Quantum computational structures: Categorical equivalence for square root qMV-algebras

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Abstract

In this paper we investigate a categorical equivalence between square root qMV-algebras (a variety of algebras arising from quantum computation) and a category of preordered semigroups.

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Introduction

Standard quantum computing is based on quantum systems with finite dimensional Hilbert spaces, specially $\mathbb{C}^2$, the two-dimensional state space of a *qbit*. A qbit state (the quantum counterpart of the classical bit) is represented by a unit vector in $\mathbb{C}^2$ and, generalizing for a positive integer n, n*-qbits* are represented by unit vectors in $\mathbb{C}^{2^n}$. They conform the information units in quantum computation. Similarly to the classical computing case, we can introduce and study the behavior of a number of *quantum logical gates* (hereafter quantum gates for short) operating on qbits. These gates are mathematically represented by *unitary operators* on the appropriate Hilbert spaces of qbits. In other words, standard quantum computation

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is mathematically founded on "*qbits-unitary operators*" and only takes into account reversible processes. This framework can be generalized to a powerful mathematical representation of quantum computation in which the qbit states are replaced by *density operators* over Hilbert spaces and unitary operators by linear operators acting over endomorphisms of Hilbert spaces called *quantum operations* [13]. The new model "*density operators-quantum operations*" also called "*quantum computation with mixed states*" [1, 16] is equivalent in computational power to the standard one but gives a place to irreversible processes as measurements in the middle of the computation.

As is well known, each density operator σ in $\mathbb{C}^2$ has a matrix representation via the Pauli matrices $\sigma = \frac{1}{2}(I + r_1\sigma_1 + r_2\sigma_2 + r_3\sigma_3)$ where:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and r_1, r_2, r_3 are real numbers s.t. $r_1^2 + r_2^2 + r_3^2 \leq 1$. We will denote by $\mathcal{D}(\mathbb{C}^2)$ the set of all density operators of $\mathbb{C}^2$. An interesting feature of density operators is the fact that any real number $0 \leq \lambda \leq 1$ uniquely determines a density operator $\rho_\lambda = \frac{1}{2}(I+(1-2\lambda)\sigma_3)$. For each $\sigma = \frac{1}{2}(I+r_1\sigma_1+r_2\sigma_2+r_3\sigma_3)$ in $\mathcal{D}(\mathbb{C}^2)$ we can associate, as dictated by the Born rule, a probability value $\mathtt{p}(\sigma)$ in the following manner: $\mathtt{p}(\sigma) = \mathtt{Tr}(\rho_1\sigma) = \frac{1-r_3}{2}$.

Recently, increasing attention has been paid to algebraic structures arising from quantum computation [2, 4, 5, 6, 14]. More precisely these structures stem from mathematical description of circuits obtained by combinations of quantum gates in the "density operators-quantum operations" model. In [2] and [4], a quantum gate system called *Poincaré irreversible quantum computational system* (for short $\mathcal{IP}$-system) was developed. This quantum gates system is the following:

- $\sigma \oplus \tau = \rho_{\mathtt{p}(\sigma)\oplus\mathtt{p}(\tau)}$ where $\mathtt{p}(\sigma) \oplus \mathtt{p}(\tau) = \min\{\mathtt{p}(\sigma) + \mathtt{p}(\tau), 1\}$ [Łukasiewicz gate]
- $\neg\rho = \sigma_x \rho \sigma_x^\dagger$ [*NOT* gate]
- $\sqrt{\rho} = \begin{pmatrix} \frac{1+i}{2} & \frac{1-i}{2} \\ \frac{1-i}{2} & \frac{1+i}{2} \end{pmatrix} \rho \begin{pmatrix} \frac{1+i}{2} & \frac{1-i}{2} \\ \frac{1-i}{2} & \frac{1+i}{2} \end{pmatrix}^\dagger$ [$\sqrt{NOT}$ gate]
- P_1, P_0, $\rho_{\frac{1}{2}}$ [Constant gates]

We can see that quantum gates $\sqrt{}$, $\neg$ are quantum operations. The Łukasiewicz quantum gate $\oplus$ is not a quantum operation but it can be *probabilistically approximated* in a uniform form by means of quantum operations [8]. Thus we may introduce the following algebraic system associated with the quantum gates known as *the Poincaré irreversible quantum computational algebra*:

$$\langle \mathcal{D}(\mathbb{C}^2), \oplus, \neg, \sqrt{}, P_0, \rho_{\frac{1}{2}}, P_1 \rangle$$

The $\mathcal{IP}$-system is an interesting quantum gates system specially for two reasons: it is related to continuous t-norms and subsequent generalizations allow to connect this system with sequential effect algebras [10], introduced to study the sequential action of quantum effects which are unsharp versions of quantum events [11, 12]. The first and more basic algebraic structure associated to the $\mathcal{IP}$-system (known as *quasi MV-algebra*) is introduced in [14] for the reduced system $\langle \oplus, \neg, P_0, P_1 \rangle$ while in [7] an equivalence between the category of quasi MV-algebras and a category of preordered semigroups is given. In [9] an algebraic structure for the full $\mathcal{IP}$-system is proposed. These algebras are known as *square root quasi MV-algebra* ($\sqrt{qMV}$-algebras for short).

In this paper we extend the result of [7] providing a categorical equivalence between square root quasi MV-algebras and a category of preordered semigroups with a unary operations. The paper is organized as follows: in Section 1 we recall some basic definitions and properties about MV-algebras and l-groups. Section 2 contains generalities on algebraic structures associated to quantum computation; more precisely, qMV-algebras and $\sqrt{qMV}$-algebras. Finally, Section 3 is dedicated to the study of the categorical equivalence of $\sqrt{qMV}$-algebras.

1 Basic Notions

We recall from [3] some basic notions about l-groups and MV-algebras, respectively. A *lattice ordered abelian group* or l-group for short, is an algebra $\langle G, +, -, \vee, \wedge, 0 \rangle$, of type $\langle 2, 1, 2, 2, 0 \rangle$, which satisfies the following conditions:

1. $\langle G, +, -, 0 \rangle$ is an abelian group,
2. $\langle G, \wedge, \vee \rangle$ is a lattice,
3. $w + (x \vee y) = (w + x) \vee (w + y)$,

4. $w + (x \wedge y) = (w + x) \wedge (w + y)$.

If x is an element of a l-group G we define the *absolute value* of x as $| x |= (x \vee 0) + (-x \vee 0) = x \vee -x$. Moreover an element $u \geq 0$ in G is a *strong unit* of G iff for each $x \in G$ there exists a natural number n such that $| x |\leq nu$. We denote by $\mathcal{LG}_u$ the category of l-groups whose objects are l-groups with strong unit and whose arrows are l-groups homomorphisms preserving strong units.

An *MV-algebra* [3] is an algebra $\langle A, \oplus, \neg, 0\rangle$ of type $\langle 2, 1, 0\rangle$ satisfying the following axioms:

1. $\langle A, \oplus, 0\rangle$ is an abelian monoid,
2. $\neg\neg x = x$,
3. $x \oplus \neg 0 = \neg 0$,
4. $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

By means of the primitive MV-algebraic functions, we can define:

$$1 = \neg 0, \qquad x \vee y = \neg(\neg x \oplus y) \oplus y,$$

$$x \odot y = \neg(\neg x \oplus \neg y), \qquad x \wedge y = \neg(\neg x \vee \neg y).$$

An important example of an MV-algebra is given by the algebra $Ł_{[0,1]} = \langle [0, 1], \oplus, \neg, 0, 1\rangle$, where $x \oplus y = \min\{x + y, 1\}$ and $\neg x = 1 - x$. It is well known that if G is an l-group and $u \in G$ is a strong unit then the interval algebra $\langle [0, u], \oplus, \neg, 0, u\rangle$ where $x \oplus y = (x + y) \wedge u$ and $\neg x = u - x$ is an MV-algebra. In this way, $Ł_{[0,1]}$ can be obtained from the l-group $\mathbf{R}$ of the real numbers with strong unit 1.

On the other hand, it is possible to construct out of an MV-algebra an l-group as we will see in what follows. Let A be an MV-algebra. A *good sequence* in A is a sequence $\mathbf{a} = (a_1, a_2, ...)$ of elements of A such that $a_i \oplus a_{i+1} = a_i$ for each $i = 1, 2...$ and there exists an integer n such that $a_i = 0$ if $i > n$. We denote by (a) the sequence $(a, 0...)$. The set of good sequences of A is noted by M_A.

Let $\mathbf{a} = (a_1, a_2, ...)$ and $\mathbf{b} = (b_1, b_2, ...)$ be arbitrary good sequences. If we consider the following operations in M_A:

M1 $\mathbf{a}+\mathbf{b} = (c_1, c_2, ...)$ where $c_i = a_i \oplus (a_{i-1} \odot b_1) \oplus (a_{i-2} \odot b_2) \oplus, ..., \oplus (a_2 \odot b_{i-2}) \oplus (a_1 \odot b_{i-1}) \oplus b_i$ for each $i = 1, 2, ...,$

M2 $\mathbf{a} \vee \mathbf{b} = (a_1 \vee b_1, a_2 \vee b_2 ...)$,

M3 $\mathbf{a} \wedge \mathbf{b} = (a_1 \wedge b_1, a_2 \wedge b_2 ...)$,

then $\langle M_A, +, \vee, \wedge, (0)\rangle$ is an abelian cancellative lattice monoid whose order is given as $\mathbf{a} \leq \mathbf{b}$ iff $a_i \leq b_i$ for each $i = 1, 2....$ This order is *translation invariant*, in the sense that $\mathbf{a} \leq \mathbf{b}$ implies that $\mathbf{a} + \mathbf{d} \leq \mathbf{b} + \mathbf{d}$ for each good sequence $\mathbf{d}$.

From the abelian lattice monoid M_A we can obtain an l-group as follows: we consider the equivalence relation "$\equiv$ " in $M_A \times M_A$ given by $(\mathbf{a}, \mathbf{b}) \equiv (\mathbf{a}', \mathbf{b}')$ iff $\mathbf{a} + \mathbf{b}' = \mathbf{a}' + \mathbf{b}$. Denoting by $[\mathbf{a}, \mathbf{b}]$ the equivalence class of $(\mathbf{a}, \mathbf{b})$, and by G_A the set of equivalence classes we can consider the following operations in G_A:

$$[\mathbf{a}, \mathbf{b}] + [\mathbf{c}, \mathbf{d}] = [\mathbf{a} + \mathbf{c}, \mathbf{b} + \mathbf{d}],$$

$$[\mathbf{a}, \mathbf{b}] \vee [\mathbf{c}, \mathbf{d}] = [(\mathbf{a} + \mathbf{d}) \vee (\mathbf{c} + \mathbf{b}), \mathbf{b} + \mathbf{d}],$$

$$[\mathbf{a}, \mathbf{b}] \wedge [\mathbf{c}, \mathbf{d}] = [(\mathbf{a} + \mathbf{d}) \wedge (\mathbf{c} + \mathbf{b}), \mathbf{b} + \mathbf{d}],$$

$$-[\mathbf{a}, \mathbf{b}] = [\mathbf{b}, \mathbf{a}],$$

$$0 = [(0), (0)].$$

In this case $\langle G_A, +, \vee, \wedge, -, 0\rangle$ is an l-group called *Chang's l-group* of the MV-algebra A and the order in G_A is given by $[\mathbf{a}, \mathbf{b}] \leq_G [\mathbf{c}, \mathbf{d}]$ iff $\mathbf{a} + \mathbf{d} \leq \mathbf{b} + \mathbf{c}$.

2 Quantum computational structures

The first and more basic algebraic structure associated to the Poincaré system was introduced in [14] for the reduced system $\langle \oplus, \neg, P_0, P_1\rangle$. This is the *quasi MV-algebra* or qMV-algebra for short. A qMV-algebra is an algebra $\langle A, \oplus, \neg, 0, 1\rangle$ of type $\langle 2, 1, 0, 0\rangle$ satisfying the following equations:

Q1. $x \oplus (y \oplus z) = (x \oplus z) \oplus y$,

Q2. $\neg\neg x = x$,

Q3. $x \oplus 1 = 1$,

Q4. $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$,

Q5. $\neg(x \oplus 0) = \neg x \oplus 0$,

Q6. $(x \oplus y) \oplus 0 = x \oplus y$,

Q7. $\neg 0 = 1$.

From an intuitive point of view, a qMV-algebra can be seen as an MV-algebra which fails to satisfy the equation $x \oplus 0 = x$. We denote by $q\mathcal{MV}$ the variety of qMV-algebras. We define the binary operations $\oplus, \odot, \vee, \wedge$ in the same way as we did for MV -algebras.

Let A be a qMV-algebra. Then we define a binary relation $\leq$ on A as follows:

$$a \leq b \ iff \ 1 = \neg a \oplus b$$

Clearly $\langle A, \leq \rangle$ is a preorder. One can also easily prove that $a \leq b$ iff $a \wedge b = a \oplus 0$ iff $a \vee b = b \oplus 0$. Moreover $a \leq a \oplus 0$ and $a \oplus 0 \leq a$. If we define $A \oplus 0 = \{x \oplus 0 : x \in A\}$ it is not very hard to see that $\langle A \oplus 0, \oplus, \neg, 0, 1 \rangle$ is an MV-algebra. An element $a \in A$ is *regular* iff $a \oplus 0 = a$. Clearly, $A \oplus 0$ is the set of regular elements. The following lemma can be easily proved:

Lemma 2.1 *Let A be a qMV-algebra. Then we have:*

1. $x \vee y = x \vee (y \oplus 0) = (x \vee y) \oplus 0$.
2. $x \wedge y = x \wedge (y \oplus 0) = (x \wedge y) \oplus 0$.
3. $x \odot y = x \odot (y \oplus 0) = (x \odot y) \oplus 0$.
4. $a \in A \oplus 0$ *iff* $\neg a \in A \oplus 0$.

□

A *square root quasi MV-algebras* or $\sqrt{q MV}$-algebra [15] is an algebra $\langle A, \oplus, \neg, \sqrt{\ }, 0, \frac{1}{2}, 1 \rangle$ of type $\langle 2, 1, 1, 0, 0, 0 \rangle$ such that:

SQ1. $\langle A, \oplus, \neg, 0, \frac{1}{2}, 1 \rangle$ is a qMV-algebra,

SQ2. $\sqrt{\neg x} = \neg \sqrt{x}$,

SQ3. $\sqrt{\sqrt{x}} = \neg x$,

SQ4. $\sqrt{x \oplus y} \oplus 0 = \sqrt{\frac{1}{2}} = \frac{1}{2}$.

We denote by $\sqrt{q\mathcal{MV}}$ the variety of $\sqrt{qMV}$-algebras. Given the MV-algebra $Ł_{[0,1]}$, the *standard* $\sqrt{qMV}$*-algebra* is built from $[0,1]\times[0,1]$ equipped with the following operations:

$$(a,b) \oplus (c,d) := (a \oplus c, \tfrac{1}{2}), \qquad 0 := (0, \tfrac{1}{2})$$

$$\neg(a,b) := (\neg a, \neg b), \qquad \tfrac{1}{2} := (\tfrac{1}{2}, \tfrac{1}{2})$$

$$\sqrt{(a,b)} := (b, \neg a), \qquad 1 := (1, \tfrac{1}{2})$$

A *quasi l-group* (shortly, *ql*-group) [7] is an algebra $\langle G, +, \vee, \wedge, -, 0\rangle$ of type $\langle 2,2,2,1,0\rangle$ such that, upon defining $G + 0 = \{x + 0 : x \in G\}$, the following conditions are satisfied:

QL1 $\langle G+0, +, \vee, \wedge, -, 0\rangle$ is an l-group,

QL2 $x + (-x) = 0$,

QL3 $-(-x) = x$,

QL4 $-(x+0) = -x + 0$,

QL5 $x + y = (x+0) + (y+0)$,

QL6 $x \vee y = (x+0) \vee (y+0)$,

QL7 $x + (y \vee z) = (x+y) \vee (x+z)$.

We denote by $q\mathcal{LG}$ the variety of ql-groups. For sake of notational clarity in what follows we will write $x - y$ instead of $x + (-y)$. We inductively define nx as follows $1x = x$ and $(n+1)x = nx + x$. It can be easily seen that a ql-group is an l-group iff it satisfies the equation $x + 0 = x$. For each element a of a ql-group G the *absolute value* of a is defined as $\mid a \mid = a \vee -a$ and it is not very hard to see that $\mid a \mid = \mid a + 0 \mid$.

Proposition 2.2 [7, Proposition 2.7] *Let G be a ql-group and let $a, b, c \in G$ then we have:*

1. $-(x \vee y) = -x \wedge -y$,

2. $-(x \wedge y) = -x \vee -y$,

3. $x \wedge y = (x+0) \wedge (y+0)$,

4. $\langle G, +\rangle$, $\langle G, \vee\rangle$ *and* $\langle G, \wedge\rangle$ *are abelian semigroups,*

5. $(x+y)+0 = x+(y+0) = x+y$,

6. $-(x+y) = -x-y$,

7. $x \vee x = x \wedge x = x+0$,

8. $x \vee (x \wedge y) = x \wedge (x \vee y) = x+0$,

9. $x+(y \wedge z) = (x+y) \wedge (x+z)$,

10. $x \vee y = x \vee (y+0) = (x \vee y)+0$,

11. $x \wedge y = x \wedge (y+0) = (x \wedge y)+0$.

Let G be a ql-group. Then we define the binary relation $\leq$ on G as

$$a \leq b \quad iff \quad a+0 = a \wedge b$$

Proposition 2.3 [7, Proposition 2.9] *Let G be a ql-group $a, b, c \in G$. Then we have:*

1. $\langle G, \leq\rangle$ *is a preorder,*

2. $a \leq b$ *iff* $b+0 = a \vee b$,

3. $a \leq a+0$ *and* $a+0 \leq a$,

4. If $a \leq b$ *then, for any* $c \in G$, $a+c \leq b+c$, $a \wedge c \leq b \wedge c$, $a \vee c \leq b \vee c$,

5. $a \wedge b \leq a \leq a \vee b$,

6. if $a \leq b$ *then* $-b \leq -a$,

7. if $0 \leq a \leq b$ *then* $0 \leq b-a \leq b$, $a+0 = a \wedge (a+b)$.

Let G be a ql-group. A function $u : G \to G$ is a *quasi unit* (q-unit for short) iff it satisfies:

1. $0 \leq u(0)$,

2. $u(x+0) = u(0) - x$,

3. If $0 \leq x \leq u(0)$ then $u(0) - u(x) = 0 + x$,

4. $uu(x) = x$.

It is not hard to verify that every *ql*-group admits a *q*-unit. For instance, it suffices to consider

$$u(x) = \begin{cases} -x, & \text{if } x \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, it can also be seen that if G is an l-group, then, for any $a \geq 0$, the function $u_a(x) = a - x$ is a quasi unit in G. It can be noticed that if $a \in G$ is a strong unit for an l-group G, then the function $u_a(x)$ is the unique function which allows us to define the negation of the MV-algebra associated to the interval $[0, a]$ of G.

Proposition 2.4 [7, Proposition 2.11] *Let G be a ql-group and u be a q-unit. Then we have:*

1 $u(0) \in G + 0$,

2 if v is a q-unit such that $v(0) = u(0)$, then for each $x \in G$ $u(x+0) = v(x+0)$,

3 if $x \leq y$ then $u(y+0) \leq u(x+0)$.

Moreover if $0 \leq x, y \leq u(0)$ then:

4 if $x \leq y$ then $u(y) \leq u(x)$,

5 $0 \leq u(x) \leq u(0)$,

6 $u(0) - x = u(x) + 0$.

For simplicity we use u_0 as abbreviation of $u(0)$. Let G be a *ql*-group. A *q*-unit u on G is said to be *strong* iff for each $x \in G$ there is an integer $n \geq 0$ such that $| x | \leq nu_0$.

3 Categorical equivalence for $\sqrt{q\mathcal{MV}}$

In order to establish a categorical equivalence for $\sqrt{qMV}$-algebras we first introduce the concept of square root ql-group.

Definition 3.1 A *square root quasi l-group* (shortly $\sqrt{ql}$-group) is an algebra $\langle G, +, \vee, \wedge, -, \sqrt{\ }, u, \frac{1}{2}, 0\rangle$ of type $\langle 2, 2, 2, 1, 1, 1, 0, 0\rangle$ satisfying the following:

SQL1 $(G, u) = \langle G, +, \vee, \wedge, -, u, \frac{1}{2}, 0\rangle$ is a ql-group with strong unit u,

SQL2 $\frac{1}{2} \vee 0 = \frac{1}{2}$,

SQL3 $\frac{1}{2} \vee u_0 = u_0$,

SQL4 $\frac{1}{2} + \frac{1}{2} = u_0$,

SQL5 $0 + \sqrt{-u_0 \vee (x \wedge u_0)} = \sqrt{\frac{1}{2}} = \frac{1}{2}$,

SQL6 $\sqrt{\sqrt{\sqrt{\sqrt{x}}}} = x$,

SQL7 for each $x \notin Reg(G)$, $-u_0 \leq x \leq 0$ or $0 \leq x \leq u_0$,

SQL8 if $0 \leq x \leq u_0$ then $u(x) = \sqrt{\sqrt{x}}$,

SQL9 if $x \notin [0, u_0]$ then $x = \sqrt{x}$.

We denote by $\sqrt{q\mathcal{LG}}_u$ the category whose objects are $\sqrt{ql}$-groups and whose arrows are $f : (G_1, u_1) \to (G_2, u_2)$ such that f is a $\langle +, \vee, \wedge, -, \sqrt{\ }, \frac{1}{2}, 0\rangle$-homomorphism and $f(u_1(0)) = u_2(0)$. If we consider the following subsets of G,

$$G_1 = \{x \in G : x \notin G + 0 \ \ and \ \ -u_0 \leq x \leq 0\}$$

$$G_2 = \{x \in G : x \notin G + 0 \ \ and \ \ 0 \leq x \leq u_0\}$$

then we have that:

$$G = (G + 0) \cup G_1 \cup G_2$$

Example 3.2 Let R be the set of real number. Consider the set

$$S_R = (R \times \{\frac{1}{2}\}) \cup ([-1, 1] \times [0, 1])$$

equipped with the following operations:

1. $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, \frac{1}{2})$,
2. $(x_1, y_1) \vee (x_2, y_2) = (x_1 \vee x_2, \frac{1}{2})$,
3. $(x_1, y_1) \wedge (x_2, y_2) = (x_1 \wedge x_2, \frac{1}{2})$,
4. $-(x, y) = (-x, 1 - y)$,
5. $0 = (0, \frac{1}{2})$,
6. $\frac{1}{2} = (\frac{1}{2}, \frac{1}{2})$,
7. $\sqrt{(x,y)} = \begin{cases} (y, 1-x), & \text{if } (0, \frac{1}{2}) \leq (x, y) \leq (1, \frac{1}{2}) \\ (x, y), & \text{otherwise} \end{cases}$
8. $u(x, y) = \begin{cases} (1-x, 1-y), & \text{if } (x, y) \in S_R + 0(x, y) \text{ or } 0 \leq x \leq 1 \\ (x, y), & \text{otherwise} \end{cases}$

then $\langle S_R, +, \vee, \wedge, -, \sqrt{\ }, u, \frac{1}{2}, 0\rangle$ is a $\sqrt{ql}$-group. In fact:

SQL1) It is not very hard to see that $\langle S_R, +, \vee, \wedge, -, \frac{1}{2}, 0\rangle$ is a ql-group and note that $S_R + 0 = R \times \{\frac{1}{2}\}$. We will see that u is a strong q-unit:

1. $0 = (0, \frac{1}{2}) \leq (1, \frac{1}{2}) = (1 - 0, 1 - \frac{1}{2}) = u(0, \frac{1}{2}) = u(0)$.
2. $u(x + 0) = u((x_1, x_2) + (0, \frac{1}{2})) = u(x_1, \frac{1}{2}) = (1 - x_1, \frac{1}{2}) = (1, \frac{1}{2}) - (x_1, x_2) = u(0, \frac{1}{2}) - (x_1, x_2) = u(0) - x$.
3. $(0, \frac{1}{2}) \leq x = (x_1, x_2) \leq (1, \frac{1}{2})$ then, $u(0) - u(x) = u(0, \frac{1}{2}) - u(x_1, x_2) = (1, \frac{1}{2}) - (1 - x_1, 1 - x_2) = (x_1, \frac{1}{2}) = (x_1, x_2) + (0, \frac{1}{2}) = x + 0$.
4. $uu(x) = u(u(x_1, x_2)) = u(1 - x_1, 1 - x_2) = (x_1, x_2) = x$, if $x = (x_1, x_2) \in S_R + 0$ or $0 \leq x_1 \leq 1$. The other case is direct.

Since $u(0) = (1, \frac{1}{2})$ and the preorder is induced by the order of the first component, for each $x \in S_R$ there is an integer n such that $| x | \leq nu(0)$. SQL2 ... SQL9) Straightforward calculation. Hence $\langle S_R, +, \vee, \wedge, -, \sqrt{\ }, u, \frac{1}{2}, 0\rangle$ is a $\sqrt{ql}$-group.

Lemma 3.3 *Let $(G, u) \in \sqrt{q\mathcal{LG}_u}$. If $x \in [0, u_0]$ then $\sqrt{x} \in [0, u_0]$.*

Proof: By Axioms SQL6 and SQL9, if $\sqrt{x} \notin [0, u_0]$ then we have that: $x = \sqrt{\sqrt{\sqrt{\sqrt{x}}}} = \sqrt{x} \notin [0, u_0]$. $\square$

Definition 3.4 Let $(G,u) \in \sqrt{q\mathcal{LG}}_u$. Consider the set

$$\sqrt{[0,u_0]} = \{x \in G : 0 \leq x \leq u_0\}$$

equipped with the following operations:

1. $x \oplus y = u_0 \wedge (x+y)$
2. $\neg x = u(x)$
3. $\sqrt{x} = \sqrt{x} \upharpoonright_{\sqrt{[0,u_0]}}$

The structure $\langle \sqrt{[0,u_0]}, \oplus, \sqrt{\ }, \neg, 0, \frac{1}{2}, u_0 \rangle$ is denoted by $\sqrt{\Gamma}(G,u)$.

Proposition 3.5 *$\sqrt{\Gamma}(G,u)$ is a $\sqrt{qMV}$-algebra.*

Proof: By Proposition 2.4-5 and Lemma 3.3 the operations are closed in $\sqrt{[0,u_0]}$. SQ1) By [7, Proposition 2.12] the reduct $\langle \sqrt{[0,u_0]}, \oplus, \neg, 0, \frac{1}{2}, u_0 \rangle$ is a qMV-algebra. We only need to prove that the basic properties of square root are satisfied. SQ2)$\sqrt{\neg x} = \sqrt{u(x)} = \sqrt{\sqrt{\sqrt{x}}} = u(\sqrt{x}) = \neg\sqrt{x}$. SQ3) By definition it is clear that $\sqrt{\sqrt{x}} = \neg x$ and $\sqrt{\frac{1}{2}} = \frac{1}{2}$. SQ4) Taking into account that $u_0 = \sqrt{\sqrt{0}}$, we have that $0 \oplus \sqrt{x \oplus y} = u_0 \wedge (0 + \sqrt{u_0 \wedge (x+y)}) = u_0 \wedge (0 + \sqrt{-u_0 \vee (u_0 \wedge (x+y))}) = u_0 \wedge \frac{1}{2} = \frac{1}{2}$. Hence $\sqrt{\Gamma}(G)$ is a $\sqrt{qMV}$-algebra.

□

If we consider the $\sqrt{ql}$-group S_R, then $\sqrt{\Gamma}(S_R)$ is a the standard quasi $\sqrt{QMV}$-algebra. The following proposition can be easily proved.

Proposition 3.6 *If we define $\sqrt{\Gamma} : \sqrt{q\mathcal{LG}}_u \to \sqrt{q\mathcal{MV}}$ such that: for each $(G,u) \in \sqrt{q\mathcal{LG}}_u$, $(G,u) \mapsto \sqrt{\Gamma}(G,u)$ and for each $\sqrt{q\mathcal{LG}}_u$-homomorphism $f : (G_1,u_1) \to (G_2,u_2)$, $f \mapsto \sqrt{\Gamma}(f) = f \upharpoonright_{[0,u_0]}$, then $\sqrt{\Gamma}$ is a functor between the category of square root ql-groups and the category of square root quasi MV-algebras.* □

We are now ready to introduce the notion of quasi good sequence, that will play an analogous role as the one played by the notion of good sequence with respect to MV-algebras.

Definition 3.7 Let A be a $\sqrt{qMV}$- algebra. A sequence $\mathbf{a} = (a_1, a_2, ...)$ of elements on A is said to be *quasi good* iff

1. $a_1 \oplus a_2 = a_1 \oplus 0$,

2. $(a_2, a_3, ...)$ is a good sequence in $A \oplus 0$.

We denote by M_A^q the set of quasi good sequences in A. Note that $(a_1, ..., a_n, ...) \in M_A^q$ iff $(a_1 \oplus 0, ..., a_n, ...)$ is good sequence in the MV-algebra $A \oplus 0$. In view of this, if $\mathbf{a} = (a_1, ..., a_n, ...)$ is a quasi good sequence, we denote by $\mathbf{a} \oplus 0 = (a_1, ..., a_n...) \oplus 0$ the good sequence $(a_1 \oplus 0, ..., a_n, ...)$ in the MV-algebra $A \oplus 0$. Note that $M_{A\oplus 0} = \{\mathbf{a} \oplus 0 : \mathbf{a} \in M_A^q\} \subseteq M_A^q$.

Let A be a $\sqrt{qMV}$-algebra and consider the structure $\langle M_A^q, +, \vee, \wedge \rangle$ defined as M1, M2, M3 in the monoid of good sequences. Then for each $\mathbf{a}, \mathbf{b}, \mathbf{c} \in M_A^q$ we have that:

$$\mathbf{a} + \mathbf{b} = (\mathbf{a} + \mathbf{b}) \oplus 0 = \mathbf{a} + (\mathbf{b} \oplus 0),$$

$$\mathbf{a} \vee \mathbf{b} = (\mathbf{a} \vee \mathbf{b}) \oplus 0 = \mathbf{a} \vee (\mathbf{b} \oplus 0),$$

$$\mathbf{a} \wedge \mathbf{b} = (\mathbf{a} \wedge \mathbf{b}) \oplus 0 = \mathbf{a} \wedge (\mathbf{b} \oplus 0).$$

if $\mathbf{a} + \mathbf{b} = \mathbf{a} + \mathbf{c}$ then $\mathbf{b} \oplus 0 = \mathbf{c} \oplus 0$,

if $\mathbf{a} + \mathbf{b} = (0)$ then $\mathbf{a} \oplus 0 = \mathbf{b} \oplus 0 = (0)$.

Taking into account the preorder $\leq$ in A, we define the binary relation $\leq$ on M_A^q as follows:

$$\mathbf{b} \leq \mathbf{a} \text{ iff } b_i \leq a_i \text{ for each } i = 1, ..., n$$

One can easily see that $\langle M_A^q, \leq \rangle$ is a preorder. Moreover for each $\mathbf{a}$, $\mathbf{b}$ in M_A^q, $\mathbf{b} \leq \mathbf{a}$ iff there exists a good sequence $\mathbf{c} \in A \oplus 0$ such that $\mathbf{b} + \mathbf{c} = \mathbf{a} \oplus 0$. The element $\mathbf{c}$ is unique and $\mathbf{c}$ is noted by $\mathbf{a} - \mathbf{b}$. We consider the following sets:

$$M_{A_1} = \{((x), \mathbf{0}) : x \notin A \oplus 0\}$$

$$M_{A_2} = \{(\mathbf{0}, (y)) : y \notin A \oplus 0\}$$

$$M(A) = (M_{A\oplus 0})^2 \cup M_{A_1} \cup M_{A_2}$$

$M(A) \subseteq M_A^q \times M_A^q$ and by Lemma 2.1-4, if $((a), \mathbf{0}) \in M_{A_1}$ then $((\neg a), \mathbf{0}) \in M_{A_1}$ and the same obtains for M_{A_2}. Also we consider the binary relation "$\equiv$" in $M(A)$ defined as follows:

$$(\mathbf{a},\mathbf{b}) \equiv (\mathbf{a}',\mathbf{b}') \ iff \ \begin{cases} \mathbf{a}+\mathbf{b}' = \mathbf{a}'+\mathbf{b}, & \text{if } \mathbf{a},\mathbf{b},\mathbf{a}',\mathbf{b}' \in M_{A\oplus 0} \\ (\mathbf{a},\mathbf{b}) = (\mathbf{a}',\mathbf{b}'), & \text{otherwise.} \end{cases}$$

Clearly $\equiv$ is reflexive and symmetric. Transitivity follows either from the fact that $(M_{A\oplus 0})^2$ is the monoid of good sequences, or by the transitivity of the equality. Thus $\equiv$ is an equivalence relation.

We denote by $[\mathbf{a},\mathbf{b}]$ the equivalence class determined by the pair $(\mathbf{a},\mathbf{b})$ and by G_A^q the set of equivalence classes. Let us define the following operations on G_A^q:

$$[\mathbf{a},\mathbf{b}] + [\mathbf{c},\mathbf{d}] = [\mathbf{a}+\mathbf{c},\mathbf{b}+\mathbf{d}],$$

$$[\mathbf{a},\mathbf{b}] \vee [\mathbf{c},\mathbf{d}] = [(\mathbf{a}+\mathbf{d}) \vee (\mathbf{c}+\mathbf{b}),\mathbf{b}+\mathbf{d}],$$

$$[\mathbf{a},\mathbf{b}] \wedge [\mathbf{c},\mathbf{d}] = [(\mathbf{a}+\mathbf{d}) \wedge (\mathbf{c}+\mathbf{b}),\mathbf{b}+\mathbf{d}],$$

$$-[\mathbf{a},\mathbf{b}] = [\mathbf{b},\mathbf{a}],$$

$$0 = [\mathbf{0},\mathbf{0}].$$

It is not very hard to see that:

1. $[\mathbf{a},\mathbf{b}] + [\mathbf{c},\mathbf{d}] = [\mathbf{a}\oplus 0,\mathbf{b}] + [\mathbf{c},\mathbf{d}] = [\mathbf{a},\mathbf{b}\oplus 0] + [\mathbf{c},\mathbf{d}]$,
2. $[\mathbf{a},\mathbf{b}] \vee [\mathbf{c},\mathbf{d}] = [\mathbf{a}\oplus 0,\mathbf{b}] \vee [\mathbf{c},\mathbf{d}] = [\mathbf{a},\mathbf{b}\oplus 0] \vee [\mathbf{c},\mathbf{d}]$,
3. $[\mathbf{a},\mathbf{b}] \wedge [\mathbf{c},\mathbf{d}] = [\mathbf{a}\oplus 0,\mathbf{b}] \wedge [\mathbf{c},\mathbf{d}] = [\mathbf{a},\mathbf{b}\oplus 0] \wedge [\mathbf{c},\mathbf{d}]$,
4. The binary operations have the form $G_A^q \times G_A^q \to G_{A\oplus 0}$,
5. $\langle G_A^q, +, \vee, \wedge, -, 0\rangle$ is a ql-group.

We will refer to $\langle G_A^q, +, \vee, \wedge, -, 0\rangle$ as *Chang's ql-group.* We can see G_A^q as the Chang's l-group $G_{A\oplus 0}$ with new *non regular elements* given by the equivalence classes $[(a),\mathbf{0}] = \{((a),\mathbf{0})\}$ and $[\mathbf{0},(b)] = \{(\mathbf{0},(b))\}$ with $a, b \notin A\oplus 0$. Thus G_A^q has a structure

$$G_A^q = G_{A\oplus 0} \cup M_{A_1} \cup M_{A_2}$$

Proposition 3.8 [7, Proposition 3.17] *Let A be a $\sqrt{qMV}$-algebra and $[\mathbf{a},\mathbf{b}] \in G_A^q$. Then the following assertions are equivalent:*

1 $0 \leq [\mathbf{a}, \mathbf{b}]$,

2 $\mathbf{b} + (0) \leq \mathbf{a} + (0)$,

3 *There exists a unique good sequence* $\mathbf{e} = (e_1, ..., e_n, 0, 0, ..)$ *in* $A \oplus 0$ *such that* $[\mathbf{a}, \mathbf{b}] + [(0), (0)] = [\mathbf{e}, (0)]$.

Moreover in the case $[\mathbf{a}, \mathbf{b}] \leq [(1), (0)]$ *there exists unique* $e_1 \in A \oplus 0$ *such that*

4 *If* $\mathbf{a}, \mathbf{b} \in G_{A+0}$ *then* $[\mathbf{a}, \mathbf{b}] = [(e_1), 0]$,

5 *otherwise,* $[\mathbf{a}, \mathbf{b}] = [(a), \mathbf{0}]$ *and* $a \oplus 0 = e_1$.

□

Proposition 3.9 *Let* A *be a* $\sqrt{qMV}$*-algebra. Consider the ql-group* G_A^q *equipped with the following operation:*

$$u([\mathbf{x}, \mathbf{y}]) = \begin{cases} [(1), (0)] - [\mathbf{x}, \mathbf{y}], & \text{if } \mathbf{x}, \mathbf{y} \in M_{A\oplus 0}, \\ [(\neg a), \mathbf{0}], & \text{if } (\mathbf{x}, \mathbf{y}) = ((a), \mathbf{0}) \in M_{A_1}, \\ [\mathbf{0}, (\neg b)], & \text{if } (\mathbf{x}, \mathbf{y}) = (\mathbf{0}, (b)) \in M_{A_2} \end{cases}$$

$$\sqrt{[\mathbf{x}, \mathbf{y}]} = \begin{cases} [(\sqrt{a}), \mathbf{0}], & [\mathbf{0}, \mathbf{0}] \leq [\mathbf{x}, \mathbf{y}] = [(a), \mathbf{0}] \leq [\mathbf{1}, \mathbf{0}] \\ [\mathbf{x}, \mathbf{y}], & \text{otherwise} \end{cases}$$

Then u *is a strong q-unit in* G_A^q *and* $\langle G_A^q, \sqrt{\ }, u, [(\frac{1}{2}), \mathbf{0}] \rangle$ *is an object in* $\sqrt{q\mathcal{LG}}_u$.

Proof: We first prove that u is a strong q-unit in G_A^q. 1) $[(0), (0)] \leq [(1), (0)] = u([(0), (0)])$. 2) $u([\mathbf{x}, \mathbf{y}] + [(0), (0)]) = u([\mathbf{x} \oplus 0, \mathbf{y} \oplus 0] + [(0), (0)]) = u([\mathbf{x} \oplus 0, \mathbf{y} \oplus 0]) = [(1), (0)] - [\mathbf{x} \oplus 0, \mathbf{y} \oplus 0] = u([(0), (0)]) - [\mathbf{x}, \mathbf{y}]$. 3) Suppose that $0 \leq [\mathbf{x}, \mathbf{y}] \leq [(1), (0)]$. Using Proposition 3.8, $[\mathbf{x}, \mathbf{y}] = [(a), \mathbf{0}]$. In this case, $u([(0), (0)]) - u([(a), \mathbf{0}]) = [(1), (0)] - [(\neg a), \mathbf{0}] = [(1), (0)] + [\mathbf{0}, (\neg a)] = [(1), (0)] + [\mathbf{0}, (\neg a \oplus 0)] = [(1), (\neg a \oplus 0)] = [(1), (\neg(a \oplus 0))] = [(1), (1) - (a \oplus 0)]$, in view of [3, Proposition 2.3.4] and the fact that $a \oplus 0 \in A \oplus 0$. On the other hand $[(0), (0)] + [(a), \mathbf{0}] = [(0), (0)] + [(a \oplus 0), \mathbf{0}] = [(a \oplus 0), \mathbf{0}]$. We need to see that $[(1), (1) - (a \oplus 0)] = [(a \oplus 0), \mathbf{0}]$, but this follows from the fact that $(1) = (1) + \mathbf{0}$ and $(1) = ((1) - (a \oplus 0)) + (a \oplus 0)$, resulting $u([(0), (0)]) - u([(a), \mathbf{0}]) = [(0), (0)] + [(a), \mathbf{0}]$. 4) By definition of u, it follows that $uu([\mathbf{x}, \mathbf{y}]) = [\mathbf{x}, \mathbf{y}]$. 5) Since $u([(0), (0)]) = [(1), (0)]$ is a strong unit of the Chang's l-group $G_{A\oplus 0}$, for each $[\mathbf{x}, \mathbf{y}] \in G_A^q \mid [\mathbf{x}, \mathbf{y}] |\leq$

$nu([(0),(0)])$ for some integer $n \geq 0$. Thus u is a strong q-unit in G_A^q. Taking into account that $G_A^q = G_{A\oplus 0} \cup M_{A_1} \cup M_{A_2}$, by Proposition 3.8, we can see that for each $x \notin Reg(G_A^q)$, $-u_0 \leq x \leq 0$ or $0 \leq x \leq u_0$. To prove that $\langle G_A^q, \sqrt{\;}\; u, [(\frac{1}{2}), \mathbf{0}]\rangle$ is an object in $\sqrt{q\mathcal{LG}_u}$ it suffices to show that if $0 \leq x \leq u_0$ then $u(x) = \sqrt{\sqrt{x}}$. In fact: let $[\mathbf{x}, \mathbf{y}] \in G_A^q$ be such that $[\mathbf{0}, \mathbf{0}] \leq [\mathbf{x}, \mathbf{y}] = [(a), \mathbf{0}] \leq [\mathbf{1}, \mathbf{0}]$. Then $\sqrt{\sqrt{[(a), \mathbf{0}]}} = \sqrt{[(\sqrt{a}), \mathbf{0}]}$. Since $[\mathbf{0}, \mathbf{0}] \leq [(\sqrt{a}), \mathbf{0}] \leq [\mathbf{1}, \mathbf{0}]$ then $\sqrt{[(\sqrt{a}), \mathbf{0}]} = [(\sqrt{\sqrt{a}}), \mathbf{0}] = [(\neg a), \mathbf{0}]$. If $[(a), \mathbf{0}]$ is regular in G_A^q then $u([(a), \mathbf{0}]) = [\mathbf{1}, \mathbf{0}] - [(a), \mathbf{0}] = [\mathbf{1}, (a)] = [(\neg a), \mathbf{0}] = \sqrt{\sqrt{[(a), \mathbf{0}]}}$. If $[(a), \mathbf{0}]$ is not regular it follows by definition of u.
□

Theorem 3.10 *Let A be a $\sqrt{qMV}$-algebra and consider the $\sqrt{q\mathcal{LG}_u}$-group $(G_A^q, u) = \langle G_A^q, \sqrt{\;}\; u, [(\frac{1}{2}), \mathbf{0}]\rangle$. Then $\varphi : A \rightarrow \sqrt{\Gamma}(G_A^q, u)$ such that $a \mapsto \varphi(a) = [(a), \mathbf{0}]$, is a $\sqrt{q\mathcal{MV}}$-isomorphism.*

Proof: By Proposition 3.8, φ is a bijection. It is not very hard to see that the restriction $\varphi \restriction_{A\oplus 0}$ is the well known MV-isomorphism from the MV-algebra $A \oplus 0$ onto the MV-algebra $\Gamma(G_{A\oplus 0}, [(1),(0)])$ see [3, Proposition 2.4.5]. Clearly $\varphi(\sqrt{x}) = [(\sqrt{x}), \mathbf{0}] = \sqrt{[(x), \mathbf{0}]}$. Let $x, y \in A$. Then $\varphi(x \oplus y) = \varphi((x\oplus 0)\oplus(y\oplus 0)) = \varphi \restriction_{A\oplus 0} ((x\oplus 0)\oplus(y\oplus 0)) = \varphi \restriction_{A\oplus 0} (x\oplus 0) \oplus \varphi \restriction_{A\oplus 0} (y \oplus 0) = [(x \oplus 0), \mathbf{0}] \oplus [(y \oplus 0), \mathbf{0}] = [(1),(0)] \wedge ([(x \oplus 0), \mathbf{0}] + [(y \oplus 0), \mathbf{0}]) = [(1),(0)] \wedge ([(x), \mathbf{0}] + [(y), \mathbf{0}]) = [(1),(0)] \wedge (\varphi(x) + \varphi(y)) = \varphi(x) \oplus \varphi(y)$. With the same argument we can see that $\varphi(x \vee y) = \varphi(x) \vee \varphi(y)$. If $x \in A\oplus 0$ then, $\varphi(\neg x) = \varphi \restriction_{A\oplus 0} (\neg x) = [(1),(0)] - [(x),(0)] = u([(x),(0)]) = \neg[(x),(0)] = \neg\varphi \restriction_{A\oplus 0} (x) = \neg\varphi(x)$. If $x \notin A\oplus 0$, $\varphi(\neg x) = [(\neg x), \mathbf{0}] = u([(x), \mathbf{0}]) = \neg\varphi(x)$. Hence φ is a $\sqrt{qMV}$-isomorphism.
□

Lemma 3.11 *Let A, B be a $\sqrt{qMV}$-algebras and $\varphi : A \rightarrow B$ be a $\sqrt{q\mathcal{MV}}$-homomorphism. If $\mathbf{a} = (a_1, a_2 ... a_n, ...)$ is a quasi good sequence in M_A^q, then $\varphi^*(\mathbf{a}) = (\varphi(a_1), \varphi(a_2), ..., \varphi(a_n), ...)$ is a quasi good sequence in M_B^q. Moreover:*

1. *φ^* define a $\langle +, \vee, \wedge, \mathbf{0}\rangle$-homomorphism $\varphi^* : M_A^q \rightarrow M_B^q$,*
2. *if we consider the $\sqrt{q\mathcal{LG}_u}$-groups (G_A^q, u_A) and (G_B^q, u_B) then the application $\varphi^\# : \langle G_A^q, u_A\rangle \rightarrow \langle G_B^q, u_B\rangle$ such that $\varphi^\#([\mathbf{a}, \mathbf{b}]) = [\varphi^*(\mathbf{a}), \varphi^*(\mathbf{b})]$ is a $\sqrt{q\mathcal{LG}_u}$-homomorphism.*

Proof: Since φ is $\sqrt{q\mathcal{MV}}$-homomorphism, we have that $\varphi(a_1) \oplus \varphi(a_2) = \varphi(a_1 \oplus a_2) = \varphi(a_1 \oplus 0) = \varphi(a_1) \oplus \varphi(0) = \varphi(a_1) \oplus 0$. Since $(a_2 ... a_n, ...)$ is a good sequence in $A \oplus 0$, $(\varphi(a_2), ..., \varphi(a_n), ...)$ is a good sequence in $B \oplus 0$ (see [3, §7.1]). Thus $\varphi^*(\mathbf{a}) \in M_B^q$. From this, it is clear that $\varphi^*(\mathbf{0}) = \mathbf{0}$.

1) We first note that $\varphi^*(\mathbf{a} \oplus \mathbf{0}) = \varphi^*(\mathbf{a}) \oplus \mathbf{0}$. Let $\triangle$ be a binary operation in M_A^q. Taking into account [3, §7.1], $\varphi^*(\mathbf{a} \triangle \mathbf{b}) = \varphi^*((\mathbf{a} \oplus 0) \triangle (\mathbf{b} \oplus 0)) = \varphi^*(\mathbf{a} \oplus 0) \triangle \varphi^*(\mathbf{b} \oplus 0) = (\varphi^*(\mathbf{a}) \oplus 0) \triangle (\varphi^*(\mathbf{b}) \oplus 0) = (\varphi^*(\mathbf{a}) \triangle \varphi^*(\mathbf{b})) \oplus 0 = \varphi^*(\mathbf{a}) \triangle \varphi^*(\mathbf{b})$. Finally φ^* is a $\langle +, \vee, \wedge, \mathbf{0} \rangle$-homomorphism. 2) Straightforward calculation. $\square$

Proposition 3.12 $\sqrt{\Xi} : \sqrt{q\mathcal{MV}} \to \sqrt{q\mathcal{LG}_u}$ *such that for each* $A \in \sqrt{q\mathcal{MV}}$, $\sqrt{\Xi}(A) = \langle G_A^q, u_A \rangle$ *and for each* $\sqrt{q\mathcal{MV}}$*-homomorphism* $\varphi : A \to B$, $\sqrt{\Xi}(\varphi) = \varphi^{\#}$ *is a functor.* $\square$

Theorem 3.13 *The composite functor* $\sqrt{\Gamma}\sqrt{\Xi} : \sqrt{q\mathcal{MV}} \to \sqrt{q\mathcal{MV}}$ *is naturally equivalent to the identity functor* $1_{\sqrt{q\mathcal{MV}}}$.

Proof: Let $\psi : A \to B$ be a $\sqrt{q\mathcal{MV}}$-homomorphism and consider $\varphi_A : A \to \sqrt{\Gamma}\sqrt{\Xi}(A)$, $\varphi_B : B \to \sqrt{\Gamma}\sqrt{\Xi}(B)$ be the $\sqrt{q\mathcal{MV}}$-isomorphisms given in Theorem 3.10. We will prove that the following diagram is commutative:

$$\begin{tikzcd} A \arrow[r, "\psi"] \arrow[d, "\varphi_A"'] & B \arrow[d, "\varphi_B"] \\ \sqrt{\Gamma}\sqrt{\Xi}(A) \arrow[r, "\sqrt{\Gamma}\sqrt{\Xi}(\psi)"'] & \sqrt{\Gamma}\sqrt{\Xi}(B) \end{tikzcd}$$

If $a \in A$ then $\varphi_B \psi(a) = [(\psi(a)), \mathbf{0}]$. On the other hand, by Lemma 3.11 $(\sqrt{\Gamma}\sqrt{\Xi}(\psi)\varphi_A)(a) = \sqrt{\Gamma}\sqrt{\Xi}(\psi)([(a), \mathbf{0}]) = [(\psi(a)), \mathbf{0}] \in \sqrt{\Gamma}\sqrt{\Xi}(B)$. Since $\sqrt{\Gamma}(\sqrt{\Xi}(\psi))$ is the restriction of $\sqrt{\Xi}(\psi)$ to $\sqrt{\Gamma}\sqrt{\Xi}(B)$, we can write $\sqrt{\Gamma}(\sqrt{\Xi}(\psi))(\varphi_A(a)) = [(\psi(a)), \mathbf{0}] = \varphi_B \psi(a)$. So the diagram is commutative.

$\square$

Let $(G, u) \in \sqrt{q\mathcal{LG}_u}$. By [3, Lemma 7.1.3], for each $0 \leq a \in G + 0$ there exists a unique good sequence $g(a) = (a_1, \ldots, a_n, 0, \ldots)$ such that $a = a_1 + \ldots + a_n$. The elements of this good sequence are inductively defined as follows $a_1 = a \wedge u_0$ and $a_{k+1} = (a - a_1 - \ldots a_k) \wedge u_0$. Thus if $a \leq u_0$ then $g(a) = (a)$. By Lemma [3, Corollary 7.1.6] the map

$$\psi_0 : G + 0 \to (G^q_{\sqrt{\Gamma}(G,u)} + [\mathbf{0}, \mathbf{0}])$$

defined by $\psi_0(a) = [g(a \vee 0), g(-a \vee 0)]$ is a $\langle +, \vee, \wedge, -, 0\rangle$-isomorphism such that $\psi_0(x) = [(x), \mathbf{0}]$ whenever $0 \leq x \leq u_0$. Using the map ψ_0 we have the following:

Proposition 3.14 *Let $(G, u) = (G+0) \cup G_1 \cup G_2 \in \sqrt{q\mathcal{LG}}_u$. Then the map $\psi : G \to G^q_{\sqrt{\Gamma}(G,u)}$ defined as follows:*

$$\psi(x) = \begin{cases} \psi_0(x), & \text{if } x \in G+0 \\ [(x), \mathbf{0}], & \text{if } x \in G_1 \ , \\ [\mathbf{0}, (x)], & \text{if } x \in G_2. \end{cases}$$

is a $\sqrt{q\mathcal{LG}}_u$-isomorphism.

Proof: Taking into account that $G^q_{\sqrt{\Gamma}(G,u)} = (G^q_{\sqrt{\Gamma}(G,u)} + [\mathbf{0}, \mathbf{0}]) \cup \mathbf{M}_{\sqrt{\boldsymbol{\Gamma}}(\mathbf{G},\mathbf{u})_\mathbf{1}} \cup \mathbf{M}_{\sqrt{\boldsymbol{\Gamma}}(\mathbf{G},\mathbf{u})_\mathbf{2}}$, the restriction $\psi \restriction_{G_i}$ is a bijection from G_i to $M_{\sqrt{\Gamma}(G,u)_i}$ for $i = 1, 2$. Thus ψ is a bijection. We first prove that $\psi(x+0) = \psi(x) + \psi(0)$. If $x \in G+0$ it follows from the fact that $\psi = \psi_0$. Suppose that $x \in G_1$. Then $\psi(x+0) = \psi_0(x+0) = [(x+0), \mathbf{0}] = [(\mathbf{x}), \mathbf{0}] + [\mathbf{0}, \mathbf{0}] = \psi(\mathbf{x}) + \psi(\mathbf{0})$. If $x \in G_2$ then, $-x \in G_1$ and $-\psi(x+0) = \psi(-(x+0)) = \psi(-x+0) = \psi(-x) + \psi(0) = -\psi(x) + \psi(0) = -(\psi(x) + \psi(0))$. Hence $\psi(x+0) = \psi(x) + \psi(0)$.

If $*$ is a binary operation then $\psi(x) * \psi(y) = (\psi(x) + \mathbf{0}) * (\psi(y) + \mathbf{0}) = (\psi(x) + \psi(0)) * (\psi(y) + \psi(0)) = \psi_0(x+0) * \psi_0(y+0) = \psi_0((x+0) * (y+0)) = \psi_0(x * y) = \psi(x * y)$.

If $x \in G+0$ then $\psi(-x) = \psi_0(-x) = -\psi_0(x) = -\psi(x)$. If $x \in G_1$ then $-x \in G_2$, thus $\psi(-x) = [\mathbf{0}, (x)] = -[(x), \mathbf{0}] = -\psi(x)$. If $x \in G_2$ we use the same argument.

Let $x \in G$. If $x \not\in [0, u_0]$ then $x = \sqrt{x}$ and $\psi(x) \not\in [[\mathbf{0}, \mathbf{0}], [(u_0), \mathbf{0}]]$. Therefore $\sqrt{\psi(x)} = \psi(x) = \psi(\sqrt{x})$. Suppose that $x \in [0, u_0]$. Clearly $\psi(x) = [(x), \mathbf{0}]$ and then $\sqrt{\psi(x)} = \sqrt{[(x), \mathbf{0}]} = [(\sqrt{x}), \mathbf{0}]$. By Lemma 3.3, $\sqrt{x} \in [0, u_0]$ and then $\psi(\sqrt{x}) = [(\sqrt{x}), \mathbf{0}]$. Thus $\psi(\sqrt{x}) = \sqrt{\psi(x)}$.

Finally ψ is a $\sqrt{q\mathcal{LG}}_u$-isomorphism.

$\square$

Theorem 3.15 *The composite functor $\sqrt{\Xi}\sqrt{\Gamma} : \sqrt{q\mathcal{LG}}_u \to \sqrt{q\mathcal{LG}}$ is naturally equivalent to the identity functor $1_{\sqrt{q\mathcal{LG}}}$.*

Proof: Let $(G,u),(H,v) \in \sqrt{q\mathcal{LG}_u}$, $f: G \to H$ be a $\sqrt{q\mathcal{LG}_u}$-homomorphism and $\psi_G : G \to G^q_{\sqrt{\Gamma}(G,u)} = \sqrt{\Xi}\sqrt{\Gamma}(G)$, $\psi_H : H \to H^q_{\sqrt{\Gamma}(H,v)} = \sqrt{\Xi}\sqrt{\Gamma}(H)$ be the isomorphisms defined in Proposition 3.14. We will see that the following diagram is commutative:

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "\psi_G"'] & H \arrow[d, "\psi_H"] \\ \sqrt{\Xi}\sqrt{\Gamma}(G) \arrow[r, "\sqrt{\Xi}\sqrt{\Gamma}(f)"'] & \sqrt{\Xi}\sqrt{\Gamma}(H) \end{tikzcd}$$

If $x \in G{+}0$ then $f(x) \in H{+}0$, thus $((\Xi\Gamma_q(f))\psi_G)(x) = (\psi_H f)(x)$ in view of [3, Theorem 7.1.7]. If $x \in G_1$ then $\psi_G(x) = [(x), \mathbf{0}]$ and $((\Xi\Gamma_q(f))\psi_G)(x) = [(f(x)),(f(0))] = [(f(x)), \mathbf{0}]$. On the other hand, since $0 \leq f(x) \leq f(u(0)) = v(0)$, $\psi_h(f(x)) = [(f(x)), \mathbf{0}]$. If $x \in G_2$ then $-x \in G_1$, since $--x = x$. Finally the diagram is commutative. $\square$

In view of the Theorems 3.10, 3.13, 3.14 and 3.15 we can establish the following result:

Theorem 3.16 *$\sqrt{q\mathcal{MV}}$ is categorically equivalent to $\sqrt{q\mathcal{LG}_u}$.* $\square$

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