INJECTIVE ENVELOPES AND LOCAL MULTIPLIER ALGEBRAS OF SOME SPATIAL CONTINUOUS TRACE C*-ALGEBRAS

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Abstract. A precise description of the injective envelope of a spatial continuous trace C*-algebra A over a Stonean space Δ is given. The description is based on the notion of a weakly continuous Hilbert bundle, which we show herein to be a Kaplansky–Hilbert module over the abelian AW*-algebra $C(\Delta)$. We then use the description of the injective envelope of A to study the first- and second-order local multiplier algebras of A. In particular, we show that the second-order local multiplier algebra of A is precisely the injective envelope of A.

Introduction

A commonly used technique in the theory of operators algebras is to study a given C*-algebra A by one or more of its enveloping algebras. Well known examples of such enveloping algebras are the enveloping von Neumann algebra A^{**} and the multiplier algebra $M(A)$. In this paper we consider two others: the local multiplier algebra $M_{\rm loc}(A)$ and the injective envelope $I(A)$, both of which have received considerable study and application in recent years (see, for example, [1, 6, 7, 9, 11, 19, 21, 22]).

The C*-algebras $M_{\rm loc}(A)$ and $I(A)$ are difficult to determine precisely, even for fairly rudimentary types of C*-algebras A. For instance, if we denote by $C_0(T)$ an abelian C*-algebra and by $K(H)$ the ideal of compact operators over H, their local multiplier algebra and injective envelope have been readily computed; but the injective envelope of $C_0(T)\otimes K(H)$ is much more difficult to describe: see [15] for an abstract description and [3, 4] for a somewhat more concrete one.

Our first goal in the present paper is to make a further contribution to the issue of the determination of $I(A)$ and $M_{\rm loc}(A)$ from A by considering continuous trace C*-algebras studied by Fell [10] that arise from continuous Hilbert bundles. The class of such algebras contains in particular all C*-algebras of the form $C_0(T)\otimes K(H)$, which we studied in [4]. Because the centres of $I(A)$ and $M_{\rm loc}(A)$ are AW*-algebras, and thus have Stonean maximal ideal spaces, we restrict ourselves in this paper to locally compact Hausdorff spaces T that are Stonean. In so doing, we establish an important first step toward a complete analysis, in the case of arbitrary T, of the C*-algebras $I(A)$, $M_{\rm loc}(A)$, and $M_{\rm loc}\left(M_{\rm loc}(A)\right)$ for spatial continuous trace C*-algebras A with spectrum T. As the passage from general T to Stonean T involves a number of technicalities, the application of the main results herein to the case of arbitrary locally compact Hausdorff spaces T will be deferred to a subsequent article.

Our second goal is to introduce and study the notion of a weakly continuous Hilbert bundle $\Omega_{\rm wk}$ relative to a continuous Hilbert bundle Ω over a locally compact Hausdorff space T. It is natural to consider Ω as a C*-module over the abelian C*-algebra $C_0(T)$; if, moreover, T is a Stonean space Δ, we then show $\Omega_{\rm wk}$ carries the structure of a faithful AW*-module over $C(\Delta)$. In this latter situation, such C*-modules are called Kaplansky–Hilbert modules and they behaviour reminds of that of Hilbert space. We study the C*-modules Ω and $\Omega_{\rm wk}$, as well as certain C*-algebras of endomorphisms of these modules, using the beautiful machinery of Kaplansky [16] in his seminal work from the early 1950s. In particular, we prove that the C*-algebra $B(\Omega_{\rm wk})$ of bounded adjointable endomorphisms of $\Omega_{\rm wk}$ is the injective envelope and second-order local multiplier algebra of the C*-algebra $K(\Omega)$ of "compact" endomorphsisms of Ω.

Assuming that $T=\Delta$, a Stonean space, and in postponing the precise definitions until the following section, the main results of this paper are summarised thusly:

- $\Omega_{\rm wk}$ is a Kaplansky–Hilbert module that contains Ω as a C*-submodule such that $\Omega^{\perp}=\{0\}$;

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- $B(\Omega_{\rm wk})$ is the injective envelope of both $K(\Omega)$ and Fell's continuous trace C*-algebra A induced by the bundle Ω;
- $B(\Omega_{\rm wk})$ is the second-order local multiplier algebra of both $K(\Omega)$ and Fell's algebra A;
- a decomposition of $\Omega_{\rm wk}$ into homogeneous submodules leads to a corresponding decomposition of (the generally non-AW*) algebra $M_{\rm loc}(A)$ but not to a decomposition of A;
- the equality $M_{\rm loc}(M_{\rm loc}(A)) = I(A)$ holds for certain type I non-separable C*-algebras, generalizing a result of Somerset [21]

1. Preliminaries

If T is a locally compact Hausdorff space and $\{H_t\}_{t\in T}$ is family of Hilbert spaces, a vector field on T with fibres H_t is a function $\nu : T \to \bigsqcup_t H_t$ in which $\nu(t) \in H_t$, for every $t \in T$. Such a vector field ν is said to be bounded if the function $t \mapsto \|\nu(t)\|$ is bounded.

Definition 1.1. *A* continuous Hilbert bundle [8] *is a triple $(T, \{H_t\}_{t\in T}, \Omega)$, where Ω is a set of vector fields on T with fibres H_t such that:*

(I) *Ω is a $C(T)$-module with the action $(f \cdot \omega)(t) = f(t)\omega(t)$;*
(II) *for each $t_0 \in T$, $\{\omega(t_0) : \omega \in \Omega\} = H_{t_0}$;*
(III) *the map $t \mapsto \|\omega(t)\|$ is continuous, for all $\omega \in \Omega$;*
(IV) *Ω is closed under local uniform approximation—that is, if $\xi : T \to \bigsqcup_t H_t$ is any vector field such that for every $t_0 \in T$ and $\varepsilon > 0$ there is an open set $U \subset T$ containing t_0 and a $\omega \in \Omega$ with $\|\omega(t) - \xi(t)\| < \varepsilon$ for all $t \in U$, then necessarily $\xi \in \Omega$.*

Dixmier and Douady [8] show that (I), (II), and (IV) can be replaced by other axioms, such as those given by Fell [10], without altering the structure that arises. For example, in the presence of the other axioms, (II) is equivalent to "$\{\omega(t_0) : \omega \in \Omega\}$ is dense in H_{t_0}, for each $t_0 \in T$"; in the presence of (IV), axiom (I) can be replaced by "Ω is a complex vector space".

We turn next to the notion of a weakly continuous Hilbert bundle. If $(T, \{H_t\}_{t\in T}, \Omega)$ is a continuous Hilbert bundle then, by the polarisation identity, the function $t \mapsto \langle\omega_1(t), \omega_2(t)\rangle$ is continuous for all $\omega_1, \omega_2 \in \Omega$. In defining $\langle\omega_1, \omega_2\rangle$ to be the map $T \to \mathbb{C}$ given by $t \mapsto \langle\omega_1(t), \omega_2(t)\rangle$, one obtains a $C(T)$-valued inner product on Ω which gives Ω the structure of an inner product module over $C(T)$.

Definition 1.2. *A vector field $\nu : T \to \bigsqcup_t H_t$ is said to be* weakly continuous *with respect to the continuous Hilbert bundle $(T, \{H_t\}_{t\in T}, \Omega)$ if the function*

$$t \longmapsto \langle\nu(t), \omega(t)\rangle$$

is continuous for all $\omega \in \Omega$. The set of all bounded weakly continuous vector fields with respect to a given Ω will be denoted by $\Omega_{\rm wk}$, that is

$$\Omega_{\rm wk} = \{\nu : T \to \bigsqcup_t H_t : \sup_t \|\nu(t)\| < \infty \text{ and } \nu \text{ is weakly continuous}\}.$$

We will call the quadruple $(T, \{H_t\}_{t\in T}, \Omega, \Omega_{\rm wk})$ a weakly continuous Hilbert bundle over T.

We remark that $\Omega_{\rm wk}$ is a $C(T)$-module under the pointwise module action, and that $\Omega \subseteq \Omega_{\rm wk}$ when T is compact (because then every continuous field on Ω is bounded). However, the function $t \mapsto \langle\nu_1(t), \nu_2(t)\rangle$ is generally not continuous for arbitrary $\nu_1, \nu_2 \in \Omega_{\rm wk}$. Thus, although $\Omega_{\rm wk}$ is, algebraically, a module over $C_b(T)$, it is not in general an inner product module over $C_b(T)$. Nevertheless, if T has the right topology—namely that of a Stonean space—then we show (Theorem 2.6) that it is possible to endow a weakly continuous Hilbert bundle with the structure of a C*-module over the C*-algebra of continuous complex-valued functions on T.

The continuous trace C*-algebras we consider herein were first studied by Fell [10]. We now recall their definition.

Assume that $\{A_t\}_{t\in T}$ is a family of C*-algebras indexed by the locally compact Hausdorff topological space T. An operator field is a map $a : T \to \bigsqcup_t A_t$ such that $a(t) \in A_t$, for each $t \in T$.

Definition 1.3. *Let $(T, \{H_t\}_{t\in T}, \Omega)$ be a continuous Hilbert bundle. An operator field $a : T \to \bigsqcup_{t\in T} K(H_t)$ is:*

(**i**) almost finite-dimensional *(with respect to Ω) if for each $t_0 \in T$ and $\varepsilon > 0$ there exist an open set $U \subset T$ containing t_0 and $\omega_1, \ldots, \omega_n \in \Omega$ such that*

(a) $\omega_1(t),\dots,\omega_n(t)$ *are linearly independent for every* $t\in U$*, and*
(b) $\|p_t a(t) p_t - a(t)\| < \varepsilon$ *for all* $t\in U$*, where* $p_t\in B(H_t)$ *is the projection with range* $\mathrm{Span}\,\{\omega_j(t)\,:\,1\le j\le n\}$*;*

(ii) weakly continuous *(with respect to* Ω*) if the complex-valued function*

$$t \longmapsto \langle a(t)\omega_1(t),\omega_2(t)\rangle$$

is continuous for every $\omega_1,\omega_2\in\Omega$*.*

Definition 1.4. ([10]) *Let* $(T,\{H_t\}_{t\in T},\Omega)$ *be a continuous Hilbert bundle. The* Fell algebra *of the Hilbert bundle* $(T,\{H_t\}_{t\in T},\Omega)$*, denoted by* $A = A(T,\{H_t\}_{t\in T},\Omega)$*, is the set of all weakly continuous, almost finite-dimensional operator fields* $a: T\to\bigsqcup_{t\in T}K(H_t)$ *for which* $t\mapsto\|a(t)\|$ *is continuous and vanishes at infinity, endowed with pointwise operations and norm*

$$\|a\| \;=\; \max_{t\in T}\|a(t)\|\,,\qquad a\in A\,.$$

We shall make repeated use of the following fact about the Fell C*-algebras: if we let A be $A = A(T,\{H_t\}_{t\in T},\Omega)$, for some continuous Hilbert bundle $(T,\{H_t\}_{t\in T},\Omega)$, then A is a continuous trace C*-algebra with spectrum $\hat{A}\simeq T$ [10, Theorems 4.4, 4.5].

2. An AW*-module Structure for $\Omega_{\rm wk}$

Assume henceforth that $T=\Delta$ is a Stonean space; that is, Δ is Hausdorff, compact, and extremely disconnected. The abelian C*-algebra $C(\Delta)$ is an AW*-algebra and so one may ask whether the C*-modules Ω and $\Omega_{\rm wk}$ are AW*-modules in the sense of Kaplansky [16]. We shall show that this is indeed true for the module $\Omega_{\rm wk}$. As a consequence of this last fact we shall get that the C*-algebra $B(\Omega_{\rm wk})$ of bounded adjointable endomorphisms of $\Omega_{\rm wk}$ is an AW*-algebra of type I.

The following lemmas are needed to describe the $C(\Delta)$-Hilbert module structure of $\Omega_{\rm wk}$.

Lemma 2.1. *Let* $f:\Delta\to\mathbb{R}$ *be a lower semicontinuous function such that there exist* $g\in C(\Delta)$ *and a meagre set* $M\subset\Delta$ *with* $f(s)=g(s)$ *for all* $s\in\Delta\setminus M$*. Then*

$$\sup_{s\in\Delta} g(s) = \sup_{s\in\Delta\setminus M} f(s) = \sup_{s\in\Delta} f(s).$$

Proof. Let $\rho = \sup\limits_{s\in\Delta\setminus M} f(s) = \sup\limits_{s\in\Delta\setminus M} g(s) \le \sup\limits_{s\in\Delta} g(s)$; then, $g(s)\le\rho$ for all $s\in\Delta\setminus M$. Because Δ is a Baire space, $\overline{\Delta\setminus M}=\Delta$; thus, by the continuity of g, $g(s)\le\rho$ for every $s\in\Delta$. A similar argument shows that $f(s)\le\rho$, for all $s\in\Delta$. □

Lemma 2.2. *Assume that* $(\Delta,\{H_s\}_{s\in\Delta},\Omega)$ *is a continuous Hilbert bundle and* $\nu\in\Omega_{\rm wk}$*. Then*

(i) *the function* $s\mapsto\|\nu(s)\|^2$ *is lower semicontinuous;*
(ii) *there is a meagre subset* $M\subset\Delta$ *and a continuous function* $h:\Delta\to\mathbb{R}_+$ *such that*
(a) $h(s)=\|\nu(s)\|^2$ *for all* $s\in\Delta\setminus M$*, and*
(b) $\|h\| = \sup\limits_{s\in\Delta\setminus M}\|\nu(s)\|^2 = \sup\limits_{s\in\Delta}\|\nu(s)\|^2$.

Proof. Let $r\in\mathbb{R}$ be fixed and consider $U_r=\{s\in\Delta\,:\,r<\|\nu(s)\|^2\}$. We aim to show that U_r is open. Choose $s_0\in U_r$. Thus, $r<\|\nu(s_0)\|^2$. By Parseval's formula, there are orthonormal vectors $\xi_1,\dots,\xi_n\in H_{s_0}$ such that $r<\sum\limits_{j=1}^{n}|\langle\nu(s_0),\xi_j\rangle|^2\le\|\nu(s_0)\|^2$. Choose any $\mu_1,\dots,\mu_n\in\Omega$ such that $\mu_j(s_0)=\xi_j$, for each j. Because $\xi_1,\dots,\xi_n$ are orthogonal, $\mu_1(s),\dots,\mu_n(s)$ are linearly independent in an open neighbourhood of s_0. Hence, by [10, Lemma 4.2], there is an open set V containing s_0 and vector fields $\omega_1,\dots,\omega_n\in\Omega$ such that $\omega_1(s),\dots,\omega_n(s)$ are orthonormal for all $s\in V$, and $\omega_j(s_0)=\xi_j$ for each j. The function

$$g(s) \;=\; \sum_{j=1}^{n}|\langle\nu(s),\omega_j(s)\rangle|^2$$

on Δ is continuous and satisfies $g(s)\le\|\nu(s)\|^2$, for every $s\in V$, and $r<g(s_0)$. Therefore, by the continuity of g, there is an open set $W\subset V$ containing s_0 such that $r<g(s)\le\|\nu(s)\|^2$ for all $s\in W$. This proves that U_r contains an open set around each of its points. That is, U_r is open.

Because every bounded nonnegative lower semicontinuous function on a Stonean space Δ agrees with a nonnegative continuous function off a meagre set M [23, Proposition III.1.7], the function $h \in C(\Delta)$ as in (**ii**) exists and satisfies $h(s) = \|\nu(s)\|^2$ for $s \in \Delta \setminus M$.

The last statement follows from Lemma 2.1. □

Let $(\Delta, \{H_t\}_{t\in\Delta}, \Omega, \Omega_{\rm wk})$ be a weakly continuous Hilbert bundle over Δ. Given $\nu \in \Omega_{\rm wk}$, the function h that arises in Lemma 2.2 will be denoted by $\langle \nu, \nu\rangle$. There is no ambiguity in so doing because if $h_1, h_2 \in C(\Delta)$ and if $h_1(s) = h_2(s)$ for all $s \notin (M_1 \cup M_2)$ for some meagre subsets M_1 and M_2, then h_1 and h_2 agree on Δ. (If not, then by continuity, h_1 and h_2 would differ on an open set U; but $\emptyset \neq U \subset M_1 \cup M_2$ is in contradiction to the fact that no meagre set in a Baire space can contain a nonempty open set.)

Now use the polarisation identity to define $\langle \nu_1, \nu_2\rangle \in C(\Delta)$, for any pair $\nu_1, \nu_2 \in \Omega_{\rm wk}$. This gives to $\Omega_{\rm wk}$ the structure of pre-inner product module over $C(\Delta)$ whereby for each $\nu_1, \nu_2 \in \Omega_{\rm wk}$ there is a meagre subset $M \subset \Delta$ such that the continuous function $\langle \nu_1, \nu_2\rangle$ satisfies

$$\langle \nu_1, \nu_2\rangle (s) \;=\; \langle \nu_1(s), \nu_2(s)\rangle\,, \qquad \forall\, s \in \Delta \setminus M\,.$$

In particular, if $\nu \in \Omega_{\rm wk}$ and $\omega \in \Omega$, then

$$\langle \nu, \omega\rangle (s) \;=\; \langle \nu(s), \omega(s)\rangle\,, \qquad \forall\, s \in \Delta\,.$$

In fact, $\Omega_{\rm wk}$ is an inner product module over $C(\Delta)$, for if $\nu \in \Omega$ satisfies $\langle \nu, \nu\rangle = 0$, then Lemma 2.2 yields $\|\nu(s)\|^2 = 0$ for all $s \in \Delta$. Therefore,

$$\|\nu\| \;=\; \|\langle \nu, \nu\rangle\|^{1/2}\,, \quad \nu \in \Omega_{\rm wk}\,,$$

defines a norm on $\Omega_{\rm wk}$, where

$$\|\nu\|^2 \;=\; \sup_{s\in\Delta} \langle \nu(s), \nu(s)\rangle \;=\; \|\langle \nu, \nu\rangle\|\,. \tag{1}$$

Recall that given a C*-algebra B, a *Hilbert C*-module over B* is a left B-module E together with a B-valued definite sequilinear map $\langle\,,\,\rangle$ such that E is complete with the norm $\|\nu\| = \|\langle \nu, \nu\rangle\|^{1/2}$ (see [17] for a detailed account on Hilbert modules).

Note that if $\nu \in \Omega_{\rm wk}$, then $|\nu|(s) := \langle \nu, \nu\rangle^{1/2}(s) \geq \|\nu(s)\|$ for $s \in \Delta$ and that $|\nu|(s) = \|\nu(s)\|$ if $s \in (\Delta \setminus M)$ for some meagre set $M \subset \Delta$ (Lemma 2.2). These facts will be used repeatedly from now on.

Proposition 2.3. *$\Omega_{\rm wk}$ is a C*-module over $C(\Delta)$ and Ω is a C*-submodule of $\Omega_{\rm wk}$.*

Proof. The only Hilbert C*-module axiom that is not obviously satisfied by $\Omega_{\rm wk}$ is the axiom of completeness. Let $\{\nu_i\}_{i\in\mathbb{N}}$ be a Cauchy sequence in $\Omega_{\rm wk}$. By the equality (1), $\{\nu_i(s)\}_{i\in\mathbb{N}}$ is a Cauchy sequence in H_s for every $s \in \Delta$. Let $\nu(s) \in H_s$ denote the limit of this sequence so that $\nu : \Delta \to \bigsqcup_{s\in\Delta} H_s$, whereby $s \mapsto \nu(s)$, is a vector field.

Choose $\omega \in \Omega$ and consider the function $g_{i,\omega} \in C(\Delta)$ given by $g_{i,\omega}(s) = \langle \omega(s), \nu_i(s)\rangle$. Let $\varepsilon > 0$. Then there is $N_\varepsilon \in \mathbb{N}$ such that $\|\nu_i - \nu_j\| < \varepsilon$, for all $i, j \geq N_\varepsilon$. Therefore, the Cauchy-Schwarz inequality yields

$$\sup_{s\in\Delta} |g_{i,\omega}(s) - g_{j,\omega}(s)| \;<\; \varepsilon\,\|\omega\|\,, \quad \forall\, i, j \geq N_\varepsilon\,.$$

Thus, the sequence $\{g_{i,\omega}\}_i$ is Cauchy in $C(\Delta)$; let $g_\omega \in C(\Delta)$ denote its limit. Observe that $g_\omega(s) = \lim_i \langle \nu_i(s), \omega(s)\rangle = \langle \nu(s), \omega(s)\rangle$, for all $s \in \Delta$. As the choice of $\omega \in \Omega$ is arbitrary, this shows that ν is weakly continuous. The Cauchy sequence $\{\nu_i\}_{i\in\mathbb{N}}$ is necessarily uniformly bounded by, say, $\rho > 0$, and then $\|\nu(s)\| \leq \rho$ for every $s \in \Delta$. That is, the function $s \to \|\nu(s)\|$ is bounded and so $\nu \in \Omega_{\rm wk}$. Finally, if $i, j \geq N_\varepsilon$, then for any $s \in \Delta$ we have $\|\nu(s) - \nu_i(s)\| \leq \|\nu(s) - \nu_j(s)\| + \|\nu_j(s) - \nu_i(s)\| \leq \|\nu(s) - \nu_j(s)\| + \varepsilon$, and so letting $j \to \infty$ yields $\|\nu(s) - \nu_i(s)\| \leq \varepsilon$ for every $s \in \Delta$. That is, $\|v - \nu_i\| \to 0$, which proves that $\Omega_{\rm wk}$ is complete.

For the case of Ω, let $\{\omega_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in Ω. For each $s \in \Delta$, $\{\omega_n(s)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in H_s; let $\omega(s)$ denote the limit. Since the limit is uniform, it is in particular locally uniform, and so $\omega \in \Omega$. Hence, Ω is complete. □

Definition 2.4. *A Hilbert C*-module E over the C*-algebra B is called a* Kaplansky–Hilbert module *if in addition B is an abelian AW*-algebra and the following three properties hold* [16, p. 842] *(Kaplansky's original term for such a module was a faithful AW*-module):*

(i) *if* $e_i \cdot \nu = 0$ *for some family* $\{e_i\}_i \subset B$ *of pairwise-orthogonal projections and* $\nu \in E$*, then also* $e \cdot \nu = 0$*, where* $e = \sup_i e_i$*;*
(ii) *if* $\{e_i\}_i \subset B$ *is a family of pairwise-orthogonal projections such that* $1 = \sup_i e_i$*, and if* $\{\nu_i\}_i \subset E$ *is a bounded family, then there is a* $\nu \in E$ *such that* $e_i \cdot \nu = e_i \cdot \nu_i$ *for all* i*;*
(iii) *if* $\nu \in E$*, then* $g \cdot \nu = 0$ *for all* $g \in B$ *only if* $\nu = 0$*.*

Remark 2.5. *The element* $\nu \in E$ *obtained in the situation described in (ii) will sometimes be denoted as* $\sum_i e_i \nu_i$*. It should be emphasized that this is not a pointwise sum.*

Theorem 2.6. $\Omega_{\rm wk}$ *is a Kaplansky–Hilbert module over* $C(\Delta)$*.*

Proof. For property (i), assume that $\nu \in \Omega_{\rm wk}$ and $\{e_i\}_i \subset C(\Delta)$ is a family of pairwise-orthogonal projections with supremum $e \in C(\Delta)$ for which $e_i \cdot \nu = 0$ for all i. Because projections in $C(\Delta)$ are the characteristic functions of clopen sets, there are pairwise-disjoint clopen sets $U_i \subset \Delta$ such that $e_i = \chi_{U_i}$. Thus, for each i, using Lemma 2.2,

$$0 \;=\; \|e_i \cdot \nu\|^2 = \max_{s\in\Delta} \langle e_i \cdot \nu, e_i \cdot \nu\rangle(s) = \sup_{s\in\Delta} \langle e_i(s)\nu(s), e_i(s)\nu(s)\rangle$$

$$= \max_{s\in\Delta} e_i(s)\;[\langle \nu,\nu\rangle(s)] = \max_{s\in U_i} \langle \nu,\nu\rangle(s)\,,$$

and so $\langle \nu,\nu\rangle(s) = 0$ for every $s \in U_i$. Let $U = \bigcup_i U_i$. The set $\overline{U}$ is clopen and $\chi_{\overline{U}} = \sup_i e_i = e$ [5, §8]. As $\langle \nu,\nu\rangle$ is a continuous function that vanishes on U, $\langle \nu,\nu\rangle$ also vanishes on $\overline{U}$. Hence,

$$\|e\cdot\nu\|^2 \;=\; \max_{s\in\Delta} e(s)\,[\langle \nu,\nu\rangle(s)] \;=\; \max_{s\in\overline{U}} \langle \nu,\nu\rangle(s) \;=\; 0\,,$$

which yields property (i).

For the proof of property (ii), assume that $\{e_i\}_i \subset C(\Delta)$ is a family of pairwise-orthogonal projections such that $1 = \sup_i e_i$ and that $\{\nu_i\}_i \subset \Omega_{\rm wk}$ is a family such that $K = \sup \|\nu_i\| < \infty$; we aim to prove that there is a $\nu \in \Omega_{\rm wk}$ such that $e_i \cdot \nu = e_i \cdot \nu_i$ for all i. As before, assume that $e_i = \chi_{U_i}$ and $U = \bigcup_i U_i$. Then $1 = \sup_i e_i$ implies that $\overline{U} = \Delta$.

For each $\omega \in \Omega$, consider the unique function $f_\omega \in C(\Delta)$ such that $e_i\, f_\omega = e_i\,\langle \omega, \nu_i\rangle$ for all i (its existence guaranteed by the fact that Δ is the Stone–Čech compactification of U). Note that for $s \in U_i$ we have that $f_\omega(s) = \langle \omega(s), \nu_i(s)\rangle$. Hence, $|f_\omega(s)| \le K\,\|\omega(s)\|$ for $s \in U$; the same inequality holds for all $s \in \Delta$ because $\overline{U} = \Delta$ and both sides of the inequality are continuous functions of s. Moreover, if $\omega_1,\,\omega_2 \in \Omega$ and $\alpha \in \mathbb{C}$ then, for $s \in U$ we get that $f_{\alpha\,\omega_1+\omega_2}(s) = \alpha\, f_{\omega_1}(s) + f_{\omega_2}(s)$ and, therefore, that $f_{\alpha\,\omega_1+\omega_2} = \alpha\, f_{\omega_1} + f_{\omega_2}$. Thus, for each $s \in \Delta$ the function $\omega(s) \mapsto f_\omega(s)$ is a bounded and linear functional on H_s. Let $\nu(s) \in H_s$ be the representing vector for this functional, yielding a vector field $\nu : \Delta \to \bigsqcup_{s\in\Delta} H_s$, $s \mapsto \nu(s)$. Since $\langle \nu(s), \omega(s)\rangle = \overline{f_\omega(s)}$, for every $\omega \in \Omega$, ν is weakly continuous. It remains to show that ν is a bounded vector field. If $s \in U$,

$$\|\nu(s)\| \;=\; \sup_{\omega\in\Omega,\ \|\omega(s)\|=1} |\langle \omega(s), \nu(s)\rangle| \;=\; \sup_{\omega\in\Omega,\ \|\omega(s)\|=1} |f_\omega(s)| \le \sup_i \|\nu_i\| \;= K\,,$$

which shows that $\|\nu(s)\|$ is uniformly bounded on U. Thus, by Lemma 2.1, the lower semicontinuous function $s \mapsto \|\nu(s)\|^2$ is bounded on Δ, since $\Delta \setminus U$ is nowhere dense. Therefore, $\nu \in \Omega_{\rm wk}$.

Now we show that $e_i \cdot \nu = e_i \cdot \nu_i$, for all i. Fix i and $s \in U_i$ and consider $\omega \in \Omega$. Then,

$$\begin{aligned}\langle \omega(s),\, e_i(s)\,\nu(s)\rangle &= \langle \omega(s),\, \nu(s)\rangle \\ &= f_\omega(s) = \langle \omega(s), \nu_i(s)\rangle \\ &= \langle \omega(s), e_i(s)\,\nu_i(s)\rangle\,.\end{aligned}$$

Since $(e_i \cdot \nu)(s) = 0 = (e_i \cdot \nu_i)(s)$ for $s \in \Delta \setminus U_i$ we conclude that $e_i \cdot \nu = e_i \cdot \nu_i$.

For the proof of property (iii), assume that $\nu \in \Omega_{\rm wk}$ satisfies $g \cdot \nu = 0$ for all $g \in C(\Delta)$. Then, in particular, $\langle \nu,\nu\rangle \cdot \nu = 0$. Hence, from $\|\nu\| = \|\langle \nu,\nu\rangle\|^{1/2}$ we conclude $\nu = 0$, which proves property (iii). □

3. Endomorphisms of Ω and $\Omega_{\rm wk}$

Throughout this section A will denote the Fell C*-algebra of the continuous Hilbert bundle $(\Delta, \{H_s\}_{s\in\Delta}, \Omega)$, as described in Definition 1.4, where Δ is Stonean. Let $B(\Omega)$ and $B(\Omega_{\rm wk})$ denote, respectively, the C*-algebras of adjointable $C(\Delta)$-endomorphisms of Ω and $\Omega_{\rm wk}$. Since, by Theorem 2.6, $\Omega_{\rm wk}$ is a Kaplansky–Hilbert AW*-module over $C(\Delta)$, $B(\Omega_{\rm wk})$ coincides with the set of all $C(\Delta)$-endomorphisms of $\Omega_{\rm wk}$ [16, Theorem 6] and is a type I AW*-algebra with centre $C(\Delta)$ [16, Theorem 7].

In the case where Ω is given by the trivial Hilbert bundle $(\Delta, \{H\}_{s\in\Delta}, C(\Delta, H))$, where H is a fixed Hilbert space, Hamana [15] proved that $B(\Omega_{\rm wk}) \cong C(\Delta)\overline{\otimes}B(H)$, the monotone complete tensor product $C(\Delta)$ and $B(H)$.

For each $\nu_1, \nu_2 \in \Omega_{\rm wk}$, consider the endomorphism Θ_{ν_1,ν_2} on $\Omega_{\rm wk}$ defined by

$$\Theta_{\nu_1,\nu_2}(\nu) \;=\; \langle \nu, \nu_2\rangle \cdot \nu_1\,, \quad \nu \in \Omega_{\rm wk}\,.$$

Let

$$F(\Omega) = \left\{ \sum_{j=1}^{n} \Theta_{\omega_j,\omega_j'} \,:\, \omega_j, \omega_j' \in \Omega \right\}, \quad F(\Omega_{\rm wk}) = \left\{ \sum_{j=1}^{n} \Theta_{\nu_j,\nu_j'} \,:\, \nu_j, \nu_j' \in \Omega_{\rm wk} \right\}.$$

If $\omega_1, \omega_2 \in \Omega$, then $\Theta_{\omega_1,\omega_2}\omega \in \Omega$ for all $\omega \in \Omega$, and so $F(\Omega) \subset B(\Omega)$. In fact, $F(\Omega)$ and $F(\Omega_{\rm wk})$ are algebraic ideals in $B(\Omega)$ and $B(\Omega_{\rm wk})$ respectively. The norm-closures of these algebraic ideals, namely $K(\Omega)$ and $K(\Omega_{\rm wk})$, are essential ideals in each of $B(\Omega)$ and $B(\Omega_{\rm wk})$—called ideals of compact endomorphisms—and the multiplier algebras of $K(\Omega)$ and $K(\Omega_{\rm wk})$ are, respectively, $B(\Omega)$ and $B(\Omega_{\rm wk})$ (see [17]).

When referring to rank-1 operators x acting on a Hilbert space H, we will use the notation $x = \xi \otimes \eta$ for such an operator—the action on $\gamma \in H$ given by $\gamma \mapsto \langle \gamma, \eta\rangle\, \xi$—and we reserve the notation $\Theta_{\xi,\eta}$ for "rank-1" operators acting on a Hilbert module.

For any C*-algebra B, we denote the injective envelope [13], [18, Chapter 15] of B by $I(B)$ (and we consider $I(B)$ as a C*-algebra rather than an operator system).

The main result of the present section is the following theorem.

Theorem 3.1. *There exist C*-algebra embeddings such that*

$$K(\Omega) \;\subseteq\; A \;\subseteq\; B(\Omega) \;\subseteq\; B(\Omega_{\rm wk}) \;=\; I(K(\Omega))\,. \tag{2}$$

In particular, $I(K(\Omega)) = I(A) = I(B(\Omega)) = B(\Omega_{\rm wk})$.

The proof of Theorem 3.1 and a description of the inclusions in (2) begin with the following set of results.

Lemma 3.2. *For every* $a \in A$ *and* $\omega \in \Omega$*, the vector field* $a \cdot \omega$ *defined by* $a \cdot \omega(s) = a(s)\omega(s)$ *is an element of* Ω.

Proof. Let $a \in A$. Then $a^*a \in A_+$ and since all fields in A are weakly continuous, for every $\omega \in \Omega$ the map $s \mapsto \|a(s)\omega(s)\| = \langle a^*a \cdot \omega(s), \omega(s)\rangle^{1/2}$ is continuous.

Suppose $s_0 \in \Delta$ and $\varepsilon > 0$. Because $H_{s_0} = \{\mu(s_0) \,:\, \mu \in \Omega\}$, there is a $\mu \in \Omega$ such that $a(s_0)\omega(s_0) = \mu(s_0)$. Since

$$\|a \cdot \omega(s) - \mu(s)\|^2 \;=\; \|a(s)\omega(s)\|^2 + \|\mu(s)\|^2 - 2{\rm Re}\,\langle a(s)\omega(s), \mu(s)\rangle$$

is continuous on Δ and vanishes at s_0, there is an open set $U \subset \Delta$ containing s_0 such that $\|a \cdot \omega(s) - \mu(s)\| < \varepsilon$ for all $s \in U$. As Ω is closed under local uniform approximation, this proves that $a \cdot \omega \in \Omega$. □

Proposition 3.3. *The map* $\varrho : A \to B(\Omega)$ *given by* $\varrho(a)\omega = a \cdot \omega$*, for* $a \in A$ *and* $\omega \in \Omega$ *is an isometric C*-homomorphism. Furthermore,* $K(\Omega) \subseteq \varrho(A) \subset B(\Omega)$ *as C*-algebras.*

Proof. It is clear that ϱ is a C*-algebra homomorphism, and so we only need to verify that it is one-to-one. To this end, assume that $\varrho(a) = 0$. Thus, $a(s)\omega(s) = 0$ for every $\omega \in \Omega$ and every $s \in \Delta$. Because $H_s = \{\omega(s) \,:\, \omega \in \Omega\}$, this implies that $a(s) = 0$ for all $s \in \Delta$, and so $a = 0$.

To show $K(\Omega) \subseteq \varrho(A) \subset B(\Omega)$ as C*-algebras, consider $\Theta_{\omega_1,\omega_2}$ with $\omega_1, \omega_2 \in \Omega$. For any $\eta_1, \eta_2 \in \Omega$, the map

$$\langle \Theta_{\omega_1,\omega_2} \cdot \eta_1, \eta_2\rangle(s) = \langle \eta_1, \omega_2\rangle(s)\, \langle \omega_1, \eta_2\rangle(s) = \langle \eta_1(s), \omega_2(s)\rangle\, \langle \omega_1(s), \eta_2(s)\rangle$$

is continuous. So $\Theta_{\omega_1,\omega_2}$ is finite dimensional and weakly continuous, which shows that $\Theta_{\omega_1,\omega_2} \in A$ and $K(\Omega) \subseteq \varrho(A)$. □

Lemma 3.4. *$\Omega^\perp = \{0\}$, with respect to the inclusion $\Omega \subset \Omega_{\mathrm{wk}}$.*

Proof. Let $\nu \in \Omega_{\mathrm{wk}}$ be such that $\langle \nu, \omega \rangle = 0$, for every $\omega \in \Omega$. That is, for every $\omega \in \Omega$ and for every $s \in \Delta$, $\langle \nu(s), \omega(s) \rangle = 0$. If $\nu \neq 0$, there exists $s_0 \in \Delta$ such that $\nu(s_0) \neq 0$. By axiom (II) in Definition 1.1, there exists $\omega \in \Omega$ such that $\omega(s_0) = \nu(s_0)$, in contradiction to $\langle \nu(s_0), \omega(s_0) \rangle = 0$. □

Lemma 3.5. *If $t_0 \in \Delta$ and $\xi \in H_{t_0}$, then there exists $\omega \in \Omega$ such that $\omega(t_0) = \xi$ and $\|\omega\| = \|\xi\|$.*

Proof. The case $\xi = 0$ is trivial. So assume that $\|\xi\| > 0$. Let $\omega' \in \Omega$ with $\omega'(t_0) = \xi$. Fix a clopen neighbourhood N of t_0 such that $N \subset \{t \in T : \|\omega'(t)\| \geq \|\omega'(t_0)\|/2\}$. Let $h'(\cdot) = \|\xi\| \cdot \|\omega'(\cdot)\|^{-1} \in C(N)$; then h' extends to a continuous function $h \in C(\Delta)$ with $h|_{\Delta \setminus N} = 0$. It is now straightforward to show that $\omega = h \cdot \omega' \in \Omega$ has the desired properties. □

Proposition 3.6. *There exists an isometric homomorphism $\vartheta : B(\Omega) \to B(\Omega_{\mathrm{wk}})$ such that for $a \in A$, $\nu \in \Omega_{\mathrm{wk}}$,*

$$(\vartheta(\varrho(a))\nu)(s) = a(s)\nu(s), \quad s \in \Delta\,. \tag{3}$$

Proof. Assume that $b \in B(\Omega)$ and $\omega \in \Omega$, $s \in \Delta$. By Lemma 3.5,

$$\begin{aligned} \|(b\,\omega)(s)\| &= \sup_{\xi \in H_s,\, \|\xi\|=1} |\langle (b\,\omega)(s), \xi \rangle| = \sup_{\eta \in \Omega,\, \|\eta\|=1} |\langle (b\,\omega)(s), \eta(s) \rangle| \\ &= \sup_{\eta \in \Omega,\, \|\eta\|=1} |\langle \omega(s), (b^*\eta)(s) \rangle| \\ &\leq \|\omega(s)\| \sup_{\eta \in \Omega,\, \|\eta\|=1} \|b^*\eta\| \leq \|\omega(s)\|\, \|b^*\| = \|\omega(s)\|\, \|b\|\,. \end{aligned}$$

Therefore, the function $\omega(s) \mapsto (b\,\omega)(s)$ is well defined and induces a bounded linear operator $b(s) \in B(H_s)$ such that $(b\,\omega)(s) = b(s)\,\omega(s)$, for $s \in \Delta$ and $\omega \in \Omega$ with $\sup_{s \in \Delta} \|b(s)\| \leq \|b\|$. Moreover,

$$\begin{aligned} \|b\| &= \sup_{\|\omega\|=1} \|b \cdot \omega\| = \sup_{\|\omega\|=1} \sup_s \|b \cdot \omega(s)\| = \sup_{\|\omega\|=1} \sup_s \|b(s)\omega(s)\| \\ &\leq \sup_{\|\omega\|=1} \sup_s \|b(s)\|\, \|\omega(s)\| \leq \sup_s \|b(s)\| \leq \|b\|\,, \end{aligned}$$

and so $\sup_{s \in \Delta} \|b(s)\| = \|b\|$. Suppose now that $\nu \in \Omega_{\mathrm{wk}}$ and $s \in \Delta$, and define a vector field $b\nu$ by $(b\,\nu)(s) = b(s)\,\nu(s)$. If $\eta \in \Omega$, then

$$\langle (b\,\nu)(s), \eta(s) \rangle \;=\; \langle \nu(s), b(s)^*\eta(s) \rangle \;=\; \langle \nu(s), (b^*\eta)(s) \rangle$$

is continuous, which shows that $b\,\nu$ is weakly continuous with respect to Ω. Since $b\,\nu$ is also uniformly bounded, we conclude that $b\,\nu \in \Omega_{\mathrm{wk}}$. It is straightforward to show that the map $\nu \mapsto b\,\nu$ is a bounded $C(\Delta)$-endomorphism of Ω_{wk} and hence it gives rise to an element of $B(\Omega_{\mathrm{wk}})$ denoted by $\vartheta(b)$. It is clear the ϑ is a C*-homomorphism. Since $\vartheta b|_\Omega = b$, $\vartheta b = 0$ implies $b = 0$ by Lemma 3.4, so ϑ is well-defined and one-to-one, and thus a C*-monomorphism. Finally, it is clear that (3) holds by construction. □

One consequence of the proof of Proposition 3.6 is that for every $b \in B(\Omega)$ there exists an operator field $\{b(s)\}_{s \in \Delta}$ acting on the Hilbert bundle $\{H_s\}_{s \in \Delta}$ such that $(b\,\omega)(s) = b(s)\,\omega(s)$, for every $s \in \Delta$. This property, however, is not shared by all elements of $B(\Omega_{\mathrm{wk}})$.

Lemma 3.7. *If $z \in B(\Omega_{\mathrm{wk}})$ and $\Theta_{\omega,\omega} z \Theta_{\mu,\mu} = 0$ for all $\omega, \mu \in \Omega$, then $z = 0$.*

Proof. For any ξ, ω, $\mu \in \Omega$ we have that

$$0 = \Theta_{\omega,\omega}\, z\, \Theta_{\mu,\mu}\, \xi = \langle \xi, \mu \rangle\, \langle z\mu, \omega \rangle\, \omega.$$

Hence, we get that

$$0 = \langle \xi, \mu \rangle\, |\langle z\mu, \omega \rangle|^2 = \langle \xi, \mu \rangle\, |\langle \mu, z^*\omega \rangle|^2.$$

We are free to choose $\xi, \mu \in \Omega$. Fix s. If $z^*\omega(s) \neq 0$, choose μ with $\xi(s) = \mu(s) = z^*\omega(s)$ and let $\xi = \mu$. Then $z^*\omega(s) = 0$ and as $s \in \Delta$ is arbitrary, $z^*\omega = 0$ for every $\omega \in \Omega$. For any $\nu \in \Omega_{\mathrm{wk}}$

and every $\omega \in \Omega$, $\langle z\nu, \omega\rangle = \langle \nu, z^*\omega\rangle = 0$. By Lemma 3.4 we conclude that $z\nu = 0$ for $\nu \in \Omega_{\rm wk}$ and hence $z = 0$. □

Proof of Theorem 3.1. We consider the embeddings $A \xrightarrow{\varrho} B(\Omega)$ and $B(\Omega) \xrightarrow{\vartheta} B(\Omega_{\rm wk})$ defined in Propositions 3.3 and 3.6. In this way, we get the inclusions in (2).

Because $B(\Omega_{\rm wk})$ is a type I AW*-algebra, it is injective [14, Proposition 5.2]. To show that $B(\Omega_{\rm wk})$ is the injective envelope $I(K(\Omega))$ of $K(\Omega)$, we need to show that the embedding $\vartheta \circ \varrho$ of A into $B(\Omega_{\rm wk})$ is rigid [18, Theorem 15.8]: that is, we aim to prove that if $\phi : B(\Omega_{\rm wk}) \to B(\Omega_{\rm wk})$ is a unital completely positive linear map for which $\phi|_{K(\Omega)} = {\rm id}_{K(\Omega)}$, then $\phi = {\rm id}_{B(\Omega_{\rm wk})}$.

Let $\phi : B(\Omega_{\rm wk}) \to B(\Omega_{\rm wk})$ be such a ucp map with $\phi|_{K(\Omega)} = {\rm id}_{K(\Omega)}$. Suppose that $z \in B(\Omega_{\rm wk})$ and $\omega, \mu \in \Omega$. Then $\Theta_{\omega,\omega} z \Theta_{\mu,\mu} = \Theta_{\langle z\mu,\omega\rangle\omega,\mu} \in K(\Omega)$. Because $K(\Omega)$ is in the multiplicative domain of ϕ, we have that $\phi(axb) = a\phi(x)b$ for all $x \in B(\Omega_{\rm wk})$ and $a, b \in K(\Omega)$. This implies that

$$\Theta_{\omega,\omega}\phi(z)\Theta_{\mu,\mu} = \phi(\Theta_{\omega,\omega} z\Theta_{\mu,\mu}) = \phi(\Theta_{\langle z\mu,\omega\rangle\omega,\mu}) = \Theta_{\langle z\mu,\omega\rangle\omega,\mu} = \Theta_{\omega,\omega} z\Theta_{\mu,\mu},$$

and so $\Theta_{\omega,\omega}(z - \phi(z))\Theta_{\mu,\mu} = 0$. Since ω, μ were arbitrary, Lemma 3.7 implies that $z - \phi(z) = 0$ and so $\phi = {\rm id}_{B(\Omega_{\rm wk})}$.

We have shown above that the the inclusion $K(\Omega) \subset B(\Omega_{\rm wk})$ is rigid. Moreover, $K(\Omega)$ is an essential ideal of $B(\Omega)$ and $K(\Omega) \subset A \subset B(\Omega)$. Hence, $I(K(\Omega)) = I(A) = I(B(\Omega)) = B(\Omega_{\rm wk})$. □

We conclude this section with a remark about the ideal $K(\Omega_{\rm wk})$ of $B(\Omega_{\rm wk})$. In type I AW*-algebras, the ideal generated by the abelian projections has a prominent role. As it happens, $K(\Omega_{\rm wk})$ is precisely this ideal.

Proposition 3.8. *The C*-algebra $K(\Omega_{\rm wk})$ coincides with the ideal $J \subset B(\Omega_{\rm wk})$ generated by the abelian projections of $B(\Omega_{\rm wk})$. In particular, $K(\Omega_{\rm wk})$ is a liminal C*-algebra with Hausdorff spectrum.*

Proof. By [16, Lemma 13], a projection $e \in B(\Omega_{\rm wk})$ is abelian if and only if there exists $\nu \in \Omega_{\rm wk}$ such that $|\nu|$ is a projection in $C(\Delta)$ and $e = \Theta_{\nu,\nu}$. Hence, $J \subseteq K(\Omega_{\rm wk})$.

To show that $K(\Omega_{\rm wk}) \subset J$, assume $\nu \in \Omega_{\rm wk}$ is nonzero. Let $\varepsilon > 0$. We will show that there is an $x_\varepsilon \in J$ such that $\|\Theta_{\nu,\nu} - x_\varepsilon\| < \varepsilon$. Let $V \subset \Delta$ be the (clopen) closure of $\{s \in \Delta : \ |\nu|(s) < \varepsilon^{1/2}/2\}$, $U = \Delta \setminus V$ (also clopen) and let $g = (1/|\nu|)\,\chi_U \in C(\Delta)_+$. Then $g|\nu| = \chi_U$ and $\|\,\chi_{\Delta\setminus U}|\nu|\,\| < \varepsilon^{1/2}$. Let $\nu' = g\cdot\nu$ so that $|\nu'| = \chi_U$. Hence, $\Theta_{\nu',\nu'} \in J$ and $\Theta_{\nu',\nu'} = g^2 \cdot \Theta_{\nu,\nu}$. Let $x_\varepsilon = |\nu|^2 \cdot \Theta_{\nu',\nu'} \in J$. Then

$$x_\varepsilon = |\nu|^2 \cdot \Theta_{\nu',\nu'} = |\nu|^2\, g^2\, \Theta_{\nu,\nu} = \chi_U\, \Theta_{\nu,\nu},$$

and

$$\begin{aligned}\|x_\varepsilon - \Theta_{\nu,\nu}\| = \|\chi_{\Delta\setminus U}\cdot\Theta_{\nu,\nu}\| &= \sup_{\xi\in\Omega_{\rm wk},\ \|\xi\|=1}\ \max_{s\in\Delta\setminus U}\ \langle\Theta_{\nu,\nu}\xi, \Theta_{\nu,\nu}\xi\rangle^{1/2}(s)\\ &= \sup_{\xi\in\Omega_{\rm wk},\ \|\xi\|=1}\ \max_{s\in\Delta\setminus U}\ \left(\langle\xi,\nu\rangle^2(s)\,|\nu|^2(s)\right)^{1/2} \le \max_{s\in\Delta\setminus U}\ |\nu|^2(s) < \varepsilon.\end{aligned}$$

As ε was arbitrary and J is closed, we conclude that $\Theta_{\nu,\nu} \in J$. The polarisation identity then shows that $\Theta_{\nu_1,\nu_2} \in J$ for all $\nu_1, \nu_2 \in \Omega_{\rm wk}$. Hence, $F(\Omega_{\rm wk}) \subset J$, and so $K(\Omega_{\rm wk}) \subseteq J$.

It remains to justify the last assertion in the statement. By the main result of [12], the ideal generated by the abelian projections in a type I AW*-algebra is liminal and has Hausdorff spectrum. Hence, this is true of $K(\Omega_{\rm wk})$. □

4. Multiplier and Local Multiplier Algebras

In the previous section we established the inclusions $K(\Omega) \subseteq A \subseteq B(\Omega) \subseteq B(\Omega_{\rm wk})$, as C*-subalgebras, and we showed that $I(A) = B(\Omega_{\rm wk})$. The present section refines these inclusions to incorporate multiplier algebras and local multiplier algebras.

Let C denote an arbitrary C*-algebra. By $M(C)$ and $M_{\rm loc}(C)$ we denote the multiplier algebra and the local multiplier algebra [2] of a C*-algebra C respectively. The second order local multiplier algebra of C is $M_{\rm loc}\left(M_{\rm loc}(C)\right)$, the local multiplier algebra of $M_{\rm loc}(C)$. By [11, Corollary 4.3], the local multiplier algebras (of all orders) of C are C*-subalgebras of the injective envelope $I(C)$ of C. In particular, $C \subseteq M_{\rm loc}(C) \subseteq M_{\rm loc}\left(M_{\rm loc}(C)\right) \subseteq I(C)$ as C*-subalgebras. By a well known theorem of Kasparov [2, Theorem 1.2.33], [17, Theorem 2.4], $M(K(\Omega)) = B(\Omega)$.

The following theorem is the main result of this section.

Theorem 4.1. *With the notations from the previous sections, the equality* $M_{\rm loc}(A) = M_{\rm loc}(K(\Omega))$ *holds and the following inclusions (as C*-subalgebras) occur:*

$$M(A) \subseteq M(K(\Omega)) = B(\Omega) \subseteq M_{\rm loc}(K(\Omega)) \subseteq M_{\rm loc}\left(M_{\rm loc}(K(\Omega))\right) = B(\Omega_{\rm wk}) . \tag{4}$$

In particular, $M_{\rm loc}\left(M_{\rm loc}(A)\right) = I(A)$.

Ara and Mathieu have presented examples of Stonean spaces Δ and trivial Hilbert bundles Ω for which the inclusion $M_{\rm loc}(K(\Omega)) \subseteq M_{\rm loc}\left(M_{\rm loc}(K(\Omega))\right)$ in (4) is proper [3, Theorem 6.13]. As a consequence of Theorem 4.1 we see that this gap can not occur for higher local multiplier algebras, i.e. for all $k \geq 2$, $M_{\rm loc}^{k+1}(K(\Omega)) = M_{\rm loc}^{k}(K(\Omega))$ — where $M_{\rm loc}^{k+1}(K(\Omega)) = M_{\rm loc}(M_{\rm loc}^{k}(K(\Omega)))$ for $k \geq 1$.

The proof of Theorem 4.1 is achieved through a number of lemmas.

Lemma 4.2. *The set*

$$F = \{\sum_{j=1}^{n} \Theta_{\omega_j,\omega_j} \,:\, n \in \mathbb{N},\ \omega_j \in \Omega\}$$

is dense in the positive cone of $K(\Omega)$.

Proof. Assume that $h \in K(\Omega)_+$ and let $\varepsilon > 0$ be arbitrary. For each $s_0 \in \Delta$ consider the positive compact operator $h(s_0) \in K(H_{s_0})$. Then there are vectors $\xi_1, \dots, \xi_{n_{s_0}} \in H_{s_0}$ such that

$$\|h(s_0) - \sum_{j=1}^{n_{s_0}} \xi_j \otimes \xi_j\| < \varepsilon .$$

Using (II) in Definition 1.1, choose $\omega_1, \dots, \omega_{n_{s_0}} \in \Omega$ such that $\omega_j(s_0) = \xi_j$, $1 \leq j \leq n_{s_0}$, and let $\kappa_{s_0} = \sum_{j=1}^{n_{s_0}} \Theta_{\omega_j,\omega_j}$. By continuity of the operator fields in A, there is an open set $U_{s_0} \subset \Delta$ containing s_0 such that $\|h(s) - \kappa_{s_0}(s)\| < \varepsilon$ for all $s \in U_{s_0}$.

This procedure leads to an open cover $\{U_s\}_{s\in\Delta}$ of Δ, from which (by compactness) there exists a finite subcover $\{U_1, \dots, U_m\}$ and corresponding fields $\kappa_i = \sum_{j=1}^{n_i} \Theta_{\omega_j^{[i]},\omega_j^{[i]}}$. Let $\{\psi_1, \dots, \psi_m\} \subset C(\Delta)$ be a partition of unity subordinate to $\{U_1, \dots, U_m\}$ and note that the equality $\psi_i \cdot \Theta_{\omega_j^{[i]},\omega_j^{[i]}} = \Theta_{\psi_i^{1/2}\cdot\omega_j^{[i]},\psi_i^{1/2}\cdot\omega_j^{[i]}}$ holds for all j and i. Hence, the field $\kappa = \sum_{i=1}^{m} \psi_i \cdot \kappa_i$ is in F, and for each $s \in \Delta$,

$$\|h(s) - \kappa(s)\| = \|\sum_{i=1}^{m} \psi_i \cdot (h - \kappa_i)(s)\| \leq \sum_{i=1}^{m} \psi_i(s)\|(h - \kappa_i)(s)\| < \varepsilon .$$

Hence, h is in the norm-closure of F. □

Lemma 4.3. *Let* $\{U_i\}_{i\in\Lambda}$ *be a family of pairwise disjoint clopen subsets of* Δ *whose union* U *is dense in* Δ*, and let* $c_i = \chi_{U_i} \in C(\Delta)$*, for each* $i \in \Lambda$*. Suppose that* $\{\omega_i\}_{i\in\Lambda}$ *is any bounded family in* Ω *and let* $\tilde\omega = \sum_{i\in\Lambda} c_i\, \omega_i \in \Omega_{\rm wk}$*, in the sense of Remark 2.5. If* $f \in C(\Delta)$ *is such that* $f(s) = 0$ *for* $s \in \Delta \setminus U$*, then* $f \cdot \tilde\omega \in \Omega$.

Proof. Fix $s_0 \in \Delta$ and let $\varepsilon > 0$. If $s_0 \in \Delta \setminus U$, then by the continuity of f and the fact that $f(s_0) = 0$ there exists an open subset $U_{s_0} \subset \Delta$ containing s_0 such that $|f(s)| < \varepsilon\|\tilde\omega\|^{-1}$ for all $s \in U_{s_0}$. Hence, the vector field $f \cdot \tilde\omega$ is within ε of the zero vector field $0 \in \Omega$ on the open set U_{s_0}.

On the other hand, if $s_0 \in U$, then there exists $j \in \Lambda$ such that $s_0 \in U_j$. By construction, $c_j \cdot \tilde\omega = c_j \cdot \omega_j$ and so $\tilde\omega(s) = \omega_j(s)$ for all $s \in U_j$. Because $\|(f \cdot \tilde\omega)(s) - (f \cdot \omega_j)(s)\| = 0$ for all $s \in U_j$, the vector field $f \cdot \tilde\omega$ is within ε of the vector field $f \cdot \omega_j \in \Omega$ on the open set U_j. Thus, by local uniform approximation property (axiom (IV) in Definition 1.1), $f \cdot \tilde\omega \in \Omega$. □

The fact that $\Omega^\perp = \{0\}$ in $\Omega_{\rm wk}$ (Lemma 3.4) suggests that Ω is somehow dense in $\Omega_{\rm wk}$. The next proposition makes this relation more explicit.

Proposition 4.4. *If* $\nu \in \Omega_{\rm wk}$ *and* $\varepsilon > 0$*, then there exist a family* $\{c_i\}_{i\in\Lambda}$ *of pairwise orthogonal projections in* $C(\Delta)$ *with supremum* 1 *and a bounded family* $\{\omega_i\}_{i\in\Lambda} \subset \Omega$ *such that* $\|\nu - \sum_{i\in\Lambda} c_i \cdot \omega_i\| < \varepsilon$.

Proof. By Lemma 2.2, the function $s \mapsto \|\nu(s)\|$ is lower semicontinuous; hence, there exists a meagre set M_ν such that the function $s \mapsto \|\nu(s)\|$ is continuous in the relative topology of $\Delta \setminus M_\nu$. Observe that $\overline{(\Delta \setminus M_\nu)} = \Delta$.

Fix $s_0 \in \Delta \setminus M_\nu$ and let $\omega \in \Omega$ be such that $\omega(s_0) = \nu(s_0)$. Since

$$\|\nu(s) - \omega(s)\|^2 \; = \; \|\nu(s)\|^2 + \|\omega(s)\|^2 - 2\mathrm{Re}\,\langle \nu, \omega\rangle(s)\,,$$

the continuity in the relative topology of $\Delta \setminus M_\nu$ guarantees the existence of an open subset U_{s_0} of Δ containing s_0 such that $\|\nu(s) - \omega(s)\| \leq \varepsilon/2$ for all $s \in (\Delta \setminus M_\nu) \cap U_{s_0}$. Hence, again by continuity we get that $\|\nu - \omega\|(s) < \varepsilon$ for all $s \in \overline{U}_{s_0}$. The set $\overline{U}_{s_0}$ is a clopen subset of Δ and $\Delta' = \Delta \setminus \overline{U}_{s_0}$ is also a Stonean space. Further, $M_\nu \cap \Delta' = M_\nu \cap (\Delta \setminus \overline{U}_{s_0})$ is a meagre set such that the function $s \mapsto \|\nu(s)\|$, for $s \in \Delta' \setminus (M_\nu \cap \Delta')$, is continuous in the relative topology.

An application of Zorn's Lemma yields a maximal family $\{(\chi_{U_i}, \omega_i)\}_{i\in\Lambda}$ such that $U_i \cap U_j = \emptyset$ for $i \neq j$ and such that $\|\chi_{U_i}(\nu - \omega_i)\| < \varepsilon$. Maximality ensures that $\overline{(\cup_{i\in I} U_i)} = \Delta$, for otherwise we can enlarge this family by the previous procedure in the Stonean space $\Delta \setminus \overline{(\cup_{i\in\Lambda} U_i)}$. If we let $c_i = \chi_{U_i}$ for $i \in \Lambda$ then it is clear by Lemma 2.2 that $\|\nu - \sum_{i\in\Lambda} c_i \cdot \omega_i\| < \varepsilon$ as for every $j \in \Lambda$ we have that $\|c_j(\nu - \sum_{i\in\Lambda} c_i \cdot \omega_i)\| = \|c_j(\nu - \omega_j)\| \leq \varepsilon$ and $\bigvee_{i\in\Lambda} c_i = 1$. □

The next result is the key step in the proof of Theorem 4.1.

Proposition 4.5. *For every abelian projection $e \in B(\Omega_{\mathrm{wk}})$ and $\varepsilon > 0$ there is an essential ideal $I \subseteq K(\Omega)$ and a multiplier $x \in M(I)$ such that $\|e - x\| < \varepsilon$.*

Proof. Assume that $e \in B(\Omega_{\mathrm{wk}})$ is an abelian projection and let $\varepsilon > 0$. Thus, by [16, Lemma 13], $e = \Theta_{\nu,\nu}$ for some $\nu \in \Omega_{\mathrm{wk}}$ for which $\langle \nu, \nu\rangle$ is a projection of $C(\Delta)$. By Proposition 4.4, there is a family $\{c_i\}_{i\in\Lambda}$ of pairwise orthogonal projections in $C(\Delta)$ with supremum 1 and a bounded family $\{\omega_j\}_{j\in\Lambda} \subset \Omega$ such that $\|\nu - \tilde\omega\| \leq \varepsilon/(2\|\nu\|)$, where $\tilde\omega = \sum_{j\in\Lambda} c_j \cdot \omega_j \in \Omega_{\mathrm{wk}}$. Each c_j is the characteristic function of a clopen set U_j and the union U of these sets U_j is dense in Δ.

Let $I = \{a \in K(\Omega) \,:\, a(s) = 0, \forall\, s \in \Delta \setminus U\}$, which is an essential ideal of $K(\Omega)$. Define $F^I \subset F \subset K(\Omega)_+$ to be the set

$$F^I \; = \; \{\sum_{i=1}^{n} \Theta_{\mu_i,\mu_i} \,:\, n \in \mathbb{N},\; \mu_i \in \Omega,\; \mu_i|_{\,\Delta\setminus U} = 0,\; i = 1, \ldots, n\}\,.$$

Suppose that $\eta \in \Omega$ satisfies $\|\eta(s)\| = 0$ for all $s \in \Delta \setminus U$, and consider $\Theta_{\eta,\eta} \in F^I$. Observe that $\Theta_{\tilde\omega,\tilde\omega}\,\Theta_{\eta,\eta} \;=\; \Theta_{\langle \eta, \tilde\omega\rangle\cdot\tilde\omega,\,\eta}$, which is an element of I because $\langle \eta, \tilde\omega\rangle(s) = \langle \eta(s), \tilde\omega(s)\rangle = 0$ for all $s \in \Delta \setminus U$ and $\langle \eta, \tilde\omega\rangle \cdot \tilde\omega \in \Omega$ by Lemma 4.3. Hence, $\Theta_{\tilde\omega,\tilde\omega}$ maps the set F^I back into I. Because F^I is dense in I_+, as we shall show below, $x = \Theta_{\tilde\omega,\tilde\omega}$ is therefore a multiplier of I. Furthermore,

$$\|e - x\| = \|\Theta_{\nu,\nu} - \Theta_{\tilde\omega,\tilde\omega}\| \leq (\|\nu\| + \|\tilde\omega\|)\,\|\nu - \tilde\omega\| \leq \varepsilon.$$

It remains to show that F^I is dense in I_+. To this end, assume $\varepsilon' > 0$ and $\kappa \in I_+$. Thus, $\kappa(s) = 0$ for all $s \in \Delta \setminus U$. Furthermore, by Lemma 4.2, there exists $h \in F$ such that $\|\kappa - h\| < \varepsilon'$. Let $\tilde h = \chi_{\Delta\setminus U} \cdot h$ and note that, as $\kappa \in I$, it is also true that $\|\kappa - \tilde h\| < \varepsilon'$. Now if h has the form $\sum_{j=1}^{n} \Theta_{\mu_j,\mu_j}$ for some $\mu_j \in \Omega$, then $\tilde h = \sum_{j=1}^{n} \Theta_{\chi_{\Delta\setminus U}\mu_j,\chi_{\Delta\setminus U}\mu_j} \in F^I$. □

Proof of Theorem 4.1. Because $K(\Omega)$ is an ideal of A, we have $M(A) \subseteq M(K(\Omega))$. Moreover, as $K(\Omega)$ is an essential ideal of A we conclude that $M_{\mathrm{loc}}(A) = M_{\mathrm{loc}}(K(\Omega))$ [2, Proposition 2.3.6]. On the other hand, by [11, Theorem 4.6], the inclusions

$$B(\Omega) = M(K(\Omega)) \;\subseteq\; M_{\mathrm{loc}}(K(\Omega)) \;\subseteq\; M_{\mathrm{loc}}\left(M_{\mathrm{loc}}(K(\Omega))\right) \subset B(\Omega_{\mathrm{wk}})$$

hold.

Therefore, we are left to show that $M_{\mathrm{loc}}\left(M_{\mathrm{loc}}(K(\Omega))\right) = B(\Omega_{\mathrm{wk}})$. By [11, Corollary 4.3], an element $z \in I(K(\Omega)) = B(\Omega_{\mathrm{wk}})$ belongs to $M_{\mathrm{loc}}(K(\Omega))$ if and only if for every $\varepsilon > 0$ there is an essential ideal $I \subseteq K(\Omega)$ and a multiplier $x \in M(I)$ such that $\|z - x\| < \varepsilon$. By Proposition 3.8, $K(\Omega_{\mathrm{wk}})$ is the (essential) ideal of $B(\Omega_{\mathrm{wk}})$ generated by the abelian projections of $B(\Omega_{\mathrm{wk}})$; thus, by Proposition 4.5, $K(\Omega_{\mathrm{wk}}) \subseteq M_{\mathrm{loc}}(K(\Omega))$. Hence, $K(\Omega_{\mathrm{wk}})$ is an essential ideal of $M_{\mathrm{loc}}(K(\Omega))$ and so $M(K(\Omega_{\mathrm{wk}})) \subseteq M_{\mathrm{loc}}\left(M_{\mathrm{loc}}(K(\Omega))\right)$. However, $B(\Omega_{\mathrm{wk}}) = M(K(\Omega_{\mathrm{wk}}))$ by Kasparov's Theorem [17, Theorem 2.4] (or by a theorem of Pedersen [20]); hence,

$$B(\Omega_{\mathrm{wk}}) = M\left(K(\Omega_{\mathrm{wk}})\right) \subseteq M_{\mathrm{loc}}\left(M_{\mathrm{loc}}(K(\Omega))\right) \;\subseteq\; B(\Omega_{\mathrm{wk}})\,,$$

which yields $M_{\rm loc}\,(M_{\rm loc}(K(\Omega))) = B(\Omega_{\rm wk})$. □

Somerset has shown that every separable postliminal (that is, type I) C*-algebra A has the property that $M_{\rm loc}(M_{\rm loc}(A)) = I(A)$ [22, Theorem 2.8]. Theorem 4.1 demonstrates that the same behavior occurs with (certain) nonseparable type I C*-algebras. Somerset's methods are different from ours in at least two ways: he employs the Baire ∗-envelope of a C*-algebra where we use the injective envelope and he uses properties of Polish spaces—spaces that arise from the separability of the algebras under study. It is reasonable to conjecture that $M_{\rm loc}(M_{\rm loc}(A)) = I(A)$ for all C*-algebras A that possess a postliminal essential ideal. To prove such a statement, it would be enough to prove it for any continuous trace C*-algebra A.

5. Effect of Decomposition into Homogeneous Modules

A Kaplansky–Hilbert module E over $C(\Delta)$ is said to be *homogeneous* [16] if if there is a subset $\{\nu_j\}_{j\in\Lambda} \subset E$ – called an *orthonormal basis* – such that $\langle \nu_i, \nu_j\rangle = 0$ for all $j \neq i$, $|\nu_j| = 1$ for all j, and $\{\nu_j\}_{j\in\Lambda}^{\perp} = \{0\}$, where for any $\nu \in E$, $|\nu|$ is the continuous real-valued function $|\nu| = \langle \nu, \nu\rangle^{1/2} \in C(\Delta)$.

Kaplansky introduced the notion of homogeneous AW*-module with the aim of reducing the study of abstract AW*-modules to the slightly more concrete setting in which the modules have an orthonormal basis. The decomposition of $\Omega_{\rm wk}$ into a direct sum of homogeneous modules affects C*-algebras of endomorphisms of $\Omega_{\rm wk}$ in different ways. In this section we show that a decomposition of $\Omega_{\rm wk}$ into a direct sum $\oplus_i E_i$ of homogeneous modules E_i leads one to consider two corresponding direct sum C*-algebras: $\oplus_i A_i$ and $\oplus_i M_{\rm loc}(A_i)$, where A_i is a subalgebra of A for all i. We prove that A need not be isomorphic to $\oplus_i A_i$, yet $M_{\rm loc}(A) \cong \oplus_i M_{\rm loc}(A_i)$. The latter result is especially interesting if one recalls that $M_{\rm loc}(A)$ is generally not an AW*-algebra [3, Theorem 6.13].

Theorem 5.1. *Let $(\Delta, \{H_s\}_{s\in\Delta}, \Omega)$ be a continuous Hilbert bundle over the Stonean space Δ. There exists a family of pairwise orthogonal projections $\{c_i\}_{i\in I} \subset C(\Delta)$ with supremum 1 such that $c_i\Omega_{\rm wk}$ is a homogeneous AW*-module over $c_iC(\Delta)$, for each $i \in I$. Furthermore, for each $i \in I$ let $c_i = \chi_{\Delta_i}$ for a clopen set Δ_i and let $\Omega_i = \{\omega|_{\Delta_i} : \omega \in \Omega\}$. Then:*

- **(i)** *$(\Delta_i, \{H_s\}_{s\in\Delta_i}, \Omega_i)$ is a continuous Hilbert bundle;*
- **(ii)** *$(\Omega_i)_{\rm wk} \cong c_i \cdot \Omega_{\rm wk}$ as C*-modules;*
- **(iii)** *$\Omega_{\rm wk} \cong \bigoplus_i (\Omega_i)_{\rm wk}$ as C*-modules;*
- **(iv)** *$B((\Omega_i)_{\rm wk}) \cong c_i \cdot B(\Omega_{\rm wk})$ as C*-algebras;*
- **(v)** *$B(\Omega_{\rm wk}) \cong \bigoplus_i B((\Omega_i)_{\rm wk})$ as C*-algebras.*

Proof. Let $B = B(\Omega_{\rm wk})$. By [16, Theorem 1], there is a family $\{c_i = \chi_{\Delta_i}\}_{i\in I} \subset C(\Delta)$ of pairwise orthogonal projections such that $1 = \sup_i\, c_i$ and $c_i\Omega_{\rm wk}$ is a homogeneous AW*-module over $c_iC(\Delta)$. Hence, the corresponding family of clopen subsets $\{\Delta_i\}_{i\in I}$ is pairwise disjoint and such that $\bigcup_{i\in I} \Delta_i$ is dense in Δ. Each Δ_i is itself a Stonean space, and it is easy to see that $C(\Delta_i) \cong c_i\, C(\Delta)$
(i) For axiom (I) in Definition 1.1, we aim to show that Ω_i is a $C(\Delta_i)$ module. Let $\omega \in \Omega$ and consider $\omega_i = \omega|_{\Delta_i}$. Choose any $f_i \in C(\Delta_i)$. As Δ_i is clopen, f_i can be extended to $F_i \in C(\Delta)$ such that $f_i = F_i|_{\Delta_i}$, and $F_i|_{\Delta\setminus\Delta_i} = 0$. The action $f_i \cdot \omega_i = (F_i \cdot \omega)|_{\Delta_i}$ gives Ω_i the structure of a $C(\Delta_i)$ module. Axioms (II) and (III) of Definition 1.1 are trivially satisfied.

For axiom (IV), let $\xi : \Delta_i \to \bigsqcup_{s\in\Delta_i} H_s$ be a vector field such that for every $s_0 \in \Delta_i$ and $\varepsilon > 0$ there is an open set $U_i \subset \Delta_i$ containing s_0 and a $\omega_i \in \Omega_i$ with $\|\omega_i(s) - \xi(s)\| < \varepsilon$ for all $s \in U_i$. Let $\Xi : \Delta_i \to \bigsqcup_{s\in\Delta} H_s$ be the vector field that coincides with ξ on Δ_i and is identically zero off Δ_i. By the definition of Ω_i, there is $\omega \in \Omega$ such that $\omega_i = \omega|_{\Delta_i}$. The set U_i is also open in Δ, and $\|\omega(s) - \Xi(s)\| < \varepsilon$ for all $s \in U_i$. If $s_0 \notin \Delta_i$ choose any open set V_i containing s_0 such that $V_i \cap U_i = \emptyset$ and let $\omega \in \Omega$ be arbitrary; then $0 = \|\chi_{\Delta_i}(s)\omega(s) - \Xi(s)\| < \varepsilon$ for all $s \in V_i$. Since $\chi_{\Delta_i} \cdot \omega \in \Omega$ and since Ω is closed under local uniform approximation, $\Xi \in \Omega$, whence $\xi \in \Omega_i$.
(ii) Let $T_i : c_i\,\Omega_{\rm wk} \to (\Omega_i)_{\rm wk}$ be given by $T_i(c_i\nu) = \nu|_{\Delta_i}$. It is clear that T_i is well defined, linear, bounded, and has trivial kernel; to show that it is onto, note that if $\nu_i \in (\Omega_i)_{\rm wk}$, then—since Δ_i is clopen—the vector field $\nu : \Delta \to \bigsqcup_{s\in\Delta} H_s$ defined by $\nu(s) = 0$, for $s \notin \Delta_i$, and $\nu(s) = \nu_i(s)$, for $s \in \Delta_i$, has the property that $\langle \omega, \nu\rangle \in C(\Delta)$, for all $\omega \in \Omega$; so $\nu \in \Omega_{\rm wk}$ and $\nu_i = T_i(c_i\nu)$.
(iii) Let $T : \Omega_{\rm wk} \to \bigoplus_i (\Omega_i)_{\rm wk}$, given by $T\nu = (T_i(c_i\nu))_{i\in I}$. The previous paragraph and Lemma 2.1 show that T is an isometry; we show now that T is onto. Suppose that $\nu' = (\nu_i)_{i\in I} \in \bigoplus_i (\Omega_i)_{\rm wk}$.

For each $i \in I$ let $\tilde{\nu}_i$ denote the vector field on Δ that coincides with ν_i on Δ_i and vanishes elsewhere. Then $\tilde{\nu}_i \in \Omega_{\rm wk}$ and $T_i(c_i\tilde{\nu}_i) = \nu_i$. Hence, if $\nu = \sum_i c_i\tilde{\nu}_i$ as in Remark 2.5, we have $\nu \in \Omega_{\rm wk}$ and $T\nu = \nu'$. Thus, $\Omega_{\rm wk}$ and $\bigoplus_i(\Omega_i)_{\rm wk}$ are isomorphic Banach spaces. Similar arguments show that $\bigoplus_i(\Omega_i)_{\rm wk}$ is a $C(\Delta)$-module and that T is module isomorphism. Hence, $\Omega_{\rm wk} \cong \bigoplus_i(\Omega_i)_{\rm wk}$ as C*-modules.
(iv) Let $\rho_i : c_i\, B(\Omega_{\rm wk}) \to B((\Omega_i)_{\rm wk})$ be given by $\rho_i(c_ib)\,T_i(c_i\nu) = (b\nu)|_{\Delta_i}$. The map is well-defined because if $c_ib_1 = c_ib_2$ then for any $\nu \in \Omega_{\rm wk}$ we have $(b_1\nu)|_{\Delta_i} = (c_ib_1\nu)|_{\Delta_i} = (c_ib_2\nu)|_{\Delta_i} = (b_2\nu)|_{\Delta_i}$. A similar computation shows that ρ_i is one-to-one, and linearity is clear. To see that ρ_i is onto, let $b_i \in B((\Omega_i)_{\rm wk})$. Consider the injection $\tilde{}: (\Omega_i)_{\rm wk} \to (\Omega_{\rm wk}$ where $\tilde{\nu_i} \in \Omega_{\rm wk}$ is the vector field that agrees with ν_i on Δ_i and is 0 elsewhere. Let $b \in B(\Omega_{\rm wk})$ be the operator given by $b\nu = \widetilde{b_i(\nu|_{\Delta_i})}$. Then $\rho_i(c_ib)(T_ic_i\nu) = (b\nu)|_{\Delta_i} = \widetilde{b_i(\nu|_{\Delta_i})}|_{\Delta_i} = b_i(\nu|_{\Delta_i}) = b_i\,(T_ic_i\nu)$, so $\rho_i(c_ib) = b_i$.
(v) Let $\rho : B(\Omega_{\rm wk}) \to \bigoplus_i B((\Omega_i)_{\rm wk})$ be the map $\rho(b) = (\rho_i(c_ib))_{i\in I}$. It is clear that ρ is a homomorphism. If $\rho(b) = 0$ for some $b \in B(\Omega_{\rm wk})$, then – as each ρ_i is one-to-one – $c_ib = 0$ for all i; this implies that $b^*b = b^*(\sup_i(c_i \cdot I))b = \sup_i(b^*c_ib) = 0$ by [14, Corollary 4.10], so $b = 0$ and ρ is one-to-one. To show that ρ is onto, let $(b_i)_i \in \bigoplus_i B((\Omega_i)_{\rm wk})$; as each ρ_i is onto, there exist operators $b^i \in B(\Omega_{\rm wk})$ with $\rho_i(c_ib^i) = b_i$. Define $b \in B(\Omega_{\rm wk})$ by $b\nu = \sum_i c_ib^i\nu$ (in the sense of Remark 2.5; that is, $c_ib\nu = c_ib^i\nu$). Then $\rho_i(c_ib)\nu|_{\Delta_i} = (c_ib\nu)|_{\Delta_i} = (c_ib^i\nu)|_{\Delta_i} = \rho_i(c_ib^i)\nu|_{\Delta_i} = b_i\nu|_{\Delta_i}$. So $\rho(b) = (b_i)_i$. □

Proposition 5.2. *Assume the notation, hypotheses, and conclusions of Theorem 5.1. Then, although $\Omega_{\rm wk} \cong \bigoplus_i(\Omega_i)_{\rm wk}$ canonically, the same is not necessarily true for Ω and $\bigoplus_i \Omega_i$. In particular, Ω can be properly contained in $\Omega_{\rm wk}$.*

Proof. Take Δ and the family of clopen subsets $\{\Delta_i\}_{i\in I}$ in Theorem 5.1 to be such that $\bigcup_{i\in I}\Delta_i \neq \Delta$. Thus, I is an infinite set. Let H be a Hilbert space with orthonormal basis $\{e_i\}_{i\in I}$ and consider the trivial Hilbert bundle $\Omega = C(\Delta, H)$ of all continuous functions $\omega : \Delta \to H$. As in Theorem 5.1, let $\Omega_i = C(\Delta_i, H)$.

For each $i \in I$, set $\omega_i \in \Omega$ with $\omega_i(s) = e_i$ for all s and consider $(\omega_i)_{i\in I} \in \bigoplus_i \Omega_i$. Under the isomorphism of Theorem 5.1, this element $(\omega_i)_{i\in I}$ is identified with $\omega = \sum_{i\in I}\chi_{\Delta_i}\cdot\tilde{\omega}_i \in \Omega_{\rm wk}$ (in the sense of Remark 2.5), where $\tilde{\omega}_i$ is any element of Ω that agrees with ω_i on Δ_i and vanishes off Δ_i. Under this identification, $\omega \notin \Omega$; that is, the function $s \mapsto \|\omega(s)\|$ fails to be continuous on Δ. We argue this by contradiction.

Assume that $s \mapsto \|\omega(s)\|$ is continuous on Δ. Because $\|\omega(s)\| = 1$ for all $s \in \cup_{i\in I}\Delta_i$, continuity implies that $\|\omega(s)\| = 1$ for $s \in \Delta$. Choose $s_0 \in \Delta \setminus (\cup_{i\in I}\Delta_i)$ and let $(s_\alpha)_{\alpha\in\Lambda} \subset \cup_{i\in I}\Delta_i$ be a net such that $s_\alpha \to s_0$. Let $\eta \in \Omega$ be the constant field $\eta(s) = \omega(s_0)$, for all $s \in \Delta$. Since $\omega \in \Omega_{\rm wk}$, we have

$$\lim_\alpha\, \langle\omega(s_\alpha), \eta(s_\alpha)\rangle \;=\; \langle\omega(s_0), \eta(s_0)\rangle \;=\; \langle\omega(s_0), \omega(s_0)\rangle \;=\; 1\,. \tag{5}$$

For each $\alpha \in \Lambda$ let $i(\alpha) \in I$ be such that $s_\alpha \in \Delta_{i(\alpha)}$. Thus, for every $\alpha \in \Lambda$, $I_\alpha = \{i(\beta) :\ \beta \in I,\ \beta \geq \alpha\}$ is an infinite set (for otherwise $s_0 \in \Delta_i$ for some $i \in I$). Therefore,

$$\lim_\alpha\, \langle\omega(s_\alpha), \eta(s_\alpha)\rangle \;=\; \lim_\alpha\, \langle e_{i(\alpha)}, \omega(s_0)\rangle \;=\; 0\,. \tag{6}$$

As (5) and (6) cannot be true simultaneously, we obtain a contradiction. Hence, $\omega \notin \Omega$. □

Our second reduction theorem below notes some consequences of Theorem 5.1.

Theorem 5.3. *Let $(\Delta, \{H_t\}_{t\in\Delta}, \Omega)$ be a continuous Hilbert bundle over the Stonean space Δ and let A denote the associated Fell algebra. Let $\{c_i\}_{i\in I} \subset C(\Delta)$ be a family of pairwise orthogonal projections with supremum 1 such that $c_i\Omega_{\rm wk}$ is a homogeneous AW*-module over $c_iC(\Delta)$, for each $i \in I$. Furthermore, for each $i \in I$ let $c_i = \chi_{\Delta_i}$ for a clopen set Δ_i and let $\Omega_i = \{\omega|_{\Delta_i} : \omega \in \Omega\}$. Then:*

- **(i)** *if A_i denotes the Fell algebra of $(\Delta_i, \{H_s\}_{s\in\Delta_i}, \Omega_i)$, then $A_i \cong c_i \cdot A$;*
- **(ii)** *$I(A_i) = B((\Omega_i)_{\rm wk})$;*
- **(iii)** *$I(A) \cong \bigoplus_{i\in I} I(A_i)$;*
- **(iv)** *$M_{\rm loc}(A) \cong \bigoplus_{i\in I} M_{\rm loc}(A_i)$.*

Proof. Let A_i denote the C*-algebra of the continuous C*-bundle $(\Delta_i, \{H_s\}_{s\in\Delta_i}, \Gamma_i)$, where Γ_i consists of all weakly continuous almost finite-dimensional operator fields $a_i : \Delta_i \to \bigsqcup_{s\in\Delta_i} K(H_s)$ such that $s \mapsto \|a_i(s)\|$ is continuous, and let $B_i = B((\Omega_i)_{\rm wk})$, a type I AW*-algebra with centre $Z(B_i) \cong C(\Delta_i)$.

(**i**) For each $a_i \in \Gamma_i$ there is an $a \in \Gamma$ such that $a_i = a|_{\Delta_i}$. To verify this, let $a : \Delta_i \to \bigsqcup_{s\in\Delta} K(H_s)$ be the operator field defined by $a(s) = a_i(s)$, for $s \in \Delta_i$, and $a(s) = 0$, for $s \notin \Delta_i$. Since Δ_i is a clopen set, the maps $s \to \|a(s)\|$ and $s \mapsto \langle a(s)\omega_1(s), \omega_2(s)\rangle$ are continuous for every $\omega_1, \omega_2 \in \Omega$. The operator field a is also locally finite-dimensional, again because Δ_i is clopen and a_i has the property on Δ_i. Hence, $a \in \Gamma$. Next, let $\pi_i : A_i \to c_iA$ be defined by $\pi_i(a_i) = c_ia$, where $a \in A$ is any operator field that restricts to a_i on Δ_i. This map is clearly well-defined, and a $*$-homomorphism.

(**ii**) By Theorem 3.1, $B((\Omega_i)_{\rm wk}) = I(A_i) = I(c_iA)$.

(**iii**) By [14, Lemma 6.2], $I(c_iA) = c_iI(A)$. Hence, $I(A_i) = B_i$ and Theorem 5.1 immediately yields $I(A) \cong \bigoplus_{i\in I} I(A_i)$.

(**iv**) We take each $M_{\rm loc}(A_i)$ to be a C*-subalgebra of $B((\Omega_i)_{\rm wk})$. First we remark that the isomorphism ρ from Theorem 5.1 sends A into $\bigoplus_i A_i$. To see why, recall that $a\nu(s) = a(s)\nu(s)$, for all $a \in A$, $\nu \in \Omega_{\rm wk}$, and $s \in \Delta$ (Proposition 3.6. Since, for a given $i \in I$, the action of $\rho_i(a)$ on $\nu_i \in (\Omega_i)_{\rm wk}$ is defined by $\nu_i \mapsto (a\nu)|_{\Delta_i}$, where $\nu \in \Omega_{\rm wk}$ is any vector with $nu|_{\Delta_i} = \nu_i$, it is easy to verify that $\rho_i(a)$ is a weakly continuous almost finite-dimensional operator field on Δ_i.

To show that $\rho\left(M_{\rm loc}(A)\right) \subseteq \bigoplus_i M_{\rm loc}(A_i)$, let $y \in M_{\rm loc}(A) \subset I(A)$ and suppose that $\varepsilon > 0$. Thus, there is an essential ideal $J \subseteq A$ and a multiplier $x \in M(J)$ such that $\|x - y\| < \varepsilon$. Further, there exists an open dense subset $U \subset \Delta$ such that

$$J \;=\; \{a \in A : a(s) = 0\,,\; s \in \Delta \setminus U\}\,. \tag{7}$$

For $i \in I$, let $U_i = \Delta_i \cap U$, which is an open dense set in Δ_i. Therefore,

$$J_i \;=\; \{a_i \in A_i :\, a(s) = 0,\; s \in \Delta_i \setminus U_i\} \tag{8}$$

is an essential ideal in A_i. We aim to show that $\rho_i(y) \in M(J_i)$. To this end, select $a_i \in J_i$. As $A_i \cong c_i \cdot A$, there is an $a \in A$ such that $a_i(s) = a(s)$ for all $s \in \Delta_i$. In particular $a \in A$ can be chosen so that $a(s) = 0$ for all $s \in \Delta \setminus \Delta_i$. Because $a_i \in J_i$, we conclude that $a(s) = 0$ for all $s \in \Delta \setminus U$; that is, $a \in J$. Therefore, $ya \in J$, which implies that $ya(s) = 0$ for all $s \in \Delta \setminus U$. In particular, $ya(s) = 0$ for all $s \in \Delta_i \setminus U_i$. The element $\rho_i(y)a_i \in B((\Omega_i)_{\rm wk})$ is in fact an operator field since $\rho_i(y)a_i = \rho_i(y)\rho_i(c_ia) = \rho_i(c_i(ya)) \in A_i$. Then, for all $s \in \Delta_i \setminus U_i$ and $\nu \in \Omega_{\rm wk}$, $[\rho_i(y)a_i](s)(T_ic_i\nu)(s) = \rho_i(y)a_i(T_ic_i\nu)(s)\rho_i(c_iya)(T_ic_i\nu)(s) = (ya)\nu|_{\Delta_i}(s) = (ya)(s)\nu|_{\Delta_i}(s) = 0$. With ν being arbitrary, we conclude that $\rho_i(y)a_i(s) = 0$, that is $\rho_i(y)a_i \in J_i$, and so $\rho_i(y)$ is a left multiplier of J_i. By a similar argument, $\rho_i(y)$ is a right multiplier of J_i, and so $\rho_i \in M(J_i)$. Thus, $\rho(y) \in \bigoplus_i M_{\rm loc}(A_i)$ and $\|\rho(x) - \rho(y)\| = \|x - y\| < \varepsilon$. As $\varepsilon > 0$ was chosen arbitrarily, this proves that $\rho(x) \in \bigoplus_i M_{\rm loc}(A_i)$.

Conversely, assume that $(x_i)_i \in \bigoplus_i M_{\rm loc}(A_i)$. For each $i \in I$, there exist an essential ideal $J_i \subset A_i$ and a $y_i \in M(J_i)$ such that $\|x_i - y_i\| < \varepsilon$ for all $i \in I$. For each $i \in I$, there exists an open dense subset $U_i \subset \Delta_i$ such that J_i is given as in (8). Define $U = \bigcup_{i\in I} U_i$, which is an open dense subset of Δ and let J be the essential ideal of A defined as in (7) (for our present choice of U). Let $y \in B(\Omega_{\rm wk})$ be such that $\rho(y) = (y_i)_i$. We now show that $y \in M(J)$.

For each $\omega \in \Omega$, we have that $y\omega \in \Omega_{\rm wk}$.

Claim: if $\omega \in \Omega$ is such that $\omega(s) = 0$ for all $s \in \Delta \setminus U$, then $y\omega \in \Omega$ and $y\omega(s) = 0$ for $s \in \Delta \setminus U$.

Consider the set $F = \{\Theta_{\omega,\omega} :\ \omega \in \Omega,\ \omega(s) = 0\ \text{ for }\ s \in \Delta \setminus U\}$, the linear span of which is dense in J_+ by Lemma 4.2. By the Claim, $y\Theta_{\omega,\omega} = \Theta_{y\omega,\omega} \in J$ for all $\omega \in \Omega$. So y is a left multiplier of J. Similarly, y is a right multiplier of J , which yields $y \in M(J)$. Hence, $(x_i)_{i\in I}$ is within ε of $\rho(y) \in \rho(M(J)) \subseteq \rho(M_{\rm loc}(A))$.

Now it remains to prove the claim. Assume that $\omega \in \Omega$ with $\omega(s) = 0$ for all $s \in \Delta \setminus U$. Let $i \in I$ and let $\omega_i = c_i\,\omega \in \Omega_i$ be the restriction of ω to Δ_i. Note that, for every $\eta_i \in \Omega_i$, $\Theta_{\omega_i,\eta_i} \in J_i$ and, hence, $y_i\Theta_{\omega_i,\eta_i} \;=\; \Theta_{y_i\omega_i,\eta_i} \in J_i$. It is also true that $y_i\omega_i \in \Omega_i$ by the following arguments. Suppose that $s_0 \in \Delta_i$ and let $\eta_i \in \Omega_i$ such that $\|\eta_i(s_0)\| = 1$. Choose a clopen subset $V_i \subset \Delta_i$ of s_0 for which $\|\eta_i(s)\| \geq 1/2$ for all $s \in V_i$ and define $f(s) \;=\; \chi_{V_i}(s)\|\eta_i(s)\|^{-2}$. Thus, $f \in C(\Delta_i)$

and so $f \cdot \eta_i \in \Omega_i$. Moreover, $y_i\Theta_{\omega_i,\eta_i}(f \cdot \eta_i) = \chi_{V_i} \cdot y_i(\omega_i) \in \Omega_i$. Thus, $y\,\omega_i$ is a local uniform limit of vectors fields in Ω_i and, hence, $y\,\omega_i \in \Omega_i$.

We now have, for all $\omega_i, \eta_i \in \Omega_i$, that $\Theta_{\omega_i,\,\eta_i} \in J_i$, $y_i\omega_i \in \Omega_i$, and $y_i\omega_i(s) = 0$ for all $s \in \Delta_i \setminus U_i$. Since $(y\omega)(s) = (y_i\,\omega_i)(s)$ for $s \in \Delta_i$, the lower semicontinuous function $s \mapsto \|(y\omega)(s)\|$ is continuous on $\bigcup_i \Delta_i$ and vanishes on $(\bigcup_i \Delta_i) \setminus U$. The key fact at this point is the following one: there exists $C > 0$ such that, if $s \in \Delta_i$ (for some $i \in I$), then $\|y\omega(s)\| \leq C\,\|\omega(s)\|$. Indeed, let $\delta > 0$ and let $W_i \subset \Delta_i$ be a clopen subset containing $s_0 \in \Delta_i$ and such that $\|\omega(s)\| \leq \|\omega(s_0)\| + \delta$ for all $s \in W_i$. Thus,

$$\|y\omega(s)\| = \|y_i\omega_i(s)\| = \|y_i\omega_i\|(s) \leq \|y_i\|\,\|\chi_{W_i} \cdot \omega_i\| \;\leq\; (\sup_i \|y_i\|)\;(\|\omega(s_0)\| + \delta),$$

for all $s \in W_i$. We aim show that the function $s \mapsto \|(y\omega)(s)\|$ is continuous on Δ. Let $s \in \Delta\setminus(\bigcup_i \Delta_i)$ and let $(s_\alpha)_\alpha \subset \bigcup_i \Delta_i$ be a net such that $s_\alpha \to s_0$ in Δ. This implies that $\lim_\alpha \|\omega(s_\alpha)\| = 0$. By lower semicontinuity of the function $s \mapsto \|(y\omega)(s)\|$,

$$0 \;\leq\; \|y\omega(s_0)\| \;\leq\; \lim_\alpha \|y\omega(s_\alpha)\| \leq C\;\|\omega(s_\alpha)\| \;= 0\;,$$

and from this it follows that the function $s \mapsto \|(y\omega)(s)\|$ is continuous on Δ and vanishes in $\Delta \setminus U$. This establishes our claim. □

Local multiplier algebras behave well under direct sums: $M_{\rm loc}(\oplus_i A_i) \cong \oplus_i\, M_{\rm loc}(A_i)$ [2, Proposition 2.3.6]. However, the isomorphism of local multiplier algebras in Theorem 5.3 cannot be established via that generic result:

Proposition 5.4. *Assume the notation, hypotheses, and conclusions of Theorem 5.3. Although ρ sends A into $\bigoplus_i A_i$, it need not be true that $A \cong \bigoplus_i A_i$.*

Proof. If Δ and Ω are as in Proposition 5.2, then (using the notation established there) we have that $\rho(\Theta_{\omega,\omega}) = (\Theta_{\omega_i,\omega_i})_{i\in I} \in \oplus_{i\in I} A_i$, but $\rho(\Theta_{\omega,\omega}) \notin \rho(A)$. □

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