COMPLETION AND AMALGAMATION OF BOUNDED DISTRIBUTIVE QUASI LATTICES

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ABSTRACT. In this note we present a completion for the variety of bounded distributive quasi lattices, and, inspired by a well-known idea of L.L. Maksimova [14], we apply this result in proving the amalgamation property for such a class of algebras.

1. INTRODUCTION

One of the basic motivations for studying the completion of a certain structure is dued to the need of filling the gaps of the original one; a leading example could be turning a partial algebra into a total one. In the case of lattice ordered structures, the most relevant examples are represented by canonical extensions and Dedekind-MacNeille completions, see [9] for a wide exposition. Nevertheless, once we move from lattice ordered structures to quasi ordered ones, i.e. structures in which the ordering relation $\leq$ is reflexive and transitive, but it fails, in general, to be antisymmetric, the classical Dedekind-MacNeille approach does not work. This observation motivated us in investigating a generalization of the classical filter-based approach to the case of *bounded distributive quasi lattices* (*bdq-lattices*), introduced by I. Chajda in [3]. The main problem in the completion of bdq-lattices lies in the fact that it may happen for elements x, y in a bdq-lattice $\mathbf{L}$ that $x \leq y$, $y \leq x$ but $x \neq y$. In this case, the (homset reformulation of the) usual notion of lattice filter is no longer useful to distinguish x and y. More precisely, it is hopeless to try to find a homomorphism f from the dbq-lattice $\mathbf{L}$ to the two-element distributive lattice $\mathbf{2}$ such that $f(x) = 1$ and $f(y) = 0$. Upon noticing that quasi-lattices lack point-regularity, and therefore filter (ideal) determinacy [10], in this paper we use a particular system of congruences which allows to obtain, out of any bdq-lattice $\mathbf{L}$, a quasi ordered space of functions $\langle E(\mathbf{L}), \tau, \preceq \rangle$, where τ is a topology on $E(\mathbf{L})$ admitting as a quotient the Priestley topology [16]. In virtue of this construction, by considering a particular subset of τ-clopen sets, we can emebed the original bdq-lattice $\mathbf{L}$ into a (functionally) complete one. Finally, as an application of the previous results, we close the paper by proving, along the style of [14], the amalgamation property for the variety of bounded distributive quasi lattices.

2. Preliminaries

In this section we recall some basic facts about bounded distributive quasi lattices and Priestley's duality for bounded distributive lattices. We start with the definition of bounded distributive quasi lattice.

Definition 2.1. [3]
  A bounded distributive quasi lattice (*bdq-lattice*) *is an algebra* $\mathbf{A} = \langle A, \vee, \wedge, 0, 1 \rangle$ *of type* $\langle 2, 2, 0, 0 \rangle$ *satisfying the following conditions:*

  (1) $\langle A, \vee \rangle$ *and* $\langle A, \wedge \rangle$ *are semigroups;*
  (2) $x \vee (x \wedge y) = x \vee x;$ $x \wedge (x \vee y) = x \wedge x;$
  (3) $x \vee (y \vee y) = x \vee y;$ $x \wedge (y \wedge y) = x \wedge y;$
  (4) $x \wedge x = x \vee x;$
  (5) $x \vee (y \wedge z) = (x \vee y) \wedge (x \wedge z);$
  (6) $x \wedge 0 = 0;$ $x \vee 1 = 1.$

Let us remark that the variety of q-lattices, introduced by I. Chajda in [3], has been deeply investigated in [4, 7], and it plays a relevant role in the context of "normally presented varieties"[5]. Moreover, a general duality theory for the class of normally presented varieties has been studied in [6], by using methods introduced in [8].

We denote by $\mathbb{BDQL}$ the variety of bdq-lattices. Let $\mathbf{A}$ be a bdq-lattice. We say that an element $a \in A$ is *regular* iff $a \vee a = a$ (and by axiom (4), also $a \wedge a = a$). The set of all idempotents of A is called the *skeleton of* A, we will denote this set by $\mathcal{R}(A)$. It can be seen that $\mathcal{R}(A)$ is closed under the bdq-lattice operations. We shall indicate the algebra whose universe is the set $\mathcal{R}(A)$ by $\mathcal{R}(\mathbf{A})$. Let $X \subseteq A$. We will denote the set of regular elements of X by $\mathcal{R}(X)$.

It can be noticed that, in general, the equations $x \wedge 1 = x$ and $x \vee 0 = x$ are not satisfied in $\mathbb{BDQL}$. However, it holds that $x \wedge 1 = x \vee 0 = x \wedge x = x \vee x$. One may readily realize that, in every bdq-lattice $\mathbf{A}$, the bdq-lattice meet and join induce a *preorder* relation $\leq$, i.e. a reflexive and transitive binary relation on A. Furthermore, the binary relation $\equiv$ defined, for $x, y \in A$, as:

$$x \equiv y \text{ iff } x \leq y \text{ and } y \leq x \tag{2.1}$$

is a congruence on $\mathbf{A}$.

It can be verified that the quotient structure $\mathbf{A}/_{\equiv}$ is a bounded distributive lattice. Furthermore, it can be seen that $\mathbf{A}/_{\equiv}$ is isomorphic to the sublattice $\mathcal{R}(\mathbf{A})$ via the map $f(x/_{\equiv}) = x \vee 0$.

Accordingly with the usual definition for lattices, we shall say that a quasi lattice $\mathbf{A} = \langle A, \wedge, \vee, 0, 1 \rangle$ is *complete* if and only if the meet and the join of any $X \subseteq A$ exist.

Example 2.1. *The bdq-lattice* **3** *is the algebra* $\langle \{0, k, 1\}, \wedge, \vee, 0, 1 \rangle$ *where the subalgebra whose universe is* $\{0, 1\}$ *is the 2-element distributive lattice*

and $0 \vee k = 1.$

$$1 \text{——} k \tag{2.2}$$
$$\vert$$
$$0$$

Let $\mathbf{A}$ be a bdq-lattice. We recall that a nonempty subset F of $\mathbf{A}$ is a *filter* if (i) $x, y \in F$ implies $x \wedge y \in F$ and (ii) $x \in F$ and $x \leq y$ imply $y \in F$. The notion of *ideal* is defined dually. An ideal I is said to be *prime* if and only if for any $x, y \in A$, if $x \wedge y \in I$, then $x \in I$ or $y \in I$. Dually, a filter F is said to be *prime* if and only if for any $x, y \in A$, if $x \vee y \in F$, then $x \in F$ or $y \in F$. Let $X \subseteq A$. We shall denote by $(X]$ $([X))$ the *ideal generated* (*filter generated*) by X in A. The notions above lead to the following

Lemma 2.1. *Let* $\mathbf{A} = \langle A, \wedge, \vee, 0, 1 \rangle$ *be a bdq-lattice and* $X \subseteq A$. *Then:*

(1) *if* X *is an ideal (filter) then* $\mathcal{R}(X)$ *is an ideal (filter) in* $\mathcal{R}(A)$;
(2) *if* X *is a prime ideal (prime filter)then* $\mathcal{R}(X)$ *is a prime ideal (prime filter) in* $\mathcal{R}(A)$;
(3) *if* X *is an ideal (filter) in* A, *then* $(\mathcal{R}(X)] = X$ $([\mathcal{R}(X)) = X)$;
(4) *if* X *is an ideal (filter) in* $\mathcal{R}(A)$, *then* $\mathcal{R}((X]) = X$ $(\mathcal{R}([X)) = X)$.

Proof. We prove the statement (3) for ideals. Let X an ideal and $x \in X$. Then $x \vee 0 \in X$ and $x \leq x \vee 0 \in \mathcal{R}(X)$. Therefore, $x \in (\mathcal{R}(X)]$. Let $x \in (\mathcal{R}(X)]$. Then, $x \leq x_1 \vee \ldots \vee x_n$, with $x_1, \ldots, x_n \in \mathcal{R}(X) \subseteq X$. Thus, $x \in X$. $\square$

Let us now state a generalized version of the *prime filter Theorem* for $\mathbb{BDQL}$:

Theorem 2.1. *Let* $\mathbf{A} = \langle A, \wedge, \vee, 0, 1 \rangle$ *be a bdq-lattice and* I, F *an ideal and a filter, respectively, such that* $I \cap F = \emptyset$. *Then, there exist a prime ideal* J *and a prime filter* G *such that* $I \subseteq J$, $F \subseteq G$ *and* $J \cap G = \emptyset$.

Proof. From Lemma 2.1 it follows that $\mathcal{R}(I)$ and $\mathcal{R}(F)$ are a filter and an ideal, respectively, in the distributive lattice $\mathcal{R}(\mathbf{A})$. Therefore, by the prime filter Theorem for distributive lattices, there is a prime ideal J' and a prime filter G' in $\mathcal{R}(\mathbf{A})$ such that $\mathcal{R}(I) \subseteq J'$, $\mathcal{R}(F) \subseteq G'$ and $J' \cap G' = \emptyset$. Exploiting Lemma 2.1, item (3), we obtain that $I = (\mathcal{R}(I)] \subseteq (J']$ and $F = [\mathcal{R}(F)) \subseteq [G')$. Suppose that $(J'] \cap [G') \neq \emptyset$. If $x \in (J'] \cap [G')$, then $x \vee 0 = x \wedge 1 \in (J'] \cap [G')$. Therefore, by Lemma 2.1-(4), $J' \cap G' \neq \emptyset$. Whence, $(J'] \cap [G') = \emptyset$. $\square$

Example 2.2. *The algebra* $\overline{\mathbf{3}}$ *is the 3-element bounded distributive q-lattice whose operations are given by the following tables:*

$\wedge$	**0**	**a**	**1**
0	0	0	0
a	0	0	0
1	0	0	1

$\vee$	**0**	**a**	**1**
0	0	0	1
a	0	0	1
1	1	1	1

$$\begin{array}{l} \mathbf{1} \\ \mid \\ \mathbf{0} \text{ ------ } \mathbf{a} \end{array} \qquad\qquad (2.3)$$

Let us notice that $\overline{\mathbf{3}}$, Example 2.2, has no nontrivial filters, whilst the non-trivial elements of its congruence lattice are the congruences whose cosets are desplayed below:

(1) $\phi = \{\{1\}, \{0, a\}\}$;
(2) $\psi = \{\{1, 0\}, \{a\}\}$.

Basically, this is dued to the fact that, in general, q-lattices lack filter-determinacy, in the sense of [10]; there exists no "reasonable" notion of filter which corresponds to the notion of congruence.

We conclude this section with a brief overview of Priestley's duality for bounded distributive lattices.

For a partially ordered set (*poset*) $\langle X, \leq \rangle$ and $Y \subseteq X$, let $\uparrow Y = \{x \in X : \exists a \in Y \ with \ a \leq x\}$ and $\downarrow Y = \{x \in X : \exists a \in Y \ with \ x \leq a\}$. If Y is the singleton $\{y\}$, then we write $\uparrow y$ and $\downarrow y$ instead of $\uparrow \{y\}$ and $\downarrow \{y\}$, respectively. A subset $Y \subseteq X$ is called *increasing* or *upset* (respectively, *decreasing* or *downset*) if $Y = \uparrow Y$ (respectively, $Y = \downarrow Y$).

We recall that a *Priestley space* is an ordered topological space $X = \langle X, \tau, \leq \rangle$ which is compact and satisfies the *Priestley separation axiom*: if $x \not\leq y$ then there is a *clopen* (closed and open) upset U such that $x \in U$ and $y \notin U$.

3. Representation of BDQL

In this section we apply the homeset approach in Priestley duality to give a completion for bounded distributive quasi lattices. Let us first recall some notations. For more details see [17].

Let $\mathbf{A}, \mathbf{B}$ be algebras in the type of bounded distributive lattices. In what follows, we shall denote by $\mathbf{B}^{\mathbf{A}}$ the set of all homomorphisms from $\mathbf{A}$ to $\mathbf{B}$. Let $\mathbf{2}$ be the two-element distributive lattice and $x, y \in A$. We define

$$F_x = \{f \in \mathbf{2}^{\mathbf{A}} : f(x) = 1\}, \qquad\qquad (3.1)$$

$$F_{\overline{y}} = \{g \in \mathbf{2}^{\mathbf{A}} : g(y) = 0\}. \qquad\qquad (3.2)$$

Then for every bounded distributive lattice $\mathbf{A}$ with dual space $\langle X, \tau, \leq \rangle$, where $X = \{f : f \in \mathbf{2}^{\mathbf{A}}\}$, we have

(1) the clopen subsets of X are the sets $F_x \cap F_{\overline{y}}$, for $x, y \in A$;
(2) the clopen increasing (decreasing) sets of X are exactly of the sets F_x $(F_{\overline{x}})$, for $x \in A$.

We continue this section by proposing a generalization of the notion of Priestley space to preorded sets. Notice that in Priestley duality the two-element lattice $\mathbf{2}$ assumes, as generator of the variety of bounded distributive lattices, a crucial role. In the framework of bounded distributive quasi

lattices, the same role will be played by $\mathbf{3}$, Example 2.1, which generates (as well as $\overline{\mathbf{3}}$, Example 2.2) $\mathbb{BDQL}$ as a variety. Let us introduce some definitions and notions.

Let $\langle X, \preceq \rangle$ be a preordered set. If we define, for any $x, y \in X$:

$$x \equiv y \text{ iff } x \preceq y \text{ and } y \preceq x,$$

then one can easily see that $\langle X/_{\equiv}, \preceq \rangle$ is a partially ordered set.

Let $\langle X, \preceq \rangle$ be a preordered set. We will say that a subset Y is *almost increasing* (*almost decreasing*) if and only if $Y/_{\equiv}$ is increasing (decreasing). It can be seen that if Y is increasing (decreasing), then it is also almost increasing (almost decreasing). Nonetheless, the converse does not hold in general. As a concrete example, consider the bdq-lattice $\mathbf{3}$ introduced in Example 2.1. It can be seen that $\{k\}$ is an almost increasing but not increasing set.

Let $\mathbf{A}$ be a bounded distributive quasi lattice. Consider

$$E(\mathbf{A}) = \{f : f \in \mathbf{3^A}\}, \tag{3.3}$$

where $\mathbf{3}$ is the three-element bdq-lattice of Example 2.1, and define for any $x \in A$:

$$F_x = \{f \in E(\mathbf{A}) : f(x) = 1\}, \tag{3.4}$$
$$F_{\overline{x}} = \{f \in E(\mathbf{A}) : f(x) = 0\}, \tag{3.5}$$
$$F_{\tilde{x}} = \{f \in E(\mathbf{A}) : f(x) = k\}. \tag{3.6}$$

Let the collection

$$\mathcal{S} := \{F_a : a \in A\} \cup \{F_{\overline{b}} : b \in A\} \cup \{F_{\tilde{c}} : c \in A\}, \tag{3.7}$$

be a subbase for a topology τ on $E(\mathbf{A})$, and let us define, for any $f, g \in E(\mathbf{A})$ and for any $a \in A$:

$$f \preceq g \text{ iff } f(a) \wedge g(a) = f(a) \vee 0. \tag{3.8}$$

Let $\mathbf{A}$ a bdq-lattice and let $x \in \mathcal{R}(A)$. We remark that, for any $f \in \mathbf{3^A}$, $f(x) \neq k$, since $f(x) = f(x \vee 0) = f(x) \vee 0 \in \{0, 1\}$.

It can be seen that:

Proposition 3.1. *Let $\mathbf{A}$ be a bounded distributive quasi lattice with $\mathcal{S}$, defined in Condition 3.7, a subbase for $E(\mathbf{A})$ and let $\tau/_{\equiv}$ be the topology induced by $\mathcal{S}/_{\equiv}$ on $E(\mathbf{A})/_{\equiv}$. Then, for any $x \in A$,*

    (1) *F_x, $F_{\tilde{x}}$ are almost increasing τ-clopen sets in $E(\mathbf{A})$;*
    (2) *$F_{\overline{x}}$ is a decreasing τ-clopen set in $E(\mathbf{A})$;*
    (3) *the projection function p is a continuous preorder preserving clopen map;*
    (4) *$\tau/_{\equiv}$ is the quotient topology;*
    (5) *$p^{-1}p(F_x)$ and $p^{-1}p(F_{\tilde{x}})$ are increasing τ-clopen sets in $E(\mathbf{A})$;*

(6) *If $x \in \mathcal{R}(\mathbf{A})$, then $p^{-1}p(F_x) = F_x$.*

Proof. (1) Let $x \in A$. Then, $p(F_x) = p(F_{\tilde{x}}) = \{f : f(x) \in k/_\equiv = 1/_\equiv\}$. Therefore, both $p(F_x), p(F_{\tilde{x}})$ are increasing sets in $E(\mathbf{A})/_\equiv$. Moreover, $\overline{F}_{\tilde{x}} = \{f : f(x) \neq k\} = \{f : f(x) = 1 \text{ or } f(x) = 0\} = F_x \cup F_{\overline{x}}$. The case $\overline{F}_x$ is similar.

(2) Let $f \in F_{\overline{x}}$ and $g \preceq f$. Then $f(x) \wedge g(x) = g(x) \vee 0 = 0$. Furthermore, $\overline{F}_{\overline{x}} = \{f : f(x) \neq 0\} = \{f : f(x) = 1 \text{ or } f(x) = k\} = F_x \cup F_{\tilde{x}}$.

(3) It is straightforward to verify that p is continuous. Moreover, it follows from the definition of $\equiv$ that p preserves the preorder. Further, if F_x is in $\mathcal{S}$, then $p(F_x) = p(F_{\tilde{x}}) = \{f : f(x) \in \{k, 1\}\} \in \mathcal{S}/_\equiv$. Therefore, p is a clopen map.

(4) Follows from the previous item.

(5) Follows from item (3).

(6) Let $x \in \mathcal{R}(\mathbf{A})$. Then, $p^{-1}p(F_x) = \{g : g(x) \leq 1 \text{ and } g(x) \geq 1\} = F_x$ since, by regularity, there is no $f \in \mathbf{3^A}$ such that $f(x) = k$. $\qquad\square$

Let us recall that a homeomorphism is a continuous one-to-one map f from a topological space X onto a topological space Y such that f^{-1} is also continuous. Let $\mathbf{A}$ be a distributive bounded quasi lattice. We shall denote by $B(\mathbf{A})$ the quotient $E(\mathbf{A})/_\equiv$. Let us notice that $f \equiv g$ in $E(\mathbf{A})$ if and only if, for any $a \in A$, $f(a) \leq g(a)$ and $f(a) \geq g(a)$, if and only if $f(a) \vee 0 = g(a) \vee 0$. If we define $E(\mathcal{R}(\mathbf{A})) = \{f : f \in \mathbf{2}^{\mathcal{R}(\mathbf{A})}\}$ and we equip it with the topology τ' induced by the restriction of the subbase $\mathcal{S}$, Condition 3.7, to $\mathcal{R}(\mathbf{A})$ and $\preceq'$ is the restriction of $\preceq$ to $E(\mathcal{R}(\mathbf{A}))$, we obtain the following:

Lemma 3.1. *Let $\mathbf{A}$ be a bounded distributive quasi lattice. Then, the map $h : \langle B(\mathbf{A}), \tau/_\equiv, \preceq_\equiv \rangle \to \langle E(\mathcal{R}(\mathbf{A})), \tau', \preceq' \rangle$ defined, for any $[f]_\equiv \in B(\mathbf{A})$ and $a \in A$, by:*

$$h([f]_\equiv(a)) = f(a \vee 0),$$

is a preorder preserving homeomorphism.

Proof. If $[f]_\equiv \preceq [g]_\equiv$, then, for any $a \in A$, $[f]_\equiv(a) \wedge [g]_\equiv(a) = [f]_\equiv(a)$, so $f(a) \wedge g(a) = f(a) \vee 0$. Surjectivity of h is immediate. Let $[f]_\equiv \neq [g]_\equiv$. Then, without loss of generality, there exists an $x \in A$ and an $f' \in [f]_\equiv$ such that $f'(x) \not\leq g(x)$. Whence, $f(x) \vee 0 = f'(x) \vee 0 \not\leq g(x) \vee 0$. Let $F_x = \{f : f(x) = 1\}$ an element of the subbase of $E(\mathcal{R}(\mathbf{A}))$. Then $h^{-1}(F_x) = \{f : f(x) \leq 1 \text{ and } f(x) \geq 1\} = \{[f]_\equiv : f(x) \in 1/_\equiv\}$, which is clopen in $B(\mathbf{A})$. Dually, if $F_x/_\equiv = \{[f]_\equiv : f(x) \in 1/_\equiv\}$, then $h(F_x/_\equiv) = \{f : f(x \vee 0) = 1\} = F_{x \vee 0}$. The case of $F_{\overline{x}}/_\equiv$ follows from the same argument. $\qquad\square$

Let $\mathbf{A}$ be a bdq-lattice. In virtue of Lemma 3.1, in what follows, when no danger of confusion is impending, we will swap sometimes from $B(\mathbf{A})$ to $E(\mathcal{R}(\mathbf{A}))$.

Finally, we obtain the following

Theorem 3.1. *If* $\mathbf{A}$ *be a bounded distributive quasi lattice, then*

$$\langle E(\mathbf{A}), \tau, \preceq \rangle$$

is a topological space with a subbase $\mathcal{S}$ *of clopen almost increasing or decreasing sets such that:*

(1) $\langle E(\mathbf{A}), \preceq \rangle$ *is a preordered set;*
(2) $\langle B(\mathbf{A}), \tau/_{\equiv}, \preceq_{\equiv} \rangle$ *is a Priestley space;*
(3) $\langle E(\mathbf{A}), \tau \rangle$ *is a compact space;*
(4) *if, for* $f, g \in E(\mathbf{A})$, $f \neq g$, *then there exists an almost increasing clopen set* U *in* τ *such that either* $f \in U$ *and* $g \notin U$, *or* $g \in U$ *and* $f \notin U$.

Proof. Let $\mathbf{A}$ be a dbq-lattice. (1) Apparently, $\langle E(\mathbf{A}), \preceq \rangle$ is a preordered set, since the q-lattice operations induce a preorder on A. Moreover, every element of $\mathcal{S}$ is an almost increasing or decreasing τ-clopen set by Proposition 3.1. (2) Follows from Proposition 3.1 and Lemma 3.1. (3) Compactness follows from the fact that the subbase $\mathcal{S}$ induces the product topology on $E(\mathbf{A})$. (4) If $f \npreceq g$, then, by item (2), there exists an increasing clopen set U in $\tau/_{\equiv}$ such that $[f]_{\equiv} \in U$ and $[g]_{\equiv} \notin U$. Whence, $f \in p^{-1}(U)$ and $g \notin p^{-1}(U)$. Let $f, g \in E(\mathbf{A})$ such that $f \preceq g$, $g \preceq f$ and $f \neq g$. Then, by Condition 3.8, there exists an $a \in A \setminus \mathcal{R}(A)$ such that $f(a) \leq g(a)$, $g(a) \leq f(a)$, but $f(a) \neq g(a)$. Consequently, either $f(a) = 1$ and $g(a) = k$, or, vice-versa, $g(a) = 1$ and $f(a) = k$. Therefore, there exist a clopen set $F_{\tilde{a}}$ such that either f or g belongs to $F_{\tilde{a}}$ and $p(F_{\tilde{a}}) = \{h : h(a) \in 1/_{\equiv} = k/_{\equiv}\}$ is increasing and clopen in $E(\mathbf{A})/_{\equiv}$. $\square$

Corollary 3.1. *Let* $\mathbf{A}$ *be a bdq-lattice, then* $\langle E(\mathbf{A}), \tau, \preceq \rangle$ *is a Hausdorff space.*

Remark 3.1. *Apparently, one may regard the space* $E(\mathbf{A})$ *as the stalk space of the bundle* $\langle E(\mathbf{A}), p, B(\mathbf{A}) \rangle$, *where the Priestley space* $E(\mathbf{A})$ *is the base space and the fibres are given, as usual, by* $\{p^{-1}(x) : x \in B(\mathbf{A})\}$ [11].

Let $\mathbf{A}$ be a bounded distributive quasi lattice. We will denote by $\mathcal{A}(E(\mathbf{A}))$ the collection of almost increasing clopen sets of $E(\mathbf{A})$.

Lemma 3.2. *Let* $\mathbf{A}$ *be a bdq-lattice. Then the structure*

$$L(E(\mathbf{A})) = \langle \mathcal{A}(E(\mathbf{A})), \wedge, \vee, \emptyset, \Phi \rangle,$$

is a complete bounded distributive quasi lattice, where, for $X, Y \in \mathcal{A}(E(\mathbf{A}))$:

- $X \wedge Y = p^{-1}(p(X) \cap p(Y))$;
- $X \vee Y = p^{-1}(p(X) \cup p(Y))$;

Proof. (1) We check some of the dbq-lattice axioms. Let $X, Y, Z \in \mathcal{A}(E(\mathbf{A}))$. Then $X \wedge (X \vee Y) = p^{-1}(p(X) \cap p(p^{-1}(p(X) \cup p(Y)))) = p^{-1}(p(X)) = p^{-1}(p(X) \cap p(X)) = X \wedge X$. $X \vee (Y \wedge Z) = p^{-1}(p(X) \cup p(p^{-1}(p(Y) \cap p(Z)))) = p^{-1}(p(X) \cup (p(Y) \cap p(Z))) = p^{-1}((p(X) \cup p(Y)) \cap (p(X) \cup p(Z))) =$

$p^{-1}(pp^{-1}(p(X) \cup p(Y)) \cap pp^{-1}(p(X) \cup p(Z))) = (X \vee Y) \wedge (X \wedge Z)$. Moreover, $X \wedge \emptyset = \emptyset$ and $X \vee \Phi = p^{-1}p(\Phi) = \Phi$. Further, completeness follows from the fact that the elements of $\mathcal{A}(\Xi)$ are clopen almost increasing sets. □

Theorem 3.2. *Let* **A** *be bounded distributive quasi lattice. Then,* **A** *is embeddable into $L(E(\mathbf{A}))$ via the mapping $\Lambda : \mathbf{A} \to L(E(\mathbf{A}))$, defined, for any $x \in A$, as follows:*

$$\Lambda(x) = \begin{cases} F_x, & \text{if } x \in \mathcal{R}(A), \\ F_{\tilde{x}}, & \text{otherwise.} \end{cases}$$

Proof. Let $x, y \in A$. If $x \not\leq y$, then, by Theorem 2.1, there are a prime ideal $[x \vee 0)$ and a prime filter $(y \vee 0]$ such that $[x \vee 0) \cap (y \vee 0] = \emptyset$ and $x \in [x \vee 0)$, $y \in (y \vee 0]$. Therefore, $\Lambda(x) \neq \Lambda(y)$. Let $x, y \notin \mathcal{R}(A)$ and $x \leq y$ and $x \geq y$ but $x \neq y$. Consider the generated prime filter $[x \vee 0)$ and set $\alpha = [x \vee 0) \setminus \{y\}$. It can be seen that the partition $\phi = \{\alpha, A \setminus [x \vee 0), \{y\}\}$ is a congruence on **A**. Therefore, under the natural morphism $f : \mathbf{A} \to \mathbf{A}/_\phi$, $f(x) = f(x \vee 0) = 1$ and $f(y) = k$. Therefore, $F_{\tilde{x}} \neq F_{\tilde{y}}$. The case $x \leq y$ and $x \geq y$ and $x \in \mathcal{R}(A)$ is a slight adaptation of the previous one. If $x, y \notin \mathcal{R}(A)$, then $\Lambda(x \wedge y) = p^{-1}p(F_{x \wedge y}) = p^{-1}(p(F_x) \cap p(F_y)) = p^{-1}(p(F_{\tilde{x}}) \cap p(F_{\tilde{y}})) = \Lambda(x) \wedge \Lambda(y)$, by the definition of $\wedge$ and Lemma 3.1. The remaining cases are proved analogously. □

4. AMALGAMATION PROPERTY

Inspired by [14], we prove in this section the amalgamation property for the variety of bounded distributive quasi lattices, as an application of the ideas proposed in the previous part of the paper.

First, an useful notion:

Definition 4.1. *Let* **A** *be a bdq-lattice. And $x, y \in A$. We set*

$$x \tau y \text{ iff } x, y \in \mathcal{R}(A) \text{ or } x = y. \tag{4.1}$$

It can be verified that τ is a congruence. Further, it can be readily checked that, for any bdq-lattice **A** and $x, y \in A/_\tau$, $x \wedge y = x \wedge y \wedge 1 = x \wedge y \wedge 0 = 0 = 1 = x \vee y$. Mimicking a result in [13], we can state without proof the following:

Lemma 4.1. *Let* **A** *be a bdq-lattice. Then* **A** *can be embedded into $\langle \mathbf{A}/_\equiv, \mathbf{A}/_\tau \rangle$ via the mapping*

$$f(x) = \langle x/_\equiv, x/_\tau \rangle.$$

A class $\mathbb{D}$ of algebras has the *congruence extension property* if and only if for any $\mathbf{A} \subseteq \mathbf{B} \in \mathbb{D}$ and any congruence θ in **A** there is a congruence θ' in **B** such that θ is the restriction of θ' to $A \times A$. In virtue of the previous lemma, by adapting an argument in [15] we obtain that

Theorem 4.1. *The variety $\mathbb{BDQL}$ has the congruence extension property.*

Let us recall that a class of algebras $\mathbb{D}$ is called *amalgamable*, if for any algebras $\mathbf{B}_0, \mathbf{B}_1, \mathbf{B}_2 \in \mathbb{D}$ and embeddings $f_1 : \mathbf{B}_0 \to \mathbf{B}_1$ and $f_2 : \mathbf{B}_0 \to \mathbf{B}_2$, there exists an algebra $\mathbf{B} \in \mathbb{D}$ and embeddings $g_1 : \mathbf{B}_1 \to \mathbf{B}$ and $g_2 : \mathbf{B}_2 \to \mathbf{B}$ such that $g_1 \circ f_1 = g_2 \circ f_2$. If $\mathbb{D}$ is amalgamable we say that $\mathbb{D}$ has the *amalgamation property*. The class $\mathbb{D}$ is said to have the *super-amalgamation property*, if it has the amalgamation property and satisfies the following additional condition: for any $b \in B_i$ and $c \in B_j$ ($\{i, j\} = \{1, 2\}$), if $g_i(b) \leq g_j(c)$ in B, then there exists $d \in B_0$ such that $b \leq f_i(d)$ in B_i and $f_j(d) \leq c$ in B_j hold.

Let $\mathbf{A}_1$, $\mathbf{A}_2$ be a bdq-lattices and $\mathbf{A}_0$ a subalgebra of them. Consider

$$W = \{\langle g_1, g_2 \rangle : g_i \in \mathbf{3}^{\mathbf{A}_i}, i \in \{1, 2\} \text{ and } g_1 \restriction_{A_0} = g_2 \restriction_{A_0}\}, \qquad (4.2)$$

if we define $\langle g_1, g_2 \rangle \preceq \langle g_1', g_2' \rangle$ iff $g_1 \preceq g_1'$ and $g_2 \preceq g_2'$, then $\langle W, \preceq \rangle$ is a preordered set.

Lemma 4.2. *W has the following properties:*
  (1) *for every $g_1 \in \mathbf{3}^{\mathbf{A}_1}$ ($g_2 \in \mathbf{3}^{\mathbf{A}_2}$) there is a $g_2 \in \mathbf{3}^{\mathbf{A}_2}$ ($g_1 \in \mathbf{3}^{\mathbf{A}_1}$) such that $\langle g_1, g_2 \rangle \in W$;*
  (2) *if $\langle g_1, g_2 \rangle \in W$ and $g_1 \preceq g_1'$ ($g_2 \preceq g_2'$), then there exists a $g_2' \in \mathbf{3}^{\mathbf{A}_2}$ ($g_1' \in \mathbf{3}^{\mathbf{A}_1}$) such that $\langle g_1', g_2' \rangle \in W$ and $g_2 \preceq g_2'$ ($g_1 \preceq g_1'$);*
  (3) *if $\langle g_1, g_2 \rangle \in W$ and $[g_1]_{\equiv} = [g_1']_{\equiv}$ but $g_1 \neq g_1'$ ($[g_2]_{\equiv} = [g_2']_{\equiv}$ but $g_2 \neq g_2'$), then there exists a $g_2' \in \mathbf{3}^{\mathbf{A}_2}$ ($g_1' \in \mathbf{3}^{\mathbf{A}_1}$) such that $[g_2]_{\equiv} = [g_2']_{\equiv}$ and $\langle g_1', g_2' \rangle \in W$ ($[g_1]_{\equiv} = [g_1']_{\equiv}$ and $\langle g_1', g_2' \rangle \in W$).*

Proof. (1) Follows from Theorem 4.1. (2) Follows from [1] Lemma 2.3. (3) If $[g_1]_{\equiv} = [g_1']_{\equiv}$ but $g_1 \neq g_1'$, then there exists an $a \in A_1 \setminus \mathcal{R}(A_1)$ such that $g_1(a) = 1$ and $g_1'(a) = k$. If $a \in A_0$ then define

$$g_2'(x) = \begin{cases} g_1'(x) & \text{if } x \in A_0 \\ \\ g_2(x), & \text{otherwise.} \end{cases}$$

In case $a \in A_1 \setminus A_0$, then set $g_2' = g_2$. It can be verified that g_2' is a homomorphism such that $[g_2]_{\equiv} = [g_2']_{\equiv}$ and $\langle g_1', g_2' \rangle \in W$. $\square$

Let $\langle K, \preceq \rangle$ a preordered set. We have already noticed that $\mathbf{A}_K = \langle \mathcal{A}(K), \wedge, \vee, \emptyset, K \rangle$, where $\mathcal{A}(K)$ is the set of almost increasing subsets of K and the operations are defined as in Lemma 3.2, is a bounded distributive quasi lattice. Let $\mathbf{A}_W$ the bdq-lattice associated to the preordered set W, we obtain the following:

Theorem 4.2. *Let $\mathbf{A}_1$, $\mathbf{A}_2$ be a bdq-lattices and $\mathbf{A}_0$ a subalgebra of them. For $i \in \{1, 2\}$, the map $f_i : \mathbf{A}_i \to \mathbf{A}_W$ defined, for any $x \in A_i$, as:*

$$f_i(x) = \begin{cases} \{\langle g_1, g_2 \rangle \in W : g_i(x) = 1\}, & \text{if } x \in \mathcal{R}(A_i) \\ \\ \{\langle g_1, g_2 \rangle \in W : g_i(x) = k\}, & \text{otherwise} \end{cases} \qquad (4.3)$$

is an embedding.

Proof. It can be seen that, for any $x \in A_i$, $f_i(x)$ is in $\mathcal{A}(W)$. Moreover, one can verify that for every $x \in A_0$, $f_1(x) = f_2(x)$. If $x, y \in A_i$ and $x \not\leq y$ there is a prime filter F such that $x \in F$ and $y \notin F$ and, consequently, a morphism $g_i \in \mathbf{3}^{\mathbf{A}_i}$ such that $g_i(x) \in \{k, 1\}$ and $g_i(y) = 0$, whence, by Lemma 4.2, there is a a morphism $g_j \in \mathbf{3}^{\mathbf{A}_j}$, $j \in \{1, 2\}$ and $i \neq j$, such that $\langle g_i, g_j \rangle \in W$. So, $\langle g_i, g_j \rangle \in f_i(x)$ and $\langle g_i, g_j \rangle \notin f_i(y)$. Let $x, y \in A_i$ and $x \equiv y$ but $x \neq y$. We can assume, without loss of generality, that $x, y \in A_i \setminus \mathcal{R}(A_i)$. Then, by using an argument similiar to the one used in the proof of Theorem 3.2, there exists a morphism $g_i \in \mathbf{3}^{\mathbf{A}_i}$ such that $g_i(x) = 1$ and $g_i(y) = k$. Whence, by Lemma 4.2-(3), there exists a $g'_j \in \mathbf{3}^{\mathbf{A}_j}$, $j \in \{1, 2\}$ and $i \neq j$, such that $\langle g'_i, g'_j \rangle \in W$ and $\langle g'_i, g'_j \rangle \in f_i(y)$ but $\langle g'_i, g'_j \rangle \notin f_i(x)$. Whence $f_i(x) \neq f_i(y)$, and, therefore, f_i is one to one. That f_i preserves the operations is along the lines of the proof of Theorem 3.2. $\square$

As a consequence of the previous result:

Theorem 4.3. *The variety* $\mathbb{BDQL}$ *has the amalgamation property.*

Let us remark that an alternative proof of the amalgamation property for bdq-lattices can be obtained by adapting an argument in [2], where the amalgamation property for the variety of quasi-MV algebras and for some relevant subclasses of the variety of $\sqrt{\prime}$ quasi-MV algebras is proved.
It is well-known that the variety of bounded distributive lattices does not have the super amalgamation property. So, there are bounded distributive lattices $\mathbf{A}_i$, $0 \leq i \leq 2$ and embeddings f_i from $\mathbf{A}_0$ to $\mathbf{A}_i$, $1 \leq i \leq 2$ such that it is impossible, in the variety of bounded distributive lattices, to find a triple $\langle \mathbf{A}, g_1, g_2 \rangle$, which is super amalgamated with the three algebras. Now, suppose that there exists such a triple in the variety of bounded distributive quasi lattices. Obviously, $g_i(A_i) \subseteq \mathcal{R}(\mathbf{A})$, since $\mathbf{A}_i$, $1 \leq i \leq 2$, is a bounded distributive lattice. On the other hand, clearly, $\mathcal{R}(\mathbf{A})$ is a bounded distributive lattice. So, the triple $\langle \mathcal{R}(\mathbf{A}), g_1, g_2 \rangle$ is super amalgamated with the three original algebras in the variety of bounded distributive lattices. A contradiction. Therefore, we can state the following:

Theorem 4.4. *The variety* $\mathbb{BDQL}$ *does not have the super amalgamation property.*

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