# POSITIVE SOLUTIONS TO OPERATOR EQUATIONS $AXB = C$

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Abstract. The operator equation $AXB = C$ has been studied by several authors, but under the extra condition that the operators $A$ and $B$ have closed range. In this article, we present different results regarding the existence of solution and also the existence of positive solution to $AXB = C$ without this extra hypothesis.

## Introduction

The main goal of this article is to study the operator equation

$$AXB = C, \tag{I}$$

where $A, B$ and $C$ are bounded linear operators defined on convenient Hilbert spaces. This kind of equations has been studied by several authors because of its multiple applications in different areas as, for example, control theory and sampling. The reader is referred to [4], [5], [16] and the references therein. However, in these works it is only considered the case in which $A, B$ and $C$ are matrices or have closed range. Our goal is to study the equation (I) with arbitrary operators $A, B$ and $C$. This consideration implies that some classical results are not longer valid. For instance, it is well known that if $A$ and $B$ have closed range then the equation (I) is solvable, i.e., there exists a bounded linear operator $D$ such that $ADB = C$, if and only if $AA'CB'B = C$ for every inner inverses, $A'$ and $B'$, of $A$ and $B$ respectively. Recall that $A'$ (non necessarily bounded) is an inner inverse of $A$ if $AA'A = A$. However, it is easy to see that this result fails if $A, B$ have not closed range. In fact, for every operator $A$ it holds $AA'AA'A = A$, but $AXA = A$ is solvable if and only if $A$ has closed range [12]. Therefore, our first aim is to determine conditions for the existence of solution of (I). For this, we first prove that the previous result still holds if $A, B$ or $C$ has closed range. Secondly, we characterize the existence of solution of (I) for arbitraries operators $A, B$ and $C$, in terms of range inclusions. These results can be found in section 2.

In section 3, we concentrate our attention on the existence of positive solutions to $AXB = C$. The existence of special classes of solutions of (I) has been considered by many authors. For example, in [13] the existence of symmetric and antisymmetric solution of the matrix equation (I) is investigated. On the other hand, the reader is referred to [4] for the existence and expression of Re-pd and Re-nnd solutions. Moreover, the existence of positive solution of the matrix equation (I) was also considered by Khatri and Mitra in [16] and for the operator equation (I) by Xu et

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al. in [16]. In this last work, the authors studied the case $A, B, B^*A^\dagger C$ with closed range and $R(B) \subseteq \overline{R(A^*)}$. Under these hypotheses they characterized the existence of positive solutions and provided the general form of these solutions. Here, we present different results related to this problem and we obtain, as an special case, the result of Xu et al.

Finally, in section 4 we investigate the equation (I) with $A, B$ with closed range and $C$ a projection with range equal to the range of $A$. First, we observe that the existence of solution of these equations is equivalent to certain angle condition between the range of $A$ and the kernel of $B$. Moreover, we prove that the solutions coincide with the inner inverses of the product $BA$. Next, we characterize different inner inverses of $BA$ in terms of these solutions. This result extends the result presented in [2] where the inner inverses of an operator are described in terms of solutions of a Douglas type equation.

## 1. Preliminaries

Along this work $\mathcal{F}, \mathcal{G}, \mathcal{H}$ and $\mathcal{K}$ denote complex Hilbert spaces with inner product $\langle \ , \ \rangle$. $L(\mathcal{H}, \mathcal{K})$ is the algebra of bounded linear operators from $\mathcal{H}$ into $\mathcal{K}$ and $L(\mathcal{H}) = L(\mathcal{H}, \mathcal{H})$. By $L(\mathcal{H})^+$ we denote the cone of positive (semidefinite) operators of $L(\mathcal{H})$, i.e., $L(\mathcal{H})^+ := \{T \in L(\mathcal{H}) : \ \langle T\xi, \xi\rangle \geq 0 \ \forall \xi \in \mathcal{H}\}$. If $T \in L(\mathcal{H}, \mathcal{K})$ then $T^*$ denotes the adjoint operator of $T$, $R(T)$ stands for the range of $T$ and $N(T)$ for its nullspace. Given a closed subspace $\mathcal{S}$ of $\mathcal{H}$, $\mathcal{Q}_\mathcal{S} := \{Q \in \mathcal{H} : \ Q^2 = Q \text{ and } R(Q) = \mathcal{S}\}$ and $Q_{\mathcal{S}//\mathcal{T}}$ denotes the operator in $\mathcal{Q}_\mathcal{S}$ with kernel $\mathcal{T}$. In particular, $P_\mathcal{S}$ denotes the orthogonal projection onto $\mathcal{S}$.

In the sequel, the symbol $T'$ stands for an arbitrary **inner inverse** of $T \in L(\mathcal{H}, \mathcal{K})$, i.e. $T' : \mathcal{D}(T') \subseteq \mathcal{K} \to \mathcal{H}$ with $R(T) \subseteq \mathcal{D}(T')$ and

$$TT'T = T. \tag{1}$$

Thus, $T' \notin L(\mathcal{K}, \mathcal{H})$, in general. Indeed, given $T \in L(\mathcal{H}, \mathcal{K})$ there exists an inner inverse of $T$, $T'$, such that $T' \in L(\mathcal{K}, \mathcal{H})$ if and only if $T$ has closed range [12]. If, in addition, $T'$ verifies

$$T'TT' = T', \tag{2}$$

then $T'$ is called a **generalized inverse** of $T$. Moreover, there exists a unique generalized inverse of $T$ which also verifies

$$(TT')^* = TT' \tag{3}$$

and

$$(T'T)^* = T'T, \tag{4}$$

which is called the **Moore-Penrose generalized inverse** of $T$ and it will be denoted by $T^\dagger$. Therefore, $T^\dagger$ is the unique generalized inverse of $T$ such that

$$T^\dagger T = P_{\overline{R(T^*)}} \ \text{ and } \ TT^\dagger = P_{\overline{R(T)}}|_{R(T) \oplus R(T)^\perp}.$$

Throughout this work the next well-known criterion due to Douglas [8] (see also Fillmore-Williams [9]) about range inclusions and factorization of operators will be crucial.

**Theorem (Douglas).** Let $A \in L(\mathcal{H}, \mathcal{K})$ and $C \in L(\mathcal{G}, \mathcal{K})$. The following conditions are equivalent:

(1) There exists $D \in L(\mathcal{G}, \mathcal{H})$ such that $AD = C$.

(2) $R(C) \subseteq R(A)$.
(3) There exists a positive number $\lambda$ such that $CC^* \leq \lambda AA^*$.

If one of these conditions holds then there exists a unique solution $\tilde{D} \in L(\mathcal{G}, \mathcal{H})$ of the equation $AX = C$ such that $R(\tilde{D}) \subseteq \overline{R(A^*)}$ and $N(\tilde{D}) = N(C)$ This solution will be called the **Douglas reduced solution**.

In the next lemma we prove that given $C \in L(\mathcal{G}, \mathcal{K})$ such that $R(C) \subseteq R(T)$ then $T^\dagger C \in L(\mathcal{G}, \mathcal{H})$, even though $T^\dagger \notin L(\mathcal{K}, \mathcal{H})$. This fact will be used frequently along this work.

**Lemma 1.1.** *Let $T \in L(\mathcal{H}, \mathcal{K})$ and $C \in L(\mathcal{G}, \mathcal{K})$ such that $R(C) \subseteq R(T)$. Then $T^\dagger C \in L(\mathcal{G}, \mathcal{H})$.*

*Proof.* Since $R(C) \subseteq R(T)$ then, by Douglas theorem, there exists $D \in L(\mathcal{G}, \mathcal{H})$ such that $TD = C$. Now, since $T^\dagger T = P_{\overline{R(T^*)}} \in L(\mathcal{H})$ then $T^\dagger C = T^\dagger TD \in L(\mathcal{G}, \mathcal{H})$. □

## 2. The existence of solutions of $AXB = C$

Given $A \in L(\mathcal{H}, \mathcal{K}), B \in L(\mathcal{F}, \mathcal{G})$ and $C \in L(\mathcal{F}, \mathcal{K})$ we shall say that the equation $AXB = C$ is **solvable** if there exists $\tilde{X} \in L(\mathcal{G}, \mathcal{H})$ such that $A\tilde{X}B = C$. The solubility of this kind of equations was first considered by R. A. Penrose [14], proving that the matrix equation $AXB = C$ is solvable if and only if $AA'CB'B = C$ for every inner inverse $A'$ of $A$ and $B'$ of $B$. The same proof works if it is considered the operator equation $AXB = C$ with $R(A)$ and $R(B)$ closed. Moreover, in the context of $C^*$-algebras the same result holds if the elements $A, B$ are regular [6]. However, as we shall prove in the next Theorem, the Penrose's assertion still holds if just one of the three operators, $A$, $B$ or $C$, has closed range. Furthermore, in Remark 2.2, we show that if neither $A, B$ nor $C$ has closed range then the result fails.

**Theorem 2.1.** *Let $A \in L(\mathcal{H}, \mathcal{K}), B \in L(\mathcal{F}, \mathcal{G})$ and $C \in L(\mathcal{F}, \mathcal{K})$. If $R(A), R(B)$ or $R(C)$ is closed then the following conditions are equivalent:*

(1) *The equation $AXB = C$ is solvable;*
(2) *$AA'CB'B = C$ for every inner inverses, $A', B'$, of $A$ and $B$, respectively;*
(3) *$R(C) \subseteq R(A)$ and $R(C^*) \subseteq R(B^*)$.*

*Proof.*

$1 \rightarrow 2$ Fist, note that if $T'$ is a generalized inner inverse of $T \in L(\mathcal{H}, \mathcal{K})$ then $(TT')^2 = TT'$ with $R(TT') = R(T)$ and $(T'T)^2 = T'T$ with $N(T'T) = N(T)$. Now, if the equation $AXB = C$ is solvable then $R(C) \subseteq R(A)$ and $N(B) \subseteq N(C)$. Therefore, it is straightforward that $AA'CB'B = C$ for every generalized inner inverses, $A', B'$, of $A$ and $B$, respectively.

$2 \rightarrow 3$ It is trivial.

$3 \rightarrow 1$ If $R(C^*) \subseteq R(B^*)$ then, by Douglas theorem, there exists $D \in L(\mathcal{K}, \mathcal{G})$ such that $B^*D = C^*$ and $N(D) = N(C^*)$ or, equivalently, $\overline{R(D^*)} = \overline{R(C)}$. So, $\overline{R(D^*)} \subseteq \overline{R(A)}$. Hence, if $R(A)$ or $R(C)$ is closed then $R(D^*) \subseteq R(A)$. Therefore, by Lemma 1.1, $A^\dagger D^* \in L(\mathcal{G}, \mathcal{H})$ and $A(A^\dagger D^*)B = C$. So, the equation $AXB = C$ is solvable. If $R(B)$ is closed then a similar proof can be done considering the Douglas reduced solution of $AX = C$. □

**Remark 2.2.** If neither $A, B$ nor $C$ has closed range then the previous result fails. For example, if $A \in L(\mathcal{H})$ has not closed range then the equation $AXA = A$ is not solvable, however, $AA'AA'A = A$ for every inner inverse, $A'$, of $A$.

In the next result we study the solubility of the equation (I) for unconstrained operators $A, B$ and $C$.

**Proposition 2.3.** *Let $A \in L(\mathcal{H}, \mathcal{K}), B \in L(\mathcal{F}, \mathcal{G})$ and $C \in L(\mathcal{F}, \mathcal{K})$. Hence, the following conditions are equivalent:*

(1) *The equation $AXB = C$ is solvable;*
(2) $R(C) \subseteq R(A)$ *and* $R((A^\dagger C)^*) \subseteq R(B^*)$;
(3) $R(C) \subseteq R(A)$ *and there exists $\tilde{Y} \in L(\mathcal{G}, \mathcal{H})$ such that $\tilde{Y}B = A^\dagger C$.*

*Moreover, if one of the previous conditions holds then every solution of $XB = A^\dagger C$ is also a solution of $AXB = C$, but given $\tilde{X} \in L(\mathcal{G}, \mathcal{H})$ such that $A\tilde{X}B = C$ then $P_{\overline{R(A^*)}}\tilde{X}$ is a solution of $XB = A^\dagger C$.*

*Proof.*
$1 \rightarrow 2$. Clearly, if there exists $\tilde{X} \in L(\mathcal{G}, \mathcal{H})$ such that $A\tilde{X}B = C$ then $R(C) \subseteq R(A)$ and so $A^\dagger C \in L(\mathcal{F}, \mathcal{H})$. Moreover, as $A^\dagger A\tilde{X}B = A^\dagger C$ then $R((A^\dagger C)^*) \subseteq R(B^*)$.

$2 \rightarrow 1$. As $R(C) \subseteq R(A)$, then $A^\dagger C \in L(\mathcal{F}, \mathcal{H})$. On the other hand, since $R((A^\dagger C)^*) \subseteq R(B^*)$ then, by Douglas theorem, there exists $\tilde{Y} \in L(\mathcal{G}, \mathcal{H})$ such that $A^\dagger C = \tilde{Y}B$. From this, $C = P_{\overline{R(A)}}|_{\mathcal{D}(A^\dagger)}C = AA^\dagger C = A\tilde{Y}B$, and the assertion follows.

$3 \leftrightarrow 2$. It is consequence of Douglas theorem.

The last part of the Proposition follows by simple computations. □

## 3. Positive solutions to $AXB = C$

The aim of this section is to study the existence and expression of the positive solutions to the operator equation $AXB = C$. This problem was first considered by C.G. Khatri and S.K. Mitra in [11]. They studied the matrix equation $AXB = C$ and provided necessary and sufficient conditions (related to the ranks of $A$ and $B$) for the existence of positive solutions. However, a similar result for operators defined on a Hilbert space seems not to be simple. Indeed, several hypotheses regarding the ranges of $A, B$, and $AB$ are needed. On the other hand, in a recent paper Xu et al. [16] studied the same problem, but for $A, B$ operators on a Hilbert space with closed range and such that $R(B) \subseteq \overline{R(A^*)}$. Under these conditions they also provided the general form of the positive solutions. Along this section we present different results related to this problem and we obtain, as an special case, the result of Xu et al. For this, the next result, due to Sebestyén [15], will be crucial.

**Theorem (Sebestyén).** Let $B, C \in L(\mathcal{H}, \mathcal{K})$ be such that the equation $XB = C$ has a solution. Hence, the equation admits a positive solution if and only if $C^*C \leq \lambda B^*C$ for some constant $\lambda \geq 0$.

**Proposition 3.1.** *Let $A \in L(\mathcal{H}, \mathcal{K})$, $B \in L(\mathcal{G}, \mathcal{H})$ and $C \in L(\mathcal{G}, \mathcal{K})$ be such that the equation $AXB = C$ is solvable. Hence, if there exists a constant $\lambda \geq 0$ such that*

$$(A^\dagger C)^*(A^\dagger C) \leq \lambda B^* A^\dagger C, \tag{5}$$

*then the equation $AXB = C$ has a positive solution.*

*Proof.* By Sebestyén's theorem, the condition (5) implies that there exists $\tilde{Y} \in L(\mathcal{H})^+$ such that $\tilde{Y}B = A^\dagger C$. Then, by Proposition 2.3, $\tilde{Y}$ is also a solution of $AXB = C$ and the assertion follows.

□

In the remainder of this section we shall consider the extra condition $R(B) \subseteq \overline{R(A^*)}$. Under this extra hypothesis we shall see that the general form of the positive solutions can be easily characterized using the matrix operator decomposition induced by $\overline{R(A^*)}$. Given a closed subspace $\mathcal{S}$ of $\mathcal{H}$ and $T \in L(\mathcal{H})$, the matrix operator decomposition of $T$ induced by $\mathcal{S}$ is

$$T = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}; \tag{6}$$

where $t_{11} = P_{\mathcal{S}}TP_{\mathcal{S}}|_{\mathcal{S}} \in L(\mathcal{S})$, $t_{12} = P_{\mathcal{S}}T(I - P_{\mathcal{S}})|_{\mathcal{S}^\perp} \in L(\mathcal{S}^\perp, \mathcal{S})$, $t_{21} = (I - P_{\mathcal{S}})TP_{\mathcal{S}}|_{\mathcal{S}} \in L(\mathcal{S}, \mathcal{S}^\perp)$ and $t_{22} = (I - P_{\mathcal{S}})T(I - P_{\mathcal{S}})|_{\mathcal{S}^\perp} \in L(\mathcal{S}^\perp)$. The next well-known theorem characterizes the positive operators in terms of its matrix decomposition. The proof of this result is based on the existence of the shorted operator [[1], Theorem 3].

**Theorem 3.2.** *Let $\mathcal{S}$ be a closed subspace of $\mathcal{H}$ and $T \in L(\mathcal{H})$ with the matrix operator decomposition induced by $\mathcal{S}$ given by (6). Then, $T \in L(\mathcal{H})^+$ if and only if*

(1) $t_{12} = t_{21}^*$,
(2) $t_{11} \geq 0$,
(3) $R(t_{12}) \subseteq R(t_{11}^{1/2})$,
(4) $t_{22} = ((t_{11}^{1/2})^\dagger t_{12})^*(t_{11}^{1/2})^\dagger t_{12} + f$, *where* $f \geq 0$.

The main result of this section is the following.

**Theorem 3.3.** *Let $A \in L(\mathcal{H}, \mathcal{K})$, $B \in L(\mathcal{G}, \mathcal{H})$ and $C \in L(\mathcal{G}, \mathcal{K})$ be such that $R(B) \subseteq \overline{R(A^*)}$. If the equation $AXB = C$ is solvable then the following conditions are equivalent:*

(1) *There exists $\tilde{X} \in L(\mathcal{H})^+$ such that $A\tilde{X}B = C$.*
(2) *There exists $\tilde{Y} \in L(\mathcal{H})^+$ such that $\tilde{Y}B = A^\dagger C$.*
(3) $B^*A^\dagger C \geq 0$ *and* $R((A^\dagger C)^*) \subseteq R((B^*A^\dagger C)^{1/2})$.

*If one of these conditions holds and we consider the matrix operator decomposition induced by $\overline{R(A^*)}$ then the general form of the positive solutions is:*

$$\tilde{X} = \begin{bmatrix} \tilde{x}_{11} & \tilde{x}_{12} \\ \tilde{x}_{12}^* & ((\tilde{x}_{11}^{1/2})^\dagger \tilde{x}_{12})^*(\tilde{x}_{11}^{1/2})^\dagger \tilde{x}_{12} + f \end{bmatrix}, \tag{7}$$

*where $\tilde{x}_{11} = P_{\overline{R(A^*)}}\tilde{Y}|_{\overline{R(A^*)}}$ with $\tilde{Y} \in L(\mathcal{H})$ a positive solution of $XB = A^\dagger C$, $R(\tilde{x}_{12}) \subseteq R(\tilde{x}_{11}^{1/2})$ and $f$ is positive.*

*Proof.*
$1 \to 2$. Let $\tilde{X} \in L(\mathcal{H})^+$ be such that $A\tilde{X}B = C$. Hence, $\tilde{Y} = P_{\overline{R(A^*)}}\tilde{X}P_{\overline{R(A^*)}} \in L(\mathcal{H})^+$ and it is easy to check that $\tilde{Y}$ is solution of $XB = A^\dagger C$.
$2 \to 3$. It follows by Sebestyén's theorem and Douglas theorem.
$3 \to 1$. If $R((A^\dagger C)^*) \subseteq R((B^*A^\dagger C)^{1/2})$ then, by Douglas theorem, there exists a constant $\lambda \geq 0$ such that $(A^\dagger C)^*(A^\dagger C) \leq \lambda B^*A^\dagger C$. Then, applying Sebestyén's

theorem, there exists $\tilde{Y} \in L(\mathcal{H})^+$ such that $\tilde{Y}B = A^\dagger C$. Now, since $R(C) \subseteq R(A)$, $A\tilde{Y}B = C$ and the assertion follows.

Finally, let us prove the general form of the positive solutions. Assume that $\tilde{X} \in L(\mathcal{H})^+$ is a solution of $AXB = C$. Then, it is straightforward that $\tilde{Y} = P_{\overline{R(A^*)}}\tilde{X}P_{\overline{R(A^*)}}$ is a positive solution of $XB = A^\dagger C$ and $\tilde{x}_{11} = \tilde{Y}|_{\overline{R(A^*)}}$. The elements $\tilde{x}_{12}, \tilde{x}_{21}$ and $\tilde{x}_{22}$ follows by Theorem 3.2.

Conversely, let $\tilde{X} \in L(\mathcal{H})$ with the matrix decomposition (7). Therefore, since $R(B) \subseteq \overline{R(A^*)}$, it holds $A\tilde{X}B = A\tilde{Y}B = C$. Moreover, by Theorem 3.2, $\tilde{X}$ is positive and so the result is proved. □

According to Theorem 3.3, it suffices to describe the positive solutions of $XB = A^\dagger C$ in order to fully characterize the positive solutions of $AXB = C$ when $R(B) \subseteq \overline{R(A^*)}$. Therefore, the following result proved in [3] will be useful to complete the characterization.

**Theorem 3.4.** *Let $A, C \in L(\mathcal{H}, \mathcal{K})$ be such that the equation $AX = C$ has a positive solution. If $R(C) \subseteq R(AC^*)$ and $cos_0(\overline{R(C^*)}, N(A)) < 1$[1] then the general positive solution is given by*

$$Y = C^*(AC^*)^\dagger C + (I - A'A)S(I - A'A)^*;$$

*where $S \in L(\mathcal{H})^+$ and $A'$ is a generalized inverse of $A$ with $A'A \in L(\mathcal{H})$.*

**Remark 3.5.** If $R(A)$ and $R(AC^*)$ are closed then the angle condition required in Theorem 3.4 is automatically verified. The reader is referred to [3] for the proof of this fact.

Therefore, as a consequence of Theorem 3.3 and Theorem 3.4, we have the following result which was also obtained by Xu et al. [[16], Theorem 5.6]. It should be stressed that they presented the expression of the general form of the positive solution in terms of the decomposition induced by $R(B)$. Here, we give the expression of positive solutions in terms of the matrix decomposition induced by $\overline{R(A^*)}$. This consideration not only simplifies the expression but also the proof. In addition, note that the operator $A^\dagger C(B^*A^\dagger C)^\dagger(A^\dagger C)^*$ which appears in the description of $\tilde{x}_{11}$ is also a particular positive solution of $AXB = C$.

**Corollary 3.6.** *Let the equation $AXB = C$ be solvable with $B$ and $B^*A^\dagger C$ with closed range and such that $R(B) \subseteq \overline{R(A^*)}$. Then, the equation $AXB = C$ has a positive solution if and only if $B^*A^\dagger C \geq 0$ and $R((A^\dagger C)^*) = R(B^*A^\dagger C)$.*

*Moreover, if we consider the matrix operator decomposition induced by $\overline{R(A^*)}$ then the general form of the positive solutions is:*

$$\tilde{X} = \begin{bmatrix} \tilde{x}_{11} & \tilde{x}_{12} \\ \tilde{x}_{12}^* & ((\tilde{x}_{11}^{1/2})^\dagger \tilde{x}_{12})^*(\tilde{x}_{11}^{1/2})^\dagger \tilde{x}_{12} + f \end{bmatrix}, \tag{8}$$

*where $\tilde{x}_{11} = A^\dagger C(B^*A^\dagger C)^\dagger(A^\dagger C)^* + P_{\overline{R(A^*)}}(I - B'B)S(I - B'B)^*|_{\overline{R(A^*)}}$ with $B'$ an inner inverse of $B$ and $S \in L(\mathcal{G})^+$; $R(\tilde{x}_{12}) \subseteq R(\tilde{x}_{11}^{1/2})$ and $f$ is positive.*

[1] $cos_0(\mathcal{S}, \mathcal{T})$ denotes the cosine of the Dixmier angle between $\mathcal{S}$ and $\mathcal{T}$, i.e, $cos_0(\mathcal{S}, \mathcal{T}) = sup\{|\langle \xi, \nu \rangle| : \xi \in \mathcal{S}, \nu \in \mathcal{T} \text{ and } \|\xi\| \leq 1, \|\nu\| \leq 1\}$. See [10] and [7] for a treatment on the theory of angles between subspaces.

We end this section by studying the equation $AXB = C$ with $R(B) = R(A^*)$. For this, we first study the existence of positive solutions of $AXA^* = C$. This problem was also studied by D. Cvetković-Ilić et al. [5], but for $A$ with closed range.

**Proposition 3.7.** *Let $A \in L(\mathcal{H}, \mathcal{K})$ and $C \in L(\mathcal{K})$ be such that $AXA^* = C$ is solvable. Then, the following conditions are equivalent:*

1. *There exists $\tilde{X} \in L(\mathcal{H})^+$ such that $A\tilde{X}A^* = C$;*
2. *$C \in L(\mathcal{K})^+$;*
3. *There exists a constant $\lambda \geq 0$ such that $(A^\dagger C)^*(A^\dagger C) \leq \lambda C$.*

*Moreover, if one of these conditions holds then there exists a unique solution $\tilde{X} \in L(\mathcal{H})^+$ such that $R(\tilde{X}) \subseteq \overline{R(A^*)}$.*

*Proof.*
$1 \to 2$ Trivial.

$2 \to 3$. Let $\tilde{X} \in L(\mathcal{H})$ be such that $C = A\tilde{X}A^*$. Since $C \in L(\mathcal{K})^+$ then $C \leq \|\tilde{X}\|AA^*$. Thus, by Douglas theorem, $R(C^{1/2}) \subseteq R(A)$ and so $\tilde{Y} = A^\dagger C^{1/2} \in L(\mathcal{K}, \mathcal{H})$. Hence, $\tilde{Y}$ is solution of $A^\dagger C = XC^{1/2}$. Therefore, applying Douglas theorem again, we obtain the inequality desired.

$3 \to 1$. Suppose that there exists a constant $\lambda \geq 0$ such that $(A^\dagger C)^*(A^\dagger C) \leq \lambda C$. Hence, by Sebestyén theorem, we obtain that the equation $XA^* = A^\dagger C$ has a positive solution. So, by Proposition 2.3, $AXA^* = C$ has a positive solution.

Finally, if $\tilde{Y}$ is a positive solution of $AXA^* = C$ then $\tilde{X} = P_{\overline{R(A^*)}}\tilde{Y}P_{\overline{R(A^*)}}$ is also a positive solution of $AXA^* = C$ and $R(\tilde{X}) \subseteq \overline{R(A^*)}$. Now, suppose that there exists $\hat{X} \in L(\mathcal{H})^+$ such that $A\hat{X}A^* = C$ and $R(\hat{X}) \subseteq \overline{R(A^*)}$. Then $A(\tilde{X} - \hat{X})A^* = 0$. So $R((\tilde{X} - \hat{X})A^*) \subseteq N(A) \cap \overline{R(A^*)} = \{0\}$. Therefore $A(\tilde{X} - \hat{X}) = 0$ and thus $R(\tilde{X} - \hat{X}) \subseteq N(A) \cap \overline{R(A^*)} = \{0\}$. Hence $\tilde{X} = \hat{X}$. □

**Proposition 3.8.** *Let $A \in L(\mathcal{H}, \mathcal{K})$, $B \in L(\mathcal{G}, \mathcal{H})$ and $C \in L(\mathcal{G}, \mathcal{K})$ be such that the equation $AXB = C$ is solvable. If $R(B) = R(A^*)$ then the equation $AXB = C$ has a positive solution if and only if $B^*A^\dagger C \geq 0$.*

*In such case, if we consider the matrix operator decomposition induced by $\overline{R(A^*)}$ then the general form of the positive solutions is:*

$$\tilde{X} = \begin{bmatrix} \tilde{x}_{11} & \tilde{x}_{12} \\ \tilde{x}_{12}^* & ((\tilde{x}_{11}^{1/2})^\dagger \tilde{x}_{12})^*(\tilde{x}_{11}^{1/2})^\dagger \tilde{x}_{12} + f \end{bmatrix}, \tag{9}$$

*where $\tilde{x}_{11} = (B^*)^\dagger (A^\dagger C)^*|_{\overline{R(A^*)}}$, $R(\tilde{x}_{12}) \subseteq R(\tilde{x}_{11}^{1/2})$ and $f$ is positive.*

*Proof.* If $AXB = C$ has a positive solution then, by Theorem 3.3, we get that $B^*A^\dagger C \geq 0$.

Conversely, since the equation $AXB = C$ is solvable then, applying Theorem 3.3, $B^*XB = B^*A^\dagger C$ is also solvable. Now, as $B^*A^\dagger C \geq 0$ then, by Proposition 3.7, there exists a unique $\tilde{X} \in L(\mathcal{H})^+$ with $R(\tilde{X}) \subseteq \overline{R(B)}$ such that $B^*\tilde{X}B = B^*A^\dagger C$. Thus, $\tilde{X}B = P_{\overline{R(B)}}\tilde{X}B = (B^*)^\dagger B^*\tilde{X}B = (B^*)^\dagger B^*A^\dagger C = P_{\overline{R(B)}}A^\dagger C = A^\dagger C$. Therefore, by Proposition 2.3, $\tilde{X} \in L(\mathcal{H})^+$ is solution of $AXB = C$.

It remains to show the general form of the positive solutions. By Theorem 3.3, it suffices to show that $(B^*)^\dagger (A^\dagger C)^*|_{\overline{R(A^*)}} = P_{\overline{R(A^*)}}Y|_{\overline{R(A^*)}}$ for every positive solution $Y$ of $XB = A^\dagger C$. Hence, let $Y \in L(\mathcal{H})^+$ be a solution of $XB = A^\dagger C$. It is straightforward that $\tilde{Y} := P_{\overline{R(A^*)}} Y P_{\overline{R(A^*)}}$ is also a positive solution of $XB = A^\dagger C$. Now, since $R(\tilde{Y}) \subseteq \overline{R(A^*)} = \overline{R(B)}$ then $\tilde{Y} = (B^*)^\dagger B^* \tilde{Y} = (B^*)^\dagger (A^\dagger C)^*$. So, $P_{\overline{R(A^*)}} Y|_{\overline{R(A^*)}} = \tilde{Y}|_{\overline{R(A^*)}} = (B^*)^\dagger (A^\dagger C)^*|_{\overline{R(A^*)}}$ and the result is proved. □

## 4. The equation $AXB = Q_{R(A)//\mathcal{T}}$

In this section we shall study the equations of the form $AXB = Q$ where $A$ and $B$ are operators with closed range and $Q \in \mathcal{Q}_{R(A)}$. We shall see that the solutions of this kind of equations coincide with the inner inverses of $BA$.

In the sequel, given $T \in L(\mathcal{H}, \mathcal{K})$ we denote by $T[i]$, $T[i,j]$, $T[i,j,k]$ and $T[i,j,k,l]$ the sets of operators f $L(\mathcal{K}, \mathcal{H})$ which satisfy equations $\{(i)\}$, $\{(i),(j)\}$, $\{(i),(j),(k)\}$ and $\{(i),(j),(k),(l)\}$ respectively; where $i,j,k,l = 1,2,3,4$.

**Theorem 4.1.** *Let $A \in L(\mathcal{G}, \mathcal{H})$ and $B \in L(\mathcal{H}, \mathcal{K})$ with closed range. The following conditions are equivalent:*

1. *$AXB = Q_{R(A)//\mathcal{T}}$ is solvable for some topological complement $\mathcal{T}$ of $R(A)$;*
2. *$cos_0(N(B), R(A)) < 1$.*

*Moreover, if one of the previous conditions holds then*

$$\{\tilde{X} \in L(\mathcal{K}, \mathcal{G}) : A\tilde{X}B \in \mathcal{Q}_{R(A)}\} = BA[1]. \tag{10}$$

*Proof.*

$1 \rightarrow 2$. If $AXB = Q_{R(A)//\mathcal{T}}$ is solvable for some topological complement $\mathcal{T}$ of $R(A)$ then $BAXBA = BA$ is solvable. So, $BA$ admits a bounded linear inner inverse and so $BA$ has closed range. On the other hand, since $AXB = Q_{R(A)//\mathcal{T}}$ is solvable then $N(B) \subseteq \mathcal{T}$ and from this we get $N(B) \cap R(A) = \{0\}$ . Then, by Theorem 1.2 of [7], we obtain that $cos_0(N(B), R(A)) < 1$.

$2 \rightarrow 1$. Since $cos_0(N(B), R(A)) < 1$ then $N(B) \dot{+} R(A)$ is closed [[7], Theorem 1.2]. Now, define $\mathcal{T} = N(B) \dot{+} (N(B) \dot{+} R(A))^\perp$. Note that $\mathcal{T}$ is closed because $cos_0(N(B), (N(B) \dot{+} R(A))^\perp) < 1$ and that $\mathcal{T} \dot{+} R(A) = \mathcal{H}$. Then $Q_{R(A)//\mathcal{T}}$ is well defined and since $N(B) \subseteq \mathcal{T}$ the assertion follows by Proposition 2.1.

Finally, suppose that $cos_0(N(B), R(A)) < 1$. Thus, if $\tilde{X} \in L(\mathcal{K}, \mathcal{G})$ is such that $A\tilde{X}B \in \mathcal{Q}_{R(A)}$, then $BA\tilde{X}BA = BA$, i.e., $\tilde{X} \in BA[1]$. Conversely, since $cos_0(N(B), R(A)) < 1$ then there exists a closed subspace $\mathcal{W}$ of $\mathcal{H}$ such that $\mathcal{W} \dot{+} N(B) = \mathcal{H}$ and $R(A) \subseteq \mathcal{W}$ (for instance, $\mathcal{W} = R(A) + (N(B) + R(A))^\perp$). Then, let $B'$ be a generalized inverse of $B$ such that $B'B = Q_{\mathcal{W}//N(B)}$. Therefore, if $\tilde{X} \in BA[1]$ then $B'(BA\tilde{X}BA) = B'BA$, and so $A\tilde{X}BA = A$ because $R(A) \subseteq \mathcal{W}$. Now, from this last equality, we obtain that $(A\tilde{X}B)^2 = A\tilde{X}B$. Moreover, $R(A) = R(A\tilde{X}BA) \subseteq R(A\tilde{X}B) \subseteq R(A)$. Therefore, $A\tilde{X}B \in \mathcal{Q}_{R(A)}$ and the equality (10) is obtained.

□

**Corollary 4.2.** *Let $A \in L(\mathcal{K}, \mathcal{H})$ and $B \in L(\mathcal{H}, \mathcal{K})$ with closed range such that $BA = (BA)^*$ and $cos_0(N(B), R(A)) < 1$. Therefore, there exists $\tilde{X} \in L(\mathcal{H})^+$ such that $A\tilde{X}B \in \mathcal{Q}_{R(A)}$ if and only if $BA \in L(\mathcal{K})^+$.*

*Proof.* Let $\tilde{X} \in L(\mathcal{H})^+$ such that $A\tilde{X}B \in \mathcal{Q}_{R(A)}$. Hence, $\tilde{X}$ is a positive solution of $BAXBA = BA$ and since $BA = (BA)^*$ then we have that $BA \in L(\mathcal{K})^+$.

Conversely, if $BA \in L(\mathcal{K})^+$ then $(BA)^\dagger \in L(\mathcal{K})^+$. Thus, as $(BA)^\dagger \in BA[1]$, by Theorem 4.1, we obtain the result. □

In the next Proposition we explore the relationship between the sets $BA[i,j,k,l]$ and the solutions of the equation $AXB \in Q_{R(A)}$. This result turns out to be an extension of Theorem 3.1 in [2] where the sets $A[i,j,k,l]$ are related with solutions of Douglas type equations.

**Proposition 4.3.** *Let $A \in L(\mathcal{G},\mathcal{H})$ and $B \in L(\mathcal{H},\mathcal{K})$ with closed range and $cos_0(N(B),R(A)) < 1$. Hence, if $\mathcal{W} = N(P_{R(BA)}B)$ then the following assertions hold:*

(i) $BA[1] = \{\tilde{X} \in L(\mathcal{K},\mathcal{G}) : A\tilde{X}B \in \mathcal{Q}_{R(A)}\}$;
(ii) $BA[1,2] = \{\tilde{X} \in L(\mathcal{K},\mathcal{G}) : A\tilde{X}B \in \mathcal{Q}_{R(A)}$ *and* $N(A\tilde{X}) = N(\tilde{X})\}$;
(iii) $BA[1,3] = \{\tilde{X} \in L(\mathcal{K},\mathcal{G}) : A\tilde{X}B = \mathcal{Q}_{R(A)//\mathcal{W}}$ *and* $N(A\tilde{X}) = R(BA)^{\perp}\}$;
(iv) $BA[1,4] = \{\tilde{X} \in L(\mathcal{K},\mathcal{G}) : A\tilde{X}B \in \mathcal{Q}_{R(A)}$ *and* $R(\tilde{X}BA) = R(A^*)\}$;
(v) $BA[1,2,3] = \{\tilde{X} \in L(\mathcal{K},\mathcal{G}) : A\tilde{X}B = \mathcal{Q}_{R(A)//\mathcal{W}}$ *and* $N(\tilde{X}) = R(BA)^{\perp}\}$;
(vi) $BA[1,2,4] = \{\tilde{X} \in L(\mathcal{K},\mathcal{G}) : A\tilde{X}B \in \mathcal{Q}_{R(A)}$ *and* $R(\tilde{X}) = R(A^*)\}$;
(vii) $BA[1,3,4] = \{\tilde{X} \in L(\mathcal{K},\mathcal{G}) : A\tilde{X}B = Q_{R(A)//\mathcal{W}}$ $N(A\tilde{X}) = R(BA)^{\perp}$ *and* $R(\tilde{X}BA) = R(A^*)\}$;
(viii) $BA[1,2,3,4] = \{\tilde{X} \in L(\mathcal{K},\mathcal{G}) : A\tilde{X}B = Q_{R(A)//\mathcal{W}},$ $R(\tilde{X}) = R(A^*)$ *and* $N(\tilde{X}) = R(BA)^{\perp}\}$.

*Proof.* First, note that since $R(A) \cap N(B) = \{0\}$ then $N(BA\tilde{X}) = N(A\tilde{X})$ for every $\tilde{X} \in L(\mathcal{K},\mathcal{G})$.

(i) It is Theorem 4.1.

(ii) Let $\tilde{X} \in BA[1,2]$. Then, by item (i), $A\tilde{X}B \in \mathcal{Q}_{R(A)}$. Furthermore, as $\tilde{X} \in BA[2]$ then $N(\tilde{X}) \subseteq N(BA\tilde{X}) \subseteq N(\tilde{X}BA\tilde{X}) = N(\tilde{X})$, i.e., $N(BA\tilde{X}) = N(\tilde{X})$ and so, $N(A\tilde{X}) = N(\tilde{X})$.

Conversely, let $\tilde{X}$ such that $A\tilde{X}B \in \mathcal{Q}_{R(A)}$ and $N(A\tilde{X}) = N(\tilde{X})$. Then, by item (i), $\tilde{X} \in BA[1]$ and $\tilde{X}BA\tilde{X} = \tilde{X}Q_{R(BA)//N(\tilde{X})} = \tilde{X}$. Therefore, $\tilde{X} \in BA[1,2]$.

(iii) Let $\tilde{X} \in BA[1,3]$. Then $BA\tilde{X} = P_{R(BA)}$ and $BA\tilde{X}B = BQ$ with $Q \in \mathcal{Q}_{R(A)}$. Thus, $N(P_{R(BA)}B) = N(BQ) = N(Q)$ where the last equality follows because $R(A) \cap N(B) = \{0\}$. In addition, $N(A\tilde{X}) = N(BA\tilde{X}) = R(BA)^{\perp}$.

Conversely, if $A\tilde{X}B = Q_{R(A)//\mathcal{W}}$ then, by item (i), $\tilde{X} \in BA[1]$ and so $BA\tilde{X} = Q \in \mathcal{Q}_{R(BA)}$. Now, as $N(BA\tilde{X}) = N(A\tilde{X}) = R(BA)^{\perp}$ then $BA\tilde{X} = P_{R(BA)}$. Therefore, $\tilde{X} \in BA[1,3]$.

(iv) If $\tilde{X} \in BA[1,4]$ then $\tilde{X}BA = P_{N(BA)^{\perp}}$. Now, since $N(BA) = N(A)$, then $\tilde{X}BA = P_{N(A)^{\perp}}$.

Conversely, let $A\tilde{X}B \in \mathcal{Q}_{R(A)}$ and $R(A^*) = R(\tilde{X}BA)$. Then, by (i) $\tilde{X} \in BA[1]$ and so $\tilde{X}BA$ is a projection with $R(\tilde{X}BA) = R(A^*)$ and $N(A) \subseteq N(\tilde{X}BA)$. Therefore, $\tilde{X}BA = P_{N(A)^{\perp}}$ and so $\tilde{X} \in BA[1,4]$.

(v) If $\tilde{X} \in BA[1,2,3]$ then, by items (ii) and (iii), $A\tilde{X}B = Q_{R(A)//\mathcal{W}}$ with $N(\tilde{X}) = R(BA)^{\perp}$. Conversely, let $A\tilde{X}B \in \mathcal{Q}_{R(A)}$ with $N(\tilde{X}) = R(BA)^{\perp}$. Then $BA\tilde{X} = P_{R(BA)}$. So $\tilde{X}BA\tilde{X} = \tilde{X}P_{R(BA)} = \tilde{X}$. Therefore $\tilde{X} \in BA[1,2,3]$.

(vi) Let $\tilde{X} \in BA[1,2,4]$. Then, by (iv), $A\tilde{X}B \in \mathcal{Q}_{R(A)}$ and $R(A^*) = R(\tilde{X}BA)$. Now, since $\tilde{X} \in BA[2]$ then $R(\tilde{X}BA) = R(\tilde{X})$ and so $R(A^*) = R(\tilde{X})$.

Conversely, if $A\tilde{X}B \in \mathcal{Q}_{R(A)}$ then $\tilde{X} \in BA[1]$ and so $\tilde{X}BA$ is a projection. Furthermore, as $R(\tilde{X}BA) \subseteq R(\tilde{X}) = R(A^*)$ and $N(A) \subseteq N(\tilde{X}BA)$, then $\tilde{X}BA =$

$P_{R(A^*)}$ and so $\tilde{X} \in BA[4]$. In addition, $\tilde{X}BA\tilde{X} = P_{R(A^*)}\tilde{X} = P_{R(\tilde{X})}\tilde{X} = \tilde{X}$, i.e., $\tilde{X} \in BA[2]$.

(vii) The assertion follows by items (iii) and (iv).

(viii) The assertion follows by items (v) and (vi). □

**Remark 4.4.** Note that item (viii) of the previous theorem provides a new way for testing the Moore-Penrose inverse of the operator product $BA$ under the condition $cos_0(N(B), R(A)) < 1$.

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