Products of orthogonal projections and polar decompositions

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Abstract

We characterize the sets $\mathfrak{X}$ of all products PQ, and $\mathfrak{Y}$ of all products PQP, where P, Q run over all orthogonal projections and we solve the problems $\arg\min\{\|P-Q\| : (P,Q) \in \mathcal{Z}\}$, for $\mathcal{Z} = \mathfrak{X}$ or $\mathfrak{Y}$. We also determine the polar decompositions and Moore-Penrose pseudoinverses of elements of $\mathfrak{X}$.

1 Introduction

Let $\mathcal{H}$ be a Hilbert space; denote by $L(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$ and by $\mathcal{P}$ the set of all orthogonal projections in $L(\mathcal{H})$: $\mathcal{P} = \{P \in L(\mathcal{H}) : P^2 = P = P^*\}$. The main goal of this paper is the study of the sets

$$\mathfrak{X} = \{PQ : P, Q \in \mathcal{P}\}$$

and

$$\mathfrak{Y} = \{PQP : P, Q \in \mathcal{P}\}.$$

In general, an operator $T \in \mathfrak{X}$ admits many factorizations like PQ. Crimmins (see comments below) proved that if $T \in \mathfrak{X}$ then $T = P_{\overline{R(T)}} P_{N(T)^\perp}$ (hereafter, $P_\mathcal{M}$ denotes the orthogonal projection onto the closed subspace $\mathcal{M}$, and $R(B)$, $N(B)$ denote the range and nullspace of B, respectively, for every operator $B \in L(\mathcal{H})$). We characterize the set $\mathfrak{X}_T = \{(P,Q) : P, Q \in \mathcal{P}, T = PQ\}$ and prove that the distinguished pair $(P_{\overline{R(T)}}, P_{N(T)^\perp}) \in \mathfrak{X}_T$ is optimal in several senses. We study a similar problem for each $S \in \mathfrak{Y}$: we characterize the set $\mathfrak{Y}_S = \{(P,Q) : P,Q \in \mathcal{P},\ S = PQP\}$ and find all pairs $(P_0, Q_0) \in \mathfrak{Y}_S$ such that $\|P_0 - Q_0\| = \min\{\|P-Q\| : (P,Q) \in \mathfrak{Y}_S\}$. We also study the polar decomposition of operators in $\mathfrak{X}$ and show that the Moore-Penrose pseudoinverse operation is a bijection between $\mathfrak{X}$ and the set $\tilde{\mathcal{Q}}$ of all closed (unbounded) projections. This bijection explains the coincidence between the set of all partial isometries which appear in the polar decomposition of oblique (i.e., not necessarily orthogonal) projections and those which appear in the polar decomposition of operators of $\mathfrak{X}$.

Products of orthogonal projections have attracted the attention of mathematicians from many different areas as functional analysis, mathematical physics, signal processing, numerical analysis, statistics, and so on. We refer the reader to recent surveys by A. Galántai [12] and A. Böttcher and I. M. Spitkovsky [5], which contain a large bibliography and several historical remarks. To their list we add a few papers which are closer to our results. I. Vidav [28] studied the polar factors of oblique projections, and obtained several results which we recently rediscovered in [8]. In a paper of H. Radjavi and J. P. Williams on products of selfadjoint operators [25] there is a proof of a theorem by T. Crimmins which characterizes the operators of $\mathfrak{X}$ in the following concise way: if $T \in L(\mathcal{H})$ then T belongs to $\mathfrak{X}$ if and only if $T^2 = TT^*T$; Crimmins also exhibited, for such T's, what we call the *canonical factorization* $T = P_{\overline{R(T)}} P_{N(T)^\perp}$. In [27] Z. Sebestyén found a condition on an operator T defined on a subspace of $\mathcal{H}$ in order to be the restriction of an orthogonal projection. We prove here that Sebestyén's condition is equivalent to Crimmins'. More recently, A. Arias and S. Gudder [2] studied, in the more general setting of von Neumann algebras, what they call *almost sharp effects*, and which are, precisely, operators like PQP, for $P,Q \in \mathcal{P}$. These effects play a role in some problems of quantum mechanics. They found a characterization of the set $\mathfrak{Y}$, which is very useful in our approach. It should be mentioned that in a complete different setting, S. Nelson and M. Neumann [20] found, for matrices, a characterization of the spectrum of elements of $\mathfrak{X}$. It turns out that their conditions can be easily translated to the Arias-Gudder's theorem. T. Oikhberg [21], [22] proved many results on operators which can be factorized as finite products of orthogonal projections. We close these comments by mentioning that some modern approaches to Heisenberg uncertainty principle, like those of Donoho and Stark [10] and Havin and Jöricke [15] (see also the survey by Folland and Sitaram [11]) are based on the compactness and spectral properties of certain products PQ, where P and Q respectively project onto time-limited and band-limited signals.

We describe the contents of the sections. Section 2 contains some preliminary results. In section 3 we study some properties of operators of $\mathfrak{X}$ and characterize the set $\mathfrak{X}_T$ for $T \in \mathfrak{X}$, and we prove that the canonical factorization $T = P_{\overline{R(T)}} P_{N(T)^\perp}$ is optimal in the following senses: if $T = P_\mathcal{M} P_\mathcal{N}$ for some closed subspaces $\mathcal{M}$, $\mathcal{N}$, then (1) $\overline{R(T)} \subseteq \mathcal{M}$ and $N(T)^\perp \subseteq \mathcal{N}$; (2) $\|(P_\mathcal{M} - P_\mathcal{N})x\| \geq \|(P_{\overline{R(T)}} - P_{N(T)^\perp})x\|$ for all $x \in \mathcal{H}$; and (3) if $R(T)$ is closed then $\|P_{\overline{R(T)}} - P_{N(T)^\perp}\| < \|P_\mathcal{M} - P_\mathcal{N}\|$ for every other $(\mathcal{M}, \mathcal{N}) \in \mathfrak{X}_T$. In section 4 we start the study of the set $\mathfrak{Y}$, by solving the problem $\arg\min\{\|P-Q\| : (P,Q) \in \mathfrak{Y}_S\}$ for each $S \in \mathfrak{Y}$. We include a theorem, whose proof is due to T. Ando, which describes, for fixed $P, Q \in \mathcal{P}$, the set $\{H \in \mathcal{P} : (PHP)^2 = PQP\}$. Section 5 is devoted to polar decompositions of elements of $\mathfrak{X}$. We characterize the set $\mathcal{J}_{\mathfrak{X}}$ (resp., $\mathfrak{X}^+$) of isometric (resp., positive) parts of operators in $\mathfrak{X}$. In particular, we prove that $\mathfrak{X} = \{V^2 : V \in \mathcal{J}_{\mathfrak{X}}\} = \mathfrak{Y}$ and the map $T \longrightarrow V$, where V is the isometric part of T, is a bijection between $\mathfrak{X}$ and $\mathcal{J}_{\mathfrak{X}}$. The situation for the positive parts is different: using Ando's theorem mentioned above, we parametrize, for every $S \in \mathfrak{Y}$, the set $\{T \in \mathfrak{X} : |T| = S\}$. In the last section we prove that the Moore-Penrose pseudoinverse of $T \in \mathfrak{X}$ is a closed unbounded oblique projection, and conversely. Using some results of Ota [23] on closed unbounded projections, we extend a well-known theorem of Penrose [24] and Greville [14], who proved this result for matrices.

2 Preliminaries

Denote $Gr(\mathcal{H})$ the Grassmannian manifold of $\mathcal{H}$, i.e., the set of all closed subspaces $\mathcal{M}$ of $\mathcal{H}$.

The *Friedrichs angle* between $\mathcal{M} \in Gr(\mathcal{H})$ and $\mathcal{N} \in Gr(\mathcal{H})$ is $\alpha(\mathcal{M},\mathcal{N}) \in [0,\pi/2]$ whose cosine is

$$c(\mathcal{M},\mathcal{N}) = \sup\{|\langle m,n\rangle| : \ m \in \mathcal{M} \ominus \mathcal{N},\ \|m\| \le 1,\ n \in \mathcal{N} \ominus \mathcal{M},\ \|m\| \le 1\},$$

where $\mathcal{M} \ominus \mathcal{N} = \mathcal{M} \cap (\mathcal{M} \cap \mathcal{N})^{\perp}$.

The *Dixmier angle* between $\mathcal{M}$ and $\mathcal{N}$ is $\alpha_0(\mathcal{M},\mathcal{N}) \in [0,\pi/2]$ whose cosine is

$$c_0(\mathcal{M},\mathcal{N}) = \sup\{|\langle m,n\rangle| : \ m \in \mathcal{M},\ \|m\| \le 1,\ n \in \mathcal{N},\ \|m\| \le 1\}.$$

It is easy to see that $c_0(\mathcal{M},\mathcal{N}) = \|P_{\mathcal{M}} P_{\mathcal{N}}\|$; we collect several well-known facts on c and c_0. The proofs can be found in the survey by F. Deutsch [9].

Theorem 2.1. *Given $\mathcal{M},\mathcal{N} \in Gr(\mathcal{H})$ the following statements hold:*

1. *$c(\mathcal{M},\mathcal{N}) < 1$ if and only if $\mathcal{M} + \mathcal{N}$ is closed if and only if $R(P_{\mathcal{M}}(I - P_{\mathcal{N}}))$ is closed;*
2. *$c_0(\mathcal{M},\mathcal{N}) < 1 \iff \mathcal{M} \cap \mathcal{N} = \{0\}$ and $\mathcal{M} + \mathcal{N}$ is closed;*
3. *$c(\mathcal{M},\mathcal{N}) = c(\mathcal{M}^{\perp},\mathcal{N}^{\perp})$, i.e., the Friedrichs angle between $\mathcal{M}$ and $\mathcal{N}$ coincides with that between $\mathcal{M}^{\perp}$ and $\mathcal{N}^{\perp}$; in particular, $\mathcal{M} + \mathcal{N}$ is closed if and only if $\mathcal{M}^{\perp} + \mathcal{N}^{\perp}$ is closed.*

We will use the well known *Krein-Krasnoselskii-Milman equality*

$$\|P - Q\| = \max\{\|P(I - Q)\|,\ \|Q(I - P)\|\}, \tag{1}$$

valid for all $P, Q \in \mathcal{P}$ (see [19], [1], [18]).

Proposition 2.2. *Given $P, Q \in \mathcal{P}$, there are four possible cases for the norms involved in Krein-Krasnoselskii-Milman equality, namely:*

1. *$\|P - Q\| < 1$ and, then, $\|P(I - Q)\| = \|Q(I - P)\| < 1$;*

2. $\|P-Q\| = \|P(I-Q)\| = 1$ *and* $\|Q(I-P)\| < 1$*;*

3. $\|P-Q\| = \|Q(I-P)\| = 1$ *and* $\|P(I-Q)\| < 1$;

4. $\|P-Q\| = \|Q(I-P)\| = \|P(I-Q)\| = 1$.

In terms of the ranges and nullspaces of P, Q*, the four possibilities read as follows:*

1. $R(P)\dot{+}N(Q) = N(P)\dot{+}R(Q) = \mathcal{H}$ *and the angles of both decompositions coincide;*

2. $R(P) + N(Q) = \mathcal{H}$*, the sum is not direct and* $N(P) + R(Q)$ *is a proper closed subspace;*

3. $N(P) + R(Q) = \mathcal{H}$*, the sum is not direct and* $R(P) + N(Q)$ *is a proper closed subspace;*

4. $N(P) + R(Q)$ *and* $R(P) + N(Q)$ *are non closed dense subspaces of* $\mathcal{H}$*.*

Recall the definition of the Moore-Penrose pseudoinverse $T^\dagger$ of $T \in L(\mathcal{H})$. This is an operator with domain $R(T) \oplus R(T)^\perp$ defined by $T^\dagger(Tx) = x$ if $x \in N(T)^\perp$ and $T^\dagger|_{R(T)^\perp} = 0$. The reader is referred to the original paper by Penrose [24] or the book by Ben-Israel and Greville [4] for properties and theorems on $T^\dagger$. We will use without explicit mention that $T^\dagger$ is bounded if and only if $R(T)$ is closed. Notice that $T^\dagger T$ and $TT^\dagger$ behaves in a different way: the first one is always bounded; indeed, it coincides with $P_{N(T)^\perp}$; however, the second is defined, and behaves like a projection, on the domain of $T^\dagger$.

3 The set of products PQ

In this section we study the sets

$$\mathfrak{X} = \{PQ : \ P,\ Q\ \in \mathcal{P}\}, \quad \mathfrak{X}_{cr} = \{T \in \mathfrak{X} : \ R(T) \text{ is closed}\}.$$

We start with a theorem that gives two alternative characterizations of the elements of $\mathfrak{X}$. The first one is due to T. Crimmins (item 2), see Radjavi and Williams [25], Theorem 8. The second (item 3) is a rewriting of a result by Z. Sebestyén for suboperators, see [27], Theorem 1.

Theorem 3.1. *For any* $T \in L(\mathcal{H})$*, the following assertions are equivalent:*

1. $T \in \mathfrak{X}$*;*

2. $T^2 = TT^*T$*;*

3. $\|Tx\|^2 = \langle Tx, x\rangle$*, for all* $x \in N(T)^\perp$*.*

In this case, $T = P_{\overline{R(T)}}P_{N(T)^\perp}$*.*

We will refer to the factorization obtained in the above theorem as the *canonical factorization* of T.

Proof. $1 \to 3$: If $T \in \mathfrak{X}$ there exist $P, Q \in \mathcal{P}$ such that $T = PQ$. Observe that $N(Q) \subseteq N(T)$ so that $N(T)^\perp \subseteq N(Q)^\perp$ and then $QP_{N(T)^\perp} = P_{N(T)^\perp}$, or $Qx = x$, for all $x \in N(T)^\perp$. Therefore, if $x \in N(T)^\perp$, then $\|Tx\|^2 = \langle T^*Tx, x\rangle = \langle QPQx, x\rangle = \langle PQx, Qx\rangle = \langle Tx, x\rangle$, as wanted.

$3 \to 2$: If $\|Tx\|^2 = \langle Tx, x\rangle$, for all $x \in N(T)^\perp$, then $\langle Ty, Ty\rangle = \langle Ty, P_{N(T)^\perp}y\rangle$, for all $y \in \mathcal{H}$, because $TP_{N(T)^\perp} = T$. Hence $\langle T^*Ty, y\rangle = \langle P_{N(T)^\perp}Ty, y\rangle$ for all $y \in \mathcal{H}$, or $T^*T = P_{N(T)^\perp}T = T^\dagger T^2$. Therefore, multiplying by T both sides of this equality, $TT^*T = TT^\dagger T^2$. But observe that $TT^\dagger$ is the orthogonal projection onto $\overline{R(T)}$, restricted to $R(T)$, and $R(T^2) \subseteq R(T)$. Then $TT^*T = T^2$.

$2 \to 1$: If $TT^*T = T^2$ then multiplying by (the possibly unbounded operator) $T^\dagger$ both sides of this equality, we get $P_{N(T)^\perp}T^*T = P_{N(T)^\perp}T$, and taking adjoints $T^*TP_{N(T)^\perp} = T^*P_{N(T)^\perp}$. Multiplying by $T^{*\dagger}$, we get $P_{N(T^*)^\perp}TP_{N(T)^\perp} = P_{N(T^*)^\perp}P_{N(T)^\perp}$. But using that $N(T^*)^\perp = \overline{R(T)}$ and that $T = P_{\overline{R(T)}}TP_{N(T)^\perp}$, it follows the equality $T = P_{\overline{R(T)}}P_{N(T)^\perp}$ so that in particular $T \in \mathfrak{X}$. ∎

It is obvious that $T^* \in \mathfrak{X}$ if $T \in \mathfrak{X}$. By the formula $T = P_{\overline{R(T)}}P_{N(T)^\perp}$, it is clear that T is determined by the closed subspaces $\overline{R(T)}$ and $N(T)$.

Theorem 3.2. *Every $T \in \mathfrak{X}$ has the following properties:*

1. $\overline{R(T)} \cap N(T) = \{0\}$*;*
2. $\overline{R(T)} \dot{+} N(T)$ *is dense;*
3. $\overline{R(T)} \dot{+} N(T) = \mathcal{H}$ *if and only if* $R(T)$ *is closed.*

Proof. 1. Let $x \in \overline{R(T)} \cap N(T)$. Then $P_{N(T)^\perp}x = 0$ and $x = P_{\overline{R(T)}}x$. Therefore, $0 = P_{N(T)^\perp}x = P_{N(T)^\perp}P_{\overline{R(T)}}x = T^*x$ so that $x \in N(T^*) = R(T)^\perp$. Thus, $x \in \overline{R(T)} \cap R(T)^\perp = \{0\}$.

2. If $T \in \mathfrak{X}$ then also $T^* \in \mathfrak{X}$. Applying 1 to T^* we get $N(T^*) \cap \overline{R(T^*)} = \{0\}$, or $R(T)^\perp \cap N(T)^\perp = \{0\}$. Taking orthogonal complements we get that $\overline{R(T)} \dot{+} N(T)$ is dense.

3. Recall from Theorem 2.1 that $\mathcal{M} + \mathcal{N}^\perp$ is closed if and only if $R(P_\mathcal{M}P_\mathcal{N})$ is closed and apply this to $\mathcal{M} = \overline{R(T)}$, $\mathcal{N} = N(T)^\perp$. Since $T = P_\mathcal{M}P_\mathcal{N}$, from 2 we get the result. ∎

Corollary 3.3. *For any P, $Q \in \mathcal{P}$ there exists only two alternatives:*

1. $R(PQ)$ *is closed and* $R(PQ) \dot{+} N(PQ) = \mathcal{H}$*; or*
2. $R(PQ)$ *is not closed and* $\overline{R(PQ)} \dot{+} N(PQ)$ *is a proper dense subspace of* $\mathcal{H}$*.*

The next result is a reformulation of the canonical factorization property.

Theorem 3.4. *Let $T \in \mathfrak{X}$. There exists a factorization $T = P_{\mathcal{M}}P_{\mathcal{N}}$ such that $\mathcal{M}\dot{+}\mathcal{N}^{\perp} = \mathcal{H}$ if and only if $R(T)$ is closed. In this case, there exists only one such factorization, namely $T = P_{R(T)}P_{N(T)^{\perp}}$, which corresponds to the decomposition $\mathcal{H} = R(T)\dot{+}N(T)$.*

Proof. Observe that, by Theorem 3.2, if $R(T)$ is closed then $R(T)\dot{+}N(T) = \mathcal{H}$ and $T = P_{R(T)}P_{N(T)^{\perp}}$.

Conversely, if $T = P_{\mathcal{M}}P_{\mathcal{N}}$ and $\mathcal{M}\dot{+}\mathcal{N}^{\perp} = \mathcal{H}$, then in particular $\mathcal{M} + \mathcal{N}^{\perp}$ is closed and, therefore, $R(T) = R(P_{\mathcal{M}}P_{\mathcal{N}})$ is closed (see [6] or [16]). The uniqueness follows from the general lemma below. ∎

Lemma 3.5. *If $\mathcal{M}\dot{+}\mathcal{N} = \mathcal{H}$, $\mathcal{M}_1\dot{+}\mathcal{N}_1 = \mathcal{H}$, $\mathcal{M} \supseteq \mathcal{M}_1$ and $\mathcal{N} \supseteq \mathcal{N}_1$ then $\mathcal{M} = \mathcal{M}_1$ and $\mathcal{N} = \mathcal{N}_1$.*

Proof. Straightforward. ∎

Remark 3.6. If $P, Q \in \mathcal{P}$ and $R(PQ)$ is closed, Theorem 3.4 and Corollary 3.3 do not imply that $R(P)\dot{+}N(Q) = \mathcal{H}$; however, it does imply that the operator $T = PQ$ admits a factorization $T = P'Q'$ such that $R(P')\dot{+}N(Q') = \mathcal{H}$.

Our next result describes all factorizations $T = P_{\mathcal{M}}P_{\mathcal{N}}$ for a given $T \in \mathfrak{X}$ and shows that the canonical factorization is optimal, in the following two senses: (1) if $T = P_{\mathcal{M}}P_{\mathcal{N}}$ then $\mathcal{M} \supseteq \overline{R(T)}$ and $\mathcal{N} \supseteq N(T)^{\perp}$ or equivalently $P_{\mathcal{M}} \geq P_{\overline{R(T)}}$ and $P_{\mathcal{N}} \geq P_{N(T)^{\perp}}$; (2) if $T = P_{\mathcal{M}}P_{\mathcal{N}}$ then $\|(P_{\mathcal{M}} - P_{\mathcal{N}})x\| \geq \|(P_{\overline{R(T)}} - P_{N(T)^{\perp}})x\|$, for all $x \in \mathcal{H}$.

Theorem 3.7. *Let $T \in \mathfrak{X}$ and $\mathcal{M}$, $\mathcal{N} \in Gr(\mathcal{H})$. Then $T = P_{\mathcal{M}}P_{\mathcal{N}}$ if and only if there exist $\mathcal{M}_1$, $\mathcal{N}_1 \in Gr(\mathcal{H})$ such that*

1. $\mathcal{M} = \overline{R(T)} \oplus \mathcal{M}_1$;
2. $\mathcal{N} = N(T)^{\perp} \oplus \mathcal{N}_1$;
3. $\mathcal{M}_1 \perp \mathcal{N}_1$;
4. $\mathcal{M}_1 \oplus \mathcal{N}_1 \subseteq R(T)^{\perp} \cap N(T)$.

Proof. By Crimmins' theorem, it holds $T = P_{\overline{R(T)}}P_{N(T)^{\perp}}$. If $T = P_{\mathcal{M}}P_{\mathcal{N}}$ then, in particular, $R(T) \subseteq \mathcal{M}$ and, since $\mathcal{M}$ is closed, $\overline{R(T)} \subseteq \mathcal{M}$. Analogously, $\mathcal{N}^{\perp} = N(P_{\mathcal{N}}) \subseteq N(T)$ and therefore $\mathcal{N} \supseteq N(T)^{\perp}$. Thus, $\mathcal{M}_1 := \mathcal{M} \ominus \overline{R(T)}$ and $\mathcal{N}_1 := \mathcal{N} \ominus N(T)^{\perp}$ are well-defined and items 1 and 2 are verified. Also, $\mathcal{M}_1 \subseteq R(T)^{\perp}$ and $\mathcal{N}_1 \subseteq N(T)$.

Now we compute $T = P_{\mathcal{M}}P_{\mathcal{N}}$, using the decompositions 1 and 2, and we get

$$P_{\overline{R(T)}}P_{N(T)^{\perp}} = T = P_{\mathcal{M}}P_{\mathcal{N}} = (P_{\overline{R(T)}} + P_{\mathcal{M}_1})(P_{N(T)^{\perp}} + P_{\mathcal{N}_1}) =$$

$$= P_{\overline{R(T)}}P_{N(T)^{\perp}} + P_{\overline{R(T)}}P_{\mathcal{N}_1} + P_{\mathcal{M}_1}P_{N(T)^{\perp}} + P_{\mathcal{M}_1}P_{\mathcal{N}_1}$$

and, after cancellation,

$$P_{\overline{R(T)}}P_{\mathcal{N}_1} + P_{\mathcal{M}_1}P_{N(T)^\perp} + P_{\mathcal{M}_1}P_{\mathcal{N}_1} = 0 \tag{2}$$

By multiplying at left equation (2) by $P_{\overline{R(T)}}$, we get $P_{\overline{R(T)}}P_{\mathcal{N}_1} = 0$, because $\mathcal{M}_1 \perp \overline{R(T)}$. From here we deduce also that $\mathcal{N}_1 \subseteq R(T)^\perp$.

We have now

$$P_{\mathcal{M}_1}P_{N(T)^\perp} + P_{\mathcal{M}_1}P_{\mathcal{N}_1} = 0 \tag{3}$$

and, by multiplying at right by $P_{N(T)^\perp}$ we get

$$P_{\mathcal{M}_1}P_{N(T)^\perp} = 0 \tag{4}$$

because $\mathcal{N}_1 \perp N(T)^\perp$; thus,

$$P_{\mathcal{M}_1}P_{\mathcal{N}_1} = 0 \tag{5}$$

and also $\mathcal{M}_1 \subseteq N(T)$. This completes the first part.

Conversely, if $\mathcal{M}_1$, $\mathcal{N}_1$ satisfies 1-4 then

$$P_\mathcal{M}P_\mathcal{N} = (P_{\overline{R(T)}} + P_{\mathcal{M}_1})(P_{N(T)^\perp} + P_{\mathcal{N}_1}) = P_{\overline{R(T)}}P_{N(T)^\perp} = T,$$

because all other products vanish. ∎

Corollary 3.8. *Let $T \in \mathfrak{X}$. Then T admits a unique factorization $T = P_\mathcal{M}P_\mathcal{N}$ if and only if $R(T)^\perp \cap N(T) = \{0\}$.*

Corollary 3.9. *Let $T \in \mathfrak{X}$. If $T = P_\mathcal{M}P_\mathcal{N}$ then $\|(P_\mathcal{M} - P_\mathcal{N})x\| \geq \|(P_{\overline{R(T)}} - P_{N(T)^\perp})x\|$ for all $x \in \mathcal{H}$.*

Proof. In fact, $P_\mathcal{M} - P_\mathcal{N} = (P_{\overline{R(T)}} - P_{N(T)^\perp}) + (P_{\mathcal{M}_1} - P_{\mathcal{N}_1})$ and the images of both terms are orthogonal so $\|P_\mathcal{M}x - P_\mathcal{N}x\|^2 = \|P_{\overline{R(T)}}x - P_{N(T)^\perp}x\|^2 + \|P_{\mathcal{M}_1}x - P_{\mathcal{N}_1}x\|^2$. ∎

In what follows, for each $T \in \mathfrak{X}$ denote $\mathfrak{X}_T := \{(P, Q) : T = PQ\}$.

Theorem 3.10. *Let $T \in \mathfrak{X}$. If $R(T)$ is not closed, then $\|P - Q\| = 1$ for all $(P, Q) \in \mathfrak{X}_T$. If $R(T)$ is closed, then $\|P_{R(T)} - P_{N(T)^\perp}\| < 1$ and $\|P - Q\| = 1$ for every other $(P, Q) \in \mathfrak{X}_T$.*

Proof. If $R(T)$ is not closed, then by Theorem 3.2, it follows that $\overline{R(T)}\dot{+}N(T)$ is a dense proper subspace of $\mathcal{H}$ and, therefore, by (1) and Theorem 2.1 $\|P_{\overline{R(T)}} - P_{N(T)^\perp}\| = 1$; by the corollary above it follows that $\|P - Q\| = 1$ for all $(P, Q) \in \mathfrak{X}_T$.

If $R(T)$ is closed, then $\mathcal{H} = R(T)\dot{+}N(T)$ then, by Theorem 2.1, $c(R(T), N(T)) = c_0(R(T), N(T)) = \|P_{R(T)}P_{N(T)}\| = \|P_{R(T)}(I - P_{N(T)^\perp})\| < 1$. Also, T^* has closed range and in the same way, we obtain that $\|P_{N(T)^\perp}P_{R(T)^\perp}\| < 1$, but $\|P_{N(T)^\perp}P_{R(T)^\perp}\| = \|(I - P_{R(T)})P_{N(T)^\perp}\|$. Applying (1), we get $\|P_{R(T)} - P_{N(T)^\perp}\| < 1$.

Finally, according to Theorem 3.4, it follows that $(P_{R(T)}, P_{N(T)^\perp})$ is the only element of $\mathfrak{X}_T$ with that property. Thus, if (P, Q) is another element of $\mathfrak{X}_T$ then $R(P) + N(Q) = \mathcal{H}$ but the sum is not direct. Therefore $\|P - Q\| = 1$. ∎

4 The set of products PQP

Denote $\mathfrak{Y} = \{PQP : \ P, Q \in \mathcal{P}\}$ and for $S \in \mathfrak{Y}$ denote $\mathfrak{Y}_S = \{(P,Q) : \ S = PQP\}$. This section is devoted to the study of these sets, following the lines of the preceding section. First, we describe the set $\mathfrak{Y}_S$ for a given $S \in \mathfrak{Y}$.

Proposition 4.1. *The set $\mathfrak{Y}_S$ is the disjoint union of all sets $\mathfrak{X}_T$, where $T \in \mathfrak{X}$ satisfies $TT^* = S$.*

Proof. If $(P,Q) \in \mathfrak{Y}_S$, then $S = PQP$, $T := PQ \in \mathfrak{X}$ and $(P,Q) \in \mathfrak{X}_T$. Conversely, if $(P,Q) \in \mathfrak{X}_T$ for some $T \in \mathfrak{X}$ such that $S = TT^*$, then $S = PQP$, i.e., $(P,Q) \in \mathfrak{Y}_S$ ∎

The set $\mathfrak{Y}$ was completely described by Arias and Gudder [2]. They proved that a positive operator $A \in L(\mathcal{H})$ belongs to $\mathfrak{Y}$ if and only if $A \leq I$ and $\dim\overline{R(A - A^2)} \leq \dim N(A)$. (Indeed, they proved a more complete result, valid for von Neumann algebras; in the case of factors, their result has the form we mentioned.)

Given $S \in \mathfrak{Y}$, we compute the norm $\|P - Q\|$ for every $(P,Q) \in \mathfrak{Y}_S$.

Theorem 4.2. *Let $S \in \mathfrak{Y}$. Then:*

1. *If $R(S)$ is not closed then $\|P - Q\| = 1$ for every pair $(P,Q) \in \mathfrak{Y}_S$.*
2. *If $R(S)$ is closed, then for each pair $(P,Q) \in \mathfrak{Y}_S$ and $T = PQ$ the following alternative holds: either $T = PQ$ is not the canonical factorization of T, and then $\|P - Q\| = 1$, or $P = P_{R(T)}$ and $Q = P_{N(T)^\perp}$, in which case $\|P_{R(T)} - P_{N(T)^\perp}\|$ is a constant < 1 which is independent of the factorization $S = TT^*$; more precisely, $\|P_{R(T)} - P_{N(T)^\perp}\| = \|P_{R(S)} - S\|^{1/2}$.*

Proof. Consider $S \in \mathfrak{Y}$. 1) If $R(S)$ is not closed then for every $T \in \mathfrak{X}$ such that $TT^* = S$, it holds that $R(T)$ is not closed; by Theorem 3.10, it follows that $\|P - Q\| = 1$ for every pair $(P,Q) \in \mathfrak{X}_T$ and so, by Proposition 4.1, the same is true for every $(P,Q) \in \mathfrak{Y}_S$.

2)If $R(S)$ is closed, fix $T \in \mathfrak{X}$ such that $TT^* = S$. By Theorem 3.10, $\|P - Q\| = 1$ for every pair $(P,Q) \in \mathfrak{X}_T$ except for the canonical pair $(P_{R(T)}, P_{N(T)^\perp})$, for which $\|P_{R(T)} - P_{N(T)^\perp}\| < 1$. Consider another $L \in \mathfrak{X}$ such that $LL^* = S$. We claim that $\|P_{R(T)} - P_{N(T)^\perp}\| = \|P_{R(L)} - P_{N(L)^\perp}\| < 1$. In order to prove this assertion, we make a series of remarks.

1. Observe that $R(S) = R(T) = R(L)$; denote $P = P_{R(S)}$.
2. If $E, F \in \mathcal{P}$ then from 1 of Propostion 2.2, it easily follows that if $\|E - F\| < 1$ then $\|E - F\| = \|E(I - F)\| = \|(I - E)F\|$.
3. Since $\|P - P_{N(T)^\perp}\| < 1$, then $\|P - P_{N(T)^\perp}\| = \|P(I - P_{N(T)^\perp}\| = \|PP_{N(T)}\|$.
4. Observe that $S = TT^* = PP_{N(T)^\perp}P = P - PP_{N(T)}P$, so that $PP_{N(T)}P = P - S$.

Thus, by items (3) and (4), it follows that $\|P - P_{N(T)^\perp}\|^2 = \|PP_{N(T)}\|^2 =$
$= \|PP_{N(T)}P\| = \|P - S\|$. ∎

Remark 4.3. The proof above shows that, if $S \in \mathfrak{Y}$ has a closed range, then the set $\mathfrak{Y}_S$ is the union of two disjoint subsets, say $\mathcal{W} = \{(P,Q) \in \mathfrak{Y}_S : R(P)\dot{+}N(Q) = \mathcal{H}\}$ and $\mathcal{Z} = \{(P,Q) \in \mathfrak{Y}_S : R(P) + N(Q) = \mathcal{H}$ and $R(P) \cap N(Q) \neq \{0\}\}$. The functional $(P,Q) \to \|P-Q\|$ takes the constant values $\|P_{R(S)} - S\|^{1/2}$ on $\mathcal{W}$ and 1 on $\mathcal{Z}$, respectively.

The following is a technical result which will be used later on:

Lemma 4.4. *Let $P \in \mathcal{P}$ and $0 \leq A \leq P$, then the following identities hold:*

$$\overline{R(P-A)} = \overline{R(P-A^2)} = \overline{R(P-A^{1/2})}$$

and

$$\overline{R(A-A^2))} = \overline{R(A(P-A))} = \overline{R(P_A - A)}.$$

Proof. Observe that the operators A, $P-A$, $P-A^2$ and $P-A^{1/2}$ are positive and commute because of the monotonicity of the positive square root; and the same holds with P_A instead of P.

Also, from $(P-A^2) = (P+A)(P-A)$ we get $N(P-A^2) = N(P-A)$: $N(P-A) \subseteq N(P-A^2)$; conversely if $(P-A^2)x = 0$ then $(P-A)x \in N(P+A) \cap \overline{R(A)} = N(P) \cap \overline{R(A)} = \{0\}$ because $N(P) \subseteq N(A)$. Taking orthogonal complement we obtain that $\overline{R(P-A^2)} = \overline{R(P-A)}$. With a similar argument, $\overline{R(P-A^{1/2})} = \overline{R(P-A)}$.

Observe that $PA = A = AP$ so $A - A^2 = A(P-A)$. To prove that $\overline{R(A(P-A))} = \overline{R(P_A - A)}$, observe that $N(A(P-A)) = N(A(P_A - A)) = N(P_A - A)$ and take orthogonal complement. ∎

Theorem 4.5. *(Ando) Let P and Q be orthogonal projections, then the matrix representation of Q in terms of P is given by*

$$Q = \begin{pmatrix} A & A^{1/2}(P-A)^{1/2}U^* \\ UA^{1/2}(P-A)^{1/2} & U(P-A)U^* + \hat{Q} \end{pmatrix}, \tag{6}$$

where $A = PQP$, U is a partial isometry with initial space $\overline{R(A(P-A))}$ and final space $\mathcal{W} \subseteq N(P)$ and $\hat{Q}$ is an orthogonal projection with $R(\hat{Q}) \subset N(P) \ominus R(U)$.

Conversely, given $P \in \mathcal{P}$, $0 \leq A \leq P$ such that $\dim \overline{R(A(P-A))} \leq \dim N(P)$, a partial isometry U with initial space $\overline{R(A(P-A))}$ and final space $\mathcal{W} \subseteq N(P)$ and an orthogonal projection $\hat{Q}$ with $R(\hat{Q}) \subseteq N(P) \ominus R(U)$ the right-hand side of (6) gives an orthogonal projection.

Proof. Given $P, Q \in \mathcal{P}$, consider the matrix representation of Q in terms of P:

$$Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}$$

Write $A := Q_{11}$ and $B := Q_{22}$. Since $Q \geq 0$, it follows that

$$0 \leq A \leq P, \;\; 0 \leq B \leq I - P \;\; \text{ and } Q_{12}^* = Q_{21}.$$

Since $Q^2 = Q$, we also have

$$Q_{12}Q_{21} = A(P - A) \quad \text{and} \quad AQ_{12} + Q_{12}B = Q_{12} \tag{7}$$

Since $Q_{12}^* = Q_{21}$, from the first equality we get

$$|Q_{21}|^2 = A(P - A) \quad \text{or } |Q_{21}| = A^{1/2}(P - A)^{1/2},$$

so, we can conclude that there is an isometry U from $\overline{R(A(P - A))}$ onto $\mathcal{W} \subseteq N(P)$ such that

$$Q_{21} = UA^{1/2}(P - A)^{1/2} \quad \text{and} \quad Q_{12} = A^{1/2}(P - A)^{1/2}U^*.$$

But applying Lemma 4.4, $\overline{R(A(P - A))} = \overline{R(P_A - A)}$.

It follows from the second identity of (7) that

$$AA^{1/2}(P - A)^{1/2}U^* + A^{1/2}(P - A)^{1/2}U^*B = A^{1/2}(P - A)^{1/2}U^*.$$

Observe that $A^{1/2}(P - A)^{1/2} = A^{1/2}(P - A)^{1/2}P_A$, by Lemma 4.4; then

$$0 = A^{1/2}(P - A)^{1/2}[U^*B - (P_A - A)U^*] = A^{1/2}(P - A)^{1/2}[U^*B - (P_A - A)U^*];$$

since $R(U^*B - (P_A - A)U^*) \subseteq \overline{R(A(P - A))}$, then

$$U^*B = (P_A - A)U^* \quad \text{and hence} \quad UU^*B = U(P_A - A)U^* = BUU^*.$$

Since $UU^* = P_U$ is an orthogonal projection and $P_UB = BP_U$, we get that

$$B = U(P_A - A)U^* + \hat{Q}$$

where $\hat{Q}$ is an orthogonal projection with $R(\hat{Q}) \subseteq N(P) \ominus R(U)$. Observe that $UP = U(P_A + P_{R(P)\ominus\overline{R(A)}}) = UP_A$, because $R(P) \ominus R(A) \subseteq N(A) \subseteq N(P_A - A) = N(U)$. Then $B = U(P - A)U^* + \hat{Q}$. Therefore we arrive at (6).

It is immediate to see that for $0 \leq A \leq P$ satisfying the dimension condition, a partial isometry U with initial space $\overline{R(A(P - A))}$ and final space $\mathcal{W} \subset N(P)$ and an orthogonal projection $\hat{Q}$ with $R(\hat{Q}) \subseteq N(P) \ominus R(U)$ the right-hand side of (6) gives an orthogonal projection. This completes the proof. ∎

As a consequence we get the following dilation result, that recovers the result [2]:

Corollary 4.6. *Given a positive contraction $A \in L(\mathcal{H})$, there exists $Q \in \mathcal{P}$ such that $A = P_AQP_A$ if and only if* $\dim \overline{R(A - A^2))} \leq \dim N(A)$.

The next result, due to T. Ando, will be useful in a characterization of the set $\mathfrak{Y}$ by means of the polar decomposition (see next section).

Corollary 4.7. *(Ando) Given $P, Q \in \mathcal{P}$, there exists $H \in \mathcal{P}$ which is a solution of*

$$(PQP)^{1/2} = PXP. \tag{8}$$

Moreover, all the orthogonal projections which are solutions of (8) are parametrized as

$$H = \begin{pmatrix} A & A^{1/2}(P - A)^{1/2}U^* \\ UA^{1/2}(P - A)^{1/2} & U(P - A)U^* + \hat{H} \end{pmatrix}$$

where $A = (PQP)^{1/2}$, U is a partial isometry with initial space $\overline{R(A(P - A))}$ and final space $\mathcal{W} \subseteq N(P)$ and $\hat{H}$ is an orthogonal projection with $R(\hat{H}) \subseteq N(P) \ominus R(U)$.

Proof. Let $A = PQP$; by the proof of the above theorem, $\dim \overline{R(P_A - A)} \le \dim N(P)$. Consider $A^{1/2}$, then $0 \le A^{1/2} \le P$. Therefore, applying Lemma 4.4, $\dim \overline{R(P_{A^{1/2}} - A^{1/2})} = \dim \overline{R(P_A - A)} \le \dim N(P)$. Finally, applying Theorem 4.5, the proof is complete. ∎

Remark 4.8. Observe that the above theorem contains an alternative proof of the result by Arias and Gudder [2] mentioned before, in the setting of Hilbert spaces.

In [20] Nelson and Neumann proved that a set $\{\lambda_1, \dots, \lambda_n\}$ is the spectrum of a $n \times n$ matrix $B = PQ$, where $P, Q \in \mathcal{P}$, if and only if $\sharp\{i : 0 < \lambda_i < 1\} \le \sharp\{i : \lambda_i = 0\}$. Since the spectrum of PQ coincides with that of PQP it follows that the result by Nelson and Neumann is the finite-dimensional version of the theorem of Arias and Gudder.

5 Polar decomposition of PQ

The *polar decomposition* of an operator $C \in L(\mathcal{H})$ is a factorization $C = V_C|C|$, where V_C is a partial isometry, $|C| = (C^*C)^{1/2}$ and $N(V_C) = N(C)$. It can be shown that this factorization exists and is unique [26]. Morever, $R(V_C) = \overline{R(C)}$, $V_C V_C^* = P_{\overline{R(C)}}$, $V_C^* V_C = P_{N(C)^\perp}$ and $C = |C^*|V_C$. In what follows, V_C will be called the *isometric part* of C and $|C|$ the *positive part* of C.

Given a subset $\mathcal{A}$ of $L(\mathcal{H})$ we consider the set $\mathcal{A}^+$ (resp., $\mathcal{J}_{\mathcal{A}}$) which consists of all positive (resp., isometric) parts of members of $\mathcal{A}$.

In [8] we characterized $\mathcal{Q}^+$ (notice that in [8], we used the more cumbersome notation $L(\mathcal{H})^+_{\mathcal{Q}}$) and $\mathcal{J}_{\mathcal{Q}}$. We apply now the results above and those of [8] to characterize $\mathfrak{X}^+$, $\mathfrak{X}^+_{cr}$, $\mathcal{J}_{\mathfrak{X}}$ and $\mathcal{J}_{\mathfrak{X}_{cr}}$.

In [8] there is a characterization of the set $\mathcal{J}_{\mathcal{Q}}$ of all partial isometries of oblique projections. More precisely, it is proven that, for a given $V \in \mathcal{J}$, there exists $E \in \mathcal{Q}$ with polar decomposition $E = V|E|$ if and only if $VP_{R(V)}$ is a positive operator with range $R(V)$. In other terms, the restriction of $VP_{R(V)}$ to $R(V)$ is a positive invertible operator in $L(R(V))$. The next result proves that the squares of such isometries exhaustes the set $\mathfrak{X}_{cr}$.

Theorem 5.1.

$$\mathfrak{X}_{cr} = \{V^2 : \ V \in \mathcal{J}_{\mathcal{Q}}\}.$$

Proof. By [14], $T \in \mathfrak{X}_{cr}$ if and only if $T^\dagger \in \mathcal{Q}$ so that we only need to prove that, if $E \in \mathcal{Q}$ has polar decomposition $E = V|E|$ then $E^\dagger = V^{*2}$, and use the general fact that V^* is the partial isometry of E^* in its polar decomposition. For this, observe that $N(E) = N(V)$ and $R(E) = R(V)$ so that $E^\dagger = P_{N(E)^\perp} P_{R(E)} = P_{N(V)^\perp} P_{R(V)} = (V^*V)(VV^*)$. By the characterization of $\mathcal{J}_{\mathcal{Q}}$, it holds $VP_{R(V)} = (VP_{R(V)})^* = P_{R(V)} V^*$, so that $V^2 V^* = VV^{*2}$. Then, $E^\dagger = V^*VV^{*2}$. But, since V^* is the Moore-Penrose inverse of V, it holds $V^*VV^* = V^*$. Thus, $E^\dagger = V^{*2}$. This proves the theorem. ∎

This result will be extended to the whole $\mathfrak{X}$, after the characterization of the set $\mathcal{J}_{\mathfrak{X}}$ in the next theorem.

Let $T \in \mathfrak{X}$ such that $T = PQ$ is the canonical factorization of T. Then the left polar decomposition of T has the form

$$T = (PQP)^{1/2} V_T. \tag{9}$$

Now we characterize the set $\mathcal{J}_{\mathfrak{X}} = \{V \in \mathcal{J} : \text{there exists } T \in \mathfrak{X} \text{ such that } V = V_T\}$, i.e., the partial isometries of the polar decompositions of elements of $\mathfrak{X}$.

Theorem 5.2. *Given $V \in \mathcal{J}$, then $V \in \mathcal{J}_{\mathfrak{X}}$ if and only if $V^2V^* \geq 0$ and $\overline{R(V^2V^*)} = R(V)$. In this case, it holds $\overline{R(V) \dot{+} N(V)} = \mathcal{H}$*

Proof. Let $V \in \mathcal{J}_{\mathfrak{X}}$, then there exists $T \in \mathfrak{X}$ such that $V = V_T$. Let $T = PQ$ be the canonical factorization of T. Recall that $P = P_{\overline{R(T)}} = P_{R(V)}$ and, by the definition of the polar decomposition, $R(V) = \overline{R(T)}$. Therefore, $V^2V^* = V(VV^*) = VP$. But, from (9) we get that $(PQP)^{1/2\dagger} T = PV = V$ so that $V = (PQP)^{1/2\dagger} PQ$ and then, $VP = (PQP)^{1/2\dagger} PQP = (PQP)^{1/2}$. Therefore

$$VP = |\, T^*| \in L(\mathcal{H})^+.$$

Moreover, $R(V^2V^*) = R(VP) = R(|\, T^*|) = R(T)$ so that $\overline{R(V^2V^*)} = R(V)$.

Conversely, suppose that $V \in \mathcal{J}$ satisfies that $V^2V^* = VP_{R(V)} \geq 0$ and that $\overline{R(VP_{R(V)})} = R(V)$. Let $A = VP_{R(V)}$ and $T = P_{R(V)}P_{N(V)^\perp} \in \mathfrak{X}$. Since A is positive, in particular $A = V^2V^* = VV^{*2}$. Then $T = (VV^*)(V^*V) = V^2V^*V = VP_{R(V)}V (= V^2) = AV$ and this is the polar decomposition of T. In fact, observe that $TT^* = AVV^*A = AP_{R(V)}A = A^2$ so that $|\, T^*| = A$; also V is a partial isometry with final space $R(V) = \overline{R(V^2V^*)} = \overline{R(A)} = \overline{R(T)}$ and nullspace $N(V) = N(T)$: $N(V) \subseteq N(T)$ and if $Tx = 0$ then $AVx = 0$; therefore $Vx \in N(A) \cap \overline{R(V)} = N(A) \cap \overline{R(A)} = \{0\}$.

The last assertion, namely that $\mathcal{H} = \overline{R(V) + N(V)} = \mathcal{H}$ if $V \in \mathcal{J}_{\mathfrak{X}}$, follows directly from Theorem 3.2, by observing that $R(V) = \overline{R(T)}$ and $N(V) = N(T)$. ∎

Given $T \in \mathfrak{X}$ with polar decomposition $T = |T^*|V$ then $T = P_{R(V)}P_{N(V)^\perp}$ is the canonical factorization of T. By the previous results, it also holds that $R(T)$ is closed if and only if $R(V) \dot{+} N(V) = \mathcal{H}$.

We have proved that if $T = V^2$ for a given $V \in \mathcal{J}_{\mathfrak{X}}$, then $T \in \mathfrak{X}$ and V is the partial isometry of T. Therefore:

Corollary 5.3. *Consider the map $\alpha : \mathcal{J}_{\mathfrak{X}} \longrightarrow L(\mathcal{H})$, $\alpha(V) = V^2$. Then α is a bijection from $\mathcal{J}_{\mathfrak{X}}$ onto $\mathfrak{X}$. In particular, $\mathfrak{X} = \{V^2 : \ V \in \mathcal{J}_{\mathfrak{X}}\}$.*

Proof. If $V \in \mathcal{J}_{\mathfrak{X}}$ then, by Theorem 5.2, $V^2V^* \geq 0$; in particular, $V^2V^* = VV^{*2}$. Then $T = (VV^*)(V^*V) \in \mathfrak{X}$; but $T = VV^{*2}V = V^2V^*V = V^2$, so that $\alpha(V) = V^2 \in \mathfrak{X}$. Let $T \in \mathfrak{X}$; if V is the partial isometry of T then, by Theorem 5.2 again, we get $V^2V^* \geq 0$ and $T = P_{\overline{R(T)}}P_{N(T)} = (VV^*)(V^*V) = VV^{*2}V = V^2V^*V = V^2 = \alpha(V)$. Thus, the isometric part of T is V, so that α is surjective and $\alpha^{-1}(T) = V$. ∎

The last Corollary extends our previous results Theorem 5.1 and [8], Theorem 5.2.

Theorem 5.4. *Let $V \in \mathcal{J}$. Then $V \in \mathcal{J}_{\mathfrak{X}}$ if and only if V has a matrix representation, in terms of the decomposition $\mathcal{H} = R(V) \oplus R(V)^{\perp}$, of the type*

$$V = \begin{pmatrix} A & (P - A^2)^{1/2}U \\ 0 & 0 \end{pmatrix} \tag{10}$$

where $P = P_V$, $0 \leq A \leq P$, $\overline{R(A)} = R(V)$, $\dim \overline{R(P - A^2)} \leq \dim R(V)^{\perp}$ and U is a partial isometry with initial space contained in $R(V)^{\perp}$ and final space $\overline{R(P - A^2)}$.

Proof. If $V \in \mathcal{J}_{\mathfrak{X}}$ then there exists $T \in \mathfrak{X}$ such that $V = V_T$. In the same way as in Theorem 5.2, if $T = PQ$ is the canonical factorization of T then

$$VP = (PQP)^{1/2} = A,$$

where $\overline{R(A)} = R(V)$ and, by Theorem 5 and Corollary 6 of [2], A satisfies that $0 \leq A \leq P$ and $\dim \overline{R(P - A)} \leq \dim N(P)$. By Lemma 4.4, $\dim \overline{R(P - A^2)} \leq \dim N(P)$.

Therefore

$$V = \begin{pmatrix} A & V_{12} \\ 0 & 0 \end{pmatrix}$$

is the matrix of V. Since $VV^* = A^2 + V_{12}V_{12}^* = P$, then $|V_{12}^*| = (P - A^2)^{1/2}$, so that $V_{12} = (P - A^2)^{1/2}U$, where U is a partial isometry with initial space contained in $R(V)^{\perp} = N(A)$ and final space $\overline{R(P - A^2)}$.

Conversely, if V has the matrix representation (10), with A and U satisfying the hypothesis of the theorem, then $VV^* = A^2 + P - A^2 = P$, so that $V \in \mathcal{J}$, $VP = A \geq 0$, $\overline{R(A)} = R(V)$ by hypothesis. Therefore, applying Theorem 5.2, it follows that $V \in \mathcal{J}_{\mathfrak{X}}$. ∎

We end this section with a characterization of the set

$$\mathfrak{X}^+ = \{A \in L(\mathcal{H})^+ : \text{there exists } T \in \mathfrak{X} \text{ such that } A = |T^*|\},$$

i.e., the positive parts of the polar decompositions of elements of $\mathfrak{X}$.

Proposition 5.5.

$$\mathfrak{X}^+ = \mathfrak{Y}.$$

Proof. Let $A \in \mathfrak{X}^+$. Then there exists $T \in \mathfrak{X}$ such that $A = (TT^*)^{1/2}$. If $T = PQ$ is the canonical factorization of T, then $A = (PQP)^{1/2}$ and applying Corollary 4.7 there exists $H \in \mathcal{P}$ such that $A = PHP$ so that $A \in \mathfrak{Y}$.

Conversely, let $A \in \mathfrak{Y}$. Then there exists $P, Q \in \mathcal{P}$ such that $A = PQP$ and we can assume that $P = P_A$. By Theorem 5 and Corollary 6 of [2], it follows that $0 \leq A \leq P$, $\dim \overline{R(P - A)} \leq \dim N(A)$ and, by Lemma 4.4, $\dim \overline{R(P - A)} = \dim \overline{R(P - A^2)}$. In this case P and A satisfy the conditions of Theorem 5.4 and we can construct an operator $T \in \mathfrak{X}$; more precisely, consider $T = AV$ with

$$V = \begin{pmatrix} A & (P - A^2)^{1/2}U \\ 0 & 0 \end{pmatrix}$$

where U is a partial isometry with initial space contained in $R(V)^{\perp}$ and final space $\overline{R(P - A^2)}$. Then $TT^* = A^2$ or $|T^*| = A$. Therefore $A \in \mathfrak{X}^+$. ∎

Corollary 5.6. *Consider the map $\beta : \mathfrak{X} \longrightarrow \mathfrak{Y}$, $\beta(T) = |T^*|$. Then the fibre of $A \in \mathfrak{Y}$ is given by*

$$\beta^{-1}(\{A\}) = \{T \in \mathfrak{X} : \ T = \begin{pmatrix} A^2 & A(P-A^2)^{1/2}U \\ 0 & 0 \end{pmatrix}\}$$

where $P = P_{\overline{R(A)}}$, U is a partial isometry with initial space contained in $N(A)$ and final space $\overline{R(P-A^2)}$.

Proof. Apply Proposition 5.5. ∎

6 On the Moore-Penrose pseudoinverse of PQ

As mentioned in the Introduction, Penrose [24] and Greville [14] proved that the Moore-Penrose pseudoinverse of an oblique matrix is a product of two orthogonal projections, and conversely. A proof of the next result, which extends their theorem to closed range operators in $\mathfrak{X}$, appears in [8].

Theorem 6.1. *Let $T \in L(\mathcal{H})$. Then $T \in \mathfrak{X}_{cr}$ if and only if there exists $E \in \mathcal{Q}$ such that $T = E^\dagger$. In symbols, $\mathfrak{X}_{cr} = \mathcal{Q}^\dagger$.*

The generalization of Penrose-Greville theorem for operators $T \in \mathfrak{X}$ with non-closed range forces the consideration of a certain class of unbounded projections. We refer the reader to the paper [23] for the properties of those projections which naturally appear in this context. In what follows, we consider the set $\tilde{\mathcal{Q}}$ of closed unbounded projections, i.e., operators E with a dense domain $\mathcal{D}(E)$ such that $\mathcal{D}(E) = N(E) \dot{+} R(E)$, $N(E)$ is closed, $R(E)$ is closed in $\mathcal{H}$ and $E(Ex) = Ex$ for all $x \in \mathcal{D}(E)$.

Theorem 6.2. *If $T \in \mathfrak{X}$ then there exists a closed unbounded projection $E : \mathcal{D}(E) \longrightarrow \mathcal{H}$ such that $T = E^\dagger$. Conversely, if E is any closed unbounded projection then there exists an element $T \in \mathfrak{X}$ such that $E^\dagger = T$. Moreover, the map $T \longrightarrow T^\dagger$ from $\mathfrak{X}$ onto $\tilde{\mathcal{Q}}$ is a bijection.*

Proof. Suppose that $T \in \mathfrak{X}$. Then (see, e.g., [4]) $E = T^\dagger$ is an unbounded pseudoinverse of T with dense domain $\mathcal{D}(E) = R(T) \oplus R(T)^\perp$, $R(E) = N(T)^\perp$ and E verifies $TET = T$, in $\mathcal{H}$, and $ETE = E$ in $\mathcal{D}(E)$. Since $R(E) = N(T)^\perp$ we get

$$P_{N(T)^\perp} Ex = Ex, \ \ \forall x \in \mathcal{D}(E). \tag{11}$$

It also holds that

$$EP_{\overline{R(T)}} x = Ex, \ \ \forall x \in \mathcal{D}(E). \tag{12}$$

In fact, if $x \in \mathcal{D}(E)$ then $Ex = E(P_{\overline{R(T)}}x + P_{R(T)^\perp}x) = EP_{\overline{R(T)}}x$ because $P_{\overline{R(T)}}x \in R(T)$ and $R(T)^\perp = N(E)$.

Observe also that $R(E) = N(T)^\perp \subseteq \mathcal{D}(E)$: if $x \in N(T)^\perp$ then $x = P_{\overline{R(T)}}x + P_{R(T)^\perp}x = P_{\overline{R(T)}}P_{N(T)^\perp}x + P_{R(T)^\perp}x = Tx + P_{R(T)^\perp}x$ so that $x \in \mathcal{D}(E)$. Therefore E^2 is well defined in $\mathcal{D}(E)$.

Finally, for $x \in \mathcal{D}(E)$, we get

$$E^2 x = EP_{N(T)^\perp} Ex = EP_{\overline{R(T)}} P_{N(T)^\perp} Ex = ETEx = Ex.$$

Observe that the first equality follows from (11) and the second from (12), because $P_{N(T)^\perp} Ex \in \mathcal{D}(E)$. We have proved that $E^2 = E$ in $\mathcal{D}(E)$; $R(E) = N(T)^\perp$ and $N(E) = R(T)^\perp$, both closed subspaces. This proves that E is an unbounded closed projection, see Lemma 3.5 of [23], namely $E = P_{N(T)^\perp // R(T)^\perp}$.

Conversely, suppose that $\mathcal{M}$ and $\mathcal{N}$ are closed subspaces such that $\mathcal{M} \dot{+} \mathcal{N}$ is a dense subspace of $\mathcal{H}$. Let $E : \mathcal{M} \dot{+} \mathcal{N} \longrightarrow \mathcal{M}$ be the (unbounded) projection with domain $\mathcal{D}(E) = \mathcal{M} \dot{+} \mathcal{N}$ onto $\mathcal{M}$ with nullspace $\mathcal{N}$. We will show that the unbounded operator E is the pseudoinverse of an element of $\mathfrak{X}$, namely, $E = (P_{\mathcal{N}^\perp} P_{\mathcal{M}})^\dagger$: in fact, $P_{\mathcal{M}} Ex = Ex$, for every $x \in \mathcal{D}(E)$ and $EP_{\mathcal{M}} = P_{\mathcal{M}}$, in $\mathcal{H}$, because $R(E) = \mathcal{M}$. Also, $R(P_{\mathcal{N}^\perp} P_{\mathcal{M}}) = R(P_{\mathcal{M}} - P_{\mathcal{N}} P_{\mathcal{M}}) \subseteq \mathcal{M} \dot{+} \mathcal{N} \subseteq \mathcal{D}(E)$. Therefore $EP_{\mathcal{N}^\perp} P_{\mathcal{M}}$ is well defined for every $x \in \mathcal{H}$ and $EP_{\mathcal{N}^\perp} P_{\mathcal{M}} = E(I - P_{\mathcal{N}}) P_{\mathcal{M}} = P_{\mathcal{M}}$, then

$$EP_{\mathcal{N}^\perp} P_{\mathcal{M}} = P_{\mathcal{M}}. \tag{13}$$

Consider $x \in R(P_{\mathcal{N}^\perp} P_{\mathcal{M}}) (\subseteq \mathcal{D}(E))$ then $x = P_{\mathcal{N}^\perp} P_{\mathcal{M}} y$, for $y \in \mathcal{H}$. Using equation (13) we get $P_{\mathcal{N}^\perp} P_{\mathcal{M}} Ex = P_{\mathcal{N}^\perp} P_{\mathcal{M}} E(P_{\mathcal{N}^\perp} P_{\mathcal{M}} y) = P_{\mathcal{N}^\perp} P_{\mathcal{M}} y = x$, then

$$P_{\mathcal{N}^\perp} P_{\mathcal{M}} Ex = x,$$

for every $x \in R(P_{\mathcal{N}^\perp} P_{\mathcal{M}})$.

On the other side, if $x \in R(P_{\mathcal{N}^\perp} P_{\mathcal{M}})^\perp = N(P_{\mathcal{M}} P_{\mathcal{N}^\perp}) = (\mathcal{N}^\perp \cap \mathcal{M}) \oplus \mathcal{N} \subseteq \mathcal{D}(E)$ then $x = y + z$, with $y \in \mathcal{N}^\perp \cap \mathcal{M}$ and $z \in \mathcal{N}$, so that $Ex = Ey = y$. Therefore,

$$P_{\mathcal{N}^\perp} P_{\mathcal{M}} Ex = P_{\mathcal{N}^\perp} Ex = P_{\mathcal{N}^\perp} Ey = P_{\mathcal{N}^\perp} y = 0.$$

This proves that

$$P_{\mathcal{N}^\perp} P_{\mathcal{M}} E = P_{\overline{R(P_{\mathcal{N}^\perp} P_{\mathcal{M}})}}, \text{ in } \mathcal{D}(E). \tag{14}$$

Equations (13) and (14) prove that $E^\dagger = P_{\mathcal{N}^\perp} P_{\mathcal{M}} \in \mathfrak{X}$. ∎

Remark 6.3. a) Observe that the domain $\mathcal{D} = R(T) \oplus R(T)^\perp$ of the operator $E = T^\dagger$ in the above theorem can be also expressed as a (not necessarily orthogonal) direct sum of two closed subspaces, more precisely $\mathcal{D} = N(T)^\perp \dot{+} R(T)^\perp = R(E) \dot{+} N(E)$: we have already proved that $N(T)^\perp \subseteq \mathcal{D}$ so that $N(T)^\perp \dot{+} R(T)^\perp \subseteq \mathcal{D}$; to prove the other inclusion we have to check that $R(T) \subseteq N(T)^\perp \dot{+} R(T)^\perp$: let $x \in R(T)$, then we can compute $T^\dagger x = Ex$ and $Ex \in N(T)^\perp$. Therefore $Ex = Ex + (I - E)x \in N(T)^\perp + R(T)^\perp$.

b) Let $T \in \mathfrak{X}$ with polar decomposition $T = V|T|$. Let us consider the operator with domain $\mathcal{D} = R(T) \oplus R(T)^\perp$, defined by

$$E = |T|^\dagger V^* |_{\mathcal{D}}.$$

Observe that $V : N(T)^\perp \longrightarrow \overline{R(T)}$ is unitary and, by construction of V, $V(R(|T|)) = R(T)$. Then, $V^*(R(T)) = R(|T|)$; also observe that $|T|^\dagger(R(|T|)) = N(T)^\perp$. Therefore, E is well-defined and $E(\mathcal{D}) = N(T)^\perp$.

If $x \in R(T)^\perp$ then $Ex = |T|^\dagger V^* x = 0$ because $R(T)^\perp = N(V^*)$. Let us see that E is the identity on $N(T)^\perp$; we have to check that $N(T)^\perp \subseteq \mathcal{D}$: if $x \in N(T)^\perp$ then $x = P_{R(T)}x + P_{R(T)^\perp}x = P_{R(T)}P_{N(T)^\perp}x + P_{R(T)^\perp}x = Tx + P_{R(T)^\perp}x \in \mathcal{D}$. Then

$$Ex = |T|^\dagger V^*(Tx + P_{R(T)^\perp}x) = |T|^\dagger V^* Tx = |T|^\dagger V^* V|T|x = |T|^\dagger |T|x = P_{N(T)^\perp}x = x.$$

Therefore, $E = P_{N(T)^\perp // R(T)^\perp}$, and its left "polar decomposition" is

$$E = |T|^\dagger V^*|_{\mathcal{D}}.$$

We can also consider $T = |T^*|V_T$ to obtain the right "polar decomposition" of E given by $E = V^*|T^*|^\dagger$, in $\mathcal{D}$.

c) Finally, observe that the Moore-Penrose pseudoinverses of positive parts of elements of $\mathfrak{X}$ are the positive parts of elements of $\tilde{\mathcal{Q}}$, i.e. $(\mathfrak{X}^+)^\dagger = \tilde{\mathcal{Q}}^+$.

In [8] the set of isometric parts of bounded oblique projections is characterized. Using this characterization, together with the construction of the (left) polar decomposition of elements of $\tilde{\mathcal{Q}}$ as above, and the fact that if $T \in \mathfrak{X}$ then $T^* \in \mathfrak{X}$, we get the following result:

Corollary 6.4.

$$\mathcal{J}_{\mathfrak{X}} = \mathcal{J}_{\tilde{\mathcal{Q}}}$$

and

$$\mathcal{J}_{\mathfrak{X}_{cr}} = \mathcal{J}_{\mathcal{Q}}.$$

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