 4.6. So it will be enough to show that (i) implies (iv).

Let $a \in \mathcal{A}$ with $a \prec b$. By Theorem 5.5, there exist unitaries $\{u_j\} \subset \mathcal{M}$ such that $a = \lim_{\mathcal{T}} E_{\mathcal{A}}(u_j b u_j^*)$. Consider the sequence of completely positive contractions $E_{\mathcal{A}}(u_j \cdot u_j^*) : \mathcal{M} \to \mathcal{A}$; by compactness in the BW topology [28, Theorem 7.4], this sequence admits a convergent (pointwise ultraweakly) subnet $\{E_{\mathcal{A}}(u_{j_k} \cdot u_{j_k}^*)\}$. Let F be the limit of such subnet. Since $a = \lim_{\mathcal{T}} E_{\mathcal{A}}(u_j b u_j^*)$ and $F(b) = \lim_{\sigma-\mathrm{wot}} E_{\mathcal{A}}(u_{j_k} b u_{j_k}^*)$, we conclude (mimicking the argument in the proof of Lemma 3.3 in [16]) that $F(b) = a$. It is easy to check that F is unital and that it preserves the trace. □

We finish this section with contractive and L^1 analogs of Theorem 5.5.

Theorem 5.8. *Let $\mathcal{A} \subset \mathcal{M}$ be a diffuse abelian von Neumann subalgebra and let $b \in \mathcal{M}^+$. Then*

$$\overline{E_{\mathcal{A}}(\{c\,b\,c^* : \|c\| \le 1\})}^{\mathcal{T}} = \{a \in \mathcal{A}^+ :\ a \prec_w b\}. \tag{5.20}$$

Proof. If $c \in \mathcal{M}$ is a contraction, then $\lambda_t(c\,b\,c^*) \le \lambda_t(b)$ [13, Lemma 2.5]. So $c\,b\,c^* \prec_w b$ and then Lemmas 5.4 and 4.3 give the inclusion "$\subset$" above.

For the reverse inclusion, the proof runs exactly as that of Theorem 5.5, but instead of using Proposition 5.1 and (3.5) to obtain a sequence of unitary operators in $\mathcal{M}$, we use (3.11) and Remark 5.2 to obtain a convenient sequence of contractions in $\mathcal{M}$. □

Remark 5.9. The positivity condition in Theorem 5.8 cannot be relaxed to selfadjointness. As a trivial example, take $b = 0$; then $-I \prec_w b$, but $c\,b\,c^* = 0$ for all c, so the set on the left in (5.20) is $\{0\}$.

Recall that $L^1(\mathcal{M}) \cap \mathcal{M}$ consists of those $x \in \mathcal{M}$ with $\tau(|x|) < \infty$, and that such elements are necessarily τ-compact.

Theorem 5.10. *Let $\mathcal{A} \subset \mathcal{M}$ be a diffuse abelian von Neumann subalgebra and let $b \in L^1(\mathcal{M}) \cap \mathcal{M}^{\rm sa}$. Then*

$$\overline{E_{\mathcal{A}}(\mathcal{U}_{\mathcal{M}}(b))}^{\|\cdot\|_1} = \{a \in L^1(\mathcal{M}) \cap \mathcal{A}^{\rm sa} : \ a \prec b, \ \tau(a) = \tau(b)\}$$

Proof. Proposition 4.6 together with Lemma 5.4 show that $E_{\mathcal{A}}(\mathcal{U}_{\mathcal{M}}(b)) \subset \{a \in \mathcal{A}^{\rm sa} : \ a \prec b, \ \tau(a) = \tau(b)\}$. Then Lemma 4.3 and the $\|\cdot\|_1$-continuity of the trace imply the inclusion of the corresponding closure.

Conversely, suppose that $a \prec b$ and $\tau(a) = \tau(b)$. First assume that $b \in \mathcal{M}^+$. Then $a \in \mathcal{A}^+$. By Theorem 5.5, there exists a sequence of unitaries $\{u_j\}$ such that $E_{\mathcal{A}}(u_j\, b\, u_j^*) \xrightarrow{\mathcal{T}} a$. Since b is positive, $\|E_{\mathcal{A}}(u_j\, b\, u_j^*)\|_1 = \tau(E_{\mathcal{A}}(u_j\, b\, u_j^*)) = \tau(b) = \tau(a) = \|a\|_1$. Then [13, Theorem 3.7] guarantees that $\|E_{\mathcal{A}}(u_j\, b\, u_j^*) - a\|_1 \to 0$.

If b is not positive, we apply Lemma 4.10 to obtain $a' \in \mathcal{A}$, $b' \in \mathcal{M}$, with

(i) $a' \prec b'$,
(ii) $\|a' - a\|_1 < \varepsilon$, $\|b' - b\|_1 < \varepsilon$;
(iii) $\tau(p^{a'}(0,\infty)) = \tau(p^{b'}(0,\infty)) = \infty$;
(iv) $\tau(p^{a'}(-\infty,0)) = \tau(p^{b'}(-\infty,0)) = \infty$;
(v) $p^{a'}(-\infty,0) + p^{a'}(0,\infty) = p^{b'}(-\infty,0) + p^{b'}(0,\infty) = I$.

Let $r_1 = p^{a'_+}(0,\infty)$, $r_2 = p^{a'_-}(0,\infty)$. The last three conditions above guarantee that we can find a unitary $v \in \mathcal{U}_{\mathcal{M}}$ with

$$v\,(p^{b'_+}(0,\infty))\,v^* = r_1, \ v\,(p^{b'_-}(0,\infty))\,v^* = r_2.$$

Let $b'' = vb'v^*$. Then $a' \prec b''$. Since both are τ-compact, we deduce that $a'_+ \prec b''_+$, $a'_- \prec b''_-$. Note that $a'_+, b''_+ \in r_1\mathcal{M}r_1$, $a'_-, b''_- \in r_2\mathcal{M}r_2$. As both $r_1, r_2 \in \mathcal{A}$ are infinite projections, the factors $r_1\mathcal{M}r_1$ and $r_2\mathcal{M}r_2$ are II_∞. So we can apply the first part of the proof to obtain unitaries $\{u_j^{(1)}\} \subset \mathcal{U}(r_1\mathcal{M}r_1)$, $\{u_j^{(2)}\} \subset \mathcal{U}(r_2\mathcal{M}r_2)$, with

$$\|E_{\mathcal{A}}(u_j^{(1)}\ b''_+\ (u_j^{(1)})^*) - a'_+\|_1 \to 0, \quad \|E_{\mathcal{A}}(u_j^{(2)}\ b''_-\ (u_j^{(2)})^*) - a'_-\|_1 \to 0$$

Since $r_1 + r_2 = I$, $r_1 r_2 = 0$, the operators $u_j = (u_j^{(1)} + u_j^{(2)})v$ are unitaries in $\mathcal{M}$. Then

$$\begin{aligned}\|E_{\mathcal{A}}(u_j\, b\, u_j^*) - a\|_1 &\leq \|E_{\mathcal{A}}(u_j\, b\, u_j^*) - E_{\mathcal{A}}(u_j\, b'\, u_j^*)\|_1 + \|E_{\mathcal{A}}(u_j\, b'\, u_j^*) - a'\|_1 + \|a' - a\|_1 \\ &\leq \|b' - b\|_1 + \|a' - a\|_1 + \|E_{\mathcal{A}}(u_j^{(1)}\, b''\, (u_j^{(1)})^*) - a'_+\|_1 \\ &\quad + \|E_{\mathcal{A}}(u_j^{(2)}\, b''\, (u_j^{(2)})^*) - a'_-\|_1 \\ &\leq 2\varepsilon + \|E_{\mathcal{A}}(u_j^{(1)}\, b''_+\, (u_j^{(1)})^*) - a'_+\|_1 + \|E_{\mathcal{A}}(u_j^{(2)}\, b''_-\, (u_j^{(2)})^*) - a'_-\|_1.\end{aligned}$$

So $\limsup_j \|E_{\mathcal{A}}(u_j\, b\, u_j^*) - a\|_1 < 2\varepsilon$, and as ε was arbitrary we conclude that $\lim_j \|E_{\mathcal{A}}(u_j\, b\, u_j^*) - a\|_1 = 0$, i.e. $a \in \overline{E_{\mathcal{A}}(\mathcal{U}_{\mathcal{M}}(b))}^{\|\cdot\|_1}$. ☐

Remark 5.11. The condition $\tau(a) = \tau(b)$ in Theorem 5.10 cannot be removed because of the $\|\cdot\|_1$-continuity of the trace τ. Actually, below we characterize the case where the trace restriction is removed but only in the case of positive operators.

Theorem 5.12. *Let $\mathcal{A} \subset \mathcal{M}$ be a diffuse abelian von Neumann subalgebra and let $b \in L^1(\mathcal{M}) \cap \mathcal{M}^+$. Then*

$$\overline{E_{\mathcal{A}}(\{c\, b\, c^* : \|c\| \leq 1\})}^{\|\cdot\|_1} = \{a \in \mathcal{A}^+ : \ a \prec_w b\} = \{a \in \mathcal{A}^+ : \ a \prec b\}.$$

Proof. If $b \in L^1(\mathcal{M}) \cap \mathcal{M}^+$ and $a \prec_w b$ then, since $\lambda_t(b) \in L^1(\mathbb{R}^+)$, we get $\lambda_t(a) \in L^1(\mathbb{R}^+)$. In particular, $a \in \mathcal{K}(\mathcal{M})^+$. Thus, the second equality is immediate from the fact that for positive τ-compact operators one has $L_t = 0$. So for the rest of the proof we focus on the first equality.

The inclusion "$\subset$" is obtained by combining the arguments at the beginning of the proofs of Theorems 5.8 and 5.10.

Conversely, let $a \prec_w b$ for some $a \in \mathcal{A}^+$ (so that $a \in \mathcal{K}(\mathcal{A})^+$). We write both a and b in terms of complete flags in $\mathcal{A}$ and $\mathcal{M}$ respectively, i.e.

$$a = \int_0^\infty \lambda_t(a)\, de_a(t), \quad b = \int_0^\infty \lambda_t(b)\, de_b(t),$$

with $e_a(t) \in \mathcal{A}$ for all t (this can be done since $\mathcal{A}$ is diffuse). Then $a \prec_w b$ means that, for any $s > 0$, $\int_0^s \lambda_t(a)\, dt \le \int_0^s \lambda_t(b)\, dt$. For each $s > 0$, let $p_s = e_a(s) \vee e_b(s)$, a finite projection. So we have $ae_a(s) \prec_w be_b(s)$ in the II_1-factor $p_s\mathcal{M}p_s$. By [3, Theorem 3.4], there exists a contraction $c_s \in p_s\mathcal{M}p_s \subset \mathcal{M}$ with

$$k_s := \tau_s(|a\, e_a(s) - E_{\mathcal{A}e_a(s)}\,(c_s\, e_b(s)\, b\, e_b(s)\, c_s^*)|) < \frac{1}{\tau(p_s)^2}.$$

The trace τ_s is given by $\tau_s = \tau/\tau(p_s)$; using the fact that $e_a(s) \in \mathcal{A}$ and that $\mathcal{A}$ is abelian, we get that $E_{\mathcal{A}\, e_a(s)}(\cdot) = e_a(s)\, E_{\mathcal{A}}(\cdot)$. So

$$\tau(|a\, e_a(s) - E_{\mathcal{A}}(e_a(s)\, c_s\, e_b(s)\, b\, e_b(s)\, c_s^*\, e_a(s))|) = \tau(p_s)\, k_s < \frac{1}{\tau(p_s)} \le \frac{1}{s}$$

(note that $p_s \ge e_a(s)$, so $\tau(p_s) \ge s$). Let $\varepsilon > 0$; fix $s > 0$ such that $s > 2/\varepsilon$ and $\int_s^\infty \lambda_t(a)\, dt < \varepsilon/2$. Put $c = e_a(s)\, c_s\, e_b(s)$, a contraction in $\mathcal{M}$. Then

$$\begin{aligned}\|a - E_{\mathcal{A}}(c\, b\, c^*)\|_1 &\le \|a - a\, e_a(s)\|_1 + \|a\, e_a(s) - E_{\mathcal{A}}(e_a(s)\, c_s\, e_b(s)\, b\, e_b(s)\, c_s^*\, e_a(s))\|_1 \\ &= \int_s^\infty \lambda_a(t)\, dt + \tau(|a\, e_a(s) - E_{\mathcal{A}}(e_a(s)\, c_s\, e_b(s)\, b\, e_b(s)\, c_s^*\, e_a(s))|) \\ &\le \frac{\varepsilon}{2} + \frac{1}{s} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

As ε was arbitrary, this shows that $a \in \overline{E_{\mathcal{A}}(\{c\, b\, c^* : \|c\| \le 1\})}^{\|\cdot\|_1}$. □

Remark 5.13. The proof of Theorem 5.12 uses a reduction to a II_1 case, under the hypothesis that the operators belong to $L^1(\mathcal{M})$. This last assumption seems to be essential for such a reduction, and there is no immediate hope of using the same idea to obtain results like Theorems 5.5 and 5.8. Conversely, one cannot expect to use those results to obtain Theorem 5.12, since convergence in measure does not imply $\|\cdot\|_1$-convergence.

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