Robust dual reconstruction systems and fusion frames *†

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Abstract

We study the duality of reconstructions systems (RS's), which are G-frames in a finite dimensional setting. After proving some basic facts about these systems and their duals, we focus on dual systems of a fixed projective RS (i.e. the analogue of a fusion frame in this context) that are optimal with respect to erasures of the RS coefficients. Finally we study the projective RS that best approximate an arbitrary reconstruction system and the existence of a dual projective system of a RS of this type.

Contents

1 Introduction

Fusion frames were introduced under the name of "frame of subspaces" in [7]. They arise naturally as a generalization of the usual frames of vectors for a Hilbert space $\mathcal{H}$; indeed frames of vectors can be treated as "one-dimensional fusion frames". Several applications of fusion frames have been studied, for example, sensor networks [10], neurology [21], coding theory [3], [4] , [15], among others. We refer the reader to [9] and the references therein for

*Keywords: Fusion frames, reconstruction systems, dual reconstruction systems.
†Mathematics subject classification (2000): 42C15, 15A60.

a detailed treatment of the fusion frame theory. Further developments can be found in [6], [8] and [22].

Given $m \in \mathbb{N}$ we denote by $\mathbb{I}_m = \{1, \dots, m\} \subseteq \mathbb{N}$. In the finite dimensional setting, a fusion frame is a sequence $\mathcal{N}_w = (w_i\,,\,\mathcal{N}_i)_{i \in \mathbb{I}_m}$ where each $w_i \in \mathbb{R}_{>0}$ and the subspaces $\mathcal{N}_i \subseteq \mathbb{C}^d$ generate $\mathbb{C}^d$. The synthesis operator of the fusion frame $\mathcal{N}_w$ is usually defined as

$$T_{\mathcal{N}_w} : \mathcal{K}_{\mathcal{N}_w} \stackrel{\text{def}}{=} \bigoplus_{i \in \mathbb{I}_m} \mathcal{N}_i \to \mathbb{C}^d \quad \text{given by} \quad T_{\mathcal{N}_w}(x_i)_{i \in \mathbb{I}_m} = \sum_{i \in \mathbb{I}_m} w_i\, x_i \ .$$

Its adjoint, the so-called analysis operator of $\mathcal{N}_w$, is given by $T^*_{\mathcal{N}_w} y = (w_i\, P_{\mathcal{N}_i}\, y)_{i \in \mathbb{I}_m}$ for $y \in \mathbb{C}^d$, where $P_{\mathcal{N}_i}$ denotes the orthogonal projection onto $\mathcal{N}_i$. The frame $\mathcal{N}_w$ induces a linear encoding-decoding scheme that can be described in terms of these operators.

However, the previous setting for the theory of fusion frames presents some technical difficulties. For example the domain of $T_{\mathcal{N}_w}$ relies strongly on the subspaces of the fusion frame. In particular, any change on the subspaces modifies the domain of the operators preventing smooth perturbations of these objects. Moreover, this kind of rigidity on the definitions implies that the notion of a dual fusion frame is not clear.

An alternative approach to the fusion frame (FF) theory comes from the theory of G-frames [24] (see also [14, 25, 26]) and its variants, namely the theory of protocols introduced in [3] and the theory of reconstruction systems considered in [17] (see also [20]), which are finite dimensional G-frames.

In this context, we fix the dimensions $\dim \mathcal{N}_i = k_i$ and consider a universal space

$$\mathcal{K} = \mathcal{K}_{m\,,\,\mathbf{k}} \stackrel{\text{def}}{=} \bigoplus_{i \in \mathbb{I}_m} \mathbb{C}^{k_i} \ .$$

If $\mathcal{N}_w = (w_i\,,\,\mathcal{N}_i)_{i \in \mathbb{I}_m}$ is a FF, it is modeled as a sequence $\mathcal{V} = \{V_i\}_{i \in \mathbb{I}_m}$ with $V_i \in L(\mathbb{C}^d\,,\,\mathbb{C}^{k_i})$ and such that $V_i^* V_i = w_i^2\, P_{\mathcal{N}_i}$ for each $i \in \mathbb{I}_m$. In general, sequences $\mathcal{V} = \{V_i\}_{i \in \mathbb{I}_m}$ such that $V_i \in L(\mathbb{C}^d\,,\,\mathbb{C}^{k_i})$ and such that they allow the construction of an encoding-decoding algorithm are called finite dimensional G-frames or reconstructions systems (RS's). Notice that the representation of a FF above as a RS is not unique, in the same manner as different vector frames can be identified with a 1-dimensional FF. The RS's that come from FF's as above are called projective RS's.

The main advantage of the RS (or more generally of the G-frames) framework with respect to the fusion frame formalism is that each (projective) RS has many RS's that are *dual* systems. In particular, the canonical dual RS remains being a RS (for details and definitions see Section 2). In contrast, it is easy to give examples of a FF such that its canonical dual is not a fusion frame (see Example 5.4 below). There exists a notion of duality among fusion frames defined by Gavruta (see [11]), where the reconstruction formula of a fixed $\mathcal{V}$ involves the FF operator $S_{\mathcal{V}}$ of $\mathcal{V}$. Nevertheless, in the context of RS's we follow [24] where the notion of dual systems can be described and characterized in a quite natural way.

The problem of computing duals that are optimal with respect to some criteria, as well as the characterization of (fusion) frames that are optimal for erasures have been widely studied in the theory of frames and G-frames (see for example [14, 24, 25, 26] and [3, 4, 5, 8, 13, 15, 17, 23] respectively). We point out that although we focus on a problem dealing with duals that are optimal for erasures, our work has a different setting. Indeed, this research is in the vein of [16] but in the more general context of G-frames, where we first fix a RS and then search for duals of this fixed RS that are optimal with respect to erasures. Note that we are interested in the case where the reconstruction error due to erasures is measured in terms of the Frobenius 2-norm we restrict our study to the finite dimensional setting. Thus, instead of working with general G-frames we deal with RS's.

Briefly, a reconstruction system arise from a usual vector frame by grouping together the elements of the frame. Thus, the coefficients involved in the encoding-decoding scheme of RS are vector valued. If $\mathcal{V}$ is a fixed projective RS, we study those dual RS's for $\mathcal{V}$ that are optimal for erasures of these vector valued coefficients. We study two different ways of measuring the performance of duals of projective RS's for the erasures of coefficients:

We first consider the mean square error (MSE), that has been considered in [15] in a probabilistic setting as opposed to our deterministic approach. We show that for every projective system there is a unique optimal dual RS for the MSE, and we give a complete characterization of this optimal dual.

In the second case we study the worst-case reconstruction error (WCRE). We show that all results of [16] can be extended (with similar techniques) to the RS setting: we give sufficient conditions on a projective system $\mathcal{V}$ which guarantee that the cannonical dual RS of $\mathcal{V}$ is optimal, and we show that several examples - in particular group RS's - satisfy these conditions.

For the case of a non projective system $\mathcal{V}$, we characterize the projective RS which is nearest to $\mathcal{V}$, in terms of a suitable distance between RS's. It is worth noticing that a projective RS may not have any projective dual (see Example 5.4). Thus, the previous results can be applied to compute approximate projective duals of a given projective RS.

On the other hand, given a projective RS it is natural to focus on dual systems that are also projective (if it is possible). Hence, it seems to be an interesting problem to characterize those projective systems $\mathcal{V}$ such that the canonical dual or some other dual system $\mathcal{W}$ is also projective and work within this class. In the last section, we present several examples of these problems.

The paper is organized as follows: In Section 2 we recall some basic facts about general reconstruction systems and we fix some of the terminology used throughout the paper. In Section 3 we study the optimal dual systems for erasures of vector coefficients. In section 4 we describe the projective RS which is nearest to a fixed RS. In Section 5 we give several examples of group RS's, and about the existence (or not) of projective dual systems.

Notations.

Given $\mathcal{H} \cong \mathbb{C}^d$ and $\mathcal{K} \cong \mathbb{C}^n$, we denote by $L(\mathcal{H}, \mathcal{K})$ the space of linear operators $T : \mathcal{H} \to \mathcal{K}$. Given an operator $T \in L(\mathcal{H}, \mathcal{K})$, $R(T) \subseteq \mathcal{K}$ denotes the image of T, $\ker T \subseteq \mathcal{H}$ the null space of T and $T^* \in L(\mathcal{K}, \mathcal{H})$ the adjoint of T. If $d \leq n$ we say that $U \in L(\mathcal{H}, \mathcal{K})$ is an isometry if $U^*U = I_{\mathcal{H}}$. In this case, U^* is called a coisometry. We denote by $\mathcal{I}(d\,,\,n)$ the set of all isometries in $L(\mathcal{H}, \mathcal{K})$.

If $\mathcal{K} = \mathcal{H}$ we denote by $L(\mathcal{H}) = L(\mathcal{H}\,,\,\mathcal{H})$, by $Gl(\mathcal{H})$ the group of all invertible operators in $L(\mathcal{H})$, by $L(\mathcal{H})^+$ the cone of positive operators and by $Gl(\mathcal{H})^+ = Gl(\mathcal{H}) \cap L(\mathcal{H})^+$.

If $T \in L(\mathcal{H})$, we denote by $\sigma(T)$ the spectrum of T, by $\operatorname{rk} T$ the rank of T, and by $\operatorname{tr} T$ the trace of T. Given $m \in \mathbb{N}$ we denote by $\mathbb{I}_m = \{1, \ldots, m\} \subseteq \mathbb{N}$ and $\mathbb{1} = \mathbb{1}_m \in \mathbb{R}^m$ denotes the vector with all its entries equal to 1.

On the other hand, $\mathcal{M}_{n,m}(\mathbb{C})$ denotes the space of complex $n \times m$ matrices. If $n = m$ we write $\mathcal{M}_n(\mathbb{C}) = \mathcal{M}_{n,n}(\mathbb{C})$, $\mathcal{G}l(n)$ the group of all invertible elements of $\mathcal{M}_n(\mathbb{C})$, $\mathcal{U}(n)$ the group of unitary matrices, $\mathcal{M}_n(\mathbb{C})^+$ the set of positive semidefinite matrices, and $\mathcal{G}l(n)^+ = \mathcal{M}_n(\mathbb{C})^+ \cap \mathcal{G}l(n)$.

If $W \subseteq \mathcal{H}$ is a subspace we denote by $P_W \in L(\mathcal{H})^+$ the orthogonal projection onto W, i.e. $R(P_W) = W$ and $\ker P_W = W^\perp$. For vectors on $\mathbb{C}^n$ we shall use the euclidean norm, but for matrices $T \in \mathcal{M}_n(\mathbb{C})$, we shall use both

1. The spectral norm $\|T\| = \|T\|_{sp} = \max\limits_{\|x\|=1} \|Tx\|$.

2. The Frobenius norm $\|T\|_2 = (\operatorname{tr} T^*T)^{1/2} = \big(\sum\limits_{i,j\in\mathbb{I}_n} |T_{ij}|^2 \big)^{1/2}$. This norm is induced by the inner product $\langle A,\ B\rangle = \operatorname{tr} B^*A$, for $A, B \in \mathcal{M}_n(\mathbb{C})$.

2 Basic framework of reconstruction systems

In what follows we consider $(m, \mathbf{k}, d)$-reconstruction systems, which are more general linear systems than those considered in [2], [3], [4], [5], [13] and [18], that also have an associated reconstruction algorithm.

Definition 2.1. Let $m, d \in \mathbb{N}$ and $\mathbf{k} = (k_1, \ldots, k_m) \in \mathbb{N}^m$.

1. We denote by $\mathcal{K} = \mathcal{K}_{m,\mathbf{k}} \stackrel{\text{def}}{=} \bigoplus_{i\in\mathbb{I}_m} \mathbb{C}^{k_i}$. Sometimes we shall write each direct summand by $\mathcal{K}_i = \mathbb{C}^{k_i}$.

2. Given a space $\mathcal{H} \cong \mathbb{C}^d$ we denote by $L(m, \mathbf{k}, d) \stackrel{\text{def}}{=} \bigoplus_{i\in\mathbb{I}_m} L(\mathcal{H}, \mathcal{K}_i) \cong L(\mathcal{H}, \mathcal{K})$. A typical element of $L(m, \mathbf{k}, d)$ is a system $\mathcal{V} = \{V_i\}_{i\in\mathbb{I}_m}$ such that each $V_i \in L(\mathcal{H}, \mathcal{K}_i)$.

3. A family $\mathcal{V} = \{V_i\}_{i\in\mathbb{I}_m} \in L(m, \mathbf{k}, d)$ is an $(m, \mathbf{k}, d)$-reconstruction system (RS) for $\mathcal{H}$ if

$$S_\mathcal{V} \stackrel{\text{def}}{=} \sum_{i\in\mathbb{I}_m} V_i^* V_i \in Gl(\mathcal{H})^+ ,$$

   i.e., if $S_\mathcal{V}$ is invertible and positive. $S_\mathcal{V}$ is called the **RS operator** of $\mathcal{V}$. In this case, the m-tuple $\mathbf{k} = (k_1, \ldots, k_m) \in \mathbb{N}^m$ satisfies that $\operatorname{tr}\mathbf{k} \stackrel{\text{def}}{=} \sum_{i\in\mathbb{I}_m} k_i \geq d$.

   We shall denote by $\mathcal{RS}(m, \mathbf{k}, d)$ the set of all $(m, \mathbf{k}, d)$-RS's for $\mathcal{H} \cong \mathbb{C}^d$.

4. The system $\mathcal{V} = \{V_i\}_{i\in\mathbb{I}_m} \in \mathcal{RS}(m, \mathbf{k}, d)$ is said to be **injective** if $V_i^* \in L(\mathcal{K}_i, \mathcal{H})$ is injective (equivalently, if $V_i V_i^* \in Gl(\mathcal{K}_i)$) for every $i \in \mathbb{I}_m$.

   We shall denote by $\mathcal{RS}_\mathcal{I}(m, \mathbf{k}, d)$ the set of all injective elements of $\mathcal{RS}(m, \mathbf{k}, d)$.

5. The system $\mathcal{V}$ is said to be **projective** if there exists a sequence $\mathbf{v} = (v_i)_{i\in\mathbb{I}_m} \in \mathbb{R}_+^m$ of positive numbers, the weights of $\mathcal{V}$, such that

$$V_i V_i^* = v_i^2 I_{\mathcal{K}_i} , \quad \text{for every} \quad i \in \mathbb{I}_m .$$

   In this case, the following properties hold:

   (a) The weights can be computed directly, since each $v_i = \|V_i\|_{sp}$.

   (b) Each $V_i = v_i U_i$ for a coisometry $U_i \in L(\mathcal{H}, \mathcal{K}_i)$. Thus $V_i^* V_i = v_i^2 P_{R(V_i^*)} \in L(\mathcal{H})^+$ for every $i \in \mathbb{I}_m$.

(c) Observe that in this case $S_{\mathcal{V}} = \sum_{i\in \mathbb{I}_m} v_i^2 \, P_{R(V_i^*)}$ as in fusion frame theory.

We shall denote by $\mathcal{P}(m, \mathbf{k}, d)$ the set of all projective elements of $\mathcal{RS}(m, \mathbf{k}, d)$.

6. The **analysis** operator of the system $\mathcal{V}$ is defined by

$$T_{\mathcal{V}} \; : \; \mathcal{H} \to \mathcal{K} = \bigoplus_{i\in \mathbb{I}_m} \mathcal{K}_i \;\; \text{given by} \quad T_{\mathcal{V}}\, x = (V_1\, x \, , \, \dots \, , \, V_m\, x) \; , \qquad \text{for} \qquad x \in \mathcal{H} \; .$$

7. Its adjoint $T_{\mathcal{V}}^*$ is called the **synthesis** operator of the system $\mathcal{V}$, and it satisfies that

$$T_{\mathcal{V}}^* : \mathcal{K} = \bigoplus_{i\in \mathbb{I}_m} \mathcal{K}_i \to \mathcal{H} \quad \text{ is given by } \quad T_{\mathcal{V}}^* \left(\, (y_i)_{i\in \mathbb{I}_m} \right) = \sum_{i\in \mathbb{I}_m} V_i^* \, y_i \; .$$

Using the previous notations and definitions we have that $S_{\mathcal{V}} = T_{\mathcal{V}}^* \, T_{\mathcal{V}}$. △

Examples 2.2 (Vector and fusion frames as RS's)**.**

1. As it was mentioned, RS's arise from usual vector frames by grouping together the elements of the frame. Therefore, it is natural to expect that in the case $\mathbf{k} = \mathbb{1}_m$, the set $\mathcal{RS}(m, \mathbf{k}, d)$ can be identified with the set of m-vector frames for $\mathcal{H} \cong \mathbb{C}^d$.

Indeed, let $\mathcal{F} = \{f_i\}_{i\in \mathbb{I}_m} \in \mathcal{H}^m$. For $i \in \mathbb{I}_m$ consider $V_i : \mathcal{H} \to \mathbb{C}$ given by $V_i(x) = \langle x, f_i\rangle$ for every $x \in \mathcal{H}$. Let $\mathcal{V}_{\mathcal{F}} = \{V_i\}_{i\in\mathbb{I}_m}$ and notice that

$$S_{\mathcal{V}_{\mathcal{F}}} = \sum_{i\in\mathbb{I}_m} V_i^* V_i = \sum_{i\in\mathbb{I}_m} \langle \, \cdot \, , f_i \rangle \, f_i = S_{\mathcal{F}} \; .$$

Thus $\mathcal{F}$ is a frame for $\mathcal{H}$ if and only if $\mathcal{V}_{\mathcal{F}} \in \mathcal{RS}(m, \mathbb{1}, d)$. Actually, $\mathcal{RS}(m, \mathbb{1}, d) = \mathcal{P}(m, \mathbb{1}, d)$ because every functional is a multiple of a coisometry. Moreover, $T_{\mathcal{V}_{\mathcal{F}}} : \mathcal{H} \to \oplus_{i\in \mathbb{I}_m} \mathbb{C} = \mathbb{C}^m$ is the usual analysis operator of $\mathcal{F}$. On the other hand, it is clear that elements in $\mathcal{RS}(m, \mathbb{1}, d)$ correspond to vector frames for $\mathcal{H}$.

2. Let $\mathcal{N}_w = (w_i \, , \mathcal{N}_i)_{i\in\mathbb{I}_m}$ be a fusion frame for $\mathcal{H} \cong \mathbb{C}^d$, with weights $w_i > 0$ and subspaces $\mathcal{N}_i \subseteq \mathcal{H}$ with $\dim \mathcal{N}_i = k_i$ for every $i \in \mathbb{I}_m$. Its fusion frame operator is

$$S_{\mathcal{N}_w} = \sum_{i\in \mathbb{I}_m} w_i^2 \, P_{\mathcal{N}_i} \in Gl(\mathcal{H})^+$$

(see [9] for a detailed exposition of fusion frames). Let $U_i \in L(\mathcal{H}, \mathbb{C}^{k_i})$ be a coisometry such that $U_i^* U_i = P_{\mathcal{N}_i}$, for every $i \in \mathbb{I}_m$. Therefore, the system $\mathcal{V} = \{V_i\}_{i\in\mathbb{I}_m} \stackrel{\text{def}}{=} \{w_i \, U_i\}_{i\in\mathbb{I}_m}$ satisfies that $S_{\mathcal{V}} = S_{\mathcal{N}_w} \in Gl(\mathcal{H})^+$. Hence $\mathcal{V} \in \mathcal{P}(m, \mathbf{k}, d)$ is a projective RS associated to $\mathcal{N}_w$. Observe that $\mathcal{V}$ has the same weights as $\mathcal{N}_w$ and it also satisfies that each $\mathcal{N}_i = R(V_i^*)$.

Conversely, given $\mathcal{V} = \{V_i\}_{i\in \mathbb{I}_m} \in \mathcal{P}(m, \mathbf{k}, d)$, the sequence $\mathcal{N}_w = \big(\|V_i\|, R(V_i^*) \big)_{i\in \mathbb{I}_m}$ is a fusion frame such that $S_{\mathcal{V}} = S_{\mathcal{N}_w}$. Nevertheless the correspondence is not one to one, since any system of coisometries $\{U_i\}_{i\in \mathbb{I}_m}$ with $(\ker U_i)^\perp = \mathcal{N}_i$ produces the same fusion frame $\mathcal{N}_w = (w_i \, , \mathcal{N}_i)_{i\in\mathbb{I}_m}$. This phenomenon is similar to the correspondence of vector frames with one dimensional fusion frames. △

Remark 2.3. In what follows we list some properties and notations about RS's:

1. Given $\mathcal{V} = \{V_i\}_{i\in\mathbb{I}_m} \in \mathcal{RS}(m, \mathbf{k}, d)$ with $S_\mathcal{V} = \sum_{i\in\mathbb{I}_m} V_i^* V_i$, then

$$\sum_{i\in\mathbb{I}_m} S_\mathcal{V}^{\ -1} V_i^* V_i = I_\mathcal{H} \ , \quad \text{and} \quad \sum_{i\in\mathbb{I}_m} V_i^* V_i \ S_\mathcal{V}^{\ -1} = I_\mathcal{H} \ . \tag{1}$$

Therefore, we obtain the reconstruction formulas

$$x = \sum_{i\in\mathbb{I}_m} S_\mathcal{V}^{-1} V_i^* (V_i x) = \sum_{i\in\mathbb{I}_m} V_i^* V_i (S_\mathcal{V}^{-1} x) \quad \text{for every} \quad x \in \mathcal{H} \ .$$

2. For every $\mathcal{V} = \{V_i\}_{i\in\mathbb{I}_m} \in \mathcal{RS}(m, \mathbf{k}, d)$, we define the system

$$\mathcal{V}^\# \stackrel{\text{def}}{=} \{V_i S_\mathcal{V}^{-1}\}_{i\in\mathbb{I}_m} \in \mathcal{RS}(m, \mathbf{k}, d) \ ,$$

called the **canonical dual** RS associated to $\mathcal{V}$. By Eq. (1), we see that

$$T_{\mathcal{V}^\#}^* T_\mathcal{V} = \sum_{i\in\mathbb{I}_m} S_\mathcal{V}^{\ -1} V_i^* V_i = I_\mathcal{H} \quad \text{and} \quad S_{\mathcal{V}^\#} = \sum_{i\in\mathbb{I}_m} S_\mathcal{V}^{-1} V_i^* V_i S_\mathcal{V}^{-1} = S_\mathcal{V}^{-1} \ .$$

Next we generalize the notion of dual RS's: △

Definition 2.4. Let $\mathcal{V} = \{V_i\}_{i\in\mathbb{I}_m}$ and $\mathcal{W} = \{W_i\}_{i\in\mathbb{I}_m} \in \mathcal{RS}(m, \mathbf{k}, d)$.

1. We say that $\mathcal{W}$ is a **dual** RS for $\mathcal{V}$ if $T_\mathcal{W}^* T_\mathcal{V} = I_\mathcal{H}$, or equivalently if

$$x = \sum_{i\in\mathbb{I}_m} W_i^* V_i x \quad \text{for every} \quad x \in \mathcal{H} \ .$$

2. We denote by $D(\mathcal{V}) \stackrel{\text{def}}{=} \{\mathcal{W} \in \mathcal{RS}(m, \mathbf{k}, d) : T_\mathcal{W}^* T_\mathcal{V} = I_\mathcal{H}\}$, the set of all dual RS's for a fixed $\mathcal{V} \in \mathcal{RS}(m, \mathbf{k}, d)$. Observe that $D(\mathcal{V}) \neq \varnothing$ since $\mathcal{V}^\# \in D(\mathcal{V})$.

△

Remark 2.5. Let $\mathcal{V} \in \mathcal{RS}(m, \mathbf{k}, d)$. Then $\mathcal{W} \in D(\mathcal{V})$ if and only if its synthesis operator $T_\mathcal{W}^*$ is a pseudo-inverse of $T_\mathcal{V}$. Indeed, $\mathcal{W} \in D(\mathcal{V}) \iff T_\mathcal{W}^* T_\mathcal{V} = I_\mathcal{H}$. Observe that the map $\mathcal{RS}(m, \mathbf{k}, d) \ni \mathcal{W} \mapsto T_\mathcal{W}^*$ is one to one. Thus, in the context of RS's each $(m, \mathbf{k}, d)$-RS has many duals that are $(m, \mathbf{k}, d)$-RS's. This is one of the advantages of the RS's setting.

Moreover, the synthesis operator $T_{\mathcal{V}^\#}^*$ of the canonical dual $\mathcal{V}^\#$ corresponds to the Moore-Penrose pseudo-inverse of $T_\mathcal{V}$. Indeed, notice that $T_\mathcal{V} T_{\mathcal{V}^\#}^* = T_\mathcal{V} S_\mathcal{V}^{-1} T_\mathcal{V}^* \in L(\mathcal{K})^+$, so that it is an orthogonal projection. Under this point of view, $\mathcal{V}^\#$ has some optimal properties that come from the theory of pseudo-inverses. △

3 Erasures and errors

Let us fix the parameters $(m, \mathbf{k}, d)$ with $\operatorname{tr} \mathbf{k} \geq d$. Hence we have the spaces $\mathcal{H} \cong \mathbb{C}^d$, $\mathcal{K}_i = \mathbb{C}^{k_i}$ and $\mathcal{K} = \bigoplus_{i\in\mathbb{I}_m} \mathcal{K}_i$. Given a subset $\mathbb{J} \subseteq \mathbb{I}_m$ of size $|\mathbb{J}| = r$, we consider

$$M_\mathbb{J} \in L(\mathcal{K}) \quad \text{given by} \quad M_\mathbb{J}\left((y_i)_{i\in\mathbb{I}_m}\right) = \left(\mathbb{1}_\mathbb{J}(i) \cdot y_i\right)_{i\in\mathbb{I}_m} \ ,$$

where $\mathbb{1}_{\mathbb{J}} : \mathbb{I}_m \to \{0,1\}$ denotes the characteristic function of the set $\mathbb{J} \subset \mathbb{I}_m$. Similarly, we consider the *packet-lost operator* $L_{\mathbb{J}} \stackrel{\text{def}}{=} M_{\mathbb{I}_m \setminus \mathbb{J}} = I_{\mathcal{K}} - M_{\mathbb{J}}$. If $\mathbb{J} = \{j\}$, we abbreviate $M_{\{j\}} = M_j$ and $L_{\{j\}} = L_j$.

Given $\mathcal{V} = \{V_i\}_{i\in\mathbb{I}_m} \in \mathcal{RS}(m, \mathbf{k}, d)$, we shall consider a "blind reconstruction" strategy in case some packets of coefficients are lost. That is, assuming that the encoded information $T_{\mathcal{V}}\, x \in \mathcal{K}$ (for some $x \in \mathcal{H}$) is altered according to the packet-lost operator $L_{\mathbb{J}}$, our reconstructed vector will be $T_{\mathcal{W}}^*\, L_{\mathbb{J}}\, T_{\mathcal{V}}\,(x)$, where $\mathcal{W} \in D(\mathcal{V})$ is some dual RS for $\mathcal{V}$. Therefore, if we fix $r \in \mathbb{I}_m$ and $\mathcal{W} \in D(\mathcal{V})$ we get the $\binom{m}{r}$-tuple

$$e(r\,,\,\mathcal{W}) = (\|I - T_{\mathcal{W}}^*\, L_{\mathbb{J}}\, T_{\mathcal{V}}\|_2)_{\mathbb{J}\in\mathcal{P}_r(\mathbb{I}_m)} \in \mathbb{R}^{\binom{m}{r}}\ ,$$

where $\mathcal{P}_r(\mathbb{I}_m) = \{\mathbb{J} \subseteq \mathbb{I}_m : |\mathbb{J}| = r\}$. This vector corresponds to the reconstruction errors (with respect to the Frobenius norm) for the erasure of r packets of coefficients, for all such possible choices. In what follows we shall consider two different measures of the reconstruction error based on the finite sequence $e(r\,,\,\mathcal{W})$ namely the mean square error and the (normalized) worst-case error.

Finally, let us mention that there are several papers that study the structure of optimal frames and fusion frames (with some further structure) for the erasures of coefficients, for example [3], [4], [5], [8], [13], [17], [23]. In contrast, we consider optimal dual RS's for a fixed RS in a similar way as considered by J. Lopez and D. Han in [16].

3.1 Minimizing the mean square error

Let $\mathcal{V} = \{V_i\}_{i\in\mathbb{I}_m} \in \mathcal{RS}(m, \mathbf{k}, d)$. In this section we consider as a measure of performance of a dual $(m, \mathbf{k}, d)$ reconstruction system $\mathcal{W}$ the mean square error (m.s.e.)

$$e_r^{(2)}(\mathcal{V}\,,\,\mathcal{W}) = \|e(r\,,\,\mathcal{W})\|_2 = \Big(\sum_{\mathbb{J}\in\mathcal{P}_r(\mathbb{I}_m)} \|I - T_{\mathcal{W}}^*\, L_{\mathbb{J}}\, T_{\mathcal{V}}\|_2^2\Big)^{1/2}$$

Let us denote by

$$e_1^{(2)}(\mathcal{V}) = \inf_{\mathcal{W}\in D(\mathcal{V})} e_1^{(2)}(\mathcal{V}, \mathcal{W}) = \inf_{\mathcal{W}\in D(\mathcal{V})} \left(\sum_{i\in\mathbb{I}_m} \|W_i^*\, V_i\|_2^2\right)^{1/2}. \tag{2}$$

We are interested in those dual RS's $\mathcal{W}$ for $\mathcal{V}$ such that $e_1^{(2)}(\mathcal{V}, \mathcal{W}) = e_1^{(2)}(\mathcal{V})$. In other words, we define the set of *1-loss optimal dual* RS's for $\mathcal{V}$ with respect to $e_1^{(2)}(\mathcal{V}, \cdot)$ as

$$D_1^{(2)}(\mathcal{V}) \stackrel{\text{def}}{=} \{\mathcal{W} \in D(\mathcal{V}) : e_1^{(2)}(\mathcal{V}\,,\,\mathcal{W}) = e_1^{(2)}(\mathcal{V})\}\ .$$

Proceeding inductively, we set $e_r^{(2)}(\mathcal{V}) = \inf\{e_r^{(2)}(\mathcal{V}\,,\,\mathcal{W}) :\ \mathcal{W} \in D_{r-1}^{(2)}(\mathcal{V})\}$ and define $D_r^{(2)}(\mathcal{V})$ as the subset of $D_{r-1}^{(2)}(\mathcal{V})$ where this infimum is attained, called the r-loss optimal dual RS's for $\mathcal{V}$ with respect to $e_r^{(2)}(\mathcal{V}, \cdot)$, in case these sets are non-empty.

Theorem 3.1. *Let $\mathcal{V} = \{V_i\}_{i\in\mathbb{I}_m} \in \mathcal{P}(m, \mathbf{k}, d)$ with weights $\mathbf{v} = (v_i)_{i\in\mathbb{I}_m}$. Then $D_1^{(2)}(\mathcal{V}) = \{\mathcal{W}_0\}$ i.e., there is a* **unique** *1-loss optimal dual RS $\mathcal{W}_0$ for the m.s.e. Moreover, if*

1. *$D \in L(\mathcal{K})$ is the block diagonal matrix $D = \bigoplus_{i\in\mathbb{I}_m} v_i^{-2}\, I_{\mathcal{K}_i}$, and*

2. $S_{\mathcal{V},D} = T_\mathcal{V}^* \, D \, T_\mathcal{V} = \sum\limits_{i\in \mathbb{I}_m} P_{R(V_i^*)} \in Gl(\mathcal{H})^+$ (*since* $S_{\mathcal{V},D} \geq (\min\limits_{i\in\mathbb{I}_m} v_i^{-2}) \cdot S_\mathcal{V} > 0$),

then the optimal system is $\mathcal{W}_0 = \{v_i^{-2} V_i \, S_{\mathcal{V},D}^{-1}\}_{i\in \mathbb{I}_m}$ *. In particular,* $T_{\mathcal{W}_0} = D \, T_\mathcal{V} \, S_{\mathcal{V},D}^{-1}$ *.*

Proof. First we check that $\mathcal{W}_0 \in D(\mathcal{V})$. Indeed, $T_{\mathcal{W}_0}^* \, T_\mathcal{V} = S_{\mathcal{V},D}^{-1} \, T_\mathcal{V}^* \, D \, T_\mathcal{V} = S_{\mathcal{V},D}^{-1} \, S_{\mathcal{V},D} = I$. Denote by $B_i = v_i^{-2} V_i \, S_{\mathcal{V},D}^{-1}$, the i-th entry of $\mathcal{W}_0$, for every $i \in \mathbb{I}_m$. Consider a dual system $\mathcal{W} = \{W_i\}_{i\in \mathbb{I}_m} \in D(\mathcal{V})$. Since each $V_i V_i^* = v_i^2 I_{\mathcal{K}_i}$, then

$$\|W_i^* V_i\|_2^2 = \operatorname{tr}(V_i^* W_i W_i^* V_i) = \operatorname{tr}(W_i W_i^* V_i V_i^*) = v_i^2 \ \operatorname{tr}(W_i W_i^*) = v_i^2 \ \|W_i^*\|_2^2 \ . \tag{3}$$

In particular, we can compute the m.s.e. for $\mathcal{W}_0$:

$$e_1^{(2)}(\mathcal{V}\,,\,\mathcal{W}_0) = \sum_{i\in \mathbb{I}_m} \|B_i^* \, V_i\|_2^2 \ \overset{(3)}{=} \ \sum_{i\in \mathbb{I}_m} v_i^2 \|B_i^*\|_2^2 = \sum_{i\in \mathbb{I}_m} v_i^{-2} \ \|S_{\mathcal{V},D}^{-1} \ V_i^*\|_2^2 \ .$$

On the other hand, for every $i \in \mathbb{I}_m$ we have

$$\begin{aligned} v_i^2 \ \|W_i^*\|_2^2 \ &= \ v_i^2 \|B_i^* + (W_i^* - B_i^*)\|_2^2 \\ &= \ v_i^{-2} \|S_{\mathcal{V},D}^{-1} \ V_i^*\|_2^2 + v_i^2 \ \|W_i^* - B_i^*\|_2^2 + 2\operatorname{Re}\Big(v_i^2 \ \operatorname{tr}\big[(W_i^* - B_i^*) B_i \big] \Big) \ . \end{aligned} \tag{4}$$

Let $t_i = \operatorname{tr}\big[(W_i^* - B_i^*) B_i\big] = v_i^{-2} \operatorname{tr}\big[(W_i^* - B_i^*) \, V_i \, S_{\mathcal{V},D}^{-1}\big]$. Then we have that

$$\sum_{i\in \mathbb{I}_m} v_i^2 \ t_i = \operatorname{tr}\Big(\sum_{i\in \mathbb{I}_m} (W_i^* - B_i^*) \, V_i \, S_{\mathcal{V},D}^{-1} \Big) = \operatorname{tr}\big[(T_\mathcal{W}^* - T_{\mathcal{W}_0}^*) \ T_\mathcal{V} \ S_{\mathcal{V},D}^{-1} \big] = 0 \ .$$

since both $\mathcal{W}$ and $\mathcal{V}^\#$ are dual RS's for $\mathcal{V}$. Therefore, summing over $\mathbb{I}_m$, the third summand of (4) vanishes and

$$\begin{aligned} e_1^{(2)}(\mathcal{V}\,,\,\mathcal{W}) = \sum_{i\in \mathbb{I}_m} \|W_i^* V_i\|_2^2 \ &\overset{(3)}{=} \ \sum_{i\in \mathbb{I}_m} v_i^2 \ \|W_i^*\|_2^2 \\ &\overset{(4)}{=} \ \sum_{i\in \mathbb{I}_m} v_i^{-2} \|S_{\mathcal{V},D}^{-1} \ V_i^*\|_2^2 + v_i^2 \ \|W_i^* - B_i^*\|_2^2 \\ &\geq \sum_{i\in \mathbb{I}_m} v_i^2 \|B_i^*\|_2^2 \overset{(3)}{=} e_1^{(2)}(\mathcal{V}\,,\,\mathcal{W}_0) \ . \end{aligned} \tag{5}$$

Therefore $\mathcal{W}_0 \in D_1^{(2)}(\mathcal{V})$. Moreover, if we take another $\mathcal{W} = \{W_i\}_{i\in \mathbb{I}_m} \in D_1^{(2)}(\mathcal{V})$, then Eq. (5) implies that $\|W_i^* - B_i^*\|_2 = 0$ for every $i \in \mathbb{I}_m$, so that $\mathcal{W} = \mathcal{W}_0$. □

Remark 3.2. As an immediate consequence of Theorem 3.1 we conclude that, for every $r \in \mathbb{I}_m$ then $D_r^{(2)}(\mathcal{V}) = \{\mathcal{W}_0\}$. That is, following the hierarchies for the definition of $D_r^{(2)}(\mathcal{V})$ as above, the r-loss optimal dual RS for $\mathcal{V}$ with respect to $e_r^{(2)}(\mathcal{V}, \cdot)$ is unique and given by $\mathcal{W}_0$, which is described in Theorem 3.1. △

We say that a system $\mathcal{V} = \{V_i\}_{i\in \mathbb{I}_m} \in \mathcal{P}(m, \mathbf{k}, d)$ is an **uniform** projective RS if the weights $\mathbf{v} = (v_i)_{i\in\mathbb{I}_m}$ of $\mathcal{V}$ satisfy that $\mathbf{v} = v\,\mathbb{1}$ for some $v > 0$.

Remark 3.3. The unique 1-loss optimal dual RS $\mathcal{W}_0 \in D_1^{(2)}(\mathcal{V})$ of Theorem 3.1, can be described in the following way: If $\mathbf{v} = (v_i)_{i\in\mathbb{I}_m}$ are the weights of the system $\mathcal{V} \in \mathcal{P}(m, \mathbf{k}, d)$, consider $\mathcal{U} = \{U_i\}_{i\in\mathbb{I}_m} = \{\frac{V_i}{v_i}\}_{i\in\mathbb{I}_m}$, which is a uniform projective RS. Then

$$S_{\mathcal{V},D} = \sum_{i\in\mathbb{I}_m} P_{R(U_i^*)} = S_{\mathcal{U}} \quad , \quad \mathcal{V} = \mathbf{v} \cdot \mathcal{U} \overset{\text{def}}{=} \{v_i U_i\}_{i\in\mathbb{I}_m} \quad \text{and} \quad \mathcal{W}_0 = \mathbf{v}^{-1} \cdot \mathcal{U}^{\#} \ , \tag{6}$$

because $\mathcal{W}_0 = \{v_i^{-2} V_i S_{\mathcal{V},D}^{-1}\}_{i\in\mathbb{I}_m} = \{v_i^{-1} U_i S_{\mathcal{U}}^{-1}\}_{i\in\mathbb{I}_m}$. △

Corollary 3.4. *Let $\mathcal{V} = \{V_i\}_{i\in\mathbb{I}_m} \in \mathcal{P}(m, \mathbf{k}, d)$. Assume that $\mathcal{V}$ is uniform. Then the unique 1-loss optimal dual RS for the m.s.e. (and hence the r-loss optimal dual RS for the m.s.e.) is the canonical dual $\mathcal{V}^{\#}$.*

Proof. If the weights of $\mathcal{V}$ are $\mathbf{v} = v\,\mathbb{1}$ then, with the notations of Remark 3.3, we have that $\mathcal{V} = v\,\mathcal{U}$, $S_{\mathcal{V}} = v^2 S_{\mathcal{U}}$ and $\mathcal{V}^{\#} = v^{-1}\mathcal{U}^{\#}$. Then we apply Eq. (6). □

Remark 3.5. Let $\mathcal{V} = \{V_i\}_{i\in\mathbb{I}_m} \in \mathcal{P}(m, \mathbf{k}, d)$ with weights $\mathbf{v} = (v_i)_{i\in\mathbb{I}_m}$. We give another interpretation of the system $\mathcal{W}_0$, the unique 1-loss optimal dual RS for $\mathcal{V}$ of Theorem 3.1:

Let $D \in L(\mathcal{K})$ be the block diagonal matrix $D = \bigoplus_{i\in\mathbb{I}_m} v_i^{-2} I_{\mathcal{K}_i}$. Consider on $\mathcal{K}$ the (perturbed) inner product given by $\langle x, y\rangle_D = \langle Dx, y\rangle$, for x , $y \in \mathcal{K}$. Then $\mathcal{K}_D \overset{\text{def}}{=} (\mathcal{K}\,,\,\langle\cdot,\cdot\rangle_D)$ becomes an inner product space (which is equivalent to $\mathcal{K}$ with its usual inner product). Given $T \in L(\mathcal{H}\,,\,\mathcal{K})$ let $T^{*_D} \in L(\mathcal{K}\,,\,\mathcal{H})$ be the adjoint operator with respect to $\langle\cdot,\cdot\rangle_D$ (on the side $\mathcal{K}$). Thus, T^{*_D} is a linear transformation such that

$$\langle Tx, y\rangle_D = \langle x, T^{*_D} y\rangle_{\mathcal{H}} \quad \text{for every} \quad x \in \mathcal{H} \ , \quad y \in \mathcal{K} \ .$$

Simple computations show that $T^{*_D} = T^* D$. Hence, if we consider the space $\mathcal{K}_D$ just as an inner product space in which the analysis of the signals coming from $\mathcal{H}$ is performed in terms of $T_{\mathcal{V}}$, then $T_{\mathcal{V}}^{*_D} = T_{\mathcal{V}}^* D$ becomes the (oblique) synthesis operator of $\mathcal{V}$.

Therefore, the (oblique) RS operator of $\mathcal{V}$ is $T_{\mathcal{V}}^{*_D} T_{\mathcal{V}} = T_{\mathcal{V}}^* D\, T_{\mathcal{V}} = S_{\mathcal{V},D}$ (as in Theorem 3.1). Finally, let us denote by $\mathcal{V}^{\#_D}$ the (oblique) D-canonical dual RS of $\mathcal{V}$, again with respect to $\mathcal{K}_D$ and notice that $\mathcal{V}^{\#_D} = \{V_i S_{\mathcal{V},D}^{-1}\}_{i\in\mathbb{I}_m}$. Then the synthesis operator of the optimal dual $\mathcal{W}_0$ of $\mathcal{V}$ coincides with the oblique synthesis operator of the D-canonical dual $\mathcal{V}^{\#_D}$. Hence, both systems $\mathcal{W}_0$ and $\mathcal{V}^{\#_D}$ produce the same (optimal) left inverse for $T_{\mathcal{V}}$. Indeed, $\mathcal{V}^{\#_D}$ has D-synthesis operator given by

$$T_{\mathcal{V}^{\#_D}}^{*_D} (y_i)_{i\in\mathbb{I}_m} = T_{\mathcal{V}^{\#_D}}^* D\, (y_i)_{i\in\mathbb{I}_m} = \sum_{i\in\mathbb{I}_m} S_{\mathcal{V},D}^{-1} V_i^* v_i^{-2} y_i = \sum_{i\in\mathbb{I}_m} \left(v_i^{-2} V_i S_{\mathcal{V},D}^{-1}\right)^* y_i \ ,$$

for every $(y_i)_{i\in\mathbb{I}_m} \in \mathcal{K}$. Therefore $T_{\mathcal{V}^{\#_D}}^{*_D} = T_{\mathcal{W}_0}^*$, since $\mathcal{W}_0 = \{v_i^{-2} V_i S_{\mathcal{V},D}^{-1}\}_{i\in\mathbb{I}_m}$. △

3.2 Minimizing the worst-case reconstruction error

Let $\mathcal{V} = \{V_i\}_{i\in\mathbb{I}_m} \in \mathcal{P}(m, \mathbf{k}, d)$. For each $\mathcal{W} = \{W_i\}_{i\in\mathbb{I}_m} \in D(\mathcal{V})$, we introduce the worst-case reconstruction error when r packets are lost with respect to the Frobenius norm:

$$e_r(\mathcal{V}\,,\,\mathcal{W}) \overset{\text{def}}{=} \|e(r\,,\,\mathcal{W})\|_\infty = \max_{\mathbb{J}\in\mathcal{P}_r(\mathbb{I}_m)} \|I - T_{\mathcal{W}}^* L_{\mathbb{J}} T_{\mathcal{V}}\|_2 \ .$$

As in the beginning of Section 3.1, we denote by

$$e_1(\mathcal{V}) = \inf_{\mathcal{W}\in D(\mathcal{V})} e_1(\mathcal{V},\mathcal{W}) = \inf_{\mathcal{W}\in D(\mathcal{V})} \max_{i\in \mathbb{I}_m} \|T^*_{\mathcal{W}}\, M_i\, T_{\mathcal{V}}\|_2 = \inf_{\mathcal{W}\in D(\mathcal{V})} \max_{i\in \mathbb{I}_m} \|W_i^*\, V_i\|_2 \ . \tag{7}$$

We define the set of *1-loss optimal dual* RS's for $\mathcal{V}$ as

$$D_1(\mathcal{V}) \stackrel{\text{def}}{=} \{\mathcal{W}\in D(\mathcal{V}) : e_1(\mathcal{V}\,,\,\mathcal{W}) = e_1(\mathcal{V})\}\ .$$

Proceeding inductively, we set $e_r(\mathcal{V}) = \inf\{e_r(\mathcal{V}\,,\,\mathcal{W}) :\ \mathcal{W}\in D_{r-1}(\mathcal{V})\}$ and define $D_r(\mathcal{V})$ as the subset of $D_{r-1}(\mathcal{V})$ where this infimum is attained, called the r-loss optimal dual RS's for $\mathcal{V}$ with respect to $e_r(\mathcal{V},\cdot)$, in case these sets are non-empty.

The following results can be viewed as a natural extension to the RS setting of the results in [16]. Their proofs use techiques similar to those of Han and Lopez.

3.6. Recall that $\mathcal{RS}(m,\mathbf{k},d) \subseteq L(m,\mathbf{k},d) = \bigoplus_{i\in\mathbb{I}_m} L(\mathcal{H}\,,\,\mathcal{K}_i) \cong L(\mathcal{H},\mathcal{K})$. If we fix an **injective** $\mathcal{V}=\{V_i\}_{i\in\mathbb{I}_m}\in\mathcal{RS}_{\mathcal{I}}(m,\mathbf{k},d)$, then the map $\|\cdot\|_{\mathcal{V}} : L(m,\mathbf{k},d)\to\mathbb{R}_+$ given by

$$\|\mathcal{W}\|_{\mathcal{V}} \stackrel{\text{def}}{=} e_1(\mathcal{V}\,,\,\mathcal{W}) = \max_{i\in\mathbb{I}_m} \|W_i^*\,V_i\|_2 \quad \text{for} \quad \mathcal{W}=\{W_i\}_{i\in\mathbb{I}_m}\in L(m,\mathbf{k},d)$$

is a norm in $L(m,\mathbf{k},d)$. Indeed, the only non trivial condition is the faithfulness. But the fact that $\mathcal{V}\in\mathcal{RS}_{\mathcal{I}}(m,\mathbf{k},d)$ (i.e. V_i is surjective for every $i\in\mathbb{I}_m$) assures that $\|\mathcal{W}\|_{\mathcal{V}}=0 \implies \|W_i^*\,V_i\|_2=0$ for every $i\in\mathbb{I}_m \implies \mathcal{W}=0$.

Since $D(\mathcal{V})$ is closed in $L(m,\mathbf{k},d)$ with the usual norm and all norms are equivalent on $L(m,\mathbf{k},d)$, then $D(\mathcal{V})$ is $\|\cdot\|_{\mathcal{V}}$-closed in $L(m,\mathbf{k},d)$. Therefore there exist elements $\mathcal{W}\in D(\mathcal{V})$ such that $\|\mathcal{W}\|_{\mathcal{V}} = \min_{\mathcal{N}\in D(\mathcal{V})} \|\mathcal{N}\|_{\mathcal{V}} = e_1(\mathcal{V})$. Indeed, the intersection of $D(\mathcal{V})$ with a fixed closed ball is a compact set. On the other hand, $D(\mathcal{V})$ is convex (actually it is an affine manifold). Since every norm is a convex map, we have proved the following result: △

Proposition 3.7. *Let $\mathcal{V}\in\mathcal{RS}_{\mathcal{I}}(m,\mathbf{k},d)$ be an* **injective** *system. Then the set $D_1(\mathcal{V})$ of 1-loss optimal dual RS's for $\mathcal{V}$ is non-empty, compact and convex.* □

Remark 3.8. The study of $D_1(\mathcal{V})$ has been considered by Han and López [16] in the particular case of $(m,\mathbb{1},d)$-RS's for $\mathcal{H}$, i.e. usual vector frames. Indeed, let $\mathcal{F}=\{f_i\}_{i\in\mathbb{I}_m}$ be a frame for $\mathcal{H}$ and let $\mathcal{V}_{\mathcal{F}}=\{V_i\}_{i\in\mathbb{I}_m}$ be its associated $(m,\mathbb{1},d)$-RS as in Example 2.2.

In this context the following optimization problem is considered in [16]: to find the $(m,\mathbb{1},d)$-RS's $\mathcal{W}^\flat\in D(\mathcal{V}_{\mathcal{F}})$ such that

$$\max_{i\in\mathbb{I}_m} \|T^*_{\mathcal{W}^\flat} M_i T_{\mathcal{V}_{\mathcal{F}}}\|_\infty = \min_{\mathcal{W}\in D(\mathcal{V}_{\mathcal{F}})} \max_{i\in\mathbb{I}_m} \|T^*_{\mathcal{W}} M_i T_{\mathcal{V}_{\mathcal{F}}}\|_\infty\ . \tag{8}$$

Notice that if $\mathcal{W}=\{W_i\}_{i\in\mathbb{I}_m}\in D(\mathcal{V}_{\mathcal{F}})$ then

$$\max_{i\in\mathbb{I}_m} \|T^*_{\mathcal{W}} M_i T_{\mathcal{V}_{\mathcal{F}}}\|_\infty = \max_{i\in\mathbb{I}_m} \|T^*_{\mathcal{W}} M_i T_{\mathcal{V}_{\mathcal{F}}}\|_2 = e_1(\mathcal{V}_{\mathcal{F}},\mathcal{W})\,,$$

since, for $i\in\mathbb{I}_m$, $T^*_{\mathcal{W}} M_i T_{\mathcal{V}_{\mathcal{F}}}$ is a rank one operator and hence its spectral norm and Frobenius norm coincide. △

Theorem 3.9. *Let $\mathcal{V}=\{V_i\}_{i\in\mathbb{I}_m}\in\mathcal{P}(m,\mathbf{k},d)$ with weights $\mathbf{v}=(v_i)_{i\in\mathbb{I}_m}$. If*

$$\|S_{\mathcal{V}}^{-1}V_i^*V_i\|_2 = v_i^2\|S_{\mathcal{V}}^{-1}P_{R(V_i^*)}\|_2 = c \quad \textit{for every} \quad i\in\mathbb{I}_m\ ,$$

then $\mathcal{V}^\#$, the canonical dual RS of $\mathcal{V}$, is the unique 1-loss optimal dual RS for $\mathcal{V}$ (and hence the r-loss optimal dual RS for every r). In other words, $D_1(\mathcal{V})=\{\mathcal{V}^\#\}$.

Proof. By Proposition 3.7, there exists some $\mathcal{W} = \{W_i\}_{i\in \mathbb{I}_m} \in D_1(\mathcal{V})$. Then

$$\|\mathcal{W}\|_{\mathcal{V}} = \max_{i\in \mathbb{I}_m} \|W_i^* V_i\|_2 \leq \max_{i\in \mathbb{I}_m} \|S_{\mathcal{V}}^{-1} V_i^* V_i\|_2 = \|\mathcal{V}^{\#}\|_{\mathcal{V}} = c \ .$$

If we denote each $V_i S_{\mathcal{V}}^{-1} = C_i$, then $\|W_i^* V_i\|_2^2 \leq c = \|C_i^* V_i\|_2^2$ for every $i \in \mathbb{I}_m$. Recall that, by Eq. (3), $\|W_i^* V_i\|_2^2 = v_i^2 \|W_i^*\|_2^2$, since $V_i V_i^* = v_i^2 I_{\mathcal{K}_i}$. Similarly, we get that each $\|C_i^* V_i\|_2^2 = v_i^2 \|C_i^*\|_2^2$. Therefore $\|W_i\|_2^2 \leq \|C_i^*\|_2^2$ for every $i \in \mathbb{I}_m$. Note that

$$\begin{aligned} \|W_i^*\|_2^2 &= \|C_i^* + (W_i^* - C_i^*)\|_2^2 \\ &= \|C_i^*\|_2^2 + \|W_i^* - C_i^*\|_2^2 + 2 \text{ Re}\Big(\text{ tr } \big[(W_i^* - C_i^*) C_i \big] \Big) \end{aligned}$$

and hence $\|W_i^* - C_i^*\|_2^2 + 2 \text{ Re}\Big(\text{ tr } \big[(W_i^* - C_i^*) C_i \big] \Big) \leq 0$, for every $i \in \mathbb{I}_m$. Finally,

$$\sum_{i\in \mathbb{I}_m} \text{tr } \big[(W_i^* - C_i^*) C_i \big] = \text{tr } \Big[(T_{\mathcal{W}}^* - T_{\mathcal{V}^{\#}}^*) T_{\mathcal{V}} S_{\mathcal{V}}^{-1} \Big] = 0 \ ,$$

since both $\mathcal{W}$ and $\mathcal{V}^{\#}$ are dual RS's for $\mathcal{V}$. Then

$$0 \leq \sum_{i\in \mathbb{I}_m} \|W_i^* - C_i^*\|_2^2 = \sum_{i\in \mathbb{I}_m} \|W_i^* - C_i^*\|_2^2 + \sum_{i\in \mathbb{I}_m} 2 \text{ Re}\Big(\text{ tr } \big[(W_i^* - C_i^*) C_i \big] \Big) \leq 0 \ ,$$

which implies that $\mathcal{W} = \{W_i\}_{i\in \mathbb{I}_m} = \{C_i\}_{i\in \mathbb{I}_m} = \mathcal{V}^{\#}$. □

A system $\mathcal{V} \in \mathcal{RS}(m, \mathbf{k}, d)$ is called a **protocol** for $\mathcal{H}$ if $S_{\mathcal{V}} = I_{\mathcal{H}}$. This notion appears in [3], [17] (see also [4], where protocols are related to C^*-encodings with noiseless subsystems).

Corollary 3.10. *Let $\mathcal{V} = \{V_i\}_{i\in \mathbb{I}_m} \in \mathcal{P}(m, \mathbf{k}, d)$ be a projective protocol for $\mathcal{H}$ (i.e. $S_{\mathcal{V}} = I$) such that $\|V_i^* V_i\|_2 = v_i^2 k_i^{1/2} = c$ for every $i \in \mathbb{I}_m$. Then $D_1(\mathcal{V}) = \{\mathcal{V}^{\#}\} = \{\mathcal{V}\}$.*

Proof. By hypothesis $S_{\mathcal{V}} = I_{\mathcal{H}}$, and hence $\|S_{\mathcal{V}}^{-1} V_i^* V_i\|_2 = \|V_i^* V_i\|_2 = c$ for every $i \in \mathbb{I}_m$. Thus, the previous theorem can be applied in this case. □

Remark 3.11. Examples of pojective protocols as in Corollary 3.10 are the equi-dimensional uniform projective protocols i.e., $\{V_i\}_{i\in\mathbb{I}_m} \in \mathcal{P}(m, k\,\mathbb{1}, d)$ that are uniform. These are the analogues of the so-called uniform fusion frames.

As a consequence Theorem 3.9 and Remark 3.8 we recover [16, Thm 2.6]. Moreover, we can conclude from the examples [16, Section 3] that the optimal dual system $\mathcal{W} \in D_1(\mathcal{V})$ may not be the canonical dual RS and may not be unique for a general $\mathcal{V} \in \mathcal{RS}(m, \mathbf{k}, d)$. △

4 Approximation by projective RS's

As we have shown in Theorems 3.1, and 3.9 , there are optimal duals RS for a fixed $\mathcal{V} \in \mathcal{P}(m, \mathbf{k}, d)$, but in general these are not projective. Although there could be some projective elements in $D(\mathcal{V})$ (we shall focus this problem in the following section), we are interested in those $(m, \mathbf{k}, d)$-projective RS's that are closest, with respect to some distance, to a fixed

$\mathcal{S} = \{S_i\}_{i\in\mathbb{I}_m} \in D(\mathcal{V})$ which has some desired properties. Given $\mathcal{W} \in \mathcal{RS}(m,\mathbf{k},d)$, we consider

$$d(\mathcal{S}\,,\,\mathcal{W}) \stackrel{\text{def}}{=} \|T_{\mathcal{S}} - T_{\mathcal{W}}\|_2 = \|T^*_{\mathcal{S}} - T^*_{\mathcal{W}}\|_2\ ,$$

the distance between their synthesis (or analysis) operators. Hence, we seek for $\mathcal{W}_0 \in \mathcal{P}(m,\mathbf{k},d)$ that minimize $d(\mathcal{S}\,,\,\mathcal{W})$ among the projective RS's. In what follows we will describe the structure of such (unique) minimizer in case $\mathcal{S}$ is an injective RS. As one would expect, its "directions" are the coisometries of the polar decompositions of the coordinate operators S_i of $\mathcal{S}$, while its weights are the "averages" of their singular values. We need first some preliminary results:

Given $k, n \in \mathbb{N}$ such that $k \le n$, we denote by

$$\mathcal{I}(k\,,\,n) = \{U \in L(\mathbb{C}^k\,,\,\mathbb{C}^n) : U^*U = I_k\}\ ,$$

the set of isometries from $\mathbb{C}^k$ into $\mathbb{C}^n$. The following result is known in the literature [1]. We include a short proof of it to keep the text self-contained.

Lemma 4.1. *Let $k, n \in \mathbb{N}$ such that $k \le n$, and let $A \in \mathcal{M}_{n,k}(\mathbb{C})$ be a full rank matrix with polar decomposition $A = U\,|A|$ with $U \in \mathcal{I}(k\,,\,n)$. Then*

$$\|A - U\|_2 \;=\; \min_{V\in\mathcal{I}(k\,,\,n)} \|A - V\|_2$$

Proof. Let $V \in \mathcal{I}(k\,,\,n)$. Then $V^*V = I_k$. Therefore

$$\begin{aligned}
\|A - V\|_2^2 &= \operatorname{tr}(A^*A) + \operatorname{tr}(I_k) - 2\,\operatorname{Re}\operatorname{tr}(A^*V)\\
&\ge \operatorname{tr}(A^*A) + k - 2\,|\operatorname{tr}(A^*V)|\\
&\ge \operatorname{tr}(A^*A) + k - 2\,\operatorname{tr}(|A|)
\end{aligned}$$

since $|\operatorname{tr}(A^*V)| \le \|V\|_{sp}\,\operatorname{tr}(|A|)$ and $\|V\|_{sp}^2 = \|V^*V\|_{sp} = \|I\|_{sp} = 1$. On the other hand, if we consider U as the polar factor above then

$$\|A - U\|_2^2 = \operatorname{tr}(A^*A) + \operatorname{tr}(I_k) - 2\,\operatorname{Re}\operatorname{tr}(A^*U) = \operatorname{tr}(A^*A) + \operatorname{tr}(I_k) - 2\,\operatorname{tr}(|A|)\ . \qquad \square$$

Recall that for every $A \in \mathcal{M}_{n,k}(\mathbb{C})$ its polar decomposition satisfies

$$A = U|A| = |A^*|U \implies U^*A = |A| \quad \text{and} \quad A^* = U^*|A^*|\ , \tag{9}$$

where $U \in \mathcal{M}_{n,k}(\mathbb{C})$ has $\ker U = \ker A$.

Proposition 4.2. *Let $\mathcal{S} = \{S_i\}_{i\in\mathbb{I}_m} \in \mathcal{RS}_{\mathcal{I}}(m,\mathbf{k},d)$. Then there exists a unique*

$$\mathcal{W}_0 \in \mathcal{P}(m,\mathbf{k},d) \quad \textit{such that} \quad d(\mathcal{S}\,,\,\mathcal{W}_0) = \min_{\mathcal{W}\in\mathcal{P}(m,\mathbf{k},d)} d(\mathcal{W}\,,\,\mathcal{S})\ , \tag{10}$$

and it is given by $\mathcal{W}_0 = \{\alpha_i\ U_i\}_{i\in\mathbb{I}_m}$ where, for each $i \in \mathbb{I}_m$, $S_i = U_i\,|S_i|$ is the polar decomposition, so that each $U_i^* \in \mathcal{I}(k_i\,,\,d)$ and $\alpha_i = \dfrac{\operatorname{tr}|S_i|}{k_i}$.

Proof. Let $\mathcal{W} \in \mathcal{P}(m, \mathbf{k}, d)$ be a system such that the minimum in (10) is attained at $\mathcal{W}$. Denote by $\mathbf{w} = (w_i)_{i \in \mathbb{I}_m} \in \mathbb{R}_+^m$ the weights of $\mathcal{W}$. Notice that

$$\|T_{\mathcal{S}}^* - T_{\mathcal{W}}^*\|_2^2 = \sum_{i=1}^m \|S_i^* - W_i^*\|_2^2 \quad \text{and} \quad \text{each} \quad W_i W_i^* = w_i^2 \, I_{\mathcal{K}_i} \ ,$$

Thus, each isometry $w_i^{-1} \, W_i^* \in \mathcal{I}(k_i \, , \, d)$ attains the minimum in the optimization problem

$$\big\| \, w_i^{-1} \, W_i^* - \frac{S_i^*}{w_i} \, \big\|_2 = \min_{X \in \mathcal{I}(k_i \, , \, d)} \big\| \, X - \frac{S_i^*}{w_i} \, \big\|_2 \ ,$$

where, by hypothesis, each $w_i^{-1} \, S_i^*$ is a full rank linear transformation. By Lemma 4.1 we get that $w_i^{-1} \, W_i^* = U_i^*$, the isometry of the polar decomposition

$$w_i^{-1} S_i^* = U_i^* \ |w_i^{-1} \, S_i^*| = w_i^{-1} \, \big(U_i^* \ |S_i^*| \big) \ \overset{(9)}{\Longrightarrow} \ S_i = U_i \ |S_i| \quad \text{and} \quad \ker U_i = \ker S_i \ ,$$

and hence $W_i = w_i \, U_i$. Next we show that each $w_i = \frac{\operatorname{tr} |S_i|}{k_i}$. Fix $i \in \mathbb{I}_m$. Then

$$\|S_i - w_i \, U_i\|_2 \ = \ \min_{\alpha > 0} \ \|S_i - \alpha \, U_i\|_2 \ .$$

Therefore $w_i \ \cdot \|U_i\|_2$ is the norm of the orthogonal projection of S_i to the line $\mathbb{R} \, U_i$, using the $\mathbb{R}$-inner product $\langle A, B\rangle = \operatorname{Re} \big[\operatorname{tr}(B^* A) \big]$. It can be computed explicitly:

$$0 \le \frac{\operatorname{tr} \, |S_i|}{\|U_i\|_2} \ \overset{(9)}{=} \ \frac{\operatorname{tr} \, \big(U_i^* \, S_i \big)}{\|U_i\|_2} = \left| \left\langle S_i \, , \, \frac{U_i}{\|U_i\|_2} \right\rangle \right| = \|P_{\mathbb{R} \, U_i}(S_i)\|_2 = w_i \ \cdot \|U_i\|_2 \ ,$$

for every $i \in \mathbb{I}_m$. Then we obtain the equalities $w_i = \|U_i\|_2^{-2} \, \operatorname{tr} |S_i| = \dfrac{\operatorname{tr} |S_i|}{k_i}$. □

Remark 4.3. Let $\mathcal{V} = \{V_i\}_{i \in \mathbb{I}_m} \in \mathcal{P}(m, \mathbf{k}, d)$ and let $\mathcal{W} = \{W_i\}_{i \in \mathbb{I}_m} \in \mathcal{P}(m, \mathbf{k}, d)$ be the nearest projective RS to $\mathcal{V}^{\#}$, as described in Proposition 4.2. Then $\mathcal{W}$ is optimal in a point-wise sense, i.e. it minimizes the Frobenius distance between each $S_{\mathcal{V}}^{-1} V_i^*$ and W_i^*. Nevertheless, we could give different criteria for finding the best dual projective RS for $\mathcal{V}$. Indeed, we pose the following optimization problems, that we consider of interest:

1. Find $\mathcal{W}_0 \in \mathcal{P}(m, \mathbf{k}, d)$ such that
$$\|T_{\mathcal{V}}^* \, T_{\mathcal{W}_0} - I\|_2 \le \|T_{\mathcal{V}}^* \, T_{\mathcal{W}} - I\|_2 \quad \text{for a fixed} \quad \mathcal{W} \in \mathcal{P}(m, \mathbf{k}, d) \ .$$

2. Find $\mathcal{W}_0 \in \mathcal{P}(m, \mathbf{k}, d)$ such that
$$\|T_{\mathcal{W}_0} \, T_{\mathcal{V}}^* - T_{\mathcal{V}} \, S_{\mathcal{V}}^{-1} \, T_{\mathcal{V}}^*\|_2 \le \|T_{\mathcal{W}} \, T_{\mathcal{V}}^* - T_{\mathcal{V}} \, S_{\mathcal{V}}^{-1} T_{\mathcal{V}}^*\|_2 \quad \text{for a fixed} \quad \mathcal{W} \in \mathcal{P}(m, \mathbf{k}, d) \ . \ \triangle$$

5 Examples

In this section we present a variety of examples related with the previous sections. First, we exhibit a family of (projective) RS's which satisfies the hypotheses of Corollary 3.4 and Theorem 3.9, the so-called group RS. In particular, the canonical dual of a group RS (which is also a group RS) is the unique r-loss optimal dual RS for the MSE as well as for the WCRE. The remaining subsections are devoted to the study of particular cases of RS where $D(\mathcal{V}) \cap \mathcal{P}(m, \mathbf{k}, d) \neq \varnothing$. The first examples show that for certain projective systems we can explicitly construct projective duals, which in general will not coincide with the canonical duals. The last example describes a Riesz RS whose unique dual RS (i.e. the canonical dual) is not projective. This leads to a characterization for Riesz RS's with projective canonical dual.

5.1 Group reconstruction systems

We begin by describing the behaviour of the group RS's: Let $\mathcal{K} \cong \mathbb{C}^k$ and $V \in L(\mathcal{H}\,,\,\mathcal{K})$. Given a unitary representation $G \ni g \mapsto U_g \in \mathcal{U}(\mathcal{H})$ of a finite group G in $\mathcal{U}(\mathcal{H})$ we say that

$$\mathcal{V}(G\,,\,V) \stackrel{\text{def}}{=} \{V\,U_g\}_{g\in G}$$

is a G**- reconstruction system** (shortly, G-RS) if $\mathcal{V}(G\,,\,V) \in \mathcal{RS}(m, k\mathbb{1}, d)$, where $m = |G|$. If $\mathcal{S} = V^*(\mathcal{K})$, this is equivalent to the fact that

$$\text{span}\Big\{ \bigcup_{g\in G} U_g(\mathcal{S}) \Big\} = \mathcal{H}\ ,$$

where $V \in L(\mathcal{H}\,,\,\mathcal{K})$ is **the base operator** for $\mathcal{V}(G\,,\,V)$. This definition of G-RS reduces to that of G-frame in the vector frames setting (see [12]).

Observe that the system $\mathcal{V}(G\,,\,V)$ is:

1. Projective (and uniform) if in addition $VV^* = v^2 I_{\mathcal{K}}$ for some $v > 0$;
2. Injective if in addition V^* is injective, in which case also $R(V\,U_g) = \mathcal{K}$ for every $g \in G$.

Notice that the RS-operator of $\mathcal{V}(G\,,\,V)$ has the following structure:

$$S_{G\,,V} \stackrel{\text{def}}{=} S_{\mathcal{V}(G\,,V)} = \sum_{g\in G} U_g^*\,V^*\,V\,U_g = \sum_{g\in G} U_{g^{-1}}\,V^*\,V\,U_g\ .$$

Fix $h \in G$. By the group structure of the representation $\{U_g\}_{g\in G}$, it follows that

$$\begin{aligned} U_h \cdot\ S_{G\,,V} &= U_h \sum_{g\in G} U_{g^{-1}} V^*VU_g \\ &= \sum_{g\in G} U_{hg^{-1}} V^*VU_g \\ &= \sum_{g\in G} U_{(gh^{-1})^{-1}} V^*VU_{gh^{-1}}U_h = S_{G\,,V}\cdot U_h\ . \end{aligned} \tag{11}$$

Then, $S_{G\,,V}$ (and therefore $S_{G\,,V}^{-1}$) commutes with the unitary representation of G. In particular, the canonical dual of a G-RS is another G-RS:

$$\mathcal{V}(G\,,\,V)^{\#} = \{V\,U_g\,S_{G\,,V}^{-1}\}_{g\in G} = \{V\,S_{G\,,V}^{-1}\,U_g\,\}_{g\in G} = \mathcal{V}(G\,,\,V\,S_{G\,,V}^{-1})\ .$$

If the base operator $V \in L(\mathcal{H},\mathcal{K})$ satisfies $VV^* = v^2\,I_{\mathcal{K}}$ then $\mathcal{V}(G\,,\,V)$ is an equi-dimensional uniform projective RS. Therefore, Corollary 3.4 implies that the 1-loss optimal dual RS for $\mathcal{V}(G\,,\,V)$ for the m.s.e. is its canonical dual $\mathcal{V}(G\,,\,V)^{\#}$.

On the other hand, if the base operator $V \in L(\mathcal{H},\mathcal{K})$ satisfies $VV^* = v^2\,I_{\mathcal{K}}$ then $\mathcal{V}(G\,,\,V)$ satisfies the hypothesis of Theorem 3.9. Indeed, if $\mathcal{V}(G\,,\,V)$ is a G-projective RS, then

$$\|(S_{G\,,V}^{-1}\,U_g^*\,V^*)(V\ U_g)\|_2 = \|\,U_g^*\,(S_{G\,,V}^{-1}\,V^*V)\,U_g\,\|_2 = \|S_{G\,,V}^{-1}\,V^*V\|_2 = c\ ,$$

since each $S_{G\,,V}^{-1}\,U_g^* = U_g^*\,S_{G\,,V}^{-1}$ for every $g \in G$ by (11). Thus, the canonical dual of such G-RS's is the unique 1- loss optimal dual for the worst-case error.

If the base operator $V \in L(\mathcal{H}, \mathcal{K})$ is surjective, so that $\mathcal{V}(G, V)$ is an injective RS then, using Proposition 4.2, the projective RS nearest to $\mathcal{V}(G, V)^{\#}$ can be computed in the following way: For every $g \in G$, we have that

$$|V\, S_{G,V}^{-1}\, U_g|^2 = U_g^*\, S_{G,V}^{-1}\, V^*\, V\, S_{G,V}^{-1}\, U_g = U_g^*\, |V S_{G,V}^{-1}|^2\, U_g \;.$$

Taking roots, we get that $|V\, U_g\, S_{G,V}^{-1}| = U_g^* |V S_{G,V}^{-1}| U_g$ for every $g \in G$. Therefore, if we consider the polar decomposition $V\, S_{G,V}^{-1} = W |V\, S_{G,V}^{-1}|$ of $V\, S_{G,V}^{-1}$, then also

$$V\, S_{G,V}^{-1}\, U_g = \big(W\, U_g\big)\, \Big(U_g^*\, |V S_{G,V}^{-1}| U_g\Big) = \big(W\, U_g\big)\, |V\, U_g\, S_{G,V}^{-1}|$$

is the polar decomposition of each entry $V\, S_{G,V}^{-1}\, U_g$ of $\mathcal{V}(G, V)^{\#}$. In conclusion, if we denote

$$w = \frac{\operatorname{tr}\, |V\, S_{G,V}^{-1}|}{k} \quad , \quad \text{then} \quad \mathcal{V}(G, w\, W) = \big\{\, w\, W\, U_g\, \big\}_{g \in G}$$

is the best projective approximation of $\mathcal{V}(G, V)^{\#}$ provided by Proposition 4.2. It is clear from the previous computations that it is again a G-RS.

5.2 Dual projective systems

Now we give an example of a system $\mathcal{V} = \{V_i\}_{i \in \mathbb{I}_m} \in \mathcal{P}(m, \mathbf{k}, d)$ with projective dual systems but such that $\mathcal{V}^{\#} \notin \mathcal{P}(m, \mathbf{k}, d)$:

Example 5.1. Let $d = 3$, $m = 2$ and $\mathbf{k} = (2, 2)$. Let V_1 and $V_2 \in L(\mathbb{C}^3, \mathbb{C}^2)$ be given by

$$V_1(x, y, z) = (y, z) \quad \text{and} \quad V_2(x, y, z) = (x, z) \quad \text{for every} \quad (x, y, z) \in \mathbb{C}^3 \;.$$

Then $\mathcal{V} = (V_1, V_2) \in \mathcal{P}(m, \mathbf{k}, d)$ with weights $\mathbb{1}_2$. If $\mathcal{S}_1 = \{e_1\}^{\perp}$ and $\mathcal{S}_2 = \{e_2\}^{\perp}$, then

$$S_{\mathcal{V}} = V_1^*\, V_1 + V_2^*\, V_2 = P_{\mathcal{S}_1} + P_{\mathcal{S}_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \;.$$

Therefore $\mathcal{V}^{\#} \notin \mathcal{P}(m, \mathbf{k}, d)$, for example because $S_{\mathcal{V}}^{-1}\, V_1^*(u, v) = (0, u, \frac{v}{2})$ for $(u, v) \in \mathbb{C}^2$. So that the entry $V_1\, S_{\mathcal{V}}^{-1}$ of $\mathcal{V}^{\#}$ is not a multiple of a coisometry.

Let $\mathcal{W} = (W_1, W_2) \in \mathcal{P}(m, \mathbf{k}, d)$ and assume that $T_{\mathcal{W}}^*\, T_{\mathcal{V}} = W_1^*\, V_1 + W_2^*\, V_2 = I_3$. Denote by $\{v_1, v_2\}$ the canonical basis of $\mathbb{C}^2$ and by $\{e_1, e_2, e_3\}$ that of $\mathbb{C}^3$. Then, easy computations using the definitions of $\mathcal{V}$ show that

$$e_3 = W_1^*\, v_2 + W_2^*\, v_2 \quad , \quad e_2 = W_1^* V_1\, e_2 = W_1^*\, v_1 \quad \text{and} \quad e_1 = W_1^* V_2\, e_1 = W_2^*\, v_1 \;. \tag{12}$$

The two last equalities show that both W_1^* and W_2^* should be isometries with weight 1. But in this case $\|W_1^*\, v_2\| = \|W_2^*\, v_2\| = 1$ and their sum also has norm one. Let $\omega = \frac{1}{2} + i\frac{\sqrt{3}}{2}$. Then $|\omega| = 1$ but $\omega + \overline{\omega} = 1$. Then we can define $W_1^*, W_2^* \in L(\mathbb{C}^2, \mathbb{C}^3)$ by

$$W_1^*(x, y) = (0, x, \omega\, y) \quad \text{and} \quad W_2^*(x, y) = (x, 0, \overline{\omega}\, y) \quad \text{for every} \quad (x, y) \in \mathbb{C}^2 \;.$$

They are isometries and satisfy the three conditions of (12). Therefore, the system $\mathcal{W} = (W_1, W_2)$ lies in $\mathcal{P}(m, \mathbf{k}, d)$ and it is a dual-RS for $\mathcal{V}$.

Nevertheless, if we consider V_1 and V_2 as operators in $L(\mathbb{R}^3, \mathbb{R}^2)$, then such a $\mathcal{W}$ can not exists. Indeed, looking at Eq. (12), we can deduce that $W_1^*\, v_2 \in \{e_2\}^{\perp}$ and $W_2^*\, v_2 \in \{e_1\}^{\perp}$. These facts, together with the equality $e_3 = W_1^*\, v_2 + W_2^*\, v_2$ imply that both $W_1^*\, v_2$ and $W_2^*\, v_2 \in \operatorname{span}\{e_3\}$, which is impossible in the real case. △

Example 5.2. We can generalize Example 5.1 in the following way: Assume that the projective system $\mathcal{V} = \{V_i\}_{i\in \mathbb{I}_m} \in \mathcal{P}(m, \mathbf{k}, d)$ satisfies that all the projections $P_i = P_{R(V_i^*)}$ are pairwise commuting. We shall see that, in this case,

$$D(\mathcal{V}) \cap \mathcal{P}(m, \mathbf{k}, d) \neq \varnothing \ .$$

Suppose first that all the weights of $\mathcal{V}$ are 1. Then $S_\mathcal{V} = \sum_{i\in\mathbb{I}_m} P_i$. The commutation hypothesis assures that, by taking all the possible intersections among the ranges of the projections P_i , we get a family of projections $(Q_j)_{j\in\mathbb{I}_n}$ in $L(\mathcal{H})$ such that

1. $Q_i\, Q_j = 0$ if $i \neq j$ and $\sum_{j\in\mathbb{I}_n} Q_j = I_\mathcal{H}$.
2. $S_\mathcal{V} = \sum_{j\in\mathbb{I}_n} r_j\, Q_j$ with $r_j \in \mathbb{I}_m$ for every $j \in \mathbb{I}_n$.
3. For every $i \in \mathbb{I}_m$ there exists $\mathbb{J}_i \subseteq \mathbb{I}_n$ such that $P_i = \sum_{j\in\mathbb{J}_i} Q_j$.

We construct the system $\mathcal{W} = \{W_i\}_{i\in \mathbb{I}_m} \in D(\mathcal{V}) \cap \mathcal{P}(m, \mathbf{k}, d)$ as follows: let $W_i = V_i\, U_i$, where $U_i = \sum_{j\in\mathbb{J}_i} \varepsilon_{ij}\, Q_j$ with

$$\varepsilon_{ij} \in \{\ 1\ ,\ \ -1\ ,\ \ \omega\ ,\ \ \overline{\omega}\ \} \quad , \qquad \text{where} \quad \omega = \frac{1}{2} + i\,\frac{\sqrt{3}}{2}\ .$$

Note that each $r_j = |\,\mathbb{S}_j\,|$, where $\mathbb{S}_j = \{i \in \mathbb{I}_m : j \in \mathbb{J}_i\}$. If $r_j = 2s_j + 1$ (the odd case) we select s_j times $\varepsilon_{ij} = -1$ and $(s_j + 1)$ times $\varepsilon_{ij} = 1$ for the r_j pairs (i, j) such that $j \in \mathbb{J}_i$.

If $r_j = 2s_j + 2$ (the even case), we select s_j times $\varepsilon_{ij} = -1$, other s_j times $\varepsilon_{ij} = 1$, plus one $\varepsilon_{ij} = \omega$ and one $\varepsilon_{ij} = \overline{\omega}$ for the r_j pairs (i, j) such that $j \in \mathbb{J}_i$. Observe that all the pairs of indexes involved are those which belong to the set

$$\mathbb{K}_\mathcal{V} = \{(i, j) \in \mathbb{I}_m \times \mathbb{I}_n : j \in \mathbb{J}_i\} = \{(i, j) \in \mathbb{I}_m \times \mathbb{I}_n : i \in \mathbb{S}_j\}\ .$$

As $|\varepsilon_{ij}| = 1$ for every $(i, j) \in \mathbb{K}_\mathcal{V}$, each factor $U_i = \sum\limits_{j\in\mathbb{J}_i} \varepsilon_{ij}\, Q_j$ acts isometrically on $R(P_i)$. Then all the entries $W_i = V_i\, U_i$ of $\mathcal{W}$ are coisometries, so that $\mathcal{W} \in \mathcal{P}(m, \mathbf{k}, d)$. Finally, observe that $V_i^*\, V_i = P_i$ and that $U_i\, P_i = U_i$ for every $i \in \mathbb{I}$. Then

$$\sum_{i\in\mathbb{I}_m} W_i^* V_i \ = \sum_{i\in\mathbb{I}_m} U_i^*\, V_i^*\, V_i = \sum_{i\in\mathbb{I}_m} U_i^* = \sum_{i\in\mathbb{I}_m} \sum_{j\in\mathbb{J}_i} \overline{\varepsilon_{ij}}\, Q_j = \sum_{j\in\mathbb{I}_n} \Big(\sum_{i\in\mathbb{S}_j} \overline{\varepsilon_{ij}} \Big)\, Q_j = I\ ,$$

since $\sum\limits_{i\in\mathbb{S}_j} \varepsilon_{ij} = 1$ for every $j \in \mathbb{I}_n$ by construction of the numbers ε_{ij} .

The general case follows from the previous case. Indeed, if $\mathcal{V}$ has weights $\mathbf{v} = (v_i)_{i\in\mathbb{I}_m}$, we replace the previous $\mathcal{W}$ by $\mathcal{W}_\mathbf{v} = \{v_i^{-2}\, W_i\}_{i\in\mathbb{I}_m}$. △

Remark 5.3 (Projective dual pairs)**.** The previous example gives a method to construct pairs of projective RS's $(\mathcal{V}, \mathcal{W})$ such that $\mathcal{W} \in D(\mathcal{V})$. Moreover, this method shows that for every choice of parameters $(m, \mathbf{k}, d)$ such that $\sum_{i\in\mathbb{I}_m} k_i \geq d$ there exist $\mathcal{V}$, $\mathcal{W} \in \mathcal{P}(m, \mathbf{k}, d)$ such that $\mathcal{W} \in D(\mathcal{V})$. Indeed, to find such a **projective dual pair** $(\mathcal{V}, \mathcal{W})$ we construct $\mathcal{V} \in \mathcal{P}(m, \mathbf{k}, d)$ in such a way that the projections $P_i = V_i^* V_i$ for $i \in \mathbb{I}_m$ are pairwise commuting (i.e. that are simultaneously diagonalizable by an orthonormal basis of $\mathcal{H}$). Then, we can apply the construction in the example above to obtain explicitly the projective dual $\mathcal{W}$.

These facts show that projective dual pairs are indeed more frequent than projective protocols i.e. RS's $\mathcal{V}$ such that $(\mathcal{V}, \mathcal{V})$ is a projective dual pair, since it is known that there are choices of parameters $(m, \mathbf{k}, d)$ for which no projective $(m, \mathbf{k}, d)$- protocols exists (see [19, Example 3.1.2.]). △

5.3 Riesz reconstruction systems

The elements of $\mathcal{RS}(m,\mathbf{k},d)$ are called Riesz RS's if $\dim\mathcal{K} = \operatorname{tr}\mathbf{k} = d = \dim\mathcal{H}$. In this case, every $\mathcal{V} = \{V_i\}_{i\in\mathbb{I}_m} \in \mathcal{RS}(m,\mathbf{k},d)$ satisfies that both the synthesis operator $T_\mathcal{V}$ and the analysis operator $T_\mathcal{V}^*$ are invertible. Also $T_{\mathcal{V}^\#}^* = S_\mathcal{V}^{-1} T_\mathcal{V}^* = T_\mathcal{V}^{-1}$, and

$$\mathcal{H} \;=\; R(V_1^*) \oplus \cdots \oplus R(V_m^*) \qquad \text{(direct sum, but not necessarily orthogonal)}\ , \tag{13}$$

because the sum gives always $\mathcal{H}$ but in this case the sum must be direct by dimensional reasons. On the other hand, if $\mathcal{V} = \{V_i\}_{i\in\mathbb{I}_m} \in \mathcal{RS}(m,\mathbf{k},d)$ is a Riesz RS for $\mathcal{H}$ then

$$D(\mathcal{V}) = \{\mathcal{V}^\#\}\ , \tag{14}$$

since the only left inverse of $T_\mathcal{V}$ is $T_\mathcal{V}^{-1} = T_{\mathcal{V}^\#}^*$. Recall that $T_\mathcal{W}^* T_\mathcal{V} = I_\mathcal{H}$ for every $\mathcal{W} \in D(\mathcal{V})$, and that the map $\mathcal{RS}(m,\mathbf{k},d) \ni \mathcal{W} \mapsto T_\mathcal{W}^*$ is one to one.

Example 5.4. Let $d = 4$, $m = 2$ and $\mathbf{k} = (2,2)$. We now construct a (necessarily) Riesz $\mathcal{V} \in \mathcal{P}(m,\mathbf{k},d)$ such that $D(\mathcal{V}) \cap \mathcal{P}(m,\mathbf{k},d) = \varnothing$. Let V_1, $V_2 \in L(\mathbb{C}^4, \mathbb{C}^2)$ be given by

$$V_1(x_1, x_2, x_3, x_4) \;= (x_1, x_2) \quad \text{and} \quad V_2(x_1, x_2, x_3, x_4) = (x_3, \tfrac{x_2 - x_4}{\sqrt{2}})\ , \tag{15}$$

for $(x_1, x_2, x_3, x_4) \in \mathbb{C}^4$. It is easy to see that $\mathcal{V} = (V_1, V_2) \in \mathcal{P}(m,\mathbf{k},d)$ with weights $(1,1)$. Let us denote by $S = \ker V_2 = \operatorname{span}\{e_1, e_2+e_4\} \subseteq \mathbb{C}^4$. Given $\mathcal{W} \in D(\mathcal{V})$, the equality

$$W_1^* V_1 + W_2^* V_2 = I_\mathcal{H}$$

implies that $W_1^* V_1 x = x$ for every $x \in S$. Then $W_1^* \in L(\mathbb{C}^2, \mathbb{C}^4)$ is completely determined as the inverse of $V_1\big|_S : S \to \mathbb{C}^2$. But we have that

$$\|V_1 e_1\| = \|e_1\| = 1 \quad \text{while} \quad \|V_1(e_2+e_4)\| = \|e_2\| = \frac{\|e_2+e_4\|}{\sqrt{2}}\ .$$

Then $V_1\big|_S$ is not a multiple of an isometry and neither is W_1^*.

We can enlarge the previous example in order to get a RS with redundancy and without projective duals. Indeed, consider $\mathcal{V}_0 = (V_1, V_2, V_3) \in \mathcal{P}(3, (2,2,2), 4)$ obtained from $\mathcal{V}$ by adding any coisometry $V_3 \in L(\mathbb{C}^4, \mathbb{C}^2)$ such that also $\ker V_3 = \mathcal{S}$. Then, arguing as before, we conclude that there is no $\mathcal{W} = (W_1, W_2, W_3) \in D(\mathcal{V}_0)$ such that W_1 is a multiple of a coisometry. △

Remark 5.5. Assume that $\mathcal{V} = \{V_i\}_{i\in\mathbb{I}_m} \in \mathcal{RS}(m,\mathbf{k},d)$ is a Riesz RS. Then, arguing as in the previous example, it is easy to see that the following conditions are equivalent:

1. $D(\mathcal{V}) \cap \mathcal{P}(m,\mathbf{k},d) \neq \varnothing$.
2. $\mathcal{V}^\# \in \mathcal{P}(m,\mathbf{k},d)$.
3. If we denote by $S_i = \bigcap_{j\neq i} \ker V_i = \big(\bigoplus_{j\neq i} R(V^*)\big)^\perp$, then $V_i\big|_{S_i} \in L(S_i, \mathcal{K}_i)$ is a multiple of an isometry, for every $i \in \mathbb{I}_m$.

These conditions are fulfilled if the sum of Eq. (13) is orthogonal. Also if every $k_i = 1$. But there exist other cases. For example, if we take the operator V_1 of Eq. (15), and consider

$$V_3(x_1, x_2, x_3, x_4) = (x_1 - x_3, x_2 - x_4) \quad \text{for} \quad (x_1, x_2, x_3, x_4) \in \mathbb{C}^4\ ,$$

then the condition 3 is satisfied by $\mathcal{V}' = (V_1, V_3) \in \mathcal{P}(2,(2,2),4)$. △

References

[1] Bhatia, Rajendra, *Matrix analysis*, Graduate Texts in Mathematics, 169. Springer-Verlag, New York, 1997. xii+347 pp.

[2] J.J. Benedetto, M. Fickus, *Finite normalized tight frames.* Adv. Comput. Math. 18, No. 2-4, 357-385 (2003).

[3] B.G. Bodmann, *Optimal linear transmission by loss-insensitive packet encoding* Appl. Comput. Harmon. Anal. 22, no. 3, (2007) 274-285.

[4] B.G. Bodmann, D.W. Kribs, V.I. Paulsen *Decoherence-Insensitive Quantum Communication by Optimal C^*-Encoding*, IEEE Transactions on Information Theory 53 (2007) 4738-4749.

[5] B.G. Bodmann, V.I. Paulsen, *Frames, graphs and erasures*, Linear Algebra Appl. 404 (2005) 118-146.

[6] P.G. Casazza, M. Fickus, *Minimizing fusion frame potential.* Acta Appl. Math. 107, No. 1-3, 7-24 (2009).

[7] P.G. Casazza, G. Kutyniok, *Frames of subspaces.* Contemp. Math. 345, 87-113 (2004).

[8] P.G. Casazza, G. Kutyniok, *Robustness of Fusion Frames under Erasures of Subspaces and of Local Frame Vectors.* Contemp. Math. 464, 149-160 (2008).

[9] P.G. Casazza, G. Kutyniok, S. Li, *Fusion Frames and Distributed Processing.* Appl. Comput. Harmon. Anal. 25, No. 1, 114-132 (2008).

[10] P.G. Casazza, G. Kutyniok, S. Li, C.J. Rozell, *Modeling Sensor Networks with Fusion Frames.* Proc. SPIE 6701, 67011M-1-67011M-11 (2007).

[11] P. Gavruta, *On the duality of fusion frames.* J. Math. Anal. Appl. 333 (2007), no. 2, 871-879.

[12] D. Han, *Classification of finite group-frames and super-frames*, Canad. Math. Bull. 50 (2007) 85 -96.

[13] R.B. Holmes, V.I. Paulsen, *Optimal frames for erasures*, Linear Algebra Appl. 377 (2004) 31-51.

[14] A. Khosravi, K. Musazadeh, *Fusion frames and g-frames*, J. Math. Anal. Appl. 342 (2008), No. 2, 1068-1083.

[15] G. Kutyniok, A. Pezeshki, R. Calderbank, T. Liu, *Robust Dimension Reduction, Fusion Frames, and Grassmannian Packings.* Appl. Comput. Harmon. Anal. 26, No. 1, 64-76 (2009).

[16] J. Lopez, D. Han, *Optimal dual frames for erasures*, Linear Algebra Appl. (2009), doi:10.1016/j.laa.2009.08.031.

[17] P. Massey, *Optimal reconstruction systems for erasures and for the q-potential.* Linear Algebra Appl. doi: 10.1016/ j.laa.2009.05.001

[18] P. Massey and M. Ruiz, *Minimization of convex functionals over frame operators*, Adv Comput Math doi:10.1007/s10444-008-9092-5.

[19] P. Massey, M. Ruiz and D. Stojanoff, *The structure of minimizers of the frame potential on fusion frames*, J. Fourier Analysis and Applications, doi:10.1007/s00041-009-9098-5.

[20] P. Massey, M. Ruiz and D. Stojanoff, *Duality in reconstruction systems*, arXiv:1007.0430v1 [math.FA].

[21] C.J. Rozell, D.H. Johnson, *Analyzing the robustness of redundant population codes in sensory and feature extraction systems.* Neurocomputing 69, 1215-1218 (2006).

[22] M. Ruiz, D. Stojanoff, *Some properties of frames of subspaces obtained by operator theory methods.* J. Math. Anal. Appl. 343, No. 1, 366-378 (2008).

[23] T. Strohmer and R.W Heath Jr., *Grassmannian frames with applications to coding and communication*, Appl. Comput. Harmon. Anal. 14 (2003) 257-275.

[24] W. Sun, *G-frames and g-Riesz bases*, J. Math. Anal. Appl. 322 (2006), No. 1, 437-452.

[25] W. Sun, *Stability of g-frames*, J. Math. Anal. Appl. 326 (2007), No. 2, 858-868.

[26] L. Zang, W. Sun, D. Chen, *Excess of a class of g-frames*, J. Math. Anal. Appl. 352 (2009), no. 2, 711-717.

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