Kinematic Integral Attitude Tracking under Input Saturation for Ascent Guidance and Control

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Abstract

A new global robust attitude tracking controller under input saturation is proposed based on a time-varying variable structure control strategy. Uniform asymptotic stability and exponential convergence conditions are given. The smooth version (i.e. not switched, a crucial matter in thrust vector control (TVC) applications) of the new controller may reject certain non-vanishing perturbations, which is a novel result in this framework. The controller is applied to the ascent guidance and control of a suborbital experimental vehicle without roll control and evaluated through simulations.

Nomenclature

q_0, q = quaternion of vehicle attitude.

$\phi, \underline{n}$ = angle and axis for the rotation associated to (q_0, q).

ω = angular velocity vector $\omega \in \mathbb{R}^3$.

q_{d0}, q_d = desired attitude quaternion.

$\delta q_0, \delta q$ = attitude tracking error quaternion.

$\delta q_{p0}, \delta q_p$ = partial quaternion to measure longitudinal attitude error.

$\phi_p, \underline{n}_p$ = angle and axis for the rotation associated to $(\delta q_{p0}, \delta q_p)$.

$\delta\omega$ = angular velocity tracking error.

$\underline{\eta}$ = states associated to non-rigid body dynamic model.

J = inertia matrix.

r = radius of the trajectory relative to earth center.

v = modulus of the velocity relative to earth fixed frame.

γ = angle between the velocity vector and the local tangent plane to earth.

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L	$=$	modulus of the lift force.
D	$=$	modulus of the drag force.
U	$=$	modulus of the thrust along the axial direction.
m	$=$	vehicle mass.
q_∞	$=$	dynamic pressure.
μ	$=$	earth gravity constant.
θ	$=$	vehicle pitch angle.
α	$=$	$\theta - \gamma$ the angle of attack on the pitch plane.
$\mathcal{T}$	$=$	Attainable torque set. Its direct time-dependence is ommited to simplify the notation.
$\mu_{\mathcal{T}}(s)$	$=$	a feasible torque modulus selection in $\mathcal{T}$ for each $s \neq \underline{0}$ and $t \geq t_0$.
$sel_{\mathcal{T}}(s)$	$=$	$\mu_{\mathcal{T}}(s)\check{s}$ is a feasible torque selection in $\mathcal{T}$, being $\check{s} = \frac{s}{\|s\|}$ for each $s \neq \underline{0}$.
$\lambda_{max}(X)$	$=$	the maximum eigenvalue of matrix X, while $\overline{\lambda}_X \geq \lambda_{max}(X(t))\ \forall\ t$.
$\lambda_{min}(X)$	$=$	the minimum eigenvalue of matrix X, while $\underline{\lambda}_X \leq \lambda_{min}(X(t))\ \forall\ t$.
$\underline{\sigma}_X, \overline{\sigma}_X$	$=$	respectively the infimum and supremum of the singular values of matrix X, over all t.
$\underline{v}\times$	$=$	is the skew matrix associated to the vector product whose first factor is $\underline{v} \in \mathbb{R}^3$.

I. Introduction

Variable structure systems (VSS) give a framework to solve regulation and tracking problems in uncertain nonlinear systems [1]. In particular, the sliding mode control design method uses a virtual output s and a proper switching logic to make attractive the manifold $s = \underline{0}$ over which the system has a desired asymptotic behaviour. Boskovic *et al.* use this approach to design a control law for attitude regulation [2] and tracking [3] that explicitly includes input saturation in the analysis. The controller proposed in [4] extends the bounded torque sets beyond the hypercubic regions considered in [2][3] by including a control allocation law and specific robustness margins for closed-loop analysis. However, none of these works address the robust performance of the regularized (i.e. not switched) version of the controller, a critical issue in thrust vector control (TVC) applications.

Several attempts to obtain a robust *and* smooth version of this controller impose conditions on the adaptation of the proportional gain which are not easily verifiable in practice [5]. Besides, only certain vanishing perturbations can be guaranteed to be rejected [6]. The smooth version proposed in [7] replaces the switching law by the hyperbolic tangent function, but the robust attitude error convergence is not guaranteed. Other approaches use integral terms with anti-wind up algorithms but the asymptotic stability of the resulting non-autonomous nonlinear system is not addressed [8].

Here, a different approach is adopted based upon the introduction of an integral term in s. The resulting regularized version features a PID-like behavior, known to reject non-vanishing perturbations such as constant torques in body frame as may appear in atmospheric rocket trajectories. In the first part two lemmas regarding the convergence of kinematic equations are shown in subsection II-A. These lemmas together with Barbalat's lemma are instrumental for the robust tracking convergence results under input saturation obtained later in the (main) subsection II-C. Based on Jurdevic-Quinn's method, some uniform asymptotic stability properties of the overall closed loop system are demonstrated within the same subsection. As shown, the approach also provides a (robust) control Lyapunov function under bounded controls, which allows computation of a meaningful $\mathcal{H}_\infty$ gain. An adaptive regularization scheme is proposed in subsection II-D. In subsection II-E, a global kinematic observer, based on [9], is used to asymptotically reject a constant bias in the gyro measurement. In section III, the proposed attitude tracking controller is applied to ascent guidance and control. In subsection III-A a convenient redefinition of the attitude kinematics error allows us to separate the longitudinal attitude error, related with the guidance command, from its roll attitude error which that might have other control requirements. This decoupling of attitude kinematics was useful in [10] for the sun-pointing of a VS-30 rocket payload (launched in 2007), and a similar approach was considered in the Ares I [11]. The strategy is evaluated through numerical simulations on a suborbital experimental vehicle that is considered part of a satellite launcher development [12], [13]. Finally, section IV is devoted to the conclusions.

II. Analysis of the Trajectory Tracking Controller

A. Attitude kinematics

Let $\mathcal{I}$ be an inertial frame, $\mathcal{S}$ the vehicle frame and $\mathcal{D}$ the desired frame. We denote by ω the angular rate of $\mathcal{S}$ relative to $\mathcal{I}$ expressed in $\mathcal{S}$. and by ω_d the angular rate of $\mathcal{D}$ relative to $\mathcal{I}$ expressed in $\mathcal{D}$, assuming ω_d and $\dot{\omega}_d$ uniformly bounded. The rotation matrix from $\mathcal{I}$ to $\mathcal{S}$, will be parameterized by the unit attitude quaternion (q_0, q), with $q_0 = \cos(\phi/2) \in \mathbb{R}$ as its scalar part and $q = \sin(\phi/2)\underline{n} \in \mathbb{R}^3$ as its vector part, where ϕ represents the rotation angle about the unit vector $\underline{n}$. The map from (q_0, q) to rotations is 2 to 1, because (q_0, q) and $(-q_0, -q)$ determine the same rotation. Let (q_{d0}, q_d) be the quaternion from $\mathcal{I}$ to $\mathcal{D}$. $I \in \mathbb{R}^{3\times 3}$ is the identity matrix and $(\cdot)^T$ denotes transposition hereafter.

The tracking error quaternion is defined for all $t \geq t_0$ as in [14]:

$$\begin{cases} \delta q_0 &= q_0 q_{d0} + q^T q_d \\ \delta q &= -q_d q_0 + (q_{d0} I - q_d \times) q \end{cases} \tag{1}$$

The associated kinematic equation with angular velocity error $\delta\omega = (\omega - \omega_d)$ is:

$$\begin{cases} \delta \dot{q}_0 &= -\frac{1}{2}\delta q^T \delta\omega \\ \delta \dot{q} &= \Gamma(\delta q_0, \delta q, \delta\omega, \omega_d) \end{cases} \tag{2}$$

where $\Gamma(\delta q_0, \delta q, \delta\omega) := \frac{1}{2}(\delta q_0 I + \delta q \times)\delta\omega + \delta q \times \omega_d$.

The following lemma will be useful in the next section (see [5] for $k \equiv 1$, and $N \equiv I$):

Lemma II.1: Let $s = N(k\Sigma\delta q + \delta\omega)$ with:

a. $\Sigma, N : \mathbb{R} \to \mathbb{R}^{3\times 3}$ symmetric and smooth matrix functions verifying $0 < \underline{\lambda}_\Sigma, \overline{\lambda}_\Sigma, \underline{\lambda}_N, \overline{\lambda}_N < +\infty$,

b. $k : \mathbb{R} \to \mathbb{R}$ has bounded derivative almost everywhere. and $|k| \leq \overline{k} < +\infty\ \forall t \geq t_0$.

c. $\exists\ t_x \geq t_0\ /\ \forall t \geq t_x\ :\ \text{sign}(\delta q_0(t)) = \text{sign}(k(t)) = \zeta_q$ (constant), $|\delta q_0(t)| > 0$ and $|k(t)| \geq \underline{k} > 0$.

If $\lim_{t\to\infty} s(t) = \underline{0}$, then $\lim_{t\to+\infty}\delta\omega(t) = \underline{0}$ and $\lim_{t\to+\infty}|\delta q_0(t)| = 1$.

Proof: See Appendix.

□

Observe that in Lemma II.1 we may select $\zeta_q \in \{-1, 1\}, t_x \geq t_0$ and $\epsilon_x \in \mathbb{R}_{>0}$ such that $\zeta_q \delta q_0(t) \geq \epsilon_x \; \forall \; t \geq t_x$, which will be assumed hereafter.

The following initial value problem is started at $t_{PID} \geq t_x \geq t_0$,

$$\begin{cases} \dot{z}_0 &= -\frac{1}{2} z^T (k_{Ip}\delta q - k_{II} z); \qquad z_0(t_{PID}) = \zeta_q \\ \dot{z} &= \Gamma(z_0, z, k_{Ip}\delta q - k_{II} z, 0); \qquad z(t_{PID}) = \underline{0} \end{cases} \tag{3}$$

with smooth matrix functions k_{Ip}, k_{II} such that $\zeta_q k_{Ip}, \zeta_q k_{II} > 0$ and $0 < \underline{\sigma}_{k_{Ip}}, \underline{\sigma}_{k_{II}}, \overline{\sigma}_{k_{Ip}}, \overline{\sigma}_{k_{II}} < +\infty$. Equation (3) gives the *kinematic integral* subsystem to be tracked as it obeys the kinematic restrictions and integrates the tracking error with forgetting factor k_{II}.

Let $(\delta q_{0z}, \delta q_z)$ be the composition of $(\delta q_0, \delta q)$ with (z_0, z), which may be seen as a tracking error between $(\delta q_0, \delta q)$ and $(z_0, -z)$. As $(z_0, -z)$ has angular velocity $-k_{Ip}\delta q + k_{II} z$ (see (3)), following (2) the new quaternion $(\delta q_{0z}, \delta q_z)$ is driven by the angular velocity error $\delta\omega_z = \omega - \omega_{dz}$ for a modified reference with angular velocity $\omega_{dz} := \omega_d - k_{Ip}\delta q + k_{II} z$. The definition of $\delta\omega$ and (1) determine:

$$\begin{cases} \delta q_{0z} &= \delta q_0 z_0 + \delta q^T z \\ \delta q_z &= z\delta q_0 + z_0 \delta q + z \times \delta q \\ \delta\omega_z &= \delta\omega + k_{Ip}\delta q - k_{II} z \end{cases} \tag{4}$$

and we will take $s = N(k\Sigma\delta q_z + \delta\omega_z)$ since t_{PID} with $k(t_{PID}^+) > 0$ to obtain $k(t_{PID}^+)\delta q_{0z} > 0$.

The following lemma relates the limiting behaviours of $(\delta q_{z0}, \delta q_z, \delta\omega_z)$ and $(\delta q_0, \delta q, \delta\omega, z_0, z)$.

Lemma II.2: If $\lim_{t\to\infty}\delta q_z(t) = \underline{0}$ and $\lim_{t\to\infty}\delta\omega_z(t) = \underline{0}$, then $\lim_{t\to+\infty}\delta\omega(t) = \underline{0}$, $\lim_{t\to+\infty}|\delta q_0(t)| = 1$ and $\lim_{t\to+\infty}|z_0(t)| = 1$.

Proof: See Appendix

□

B. Vehicle's dynamic model

The main body dynamics is considered as a perturbed rigid body:

$$J\dot{\omega} \;=\; -\omega \times J\omega + \nu(J,\omega) + T_c + T_d \tag{5}$$

where $T_c \in \mathcal{T}$ is the control torque $\mathcal{T}$, T_d is the disturbance torque, $J = J^T > 0$ is the inertia matrix function with $0 < \underline{\lambda}_J, \overline{\lambda}_J < +\infty$ and $\overline{\sigma}_{\dot{J}} < +\infty$. $\nu(J,\omega)$ gives the terms associated with the variable mass, which may be bounded linearly with $\|\omega\|$ while $\frac{\|\nu(J,\omega)\|}{\|\omega\|}$ is integrable as the rocket mass is finite (see [15]). Also $\nu(J,\omega)$ includes the term $-\dot{J}\omega$, where $\dot{J}J < 0$ for $\dot{J} \neq \mathbf{0}$ in a typical rocket application. Additional differential equations may be useful:

$$\dot{\underline{\eta}} = f_\eta(t, \underline{\eta}, q, \omega, T_c, J, T_d, \omega_{Ai}, \zeta_{Ai}, \omega_{Fj}, \zeta_{Fj}, \omega_{Sk}, \zeta_{Sk}, \ldots) \tag{6}$$

for (second order) dynamics of actuators, slosh and flexibility modes, being (ω_*,ζ_*) their natural frequencies and damping factors. The associated perturbation terms are included in T_d (see [16][17]).

C. Attitude tracking law design

The following (negative) feedback control law will be considered:

$$T_c(s) \;=\; \mathrm{sel}_{\mathcal{T}}(-s) \tag{7}$$

where for each $s \neq \underline{0}$ the torque selection function $\mathrm{sel}_{\mathcal{T}}(s)$ generates a torque in $\mathcal{T}$ along s with modulus $\mu_{\mathcal{T}}(s) : \mathbb{R}^3 \to \mathbb{R}_{>0}$, i.e. $\mathrm{sel}_{\mathcal{T}}(s) = \mu_{\mathcal{T}}(s)\check{s}$, where $\check{s} = s/\|s\|$ hereafter. We assume that $\mu_{\mathcal{T}}(s)$ realizes the maximum feasible modulus for each $s \neq \underline{0}$ and $\mathcal{T}$ is a convex bounded absorbing set for all $t \geq t_0$ (see [18][19]). Hence $\mathrm{sel}_{\mathcal{T}}(s)$ is continuous for $s \neq \underline{0}$, a mandatory condition for a TVC and desirable for RCS (Reaction Control System) duty cycles generation [10].

Let $\dot{\hat{\omega}}_\tau$ be a bounded angular acceleration to be defined for each law; hence we state the following:

Definition II.1: Let $\rho_\tau := \inf_{s\neq\underline{0}, T_d, \dot{\hat{\omega}}_\tau, J, t\geq t_0} \left\{-\check{s}^T\left(\mathrm{sel}_{\mathcal{T}}(-s) + T_d - J\dot{\hat{\omega}}_\tau\right)\right\}$ a robustness margin.

Assumption II.1: There exists a bounded $\hat{\omega} \in \mathbb{R}_{\geq 0}$ and a smooth matrix function $M_\omega > \epsilon_M I$, $\epsilon_M \in \mathbb{R}_{>0}$, such that $\forall t \geq t_0$, $\|\omega\| > \hat{\omega}$ implies $\omega^T J\dot{M}_\omega J\omega + 2\omega^T JM_\omega(\dot{J}\omega - \omega \times J\omega + \nu(J,\omega)) \leq 0$.

Previous assumption is used to show that $V_\omega := \omega^T J M_\omega J\omega$ is uniformly bounded. In particular, if $M_\omega = c_I I + c_J J^{-1}$ with c_I, c_J bounded, real and smooth scalar functions with positive lower bounds such that $\omega^T J\dot{M}_\omega J\omega + 2\omega^T JM_\omega(\dot{J}\omega + \nu(J,\omega)) \leq 0 \;\forall\; \omega \in \mathbb{R}^3$, then $\hat{\omega} = 0$.

We now state a new global and robust asymptotic attitude tracking result under input saturation for the tracking errors $(\delta q_0, \delta q)$ and $\delta\omega$. Uniform asymptotic stability properties are given later.

Lemma II.3: Suppose that $\rho_\tau > 0$ with $\dot{\tilde{\omega}}_\tau := J^{-1}(J\dot{\omega}_d + \omega_d \times J\omega_d - \nu(J,\omega_d))$, there exists s defined as in Lemma II.1 such that the bounded control $T_c(s) = \mathrm{sel}_{\mathcal{T}}(-s)$ determines a bounded ω, $\lim_{t\to+\infty}\delta\omega(t) = \underline{0}$ and $\lim_{t\to+\infty}|\delta q_0(t)| = 1$. An update law for k is given by (11).

Proof: Let $N = k_v MJ$ with $k_v \in \mathbb{R}_{>0}$ and $M : \mathbb{R} \to \mathbb{R}^{3\times 3}$ a positive definite and smooth matrix function with uniformly bounded derivative. Following the same analysis in [10] with the Lyapunov function $V_\omega = \omega^T JM_\omega J\omega$ and M_ω as in Assumption II.1, it will be shown at the end of this proof that ω is uniformly bounded provided $\rho_\tau > 0$ and $|k| \leq \overline{k}$. Under this hypothesis, we now consider the Barbalat's function for $g, h > 0$ to be determined in the implementation:

$$B = \frac{1}{2}\delta\omega^T JMJ\delta\omega + \frac{g}{2}(1 - \delta q_0^2) + h|k| \tag{8}$$

By derivation over the trajectories of the system,

$$\begin{aligned}
\dot{B} &= \frac{1}{2}\delta\omega^T J\dot{M}J\delta\omega + \delta\omega^T JM\left(\dot{J}\omega - \omega\times J\omega + \nu(J,\omega) + T_c(s) + T_d - \frac{d}{dt}[J\omega_d]\right) \\
&\quad -\frac{g}{2}2\delta q_0\delta\dot{q}_0 + h\mathrm{sign}(k)\dot{k} \qquad (9)\\
&= \frac{1}{2}\delta\omega^T J\dot{M}J\delta\omega + (k_v^{-1}s - kMJ\Sigma\delta q)^T(T_c(s) + T_d - J\dot{\tilde{\omega}}_\tau) + \delta\omega^T JM(\psi(\omega,\omega_d,J) + \dot{J}\delta\omega) \\
&\quad +\frac{g}{2}\delta q_0\delta q^T\delta\omega + h\mathrm{sign}(k)\dot{k} \qquad (10)
\end{aligned}$$

where $\psi(\omega,\omega_d,J) = -\omega\times J\omega + \omega_d\times J\omega_d + \nu(J,\omega) - \nu(J,\omega_d)$.

$$\begin{aligned}
\dot{k} &= h^{-1}\mathrm{sign}(k)\left(-\frac{1}{2}\delta\omega^T J\dot{M}J\delta\omega + k\delta q^T\Sigma JM(T_c(s) + \hat{T}_d + J\dot{\tilde{\omega}}_\tau)\right. \\
&\quad \left.-\frac{g}{2}\delta q_0\delta q^T\delta\omega - f\|\delta q\| - \delta\omega^T JM(\psi(\omega,\omega_d,J) + \dot{J}\delta\omega)\right) \qquad (11)
\end{aligned}$$

for $\text{sign}(k(t_0)) = \text{sign}(\delta q_0(t_0))$. Here $\hat{T}_d$ is an estimator of the disturbance torque $T_d = \hat{T}_d + \triangle_{\hat{T}_d}$, where $\|\triangle_{\hat{T}_d}\|$ is bounded by a real function $b_{\triangle_{\hat{T}_d}}$. By Definition II.1 and (7):

$$\begin{aligned} \dot{B} &= k_v^{-1} s^T (T_c(s) + T_d - J\dot{\omega}_\tau) - f\|\delta q\| - k\delta q^T \Sigma J M \triangle_{\hat{T}_d} \\ &= -\mu_{\mathcal{T}}(-s) k_v^{-1}\|s\| + k_v^{-1} s^T (T_d - J\dot{\omega}_\tau) - f\|\delta q\| - k\delta q^T \Sigma J M \triangle_{\hat{T}_d} \\ &\leq -\rho_\tau k_v^{-1}\|s\| - (f - |k|\lambda_{max}(\Sigma)\lambda_{max}(J)\lambda_{max}(M) b_{\triangle_{\hat{T}_d}})\|\delta q\| \end{aligned} \tag{12}$$

Taking $f = \xi + |k|\lambda_{max}(\Sigma)\lambda_{max}(J)\lambda_{max}(M) b_{\triangle_{\hat{T}_d}}$, where $\xi \in \mathbb{R}_{>0}$, we obtain:

$$\dot{B} \leq -\rho_\tau k_v^{-1}\|s\| - \xi\|\delta q\| \leq 0 \tag{13}$$

hence $0 \leq B(\infty) \leq B(t_0) < +\infty$. Notice that (13) also holds imposing $\dot{k} = 0$ for $|k| = \overline{k} > |k(t_0)|$. Hence B, ω_d, $\dot{\omega}_d$, J, M, ω (under Assumption II.1) and $\dot{k}$ are bounded. Integrating $\dot{B}$:

$$\int_{t_0}^{\infty} |\dot{B}| dt = -\int_{t_0}^{\infty} \dot{B} dt = B(t_0) - B(\infty) < \infty$$

hence $\dot{B} \in \mathcal{L}^1$ and therefore s and δq belongs to $\mathcal{L}^1$. Since $T_c \in \mathcal{L}^\infty$, assuming $\dot{\Sigma}$, $\dot{M}$ uniformly bounded we obtain $\dot{s}$, $\delta\dot{q}$, $\dot{\omega} \in \mathcal{L}^\infty$ and hence s, δq and $(\omega - \omega_d)$ are uniformly continuous. Using the Barbalat's lemma [1], uniform continuity of functions in $\mathcal{L}^p$ $(p \neq \infty)$ implies the asymptotic convergence towards zero, i.e. $s \to \underline{0}$ and $\delta q \to \underline{0}$. Under the gain sign selection $\text{sign}(k) = \text{sign}(\delta q_0)$ and considering that $\delta\dot{q}_0$ is bounded and B is continuous at each time t_i of switching while $\dot{B}(t_i) < -\xi < 0$, there is a finite number of crossings of δq_0 through $|\delta q_0| < \epsilon_x \in \mathbb{R}_{>0}$ before the time t_x and $|k(t)| \geq \underline{k} > 0$ can be guaranteed (see [4] and following Remark). Hence, Lemma II.1 gives the claim.

Finally, lets see that ω is uniformly bounded under Assumption II.1 and for $T_c = \text{sel}_{\mathcal{T}}(-s)$. Let $M_\omega = c_\omega M$ with c_ω a bounded smooth and real scalar function with a positive lower bound. As $\dot{V}_\omega = \omega^T J\dot{M}_\omega J\omega + 2\omega^T J M_\omega(\dot{J}\omega - \omega \times J\omega + \nu(J,\omega)) + 2\omega^T J M_\omega(T_d + \text{sel}_{\mathcal{T}}(-k_v M J(k\Sigma\delta q + \omega - \omega_d)))$, after finite time (function of initial conditions) ω is bounded as $\|\omega\| \leq \epsilon_\omega + \max\left\{\hat{\omega}, (\overline{k\lambda_\Sigma} + \hat{\omega}_d)\chi\right\}$, for any $\epsilon_\omega \in \mathbb{R}_{>0}$ and certain $1 < \chi < +\infty$ (as $\lim_{\|\underline{v}_1\| \to +\infty} \underline{v}_1^T(T_d + \text{sel}_{\mathcal{T}}(-(\underline{v}_1 + \underline{v}_2))) < 0$ for $\rho_\tau \in \mathbb{R}_{>0}$ and $\|\underline{v}_2\|$ uniformly bounded), $\hat{\omega}_d := \max_{t \geq t_0}\{\|\omega_d\|\}$ and $\hat{\omega}$ as in Assumption II.1. This also guarantees a bounded $\omega\ \forall\ t > t_0$, as during this finite time it obeys (5) with bounded T_c, T_d (due to $\rho_\tau > 0$), while ν and the Coriolis term are respectively bounded in norm linearly with $\|\omega\|$ and $\|\omega\|^2$, being therefore integrable along any finite time.

□

Remark *II.1:* There are several ways to guarantee the controller restrictions using (11). In particular,

a. For $\omega_d = 0$ after finite time and being $\frac{d}{dt}|k|$ bounded, we may choose h large enough to guarantee $|k| > \underline{k}\ \forall\ t \leq t_x$. Also, it will be shown feasible to keep $k(t)$ constant after t_x.

b. To guarantee the upper bound on $|k| \leq \overline{k}$ for any given $\overline{k} > |k(t_0)|$, we may set $\dot{k} = 0$ when $|k| = \overline{k}$, which preserves (13).

c. The sign of k might be selected using some hysteresis law inside $|\delta q_0| \leq \epsilon_x$ to determine a finite number of switches in $\text{sign}(k)$ before t_x, guaranteeing $\text{sign}(k)\delta q_0 > \epsilon_x\ \forall\ t$ such that $|\delta q_0| > \epsilon_x$.

d. Take $\dot{\Sigma}$, $\dot{M}$ uniformly bounded.

e. An estimator $\hat{T}_d$ of the disturbance torque T_d may be used to reduce the uncertainty (otherwise, set $\hat{T}_d = \underline{0}$).

Assumption II.1 and the hypothesis in Lemma II.1 complete the constraints of the controller proposed in Lemma II.3.

Remark *II.2:* By redefining ρ_τ, measurement and model uncertainties may be handled as in [10].

Property II.1: For s defined as in Lemma II.1, with $\zeta_q \delta q_0 > \epsilon_x$ and $NJ = JN$ after $t_x \geq t_0$, there exist a control Lyapunov function V (as defined in [20] for unbounded controls) for the attitude tracking problem $\forall t \geq t_x$. Moreover, $T_c = \text{sel}_{\mathcal{T}}(-s)$ is the direction-preserving continuous pointwise minimizer of $\dot{V}\ \forall\ s \neq \underline{0}, t > t_x, T_c \in \mathcal{T}$. Under bounded controls in $\mathcal{T}$ and all feasible disturbances T_d, the closed loop system is uniformly asymptotically stable and the control law $T_c = \text{sel}_{\mathcal{T}}(-s)$ is the direction-preserving continuous pointwise minimizer of $\dot{V}\ \forall\ s \neq \underline{0}, t > t_x, T_c \in \mathcal{T}$ if $\rho_\tau > 0$ with $\dot{\hat{\omega}}_\tau = \dot{\hat{\omega}}_{\tau 0} - J^{-1}\psi - k\Sigma\Gamma - a_1 \delta q_0 (NJ)^{-1}\delta q$, $\dot{\hat{\omega}}_{\tau 0} := J^{-1}(J\dot{\omega}_d + \omega_d \times J\omega_d - \nu(J, \omega_d))$, $a_1 > a_0 \in \mathbb{R}_{>0}$ is a real and non-increasing function, $\psi(\omega, \omega_d, J) = -\omega \times J\omega + \omega_d \times J\omega_d + \nu(J, \omega) - \nu(J, \omega_d)$ and N is such that $\dot{N}J + N\dot{J} \leq 0$. Under Definition 3.8 in [21], V is called robust control Lyapunov function. For $\mathcal{T}$ spherical $\forall t > t_x$, $T_c = \mathrm{sel}_{\mathcal{T}}(-s)$ is the direction-unrestricted pointwise minimizer of $\dot{V}$.

Proof: Let $\underline{x} = [\delta q^T \ \delta\omega^T]^T$ and $V = \frac{1}{2}\underline{x}^T P \underline{x}$ with:

$$P \ := \begin{bmatrix} A & k\Sigma NJ \\ k(\Sigma NJ)^T & NJ \end{bmatrix}, \ NJ > 0, \ A - k^2\Sigma NJ(NJ)^{-1}(\Sigma NJ)^T > 0 \tag{14}$$

which guarantees $P > 0$ through the Schur complements, and let $0 < \underline{\lambda}_P \leq \lambda_{min}(P)$ and $\overline{\lambda}_P \leq \infty \ \forall \ t > t_x$. Hence $\underline{\lambda}_P\|\underline{x}\|^2 \leq 2V \leq \overline{\lambda}_P\|\underline{x}\|^2$. Notice that as $N, J > 0$ commute, $NJ > 0$ and $A - k^2\Sigma NJ(NJ)^{-1}(\Sigma NJ)^T = A - k^2\Sigma NJ\Sigma$.

Lets define $F(t, \underline{x}), G(t, \underline{x})$ such that $\dot{\underline{x}} = F(t, \underline{x}) + G(t, \underline{x})(T_c + T_d)$; hence:

$$F \ = \begin{bmatrix} \Gamma(\delta q_0, \delta q, \delta\omega, \delta\omega_d) \\ J^{-1}\psi(\omega, \omega_d, J) - \dot{\tilde{\omega}}_{\tau 0} \end{bmatrix}, \quad G = \begin{bmatrix} \mathbf{0} \\ J^{-1} \end{bmatrix} \tag{15}$$

It may be verified that $(\nabla_x V G)^T = N(k\Sigma\delta q + \delta\omega) = s$, hence $T_c = \mathrm{sel}_{\mathcal{T}}(-s)$ is the direction-preserving pointwise minimizer of $\dot{V}$ and continuous $\forall \ s \neq \underline{0}$, being also the direction-unrestricted minimizer if $\mathcal{T}$ were spherical $\forall \ t > t_0$. Let V_0 be the restriction of V to $\delta\omega = -k\Sigma\delta q$, hence $2V_0 = \delta q^T A\delta q + k^2\delta q^T\Sigma NJ\Sigma\delta q - 2k^2\delta q\Sigma NJ\Sigma\delta q = \delta q^T P_0\delta q$ where $P_0 = A - k^2\Sigma NJ\Sigma > 0$. Assume $\overline{\lambda}_{\dot{P}_0} \leq 0 \ \forall \ t > t_x$. By derivation over the trajectories:

$$\begin{aligned} 2\dot{V}_0 \ &\leq \ \overline{\lambda}_{\dot{P}_0}\|\delta q\|^2 + 2\delta q^T P_0\Gamma(\delta q_0, \delta q, -k\Sigma\delta q, \omega_d) \\ &= \ \overline{\lambda}_{\dot{P}_0}\|\delta q\|^2 - k\delta q_0\delta q^T P_0\Sigma\delta q - k\delta q^T P_0(\delta q \times \Sigma\delta q) + 2\delta q^T P_0(\delta q \times \omega_d) \\ &\leq \ -(-\overline{\lambda}_{\dot{P}_0} + k\delta q_0\underline{\lambda}_{P_0\Sigma})\|\delta q\|^2 - k\delta q^T((P_0\delta q) \times \Sigma\delta q) + 2\delta q^T P_0(\delta q \times \omega_d) \end{aligned}$$

Let $A = aI + k^2\Sigma NJ\Sigma$, $a = a(t) > 0$ and $\dot{a} = -b = -b(t) \leq 0$. Hence $P_0 = aI$ and:

$$\dot{V}_0 \ \leq \ -\frac{1}{2}(b + ak\delta q_0\underline{\lambda}_\Sigma)\|\delta q\|^2 < 0 \ \text{ for } \ \delta q \neq \underline{0}, \ k\delta q_0 > 0 \tag{16}$$

As a may take large enough values, this guarantees also $P > 0$ without more restrictions for the matrices involved in the definition of s. Hence as V is positive definite and on the surface $s = \underline{0}$ (i.e.

where $(\nabla_x VG)^T = \underline{0}$) we showed that $\dot{V} < 0$, V is a control Lyapunov function for the non-autonomous system with unbounded controls (see [20]).

Denoting $\psi \equiv \psi(\omega, \omega_d, J)$ and $\Gamma \equiv \Gamma(\delta q_0, \delta q, \delta\omega, \delta\omega_d)$ to simplify the notation, we obtain:

$$
\begin{aligned}
(\nabla_x VF)^T &= \Gamma^T A\delta q + (J^{-1}\psi - \dot{\bar{\omega}}_{\tau 0})^T JN\Sigma k\delta q + k\Gamma^T \Sigma NJ\delta\omega + (J^{-1}\psi - \dot{\bar{\omega}}_{\tau 0})^T JN\delta\omega \\
&= (J^{-1}\psi - \dot{\bar{\omega}}_{\tau 0})^T Js + \Gamma^T((aI + k^2\Sigma NJ\Sigma)\delta q + k\Sigma JN\delta\omega) \\
&= (J^{-1}\psi - \dot{\bar{\omega}}_{\tau 0})^T Js + \Gamma^T(a\delta q + k\Sigma Js) \qquad (17) \\
&= (J^{-1}\psi - \dot{\bar{\omega}}_{\tau 0})^T Js + k\Gamma^T\Sigma Js + \frac{1}{2}a\delta q_0 \delta q^T \delta\omega \\
&= (J^{-1}\psi - \dot{\bar{\omega}}_{\tau 0})^T Js + k\Gamma^T\Sigma Js + \frac{1}{2}a\delta q_0 \delta q^T (N^{-1}s - k\Sigma\delta q) \\
&= -\dot{\bar{\omega}}_\tau^T Js - \frac{1}{2}ak\delta q_0 \delta q^T \Sigma\delta q
\end{aligned}
$$

with $a = 2a_1$. Notice that $\dot{\bar{\omega}}_\tau$ is uniformly bounded if ω is uniformly bounded, as $\|\Gamma\| \le \frac{1}{2}\|\delta\omega\| + \|\omega_d\|$ and $\|\psi\|$ may be bounded as a function of $\|\delta\omega\|$. This holds under Assumption II.1, but implies that the evaluation of $\rho_\tau > 0$ also depend on initial conditions.

Let $\frac{d}{dt}(NJ) = \dot{N}J + N\dot{J} \le 0$ be such that $\frac{1}{2}\underline{x}^T \dot{P}\underline{x} \le -b_*\|\delta q\|^2 \le 0$ for $0 \le b_*(t) \le \frac{1}{2}a\underline{k}\delta q_0 \underline{\lambda}_\Sigma - \epsilon_k$ with a constant $\epsilon_k \in \mathbb{R}_{>0}\ \forall\ t > t_x$.

Hence $\dot{V} \le -b_*\|\delta q\|^2 - \frac{1}{2}ak\delta q_0\delta q^T\Sigma\delta q + s^T(T_c + T_d - J\dot{\bar{\omega}}_\tau) \le -\epsilon_k\|\delta q\|^2 - \rho_\tau\|s\| < 0$ if $\rho_\tau > 0\ \forall\ \underline{x} \ne \underline{0}$, as $\underline{x} = H[\delta q^T s^T]^T$ where H has a positive lower bound for its singular values due to $\underline{\lambda}_N, \underline{\lambda}_J > 0$. This proves the uniform asymptotic stability of the origin of the closed loop system using the Jurdevic-Quinn condition (Theorem 3.2 in [20]) as $\dot{V} \le -c(\|\underline{x}\|)$ for a class-$\mathcal{K}$ function (continuous strictly increasing real functions such that $c(0) = 0$) $c(\|\underline{x}\|) \le \epsilon_k\|\delta q\|^2 + \rho_\tau\|s\|$. Also makes V a robust control Lyapunov function (see Definition 3.8 in [21]) as the inequality holds over all feasible disturbances and bounded controls, guaranteeing the robust uniform asymptotic stability towards $\delta q = \underline{0}$ and $\delta\omega = \underline{0}$.

□

Remark *II.3:* Equation (16) also gives $\dot{V}_0 \le -c_{V_0}\|\delta q\|^2 \le -\frac{c_{V_0}}{\bar{\lambda}_{P_0}}V_0$ for certain $c_{V_0} > 0$; hence for $t > t_x$ and small enough $\|s\|$, the convergence is exponential. Moreover, as $s \longrightarrow \underline{0}\ \exists t_s < +\infty$ after which tracking errors converge exponentially.

Now we consider the kinematic integral tracking errors to obtain the PID-like control structure.

Lemma II.4: Suppose that $\rho_\tau > 0$ with $\dot{\tilde{\omega}}_\tau := J^{-1}(J\dot{\omega}_{dz} + \omega_{dz} \times J\omega_{dz} - \nu(J, \omega_{dz}))$ and $\zeta_q \delta q_0 \geq \epsilon_x > 0$, $\delta q_{z0} \geq \epsilon_y > 0$, $\zeta_q z_0 \geq \epsilon_z > 0 \; \forall \; t \geq t_{PID} \geq t_x$. There exist s such that $T_c(s) = \mathrm{sel}_{\mathcal{T}}(-s)$ determines a bounded ω, $\lim_{t\to+\infty}\delta\omega(t) = \underline{0}$, $\lim_{t\to+\infty}|\delta q_0(t)| = 1$ and $\lim_{t\to+\infty}|z_0(t)| = 1$. An update law for k is given by (18).

Proof: Under $\rho_\tau > 0$ with $\dot{\tilde{\omega}}_\tau := J^{-1}(J\dot{\omega}_{dz} + \omega_{dz} \times J\omega_{dz} - \nu(J, \omega_{dz}))$, the control law given in Lemma II.3 may be applied with the new errors $(\delta q_{0z}, \delta q_z)$ and $\delta\omega_z$ by taking $s = k_v MJ(k\Sigma\delta q_z + \delta\omega_z)$. Let $B = \frac{1}{2}\delta\omega_z^T JMJ\delta\omega_z + \frac{g}{2}(1 - \delta q_{z0}^2) + h|k|$ and the adjustment law:

$$\begin{aligned} \dot{k} &= h^{-1}\mathrm{sign}(k)\Big(-\frac{1}{2}\delta\omega_z^T J\dot{M}J\delta\omega_z + k\delta q_z^T \Sigma JM(T_c(s) + \hat{T}_d - J\dot{\tilde{\omega}}_\tau) \\ &\quad -\frac{g}{2}\delta q_{0z}\delta q_z^T\delta\omega_z - f\|\delta q_z\| - \delta\omega_z^T JM(\psi(\omega, \omega_{dz}, J) + \dot{J}\delta\omega)\Big) \end{aligned} \tag{18}$$

with $\psi(\omega, \omega_{dz}, J)$ defined as in (10). As Lemma II.3 guarantees $\delta q_z \longrightarrow \underline{0}$ and $\delta\omega_z \longrightarrow \underline{0}$, Lemma II.2 gives the claim.

□

Property II.2: For $s = N(k\Sigma\delta q_z + \omega_z)$ with $\zeta_q\delta q_0 \geq \epsilon_x > 0$, $\delta q_{0z} \geq \epsilon_y > 0$, $\zeta_q z_0 \geq \epsilon_z > 0$, $k \geq \underline{k}$, $NJ = JN$, $\frac{d}{dt}(NJ) < 0$ after $t_{PID} \geq t_x \geq t_0$, there exists a (robust) (control) Lyapunov function V for the kinematic integral attitude tracking problem, while $T_c = \mathrm{sel}_{\mathcal{T}}(-s)$ inherits all the properties listed in Property II.1. In particular, under bounded control the uniform asymptotic stability requires $\rho_\tau > 0$ with $\dot{\tilde{\omega}}_\tau = \dot{\tilde{\omega}}_{\tau 0} - J^{-1}\psi - k\Sigma\Gamma - a_1\delta q_{0z}(NJ)^{-1}\delta q_z$, $\dot{\tilde{\omega}}_{\tau 0} := J^{-1}(J\dot{\omega}_{dz} + \omega_{dz} \times J\omega_{dz} - \nu(J, \omega_{dz}))$, $\psi(\omega, \omega_{dz}, J) = -\omega \times J\omega + \omega_{dz} \times J\omega_{dz} + \nu(J, \omega) - \nu(J, \omega_{dz})$. The following equation gives a sufficient condition to select feasibles k_{II}, k_{Ip} by imposing $k_{Ip} = c_{Ip}I$ with $c_{Ip}\zeta_q > \epsilon_{Ip} > 0 \; \forall \; t > t_{PID} \geq t_x$ and $\zeta_q k_{II} > 0$ such that:

$$-\dot{a}_z I + \frac{a_z}{2}\left(z_0 k_{II} + c_{Ip}\delta q_0 I\right) - \frac{a_z^2 c_{Ip}^2}{16 a_1 k \delta q_{z0}}\Sigma^{-1} - Q_z = \mathbf{0} \tag{19}$$

for some smooth real function $a_z(t) \geq \epsilon_{a_z} \in \mathbb{R}_{>0}$ and a smooth real matrix function $Q_z > 0$ such that $\underline{\lambda}(Q_z) > 0$ over all $t > t_{PID} > t_x$.

Proof: Without loss of generality, we consider the state-space representation of the kinematic integral attitude tracking system with $\underline{x} := [z^T \ \delta q_z^T \ \delta\omega_z^T]^T$ (which shares the equilibrium and is diffeomorphic to $[z^T \ \delta q^T \ \delta\omega^T]^T$ through (4)) and $V = \frac{1}{2}\underline{x}^T P \underline{x}$ with:

$$P \ := \begin{bmatrix} a_z I & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & A & k\Sigma NJ \\ \mathbf{0} & k(\Sigma NJ)^T & NJ \end{bmatrix}, \ NJ > 0, \ A - k^2\Sigma NJ(NJ)^{-1}(\Sigma NJ)^T > 0$$

where $a_z = a_z(t) > \epsilon_{a_z} \in \mathbb{R}_{>0}$ is a smooth real function and $0 < \underline{\lambda}_P\|\underline{x}\|^2 \leq 2V \leq \overline{\lambda}_P\|\underline{x}\|^2 < +\infty$ for $\underline{x} \neq \underline{0}$. Also, $(\nabla_x VG)^T = N(k\Sigma\delta q_z + \delta\omega_z) = s$ and $T_c = \mathrm{sel}_{\mathcal{T}}(-s)$ has the claimed properties.

Let $A = aI + k^2\Sigma NJ\Sigma$, hence under hypothesis and for $a = 2a_1$ nonincreasing:

$$\begin{aligned} \dot{V} &\leq -\delta q_z^T\Sigma\delta q_z\delta q_{z0}ka_1 - \rho_\tau\|s\| + \dot{a}_z z^T z + \frac{a_z z_0}{2} z^T(k_{Ip}\delta q - k_{II}z) \\ &= -\delta q_z^T\Sigma\delta q_z\delta q_{z0}ka_1 - \rho_\tau\|s\| + \dot{a}_z z^T z - \frac{a_z z_0}{2} z^T k_{II} z + \frac{a_z z_0}{2} z^T k_{Ip}\delta q \end{aligned} \quad (20)$$

Notice that (4) is invertible and for $|z_0| > \epsilon_z > 0$ we may extract $\delta q = \frac{1}{z_0^2}(z_0 I - z\times)(\delta q_z - z\delta q_0)$. By completing squares and setting $k_{Ip} = c_{Ip}I$ $(c_{Ip}\zeta_q > 0)$, (20) becomes:

$$\begin{aligned} \dot{V} &\leq -\delta q_z^T\Sigma\delta q_z\delta q_{z0}ka_1 - \rho_\tau\|s\| + \dot{a}_z z^T z - \frac{a_z z_0}{2} z^T k_{II} z + \frac{a_z c_{Ip}}{2} z^T(\delta q_z - z\delta q_0) \\ &= -\delta q_z^T\Sigma\delta q_z\delta q_{z0}ka_1 - \rho_\tau\|s\| + \dot{a}_z z^T z - \frac{a_z z_0}{2} z^T k_{II} z + \frac{a_z c_{Ip}}{2} z^T\delta q_z - \frac{a_z c_{Ip}\delta q_0}{2} z^T z \\ &= -\rho_\tau\|s\| - \begin{bmatrix} z \\ \delta q_z \end{bmatrix}^T \begin{bmatrix} -\dot{a}_z I + \frac{a_z z_0}{2}k_{II} + \frac{a_z c_{Ip}\delta q_0}{2} I & -\frac{a_z c_{Ip}}{4} I \\ -\frac{a_z c_{Ip}}{4} I & a_1 k\Sigma\delta q_{z0} \end{bmatrix} \begin{bmatrix} z \\ \delta q_z \end{bmatrix} \\ &= -\rho_\tau\|s\| - \begin{bmatrix} z \\ \delta q_z \end{bmatrix}^T M_z \begin{bmatrix} z \\ \delta q_z \end{bmatrix} \end{aligned} \quad (21)$$

For $(\nabla_x VG)^T = N(k\Sigma\delta q_z + \delta\omega_z) = s = \underline{0}$, we may guarantee $\dot{V} < 0$ for $[z^T, \delta q_z^T]^T \neq \underline{0}$ provided a sign definite quadratic form in (21), where $a_1 k\Sigma\delta q_{z0} > a_0\underline{k}\underline{\lambda}_\Sigma\epsilon_y > 0$ and its Schur complement is positive provided (19). Notice that (19) may be solved for $a_z > \epsilon_{a_z} > 0, \dot{a}_z \leq 0$ considering that $z_0 k_{II} > 0$ and $c_{Ip}\delta q_0 > 0$. Moreover, if $\dot{a}_z \leq \frac{a_z|z_0|}{2}\underline{\sigma}_{k_{II}}$, $\lambda_{min}(Q_z)$ may be maximized taking $c_{Ip} = c^*_{Ip} := \frac{4a_1 k\delta q_0 \underline{\lambda}_\Sigma \delta q_{z0}}{a_z}$, which is obtained from (19) by making zero its first partial derivative with c_{Ip}. By substitution in (21), $\dot{V} \leq -\rho_\tau\|s\| - \underline{\lambda}_{M_z}\|[z^T \;\; \delta q_z^T]^T\|^2 \; \forall t > t_{PID}$. Moreover, $\underline{\lambda}_{M_z} \geq f_0(\delta q_0)\delta q_0 a_0 \underline{k}\delta q_{z0}\underline{\lambda}_\Sigma$ with $f_0(\delta q_0) = \delta q_0 + \frac{1}{2\delta q_0} + \sqrt{\left(\delta q_0 + \frac{1}{2\delta q_0}\right)^2 - 1}$. This is obtained from $M_z \geq \begin{bmatrix} \frac{a_z c^*_{Ip}\delta q_0}{2} I & -\frac{a_z c^*_{Ip}}{4} I \\ -\frac{a_z c^*_{Ip}}{4} I & a_1 \underline{k}\underline{\lambda}_\Sigma \delta q_{z0} I \end{bmatrix}$ in (21) by using $-\dot{a}_z I + \frac{a_z z_0}{2} k_{II} \geq \mathbf{0}$, $I\underline{\lambda}_\Sigma \leq \Sigma$ and rearranging the resulting matrix as block-diagonal equal blocks $B_{M_z} = \begin{bmatrix} 2\delta q_0 & -1 \\ -1 & \delta q_0^{-1} \end{bmatrix} \delta q_0 a_1 k \delta q_{z0}\underline{\lambda}_\Sigma$ whose eigenvalues give previous bound. The robust uniform asymptotic stability follows as in Property II.1.

□

Remark *II.4:* Let $V_z := 2V - 2a_z\|z\|^2 = v_z^T P_2 v_z$, $u_z := k\Sigma\delta q_z + \delta\omega_z$ with all definitions as in the proof of Property II.2 and $v_z := \begin{bmatrix} k\Sigma\delta q_z \\ \delta\omega_z \end{bmatrix}$, $P_2 := \begin{bmatrix} a(k\Sigma)^{-2} + NJ & NJ \\ (NJ)^T & NJ \end{bmatrix}$.

a. If $V_z(t_{PID}) < \underline{\lambda}_{P_2}\underline{\sigma}_K^2(1-\epsilon_y^2)^2$ and $\delta q_{0z}(t_{PID}) > \epsilon_y > 0$, then $\delta q_{0z}(t_{PID}) > \epsilon_y > 0 \; \forall t > t_{PID}$.

b. Moreover, taking $k_{II} := \frac{k_{II0}}{|z_0|}$ such that $\zeta_q k_{II0} > \zeta_q k_{Ip}\epsilon_z \frac{\sqrt{1-\epsilon_x^2}}{\sqrt{1-\epsilon_z^2}}$, guarantees $\zeta_q z_0 > \epsilon_z > 0$ and $\zeta_q \delta q_0 > \epsilon_x > 0 \forall t > t_{PID}$ provided $\epsilon_y \geq \epsilon_x + \sqrt{1-\epsilon_x^2}\sqrt{1-\epsilon_z^2}$.

Hence the hypotheses of Property II.2 may be assured *a priori.*

Proof: See Appendix

□

Let $k_p = kk_v\Sigma$, $k_v \in \mathbb{R}_{>0}$ and $N = k_v MJ$. After t_{PID}, s becomes:

$$
\begin{aligned}
s &= MJ(k_p(z\delta q_0 + z_0\delta q + z \times \delta q) + k_v(\delta\omega + k_{Ip}\delta q - k_{II}z)) \\
&= MJ((k_p z_0 I + k_p z \times + k_v k_{Ip})\delta q + k_v\delta\omega + (k_p\delta q_0 - k_v k_{II})z) \qquad (22)
\end{aligned}
$$

which gives a structure similar to a PID controller, provided appropriate sign definite matrices $K_I(\delta q_0) := (k_p\delta q_0 - k_v k_{II})k_{Ip}$ (where k_{Ip} appears through (3)) and $K_p(z_0, z) := k_p z_0 I + k_p z \times + k_v k_{Ip}$. Notice that in K_p the second term is skew-symmetric, hence the matrix preserves its sign condition as sign(z_0) is constant. On K_I this determines the condition $\zeta_q k_{II} < \zeta_q k_p \frac{\delta q_0}{k_v}$.

Remark *II.5:* A min-norm robust control may be also obtained using previous designs and preserving the uniform attractivity and asymptotic stability properties if the following law is taken instead. For a given estimator $\hat{\rho}_{\mathcal{T}}(s)$ of the robustness margin at each s and $t \geq t_0$:

$$T_c(s) \quad = \quad -(1+\epsilon_\kappa)(\mu_{\mathcal{T}}(-s) - \hat{\rho}_{\mathcal{T}}(-s))\check{s} \tag{23}$$

for certain smooth function $\epsilon_\kappa > 0$ $\forall t \geq t_0$, and $\hat{\rho}_{\mathcal{T}}(s) \leq \mu_{\mathcal{T}}(s) - \epsilon_{\hat{\rho}}$ $\forall$ $s \neq \underline{0}$. Here the discontinuous step depends on certain estimated disturbance size $\mu_{\mathcal{T}}(-s) - \hat{\rho}_{\mathcal{T}}(-s)$. Notice that for small enough s, δq_z and z, the condition on $\dot{V}$ would only be assured by a discontinuous control. To measure the effect of a regularization $\tilde{T}_c$ of T_c, given in (23), on stability Property II.2, define $\kappa_\tau := \frac{\tilde{T}_c^T T_c}{1+\epsilon_\kappa}$. Also let $\ell_\tau := 1 + \epsilon_\kappa - \kappa_\tau$ if $\kappa_\tau \leq 1 + \epsilon_\kappa$, or $\ell_\tau := 0$ otherwise. Hence $\dot{V} \leq -\epsilon_\kappa(\mu_{\mathcal{T}}(-s) - \hat{\rho}_{\mathcal{T}}(-s))\|s\| - \underline{\lambda}_{M_z}\|[z^T\ \delta q_z^T]^T\|^2 + \ell_\tau(\mu_{\mathcal{T}}(-s) - \hat{\rho}_{\mathcal{T}}(-s))\|s\|$ and $\forall$ $\bar{t} \geq \underline{t} \geq t_{PID}$:

$$\int_{\underline{t}}^{\bar{t}} \|[z^T\ \delta q_z^T]^T\|^2 dt \leq \gamma_\tau^2 \int_{\underline{t}}^{\bar{t}} |d_\tau|^2 dt + \frac{1}{\underline{\lambda}_{M_z}} \left(V(\underline{x}(\underline{t})) - V(\underline{x}(\bar{t}))\right) \tag{24}$$

where $d_\tau := \sqrt{\ell_\tau \|s\|}$ is a signal that measures the non-rejected disturbances and is associated by (24) to the $\mathcal{H}_\infty$ gain $\gamma_\tau^2 := \frac{1}{\underline{\lambda}_{M_z}} \left(\sup_{s\neq\underline{0}} \{\mu_{\mathcal{T}}(s) - \hat{\rho}_{\mathcal{T}}(s)\} + \epsilon_{\hat{\rho}}\right)$. Given $\epsilon_x \leq \zeta_q \delta q_0$ and $\epsilon_y \leq \delta q_{z0}$, a gain lower than γ_τ may be guaranteed by bounding $\underline{\lambda}_{M_z}$ as in Property II.2:

$$\underline{k\lambda_\Sigma} \geq \frac{1}{\underline{f}_0 a_0 \gamma_\tau^2 \epsilon_x \epsilon_y} \left(\sup_{s\neq\underline{0}} \{\mu_{\mathcal{T}}(s) - \hat{\rho}_{\mathcal{T}}(s)\} + \epsilon_{\hat{\rho}} \right) \tag{25}$$

provided $\dot{a}_z \leq \frac{a_z|z_0|}{2} \underline{\sigma}_{k_{II}}$ and $\underline{f}_0 := \min_{\epsilon_x < \delta q_0 < 1} f_0(\delta q_0)$ as in the proof of Property II.2.

Remark *II.6:* The free parameter a_z allows us to select $\zeta_q k_{Ip}$ as large as desired under discontinuous control. However, it is restricted under a smooth control, as the Routh absolute stability criterion shows for the axisymmetric rocket [22] linearized around constant inertia and Mach number, zero angle of attack and zero roll rate, which determines for (constant) $K_p^0 := k_p + k_v k_{Ip} \zeta_q$, $K_I^0 := (k_p \zeta_q - k_v k_{II}) k_{Ip}$ and $M = I$:

$$\frac{m_\alpha}{\bar{\lambda}_J} + \frac{K_I^0}{k_v - \frac{m_q}{\bar{\lambda}_J}} < \frac{K_p^0}{2} < \frac{\omega_{Ai}^2}{2}, \quad \frac{m_q}{\bar{\lambda}_J} < k_v \tag{26}$$

for $m_\alpha = C_{m\alpha} q_\infty S_{ref} D$, $m_q = C_{mq} q_\infty S_{ref} D$, where S_{ref} is the reference surface, D the diameter, $C_{m\alpha}$ and C_{mq} are the pitch and pitch damping aerodynamic coefficients.

D. Adaptive regularization

Consider a linear saturation with slope δ_s and the Barbalat function modified as in [7]:

$$B = \frac{1}{2}\delta\omega_z^T JMJ\delta\omega_z + \frac{g}{2}(1 - \delta q_{z0}^2) + h|k| + \delta_s \tag{27}$$

with $T_c(s) = -s/\delta_s$ for $-s/\delta_s \in \mathcal{T}$; otherwise $T_c(s) = \text{sel}_{\mathcal{T}}(-s)$. By derivation of (27):

$$\dot{B} \ \leq \ k_v^{-1} s^T \left(T_c(s) + T_d - J\dot{\tilde{\omega}}_\tau\right) - \xi\|\delta q_z\| + \dot{\delta}_s \tag{28}$$

The convergence is guaranteed for $-s/\delta_s \notin \mathcal{T}$. Otherwise, consider $\delta_s > 0$ as proposed in [5][7]:

$$\dot{\delta}_s \ = \ -\sigma_0\|s\|^{r_0}\exp(-\sigma_1\|s\|^{r_1}) + \sigma_2\|\delta q_z\|; \quad \delta_s(t_0) = 1 \tag{29}$$

As $\delta_s > 0 \ \forall t \geq t_0$, taking $\sigma_2 < \xi$ assures a negative last term in (28) for small s and greater control authority for smaller tracking errors.

To avoid chatter, a minimum value for δ_s is imposed, which determines a maximum bandwidth for the control law and the disturbances to be rejected. However, the integral term allows now to reject slow enough disturbances, which was not guaranteed by the smooth version of variable structure controllers based on PD-like surfaces. In fact, $\delta q, \delta\omega \longrightarrow \underline{0}$ implies $s/\delta_s \longrightarrow \underline{0}$ with a PD surface and $\delta_s > \underline{\delta}_s > 0$, hence it is not compatible with the rejection of a constant torque $T_{d\infty} \in \mathbb{R}^3$. Instead, s in (22) allows $\delta q, \delta\omega \longrightarrow \underline{0}$ while $s/\delta_s \longrightarrow T_{d\infty} \neq \underline{0}$ with $\delta_s > \underline{\delta}_s > 0$.

As $s(t_{PID}^+) - s(t_{PID}^-) = N(k_{Ip} + (k(t_{PID}^+)\zeta_q - k(t_{PID}^-))\Sigma)\delta q \leq \epsilon_s\|\delta q\|$ for certain $\epsilon_s > 0$, the step magnitude may be guaranteed easily. Moreover, taking $k_{Ip}(t_{PID}) = \mathbf{0}$ (redefining $\underline{\sigma}_{k_{Ip}}$ after $t_{PID} + \epsilon_t$ with $\epsilon_t > 0$) and $k(t_{PID}^+) = |k(t_{PID}^-)|$, s remains continuous.

E. Angular velocity error measurement, estimation and bounds

Some slowly-varying errors may be estimated and rejected, such as the gyro alignment, bias and scale factor. For a constant bias ω_b, consider the measurement $\omega_\triangle = \omega + \omega_b$, where ω will be estimated through $\hat{\omega} = \omega_{\triangle} - \hat{\omega}_b$. The following observer assures $\hat{\omega}_b \longrightarrow \omega_b$ globally [9]:

$$\dot{\hat{q}} = \frac{1}{2}Q(\hat{q}_0, \hat{q})R^T(\delta q_{0o}, \delta q_o)\left(\hat{\omega} + k_o\delta q_o \text{sign}(\delta q_{0o})\right) \tag{30}$$

$$\dot{\hat{\omega}}_b = -\frac{1}{2}\delta q_o \text{sign}(\delta q_{0o}) \tag{31}$$

where $Q(\cdot,\cdot)$ is the matrix associated to the kinematic equation $[\dot{q}_0, \dot{q}]^T = Q(q_0, q)\omega$ and $R(\delta q_{0o}, \delta q_o)$ is the rotation matrix of the observation error quaternion $(\delta q_{0o}, \delta q_o)$, obtained as (q_0, q) composed with $(\hat{q}_0, -\hat{q})$.

To guarantee an uniformly bounded $\dot{\omega}$ for Lemma II.4, ω_{dz} and $J\dot{\omega}_{dz}$ may be bounded as:

$$\max_{t\geq t_{PID}} \|\omega_{dz}\| \leq \max_{t\geq t_{PID}} \{\|\omega_d\|\} + \overline{\sigma}_{k_{Ip}} + \overline{\sigma}_{k_{II}} \tag{32}$$

$$\max_{t\geq t_{PID}} \|J\dot{\omega}_{dz}\| \leq \max_{t\geq t_{PID}} \left\{\|J\dot{\omega}_d\| + \|J\dot{k}_{Ip}\delta q\| + \|Jk_{Ip}\delta\dot{q}\| + \|J\dot{k}_{II}z\| + \|Jk_{II}\dot{z}\|\right\} \tag{33}$$

where $\|\delta q\|, \|z\| \leq 1$, $\|\delta\dot{q}\| \leq \frac{1}{2}\|\delta\omega\| + \|\omega_d\| \leq \frac{1}{2}(\|\delta\omega_z\| + \overline{\sigma}_{k_{Ip}} + \overline{\sigma}_{k_{II}}) + \|\omega_d\|$, $\|\dot{z}\| \leq \frac{1}{2}(\overline{\sigma}_{k_{Ip}} + \overline{\sigma}_{k_{II}})$. Notice that $\|\delta\omega_z\|$ is bounded as a function of initial conditions through (8) as $\|\delta\omega_z\| \leq \frac{\sqrt{2B(t_{PID})}}{\underline{\lambda}_J}$, while for the PD case with $k_{Ip} = 0$ it could be bounded globally.

Here $\omega_d = \omega_{di}^d$ has been assumed (angular velocity of $\mathcal{D}$ relative to $\mathcal{I}$, written in $\mathcal{D}$), but $\omega_d := \omega_{di}^s = R^T(\delta q_0, \delta q)\omega_{di}^d$ can be taken instead, computing previous bounds accordingly.

III. Performance evaluation with closed loop simulations

A. *The partial quaternion and its kinematic integral*

As the attitude controller will be applied to ascent guidance (related to the pointing of vehicle's longitudinal axis), we separate the longitudinal axis kinematics from the (perhaps uncontrollable) roll motion, which might have different restrictions and objectives.

Let $C = [\underline{c}_1,\ \underline{c}_2,\ \underline{c}_3]$ be the rotation matrix from $\mathcal{I}$ to $\mathcal{S}$, associated to the quaternion (q_0, q). Without loss of generality, assume that we want $\underline{c}_3$ pointing towards $\underline{e}_3 = [0\ 0\ 1]^T$. The *partial quaternion* [10] that measures this error is defined as the rotation that makes $\underline{c}_3 \longrightarrow \underline{e}_3$ with null motion around $\underline{c}_3$.

This means a rotation around the axis:

$$\underline{n}_p = \frac{\underline{c}_3 \times \underline{e}_3}{\|\underline{c}_3 \times \underline{e}_3\|} \tag{34}$$

by angle $\phi_p = \text{acos}(\underline{c}_3^T \underline{e}_3)$. Notice that the third component in $\underline{n}_p$ is zero. We will use the following notation for the partial quaternion:

$$\begin{cases} \delta q_{p0} = \cos(\phi_p/2) \\ \delta q_p = \sin(\phi_p/2)\underline{n}_p \end{cases} \tag{35}$$

With the restriction $\delta \dot{q}_p^T \underline{e}_3 = 0$ and for $\delta q_{p0} > \epsilon_{qp0} > 0$, $\delta\omega_{p3}$ is replaced by:

$$\delta\omega_{p3} = \frac{1}{\delta q_{p0}} \left(-\delta q_{p1}(\omega_2 + \omega_{d2}) + \delta q_{p2}(\omega_1 + \omega_{d1})\right)$$
$$\Longrightarrow \begin{cases} \delta \dot{q}_{p0} = -\frac{1}{2}(\delta q_{p1}\delta\omega_{p1} + \delta q_{p2}\delta\omega_{p2}) \\ \delta \dot{q}_p = \Gamma_p(\delta q_{p0}, \delta q_p, \delta\omega_p, \omega_d) \end{cases} \tag{36}$$

where $\delta\omega_{p1} = \delta\omega_1$, $\delta\omega_{p2} = \delta\omega_2$ and $\Gamma_p = [\Gamma_{p1}\ \Gamma_{p2}\ 0]^T$ with:

$$\Gamma_{p1}(\delta q_{p0}, \delta q_p, \delta\omega_p, \omega_d) = \frac{1}{2}\left(\delta q_{p0} + \frac{\delta q_{p2}^2}{\delta q_{p0}}\right)\delta\omega_{p1} - \frac{\delta q_{p1}\delta q_{p2}}{2\delta q_{p0}}\delta\omega_{p2} + \frac{\delta q_{p2}^2}{\delta q_{p0}}\omega_{d1} - \frac{\delta q_{p1}\delta q_{p2}}{\delta q_{p0}}\omega_{d2} + \delta q_{p2}\omega_{d3}$$
$$\Gamma_{p2}(\delta q_{p0}, \delta q_p, \delta\omega_p, \omega_d) = \frac{1}{2}\left(\delta q_{p0} + \frac{\delta q_{p1}^2}{\delta q_{p0}}\right)\delta\omega_{p2} - \frac{\delta q_{p1}\delta q_{p2}}{2\delta q_{p0}}\delta\omega_{p1} + \frac{\delta q_{p1}^2}{\delta q_{p0}}\omega_{d2} - \frac{\delta q_{p1}\delta q_{p2}}{\delta q_{p0}}\omega_{d1} + \delta q_{p1}\omega_{d3}$$

For $\omega_d^T \underline{e}_3 = \omega_3$ the desired roll velocity is the actual velocity under free roll motion.

The kinematic integration of δq_p is given by $z_3 \equiv 0$, $z_0 = \zeta_q\sqrt{1 - z_1^2 - z_2^2}$, $\zeta_q z_0 > \epsilon_z > 0$ and:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \left(\frac{z_0}{2}I + \frac{D(z_2^2, z_1^2)}{2z_0}\right)\left(k_{Ip}\begin{bmatrix} \delta q_{p1} \\ \delta q_{p2} \end{bmatrix} - k_{II}\begin{bmatrix} \delta z_1 \\ \delta z_2 \end{bmatrix}\right) - \frac{z_1 z_2}{2z_0}\left(k_{Ip}\begin{bmatrix} \delta q_{p2} \\ \delta q_{p1} \end{bmatrix} - k_{II}\begin{bmatrix} \delta z_2 \\ \delta z_1 \end{bmatrix}\right)$$

where $z(t) = \underline{0}\ \forall\ t \leq t_{PID}$, $D(a, b) \in \mathbb{R}^{2\times2}$ is the diagonal matrix with elements a, b, and I is the identity in $\mathbb{R}^{2\times2}$ here. Notice that (z_0, z) is part of the reference, hence we can avoid the condition $z_0 = 0$ by appropriate selection of t_{PID} and $k_{Ip}(t), k_{II}(t) \in \mathbb{R}^{2\times2}$ for this particular case.

Remark *III.1:* As lemmas II.1, II.2, II.3 and II.4 were obtained using the equation for $\delta\dot{q}_0$ which has the same expression than for $\delta\dot{q}_{p0}$, they also apply to the partial quaternion kinematics. The proof of Property II.1 (and therefore Property II.2) holds for partial quaternions as (16) is also obtained as $\delta q_p^T P_0(\delta q_p \times \underline{v}) = 0$ and $\delta q_p^T((P_0\delta q_p) \times \underline{v}) = 0$ for any $\underline{v} \in \mathbb{R}^3$ under $P_0 = aI$.

B. Ascent guidance law for the evaluation of the controller

Consider the usual ascent model [23]:

$$\begin{cases} \dot{r} &= v\sin(\gamma) \\ \dot{v} &= \frac{U\cos(\alpha)-D}{m} - \frac{\mu\sin(\gamma)}{r^2} \\ \dot{\gamma} &= \frac{U\sin(\alpha)+L}{mv} + \left(\frac{v}{r} - \frac{\mu}{vr^2}\right)\cos(\gamma) \end{cases} \quad (37)$$

For $\alpha(t) = 0 \; \forall \; t$, this trajectory is known as gravity turn. Let r^*, v^*, γ^*, α^* be the variables of the reference trajectory, and $e_r = r - r^*$ the ascent guidance error (the altitude difference).

It may be verified that $\ddot{r} = f_r + g_r \sin(\alpha)$, with $g_r = \cos(\gamma)\frac{U}{m}$ and $f_r = \dot{v}\sin(\gamma) + \cos(\gamma)\frac{L}{m} + \frac{v^2\cos(\gamma)}{r} + \frac{\mu\cos(\gamma)}{vr^2}$. The desired trajectory satisfies $\ddot{r}^* = f_r^* + g_r^* \sin\alpha^*$, where $\alpha^* = 0$. The guidance command, given without proof, implements $s_r = k_r e_r + k_{\dot{r}}\dot{e}_r$ under input saturation:

$$\alpha_p \;=\; \mathrm{sat}_{\alpha_{top}}\left(\mathrm{asin}\left(\frac{g_r^*\sin(\alpha^*)}{g_r}\right) + \alpha_{eq} + \Lambda_r \mathrm{sign}(-s_r)\right) \quad (38)$$

where sat$(\cdot)$ is a linear saturation with bound α_{top} given by the vehicle structure as:

$$\alpha_{top} \;=\; \min\left\{\alpha_{max}, \frac{(\alpha q_\infty)_{max}}{q_\infty}\right\} \quad \text{with} \quad \alpha_{max}, (\alpha q_\infty)_{max} \in \mathbb{R}_{>0} \quad (39)$$

where α_{eq} and Λ_r are determined by the implementation. A low pass filter is used to smooth α_p (see [23]).

C. Simulation results of the guidance and control loops

Here we evaluate the control law for the ascent guidance of a suborbital rocket using TVC.

Simulation results are provided for a constant wind profile of 25m/s in the direction of the trajectory plane, being the worst case for the ascent altitude error. The results for the attitude PD and PID laws are shown in Figures 1 and 2 (where θ_{TVC} stands for the angle between the thruster centerine and the vehicle longitudinal axis), while the Figure 3 shows the angular velocity components. A comparison between open loop and closed loop ascent guidance using the proposed attitude tracking controller is given in Figure 4. The bias estimator is used since 9.5 seconds in these simulation, and is shown in

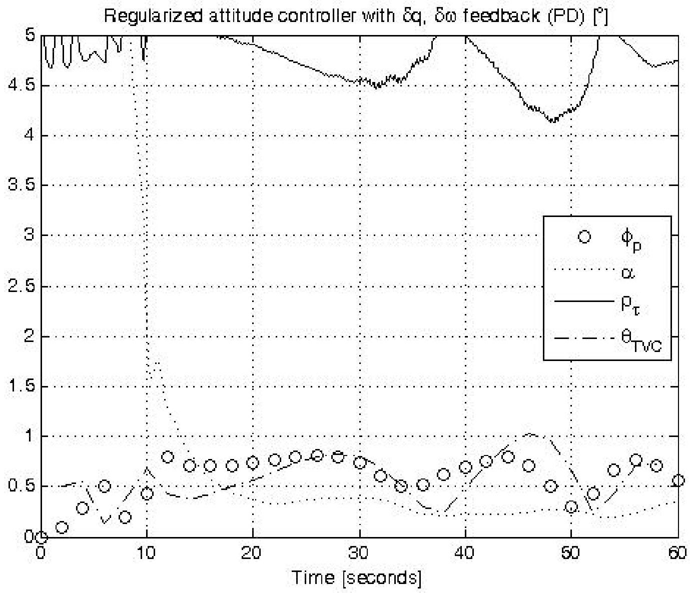

Fig. 1. Attitude tracking control performance without integrator.

Figure 5. A similar update is needed if a INS-GPS were used instead, with justifies this simulated scenario. The sign conditions for $\delta q_0, \delta q_{z0}$ and z_0 are always fulfilled using the partial quaternion. The vehicle has a free roll motion induced by the fins cant angle and there is no roll control system. The parameters were $k_v = 3$, $k(0) = 20$, $\Sigma = I/k_v$, $M = I$, $k_{Ip} = 2I$, $k_{II} = 0.3I$, $h = 1 \cdot 10^{11}$, $g = 5 \cdot 10^8$, $\xi = 100$, $b_{\triangle_{\hat{T}_d}} = 0.33\|\hat{T}_d\| + 1000Nm$. Also, the uniform asymptotic stability condition (19) is fulfilled by any constant $0 < a_z < 2k(t)\delta q_{0z}(0.1z_0 + 0.66\delta q_0)$ for $a_1 = a_0 = 1$. The norm of the control also satisfies the robust uniform asymptotic stability condition for the smoother control (23) with $\kappa_\tau > 1$ (saturated to 4 in the graphic) almost all the time, implementing (29) with $\sigma_0 = 1$, $\sigma_1 = 0.0025$, $r_0 = 0$, $r_1 = \frac{1}{2}$, $\sigma_2 = 20$. Also, $\frac{1}{3} < \delta_s < 1$ was imposed using (26), allowing $\omega_{Ai} > \sqrt{\frac{k+3*0.3}{\delta_s}}\ \forall\ t > t_0$ under maximal excursion ($\pm 5°$).

A wind jet of $35m/s$ between $5000m$ and $5150m$ along the horizontal direction was simulated as a disturbance, within a constant wind profile of $-12.5m/s$ in other altitudes along the same direction; the same PID control law was selected and the result shows a saturation of the control torque through the gimbal angle bound of $\pm 5°$ satisfies also the TVC dynamic restrictions [12]. The simulations consider a wind speed/altitude slopes, with a maximum speed magnitude of $30m/s$, and different wind velocities

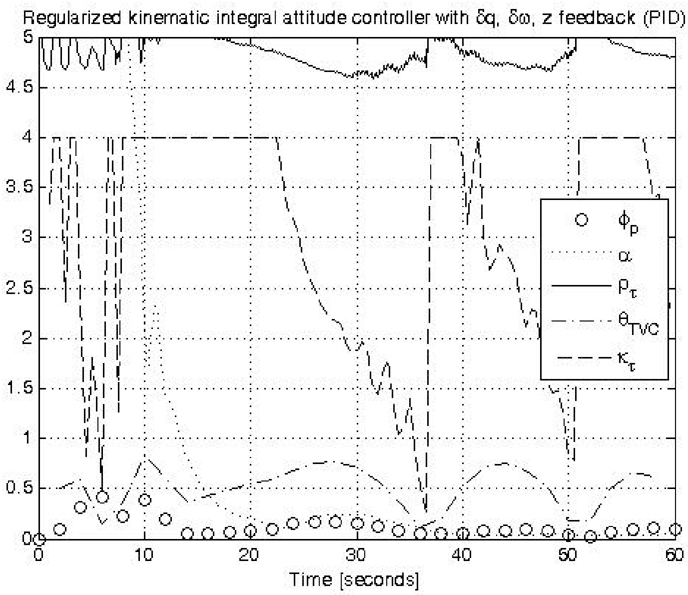

Fig. 2. Kinematic integral attitude tracking control performance (in degrees, except κ_τ).

profiles at other altitudes with a magnitude of $12.5m/s$. The dominant term to guarantee $\rho_\tau > 0$ is given by the aerodynamic torques, as pointed out in [17]. Using a model of aerodynamic first derivatives, lets evaluate a time-varying robustness margin as:

$$\rho_\tau(t) \simeq -\check{s}^T \left(T_c(s) + C_{L\alpha}(z_{cp} - z_{cm}) q_\infty S_{ref} \underline{u}_{\alpha\beta} + T_{\overline{d}} - J\dot{\check{\omega}}_\tau \right) \tag{40}$$

where $C_{L\alpha}$ is the lift curve slope coefficient, $z_{cp} - z_{cm}$ is the difference between the centers of pressure and mass measured from nose, S_{ref} is the reference surface, $\underline{u}_{\alpha\beta}$ is a unitary vector depending on the wind incidence angles, and $T_{\overline{d}}$ is the sum of other disturbances simulated but not detailed here (for instance higher order aerodynamics, flexible body and sloshing). $T_{\overline{d}}$ also includes the term associated to the bias observer (and other navigation errors in general) as $T_c(s_\triangle) - T_c(s)$, which is asymptotically convergent under (30) and (31). To translate this margin directly into a TVC margin and avoid the definition of specific vehicle details in (40) which are not relevant here for the controller evaluation, the simulation output was directly expressed in equivalent TVC angles in Figure 6, where the positiveness of margin ρ_τ is estimated as the minimum of $\rho_\tau(t)$ over t and worst case wind profiles. Figure 7 shows a particular simulation where the control actually saturates and the maximum attack angle is near

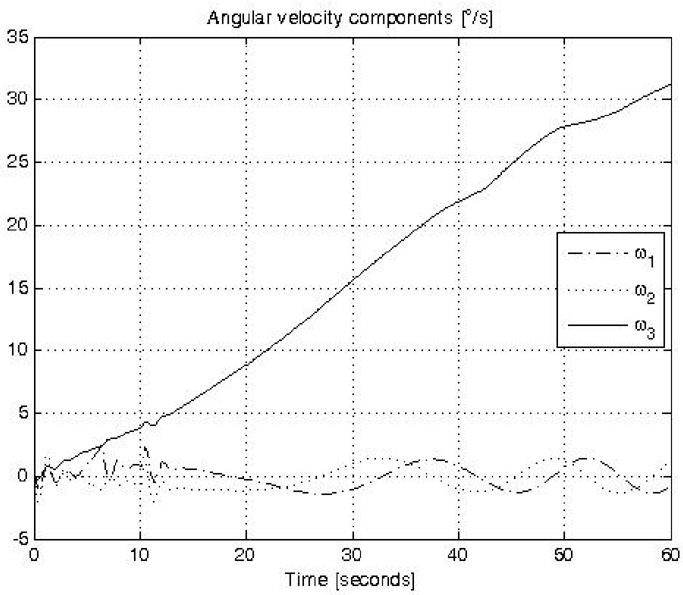

Fig. 3. Body angular velocity components.

4° during the windshear between $5000m$ and $5150m$. For these simulations the minimum relation between the maximum feasible torque in pitch and yaw, and its corresponding inertia moment was $0.6 rad/s^2$. The execution time on the flight control computer tests was lower than $4ms$ for the control, bias observer, altitude and azimuth (not detailed here) guidance loops and the gravity turn trajectory generator. The control frequency was $10Hz$, located between the slosh and flexible modes.

IV. Conclusions

A path to overcome the drawbacks identified in [6] for robust attitude control under input saturation was proposed. A new control law is defined and analysed using nonlinear stability tools. The result has important practical applications, considering here the ascent atmospheric phase of a suborbital rocket. Our approach follows the robust adaptive control started in [3], improving its smooth version. Future work will deal with disturbance rejection optimization.

Appendix

Proof of Lemma II.1: Observe that $\forall t \geq t_0$, $u := k\Sigma\delta q + \delta\omega$ verifies $\underline{\lambda}_N\|u\| \leq \|s\| \leq \overline{\lambda}_N\|u\|$ hence $s \longrightarrow \underline{0} \ \Leftrightarrow u \longrightarrow \underline{0}$. Consider $t \geq t_x$ hereafter and let $K := k\Sigma = \zeta_q|k|\Sigma$, hence $0 < \underline{\sigma}_K = \underline{k\lambda_\Sigma} \leq$

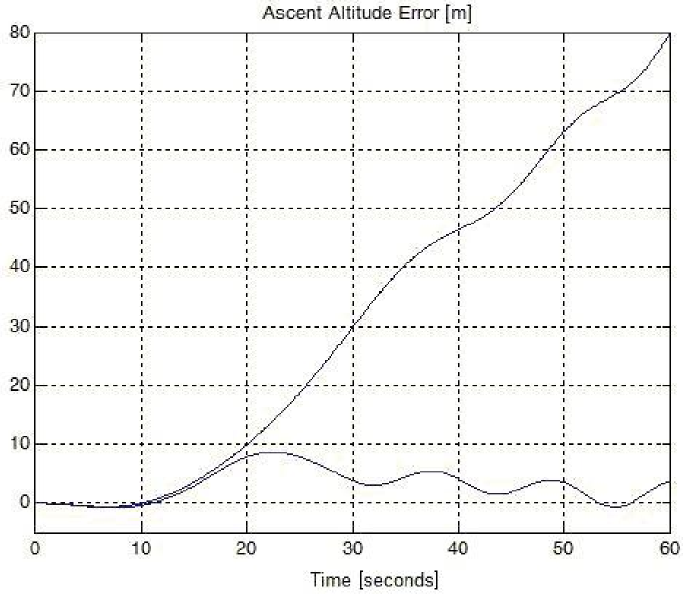

Fig. 4. Altitude guidance error in open loop (upper) and closed loop (lower), in meters.

$\overline{\sigma}_K = \overline{k\lambda_\Sigma} < +\infty$. From (2), $\delta\dot{q}_0 = -\frac{1}{2}\delta q^T \delta\omega = \frac{1}{2}\delta q^T(K\delta q - u)$, and since $|\delta q_0| = \zeta_q \delta q_0$, $\frac{d}{dt}|\delta q_0| = \zeta_q \delta\dot{q}_0$ and:

$$\zeta_q \delta\dot{q}_0 \geq \frac{1}{2}(\underline{\sigma}_K(1-\delta q_0^2) - \|u\|) = \frac{\underline{\sigma}_K}{2}\left(1 - \frac{\|u\|}{\underline{\sigma}_K} - \delta q_0^2\right) \tag{41}$$

Take any $0 < \hat{u} < \underline{\sigma}_K$. As $u \longrightarrow \underline{0}$, $\exists\ T(\hat{u}) \geq t_x$ such that $\|u\| < \hat{u}\ \forall\ t > T(\hat{u})$. The proof continues as in [5] (where $\zeta_q = 1$); an alternative proof is given next.

First suppose that $\delta q_0^2(T) < 1 - \frac{\hat{u}}{\underline{\sigma}_K} - \delta_0$ for some $0 < \delta_0 < 1 - \frac{\hat{u}}{\underline{\sigma}_K}$. While this inequality holds, $|\delta q_0|$ is increasing by (41) with derivative $\frac{d}{dt}|\delta q_0| > \frac{\underline{\sigma}_K \delta_0}{2} > 0$, hence at $t = T' = T + \frac{2}{\underline{\sigma}_K \delta_0}$, we obtain $\delta q_0^2(t) \geq 1 - \frac{\hat{u}}{\underline{\sigma}_K} - \delta_0$ which remains valid $\forall\ t > T'$ as $|\delta q_0|$ is increasing when the equality holds. Hence, we have guaranteed $\delta q_0^2 \geq 1 - \frac{\hat{u}}{\underline{\sigma}_K} - \delta_0\ \forall\ t \geq T'$ and any $\delta q_0(T)$. Given any $b_q, b_\omega > 0$ there exist $\hat{u}, \delta_0, T'$ as before and such that $\|\delta q\| = \sqrt{1 - \delta q_0^2} \leq \sqrt{\frac{\hat{u}}{\underline{\sigma}_K} + \delta_0} \leq b_q$ and $\|\delta\omega\| = \|u - K\delta q\| \leq \hat{u} + \overline{\sigma}_K\sqrt{\frac{\hat{u}}{\underline{\sigma}_K} + \delta_0} \leq b_\omega\ \forall\ t \geq T'$. This is equivalent to state $\lim_{t\to+\infty}\delta\omega(t) = \underline{0}$, $\lim_{t\to+\infty}|\delta q_0(t)| = 1$.

□

Proof of Lemma II.2: By definition of δq_z, $\delta q_z \longrightarrow \underline{0} \iff \delta q \longrightarrow -z \iff u := -k_{Ip}(\delta q + z) \longrightarrow \underline{0}$.

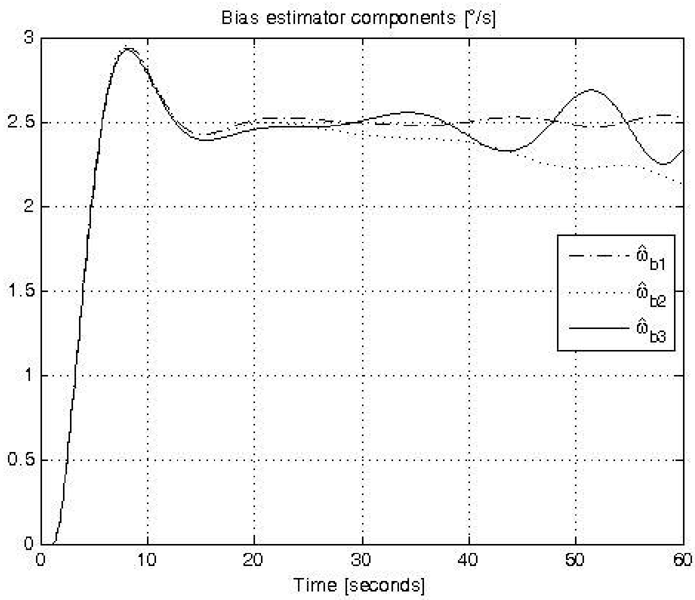

Fig. 5. Bias estimation for a bias value of $2.5^o/s$ on each axis.

Let $K = k_{Ip} + k_{II}$ and $\underline{\sigma}_K := \inf_t \{\underline{\lambda}(\zeta_q K)\}$; the first equation in (3) becomes:

$$\dot{z}_0 \;=\; \frac{1}{2}\left(z^T K z + z^T u\right) \implies \zeta_q \dot{z}_0 \geq \frac{\underline{\sigma}_K}{2}\left(1 - \frac{\|u\|}{\underline{\sigma}_K} - z_0^2\right) \tag{42}$$

The same arguments after equation (41) determine $|z_0| \longrightarrow 1$. Since $\|\delta q\| \leq \|\delta q + z\| + \|z\|$ it follows that $\delta q \longrightarrow \underline{0}$ and $|\delta q_0| \longrightarrow 1$. Using (4), we also obtain $\delta\omega \longrightarrow \delta\omega_z \longrightarrow \underline{0}$.

□

Proof of Remark II.4: Consider (41) with variables δq_{z0} and u_z. For $\delta q_{0z} = \epsilon_y$ we have $\delta \dot{q}_{0z} > 0$ under $\epsilon_y^2 < 1 - \frac{\|u_z\|}{\underline{\sigma}_K} \iff \|u_z\| < \underline{\sigma}_K(1-\epsilon_y^2)$. As $\|u_z\| \leq \|v_z\|$ and $\dot{V}_{PID} \leq 0$ (see (20)), $V_z(t_{PID}) \geq v_z^T P_2 v_z \geq \|u_z\|^2 \underline{\lambda}_{P_2}\ \forall\ t \geq t_{PID}$. Hence if $V_z(t_{PID}) < \underline{\lambda}_{P_2}\underline{\sigma}_K^2(1-\epsilon_y^2)^2$, then $\dot{q}_{z0} > 0$ whenever $\delta q_{z0} = \epsilon_y$, which assures $\delta q_{z0} \geq \epsilon_y\ \forall\ t \geq t_{PID}$. The second claim follows from (3) observing that under hypothesis the damping term $-\frac{k_{II0}}{|z_0|} z$ dominates $k_{Ip}\delta q\ \forall \|\delta q\| \leq \sqrt{1-\epsilon_x^2}$ and $|z_0| \geq \epsilon_z$. For $\delta q_{z0} \geq \epsilon_y, |z_0| \geq \epsilon_z$ and using (4), $\delta q_0 \geq \epsilon_x \impliedby \delta q_0 z_0 = \delta q_{z0} - \delta q^T z \geq \epsilon_x \impliedby \epsilon_y \geq \epsilon_x + \sqrt{1-\epsilon_x^2}\sqrt{1-\epsilon_z^2}$.

□

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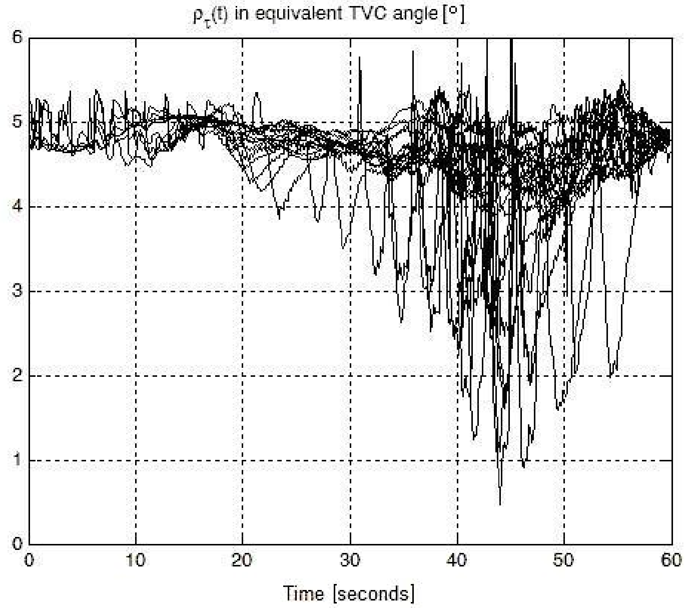

Fig. 6. ρ_τ as a function of time, expressed as TVC equivalent angle for worst case wind profiles.

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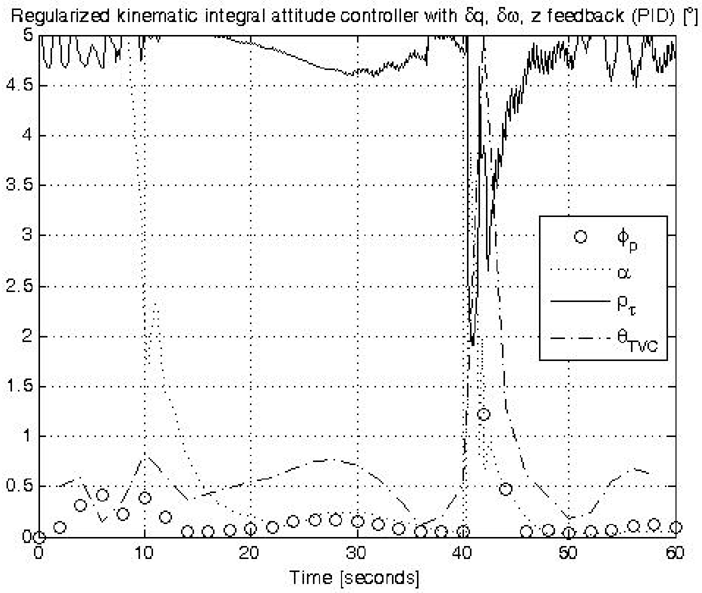

Fig. 7. Kinematic integral attitude tracking under windshear and saturation.

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