Abstract splines in Krein spaces

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Abstract

We present generalizations to Krein spaces of the abstract interpolation and smoothing problems proposed by Atteia in Hilbert spaces. Let $\mathcal{H}$, $\mathcal{K}$ be two Krein spaces and $\mathcal{E}$ a Hilbert space. Given (bounded) surjective operators $T : \mathcal{H} \to \mathcal{K}$ and $V : \mathcal{H} \to \mathcal{E}$ and a fixed $z_0 \in \mathcal{E}$, we study the $x_0 \in \mathcal{H}$ such that $Vx_0 = z_0$ and $[Tx_0, Tx_0]_{\mathcal{K}} = \min\{[Tx, Tx]_{\mathcal{K}} : V(x) = z_0\}$. We also study the following problem, given $\rho > 0$ and fixed $z_0 \in \mathcal{E}$, find $x_0 \in \mathcal{H}$ such that $[Tx_0, Tx_0]_{\mathcal{K}} + \rho\|Vx_0 - z_0\|^2_{\mathcal{E}} = \min_{x\in\mathcal{H}} \left([Tx, Tx]_{\mathcal{K}} + \rho\|Vx - z_0\|^2_{\mathcal{E}}\right)$.

1 Introduction

Since I. J. Schoenberg introduced the spline functions [21], they have became an important notion in several branches of mathematics such us approximation theory, statistics, numerical analysis and partial differential equations, among others. Recently, they have been useful to solve some practical issues in signal and image processing, computer graphics, learning theory and other applications.

In the sixties, a Hilbert space formulation of spline functions, known as abstract splines, was introduced by M. Atteia [3] and extended by several authors, see for instance [2, 6, 17, 20]. Given Hilbert spaces $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{E}$, consider (bounded) surjective operators $T : \mathcal{H} \to \mathcal{K}$ and $V : \mathcal{H} \to \mathcal{E}$. The abstract interpolation problem in Hilbert spaces can be stated as follows: fixed $z_0 \in \mathcal{E}$, find $x_0 \in \mathcal{H}$ such that $Vx_0 = z_0$ and

$$\|Tx_0\|^2_{\mathcal{K}} = \min\{\|Tx\|^2_{\mathcal{K}} : Vx = z_0\}.$$

On the other hand, given $\rho > 0$ and fixed $z_0 \in \mathcal{E}$, the abstract smoothing problem consists in minimizing the function $F : \mathcal{H} \to \mathbb{R}^+$ defined by $F(x) = \|Tx\|^2_{\mathcal{K}} + \rho\|Vx - z_0\|^2_{\mathcal{E}}$. For a complete exposition on these subjects see the book of Atteia [4] and the survey by R. Champion et al. [11].

In this work, mainly motivated by the ideas of Atteia, we present generalizations of the abstract interpolation and smoothing problems to Krein spaces. More precisely, let $\mathcal{H}$, $\mathcal{K}$ be two Krein spaces and $\mathcal{E}$ a Hilbert space. Given (bounded) surjective operators $T : \mathcal{H} \to \mathcal{K}$ and $V : \mathcal{H} \to \mathcal{E}$, we study the following problems:

Problem (interpolating spline)**.** Fixed $z_0 \in \mathcal{E}$, find $x_0 \in \mathcal{H}$ such that $Vx_0 = z_0$ and

$$[Tx_0, Tx_0]_{\mathcal{K}} = \min\{[Tx, Tx]_{\mathcal{K}} : V(x) = z_0\},$$

Problem (smoothing spline)**.** Given $\rho > 0$ and fixed $z_0 \in \mathcal{E}$, find $x_0 \in \mathcal{H}$ such that

$$[Tx_0, Tx_0]_{\mathcal{K}} + \rho\|Vx_0 - z_0\|^2_{\mathcal{E}} = \min_{x\in\mathcal{H}} \left([Tx, Tx]_{\mathcal{K}} + \rho\|Vx - z_0\|^2_{\mathcal{E}}\right).$$

Spline functions in indefinite metric spaces have already been studied in [9] to solve numerical aspects related to learning theory problems. Although the problems presented there are different from those studied in this work, they are closely related. In [8] another version of the smoothing problem is studied. Given $z_0 \in \mathcal{E}$, instead of finding the minimum of $F_\rho(x) = [Tx, Tx]_{\mathcal{K}} + \rho\|Vx - z_0\|^2_{\mathcal{E}}$, the authors are interested in stabilizing it.

The paper is organized as follows: Section 2 contains the preliminaries. In Section 3 we study the interpolating problem, we give necessary and sufficient conditions for the existence (and uniqueness) of solutions of this problem, and characterize them. Also, given a frame $\{f_n\}_{n\in\mathbb{N}}$ for the Hilbert space $\mathcal{E}$, we give conditions to obtain different frames for $|\mathcal{H}|$ (the Hilbert space associated to $\mathcal{H}$) composed by interpolating splines corresponding to the family $\{f_n\}_{n\in\mathbb{N}}$.

Section 4 is devoted to the study of the smoothing problem: after characterizing the set of solutions of this problem, we show that it is related to the set of solutions of an interpolating problem for a certain $z_\rho \in \mathcal{E}$. As it was studied by Atteia in Hilbert spaces, we analyze the convergence of the solutions of the smoothing problem to the solutions of the interpolating problem as ρ goes to infinity.

In section 5 we extend to Krein spaces a variational problem studied by A. I. Rozhenko and V. A. Vasilenko in [19], which mixes both interpolating and smoothing problems.

2 Preliminaries

Along this work $\mathcal{E}$ denotes a complex (separable) Hilbert space. If $\mathcal{F}$ is another Hilbert space then $L(\mathcal{E},\mathcal{F})$ is the algebra of bounded linear operators from $\mathcal{E}$ into $\mathcal{F}$, $L(\mathcal{E}) = L(\mathcal{E},\mathcal{E})$ and denote by $\mathcal{Q}$ the set of (oblique) projections, i.e. $\mathcal{Q} = \{Q \in L(\mathcal{E}) \ : \ Q^2 = Q\}$. If $T \in L(\mathcal{E},\mathcal{F})$ then $T^* \in L(\mathcal{F},\mathcal{E})$ denotes the adjoint operator of T, $R(T)$ stands for the range of T and $N(T)$ for its nullspace. Also, if $T \in L(\mathcal{E},\mathcal{F})$ has closed range, $T^\dagger$ denotes the Moore-Penrose inverse of T.

If $\mathcal{S}$ and $\mathcal{T}$ are two (closed) subspaces of $\mathcal{E}$, denote by $\mathcal{S} \dot{+} \mathcal{T}$ the direct sum of $\mathcal{S}$ and $\mathcal{T}$, $\mathcal{S} \oplus \mathcal{T}$ the (direct) orthogonal sum of them and $\mathcal{S} \ominus \mathcal{T} := \mathcal{S} \cap (\mathcal{S} \cap \mathcal{T})^\perp$. If $\mathcal{E} = \mathcal{S} \dot{+} \mathcal{T}$, the oblique projection onto $\mathcal{S}$ along $\mathcal{T}$, $P_{\mathcal{S}//\mathcal{T}}$, is the unique $Q \in \mathcal{Q}$ with $R(P_{\mathcal{S}//\mathcal{T}}) = \mathcal{S}$ and $N(P_{\mathcal{S}//\mathcal{T}}) = \mathcal{T}$. In particular, $P_\mathcal{S} := P_{\mathcal{S}//\mathcal{S}^\perp}$ is the orthogonal projection onto $\mathcal{S}$.

2.1 Krein spaces

In what follows we present the standard notation and some basic results on Krein spaces. For a complete exposition on the subject (and the proofs of the results below) see the books by J. Bognár [5] and T. Ya. Azizov and I. S. Iokhvidov [14], the monographs by T. Ando [1] and by M. Dritschel and J. Rovnyak [13] and the paper by J. Rovnyak [18].

Definition. An indefinite metric space $(\mathcal{H}, [\ ,\ ])$ is a *Krein space* if it can be decomposed as a direct (orthogonal) sum of a Hilbert space and an anti Hilbert space, i.e. there exist subspaces $\mathcal{H}_\pm$ of $\mathcal{H}$ such that $(\mathcal{H}_+, [\ ,\ ])$ and $(\mathcal{H}_-, -[\ ,\ ])$ are Hilbert spaces,

$$\mathcal{H} = \mathcal{H}_+ \dot{+} \mathcal{H}_-, \tag{2.1}$$

and $\mathcal{H}_+$ is orthogonal to $\mathcal{H}_-$ respect to the indefinite metric. Sometimes we use the notation $[\ ,\ ]_\mathcal{H}$ instead of $[\ ,\ ]$ to emphasize the Krein space considered.

A pair of subspaces $\mathcal{H}_\pm$ as in Eq. (2.1) is called a *fundamental decomposition* of $\mathcal{H}$. Given a Krein space $\mathcal{H}$ and a fundamental decomposition $\mathcal{H} = \mathcal{H}_+ \dot{+} \mathcal{H}_-$, the direct (orthogonal) sum of the Hilbert spaces $(\mathcal{H}_+, [\ ,\ ])$ and $(\mathcal{H}_-, -[\ ,\ ])$ is denoted by $(|\mathcal{H}|, \langle\ ,\ \rangle)$. Observe that the indefinite metric of $\mathcal{H}$ and the inner product of $|\mathcal{H}|$ are related by means of a *fundamental symmetry* $J \in L(\mathcal{H})$: given a fundamental decomposition $\mathcal{H} = \mathcal{H}_+ \dot{+} \mathcal{H}_-$, let $P_\pm$ be the orthogonal projection onto $\mathcal{H}_\pm$ and consider $J = P_+ - P_-$, then

$$[\,x, y\,] = \langle\, Jx, y\,\rangle, \quad x, y \in \mathcal{H}.$$

If $\mathcal{H}$ and $\mathcal{K}$ are Krein spaces then $L(\mathcal{H},\mathcal{K})$ stands for $L(|\mathcal{H}|, |\mathcal{K}|)$, and $L(\mathcal{H})$ for $L(|\mathcal{H}|)$. Given $T \in L(\mathcal{H},\mathcal{K})$, the J-adjoint operator of T is the unique operator $T^\# \in L(\mathcal{K},\mathcal{H})$ such that

$$[\,Tx, y\,]_\mathcal{K} = [\,x, T^\# y\,]_\mathcal{H}, \quad x \in \mathcal{H}, y \in \mathcal{K}.$$

Notice that T^* and $T^\#$ are related by $T^\# = J_\mathcal{H} T^* J_\mathcal{K}$, where $J_\mathcal{H}$ and $J_\mathcal{K}$ are the fundamental symmetries associated to $\mathcal{H}$ and $\mathcal{K}$, respectively. An operator $T \in L(\mathcal{H})$ is said to be J-selfadjoint if $T = T^\#$.

Given a subspace $\mathcal{S}$ of a Krein space $\mathcal{H}$, the *J-orthogonal companion* to $\mathcal{S}$ is defined by

$$\mathcal{S}^{[\perp]} = \{x \in \mathcal{H} : [\,x, s\,] = 0 \text{ for every } s \in \mathcal{S}\}.$$

Observe that $\mathcal{S}^{[\perp]}$ and $\mathcal{S}^{\perp}$ are related by $\mathcal{S}^{[\perp]} = J(\mathcal{S}^{\perp}) = J(\mathcal{S})^{\perp}$. Also, notice that if $T \in L(\mathcal{H}, \mathcal{K})$ and $\mathcal{S}$ is a closed subspace of $\mathcal{K}$ then

$$T^{\#}(\mathcal{S})^{[\perp]_{\mathcal{H}}} = T^{-1}(\mathcal{S}^{[\perp]_{\mathcal{K}}}). \tag{2.2}$$

A subspace $\mathcal{S}$ of $\mathcal{H}$ is non degenerated if $\mathcal{S} \cap \mathcal{S}^{[\perp]} = \{0\}$. A vector $x \in \mathcal{H}$ is *J-positive* if $[\,x, x\,] > 0$. A subspace $\mathcal{S}$ of $\mathcal{H}$ is J-positive if every $x \in \mathcal{S}$, $x \neq 0$, is a J-positive vector. A subspace $\mathcal{S}$ of $\mathcal{H}$ is said to be *uniformly J-positive* if there exists $\alpha > 0$ such that

$$[\,x, x\,] \geq \alpha \|x\|^2, \qquad \text{for every } x \in \mathcal{S},$$

where $\|\;\|$ stands for the norm of the associated Hilbert space $|\mathcal{H}|$. J-nonnegative, J-neutral, J-negative and J-nonpositive vectors (and subspaces) are defined analogously. A subspace $\mathcal{S}$ of $\mathcal{H}$ is uniformly J-negative if there exists $\alpha > 0$ such that $[\,x, x\,] \leq -\alpha\|x\|^2$ for every $x \in \mathcal{S}$. Notice that if $\mathcal{S}$ is a J-definite subspace of $\mathcal{H}$ then it is non degenerated.

Definition. Let $\mathcal{H}$ be a Krein space with fundamental symmetry J. A subspace $\mathcal{S}$ of $\mathcal{H}$ is called *regular* if $\mathcal{S}$ is the range of a J-selfadjoint projection.

Proposition 2.1 (Cor. 7.17 in [14])**.** *Let $\mathcal{H}$ be a Krein space with fundamental symmetry J and $\mathcal{S}$ a J-nonnegative closed subspace of $\mathcal{H}$. Then, $\mathcal{S}$ is regular if and only if $\mathcal{S}$ is uniformly J-positive.*

Corollary 2.2 (Thm. 8.4 in [5])**.** *Let $\mathcal{H}$ be a Krein space with fundamental symmetry J and $\mathcal{S}$ a closed uniformly J-positive subspace of $\mathcal{H}$. If Q is the J-selfadjoint projection onto $\mathcal{S}$ then, given $x \in \mathcal{H}$,*

$$[\,x - Qx, x - Qx\,] = \min_{y \in \mathcal{S}} [\,x - y, x - y\,].$$

2.2 Angles between subspaces and reduced minimum modulus

Let $\mathcal{S}$ and $\mathcal{T}$ be two closed subspaces of a Hilbert space $\mathcal{E}$.

Definition. The cosine of the *Friedrichs angle* between $\mathcal{S}$ and $\mathcal{T}$ is defined by

$$c(\mathcal{S}, \mathcal{T}) = \sup\{|\,\langle\, x, y\,\rangle\,| : x \in \mathcal{S} \ominus \mathcal{T}, \|x\| = 1,\, y \in \mathcal{T} \ominus \mathcal{S}, \|y\| = 1\}.$$

On the other hand, the cosine of the *Dixmier angle* between $\mathcal{S}$ and $\mathcal{T}$ is defined by

$$c_0(\mathcal{S}, \mathcal{T}) = \sup\{|\,\langle\, x, y\,\rangle\,| : x \in \mathcal{S}, \|x\| = 1,\, y \in \mathcal{T}, \|y\| = 1\}.$$

If $\mathcal{S} \cap \mathcal{T} = \{0\}$ the above definitions coincide, but observe that $c(\mathcal{S}, \mathcal{T}) \leq c_0(\mathcal{S}, \mathcal{T})$ in general. It is well known that

$$c(\mathcal{S}, \mathcal{T}) < 1 \quad \Leftrightarrow \quad \mathcal{S} + \mathcal{T} \text{ is closed} \quad \Leftrightarrow \quad \mathcal{S}^{\perp} + \mathcal{T}^{\perp} \text{ is closed} \quad \Leftrightarrow \quad c(\mathcal{S}^{\perp}, \mathcal{T}^{\perp}) < 1.$$

Furthermore, if $P_{\mathcal{S}}$ and $P_{\mathcal{T}}$ are the orthogonal projections onto $\mathcal{S}$ and $\mathcal{T}$, respectively, then $c(\mathcal{S}, \mathcal{T}) < 1$ if and only if $(I - P_{\mathcal{S}})P_{\mathcal{T}}$ has closed range, or equivalently, $(I - P_{\mathcal{T}})P_{\mathcal{S}}$ has closed range. See [12] for further details.

Also, recall the following well known result about the product of operators with closed ranges [7, 15].

Proposition 2.3. *Given a Hilbert space $\mathcal{H}$, let $A, B \in L(\mathcal{H})$ be closed range operators. Then, AB has closed range if and only if $c(R(B), N(A)) < 1$.*

The next definition is due to T. Kato, see [16, Ch. IV, § 5] for a complete exposition on this subject.

Definition. The *reduced minimum modulus* $\gamma(T)$ of an operator $T \in L(\mathcal{E})$ is defined by

$$\gamma(T) = \inf\{\|Tx\| : \|x\| = 1;\, x \in N(T)^{\perp}\}.$$

It is well known that $\gamma(T) = \gamma(T^*) = \gamma(T^*T)^{1/2}$. Also, it can be shown that an operator T has closed range if and only if $\gamma(T) > 0$. In this case, $\gamma(T) = \|T^{\dagger}\|^{-1}$.

2.3 Bases and frames for Hilbert spaces

Recall that a sequence $\{f_n\}_{n\in\mathbb{N}}$ in a Banach space X is called a *Schauder basis* of X if for every $x \in X$ there is a unique sequence of scalars $\{c_n\}_{n\in\mathbb{N}}$ so that $x = \sum_{n=1}^{\infty} c_n f_n$, where the series converges in the norm topology.

A vector sequence $\{f_n\}_{n\in\mathbb{N}}$ in a Hilbert space $\mathcal{E}$ is a *frame* if there exist constants $0 < A < B$ such that

$$A\|z\|^2 \leq \sum_{n=1}^{\infty} |\langle z, f_n \rangle|^2 \leq B\|z\|^2, \qquad \text{for every } z \in \mathcal{E}. \tag{2.3}$$

On the other hand, $\{f_n\}_{n\in\mathbb{N}}$ is a *Riesz basis* if there exist constants $0 < A < B$ such that

$$A\sum_{n=1}^{m} |c_n|^2 \leq \left\| \sum_{n=1}^{m} c_n f_n \right\|^2 \leq B\sum_{n=1}^{m} |c_n|^2, \tag{2.4}$$

for all finite sequences $c_1, \ldots, c_m$.

Then, a (Schauder) basis $\{f_n\}_{n\in\mathbb{N}}$ (of a Hilbert space $\mathcal{E}$) is a Riesz basis if and only if it is unconditional (meaning that if $\sum_{n=1}^{\infty} c_n f_n$ converges for some coefficients $\{c_n\}_{n\in\mathbb{N}}$, then it actually converges unconditionally) and $0 < \inf_n \|f_n\| \leq \sup_n \|f_n\| < \infty$.

Observe that there is a close connection between frames and Riesz bases:

1. $\{f_n\}_{n\in\mathbb{N}}$ is a Riesz basis if and only if $\{f_n\}_{n\in\mathbb{N}}$ is a frame and $\sum_{n=1}^{\infty} c_n f_n = 0 \;\Rightarrow\; c_n = 0$ for every $n \in \mathbb{N}$.

2. If $\{f_n\}_{n\in\mathbb{N}}$ is a Riesz basis, then the numbers A, B appearing in (2.4) are actually frame bounds.

See [10, 22] for further details on this subject.

3 Definitions and basic results

Let $\mathcal{H}$, $\mathcal{K}$ be Krein spaces with fundamental symmetries $J_{\mathcal{H}}$ and $J_{\mathcal{K}}$, respectively, and consider a Hilbert space $\mathcal{E}$. Troughout this work, the operators $T \in L(\mathcal{H}, \mathcal{K})$ and $V \in L(\mathcal{H}, \mathcal{E})$ are surjective. Consider the following generalization of the interpolating spline problem [3]:

Problem 1. Given $z_0 \in \mathcal{E}$, find $x_0 \in V^{-1}(\{z_0\})$ such that

$$[\,Tx_0, Tx_0\,]_{\mathcal{K}} = \min\{[\,Tx, Tx\,]_{\mathcal{K}} : \; Vx = z_0\}. \tag{3.1}$$

Definition. Any element $x_0 \in V^{-1}(\{z_0\})$ satisfying Eq. (3.1) is called an *indefinite abstract spline or* (T, V)*-interpolant* to $z_0 \in \mathcal{E}$. The set of (T, V)-interpolants to z_0 is denoted by $sp(T, V, z_0)$.

Considering $V^\dagger$, the Moore-Penrose inverse of V, we can restate the problem as:

Problem 2. Fixed $z_0 \in \mathcal{E}$, find $u_0 \in N(V)$ such that

$$[\,T(V^\dagger z_0 + u_0), T(V^\dagger z_0 + u_0)\,]_{\mathcal{K}} = \min\{[\,T(V^\dagger z_0 + u), T(V^\dagger z_0 + u)\,]_{\mathcal{K}} : \; u \in N(V)\}. \tag{3.2}$$

Lemma 3.1. *Given $z_0 \in \mathcal{E}$, $x_0 \in V^{-1}(\{z_0\})$ is a (T, V)-interpolant to z_0 if and only if $T(N(V))$ is a $J_{\mathcal{K}}$-nonnegative subspace of $\mathcal{K}$ and $Tx_0 \in T(N(V))^{[\perp]_{\mathcal{K}}}$.*

Proof. Suppose that $x_0 \in \mathcal{H}$ is a (T, V)-interpolant to z_0. Then, for every $u \in N(V)$ and $\alpha \in \mathbb{R}$,

$$[\,Tx_0, Tx_0\,] \leq [\,T(x_0 + \alpha u), T(x_0 + \alpha u)\,] = [\,Tx_0, Tx_0\,] + 2\alpha\,\mathrm{Re}[\,Tx_0, Tu\,] + \alpha^2[\,Tu, Tu\,].$$

Therefore, $2\alpha\,\mathrm{Re}[\,Tx_0, Tu\,] + \alpha^2[\,Tu, Tu\,] \geq 0$ for every $\alpha \in \mathbb{R}$, and a standard argument shows that $\mathrm{Re}[\,Tx_0, Tu\,] = 0$. Analogously, if $\beta = i\alpha$, $\alpha \in \mathbb{R}$, it follows that $\mathrm{Im}[\,Tx_0, Tu\,] = 0$. Then, $[\,Tx_0, Tu\,] = 0$ and $[\,Tu, Tu\,] \geq 0$ for every $u \in N(V)$.

Conversely, suppose that $T(N(V))$ is a $J_{\mathcal{K}}$-nonnegative subspace of $\mathcal{K}$ and there exists $x_0 \in \mathcal{H}$ such that $Tx_0\,[\perp]_{\mathcal{K}}\,T(N(V))$. If $u_0 = x_0 - V^{\dagger}z_0 \in N(V)$ then, for every $u \in N(V)$,

$$[\,T(V^{\dagger}z_0 + u), T(V^{\dagger}z_0 + u)\,] = [\,T(V^{\dagger}z_0 + u_0), T(V^{\dagger}z_0 + u_0)\,] + [\,T(u - u_0), T(u - u_0)\,] \geq [\,Tx_0, Tx_0\,].$$

Therefore, x_0 is a (T, V)-interpolant to z_0.

□

Corollary 3.2. *Suppose that $T(N(V))$ is a $J_{\mathcal{K}}$-nonnegative subspace of $\mathcal{K}$ and let $z_0 \in \mathcal{E}$. Then,*

$$sp(T, V, z_0) = (V^{\dagger}z_0 + N(V)) \cap T^{\#}T(N(V))^{[\perp]_{\mathcal{H}}}.$$

Proof. Given $z_0 \in \mathcal{E}$, suppose that $x_0 \in \mathcal{H}$ is a (T, V)-interpolant to z_0. Then, $u_0 = x_0 - V^{\dagger}z_0 \in N(V)$ and by the above lemma, $Tx_0 \in T(N(V))^{[\perp]_{\mathcal{K}}}$, or equivalently by (2.2), $x_0 \in T^{\#}T(N(V))^{[\perp]_{\mathcal{H}}}$. Therefore, $x_0 \in (V^{\dagger}z_0 + N(V)) \cap T^{\#}T(N(V))^{[\perp]_{\mathcal{H}}}$.

On the other hand, if $u \in N(V)$ is such that $x = V^{\dagger}z_0 + u \in T^{\#}T(N(V))^{[\perp]_{\mathcal{H}}}$, then $Tx \in T(N(V))^{[\perp]_{\mathcal{K}}}$ and $Vx = z_0$. So, applying Lemma 3.1, it follows that $x \in sp(T, V, z_0)$. □

Lemma 3.3.

1. *If $T(N(V))$ is non degenerated, then $N(V) \cap T^{\#}T(N(V))^{[\perp]} = N(V) \cap N(T)$.*
2. *If $T(N(V))$ is regular, then $\mathcal{H} = N(V) + T^{\#}T(N(V))^{[\perp]}$.*

Proof. 1. By (2.2), $N(T) \subseteq T^{\#}T(N(V))^{[\perp]}$, so that $N(T) \cap N(V) \subseteq N(V) \cap T^{\#}T(N(V))^{[\perp]}$. Conversely, if $x \in N(V) \cap T^{\#}T(N(V))^{[\perp]}$ then $Tx \in T(N(V)) \cap T(N(V))^{[\perp]} = \{0\}$. Thus, $x \in N(V) \cap N(T)$.
2. If $T(N(V))$ is a regular subspace of $\mathcal{K}$ then $\mathcal{K} = T(N(V)) + T(N(V))^{[\perp]}$. Therefore, $\mathcal{H} = T^{-1}(T(N(V))) + T^{-1}(T(N(V))^{[\perp]}) = N(V) + T^{\#}T(N(V))^{[\perp]}$ (see Eq. (2.2)). □

If $T(N(V))$ is a regular subspace of $\mathcal{K}$ then $\mathcal{H} = N(V) + T^{\#}T(N(V))^{[\perp]}$, but this may not be a direct sum. Therefore, there is a family of subspaces of $T^{\#}T(N(V))^{[\perp]}$ which are complementary to $N(V)$. Along this work, if $T(N(V))$ is a regular subspace of $\mathcal{K}$ we will consider the following projection:

$$Q_0 = P_{N(V)//T^{\#}T(N(V))^{[\perp]} \ominus N(V)}. \tag{3.3}$$

Proposition 3.4. *Suppose that $T(N(V))$ is a closed subspace of $\mathcal{K}$. Then, the set $sp(T, V, z) \neq \varnothing$ for every $z \in \mathcal{E}$ if and only if $T(N(V))$ is $J_{\mathcal{K}}$-uniformly positive. In this case, $sp(T, V, z)$ is an affine manifold parallel to $N(V) \cap N(T)$.*

Proof. Suppose that $T(N(V))$ is a closed $J_{\mathcal{K}}$-uniformly positive subspace of $\mathcal{K}$. Then, by Proposition 2.1, $T(N(V))$ is a regular subspace of $\mathcal{K}$. Therefore, $Q_0 \in \mathcal{Q}$ with $R(Q_0) = N(V)$ and $N(Q_0) \subseteq T^{\#}T(N(V))^{[\perp]_{\mathcal{K}}}$ (see Lemma 3.3).

Fixed $z \in \mathcal{E}$ and let $x = (I - Q_0)V^{\dagger}z \in \mathcal{H}$. Then, $Vx = z$ and $Tx \in T(N(V))^{[\perp]_{\mathcal{K}}}$. So, by Lemma 3.1, $x \in sp(T, V, z)$, i.e. $sp(T, V, z) \neq \varnothing$ for every $z \in \mathcal{E}$.

Conversely, suppose that $sp(T, V, z) \neq \varnothing$ for every $z \in \mathcal{E}$. Then, as a consequence of Lemma 3.1, $T(N(V))$ is a $J_{\mathcal{K}}$-nonnegative subspace of $\mathcal{K}$. Furthermore, for each $z \in \mathcal{E}$, there exists a vector $x_z \in \mathcal{H}$ such that $Vx_z = z$ and $Tx_z \in T(N(V))^{[\perp]_{\mathcal{K}}}$. Since $V^{\dagger}z = (V^{\dagger}z - x_z) + x_z$ and $V(V^{\dagger}z - x_z) = 0$ for every $z \in \mathcal{E}$, it is easy to see that $N(V)^{\perp} \subseteq N(V) + T^{\#}T(N(V))^{[\perp]_{\mathcal{K}}}$. Therefore, $\mathcal{H} = N(V) + T^{\#}T(N(V))^{[\perp]_{\mathcal{K}}}$ and $\mathcal{K} = T(N(V)) + T(N(V))^{[\perp]_{\mathcal{K}}}$. So, $T(N(V))$ is a regular $J_{\mathcal{K}}$-nonnegative subspace of $\mathcal{K}$, i.e. $T(N(V))$ is a uniformly $J_{\mathcal{K}}$-positive subspace of $\mathcal{K}$.

Assuming that $T(N(V))$ is $J_{\mathcal{K}}$-uniformly positive, observe that, if $x_1, x_2 \in sp(T, V, z)$ then, applying Lemma 3.3,

$$x_1 - x_2 \in N(V) \cap T^{\#}T(N(V))^{[\perp]_{\mathcal{H}}} = N(V) \cap N(T).$$ □

Corollary 3.5. *Suppose that $T(N(V))$ is a closed $J_{\mathcal{K}}$-uniformly positive subspace of $\mathcal{K}$ and $N(T) \cap N(V) = \{0\}$. Then, given $z \in \mathcal{E}$, $sp(T, V, z)$ is a singleton. More precisely,*

$$sp(T, V, z) = \{(I - Q_0)V^{\dagger}z\}.$$

In what follows we show that the set $sp(T,V,z_0)$ can be parametrized by means of a family of projections with range $N(V)$.

Proposition 3.6. *Suppose that $T(N(V))$ is a closed $J_{\mathcal{K}}$-uniformly positive subspace of $\mathcal{K}$. Given $z_0 \in \mathcal{E}$, $x \in sp(T,V,z_0)$ if and only if there exists $Q \in \mathcal{Q}$ with $R(Q) = N(V)$ and $N(Q) \subseteq T^{\#}T(N(V))^{[\perp]}$ such that $x = (I-Q)V^{\dagger}z_0$.*

To prove the above proposition, we need the following lemma.

Lemma 3.7. *Suppose that $T(N(V))$ is a regular subspace of $\mathcal{K}$. Let $Q \in \mathcal{Q}$. Then, $R(Q) = N(V)$ and $N(Q) \subseteq T^{\#}T(N(V))^{[\perp]}$ if and only if there exists an operator $Z \in L(\mathcal{H})$ such that $N(V) \subseteq N(Z)$, $R(Z) \subseteq N(V) \cap N(T)$ and $Q = Q_0 + Z$.*

Proof. If $Q \in L(\mathcal{H})$ is a projection with $R(Q) = N(V)$ and $N(Q) \subseteq T^{\#}T(N(V))^{[\perp]}$, let $Z = Q - Q_0$. Since $R(Q) = R(Q_0) = N(V)$ it is trivial that $N(V) \subseteq N(Z)$. On the other hand, consider $y = Zx \in R(Z)$: $y = Qx - Q_0x \in N(V)$ and $y = (I-Q_0)x - (I-Q)x \in T^{\#}T(N(V))^{[\perp]}$. Then $y \in N(V) \cap T^{\#}T(N(V))^{[\perp]}$.

Conversely, given $Z \in L(\mathcal{H})$ with $N(V) \subseteq N(Z)$ and $R(Z) \subseteq N(V) \cap N(T)$, consider $Q = Q_0 + Z$. Then, $Q^2 = Q$ because $Z^2 = 0$, $Q_0Z = Z$ and $ZQ_0 = 0$. It is easy to see that $R(Q) \subseteq N(V)$ and, if $x \in N(V)$ then $Qx = Q_0x = x$. Therefore, $R(Q) = N(V)$. Finally, observe that if $x \in N(Q)$ then $x = (I-Q)x = (I-Q_0)x - Zx \in T^{\#}T(N(V))^{[\perp]}$, because $N(Q_0), R(Z) \subseteq T^{\#}T(N(V))^{[\perp]}$. □

Proof. (of Proposition 3.6) If $x = (I-Q)V^{\dagger}z_0$, where $Q \in \mathcal{Q}$ with $R(Q) = N(V)$ and $N(Q) \subset T^{\#}T(N(V))^{[\perp]}$, it is easy to see that $Vx = z_0$ and $Tx \in T(N(V))^{[\perp]}$. Then, by Lemma 3.1, $x \in sp(T,V,z_0)$.

Conversely, as a consequence of Proposition 3.4, $sp(T,V,z_0) = (I-Q_0)V^{\dagger}z_0 + N(V) \cap N(T)$ because $(I-Q_0)V^{\dagger}z_0 \in sp(T,V,z_0)$. Then, if $x \in sp(T,V,z_0)$ there exists $u \in N(V) \cap N(T)$ such that $x = (I-Q_0)V^{\dagger}z_0 + u$. So, consider $Z \in L(\mathcal{H})$ such that $Z(V^{\dagger}z_0) = -u$ and $Zy = 0$ if $y \perp V^{\dagger}z_0$. Then,

$$x = (I-Q_0)V^{\dagger}z_0 - ZV^{\dagger}z_0 = (I-(Q_0+Z))V^{\dagger}z_0,$$

$N(V) \subseteq N(Z)$ and $R(Z) \subseteq N(V) \cap N(T)$. Therefore, by the above proposition, $Q = Q_0 + Z \in \mathcal{Q}$ with $R(Q) = N(V)$ and $N(Q) \subset T^{\#}T(N(V))^{[\perp]}$. □

3.1 Frames of indefinite abstract splines

In what follows, recall that $T \in L(\mathcal{H},\mathcal{K})$ and $V \in L(\mathcal{H},\mathcal{E})$ are surjective operators and suppose that $T(N(V))$ is a closed $J_{\mathcal{K}}$-uniformly positive subspace of $\mathcal{K}$.

Proposition 3.8. *Given a sequence $\{f_n\}_{n\in\mathbb{N}}$ in $\mathcal{E}$, suppose that there exists a frame $\{g_n\}_{n\in\mathbb{N}}$ for $\mathcal{W} = T^{\#}T(N(V))^{[\perp]}$ such that $g_n \in sp(T,V,f_n)$ for every $n \in \mathbb{N}$. Then, $\{f_n\}_{n\in\mathbb{N}}$ is a frame for $\mathcal{E}$.*

Proof. If $g_n \in sp(T,V,f_n)$ then, by Proposition 3.6, there exists $Q_n \in \mathcal{Q}$ with $R(Q_n) = N(V)$ and $N(Q_n) \subseteq \mathcal{W}$, such that $g_n = (I-Q_n)V^{\dagger}f_n$. Since $V(I-Q_n)V^{\dagger} = I_{\mathcal{E}}$ for every $n \in \mathbb{N}$, it is easy to see that

$$\sum_{n=1}^{\infty} |\langle z, f_n \rangle|^2 = \sum_{n=1}^{\infty} |\langle V^*z, (I-Q_n)V^{\dagger}f_n \rangle|^2 = \sum_{n=1}^{\infty} |\langle P_{\mathcal{W}}V^*z, g_n \rangle|^2, \quad \text{for every } z \in \mathcal{E},$$

since $P_{\mathcal{W}}(I-Q_n) = (I-Q_n)$. Therefore, if $\{g_n\}_{n\in\mathbb{N}}$ is a frame for $\mathcal{W}$ with frame bounds $0 < A < B$,

$$A\|P_{\mathcal{W}}V^*z\|^2 \leq \sum_{n=1}^{\infty} |\langle z, f_n \rangle|^2 \leq B\|P_{\mathcal{W}}V^*z\|^2 \leq B\|V\|^2\|z\|^2,$$

for every $z \in \mathcal{E}$. But $\|P_{\mathcal{W}}V^*z\|^2 \geq \gamma(P_{\mathcal{W}}V^*)^2\|z\|^2 = \gamma(VP_{\mathcal{W}})^2\|z\|^2$. Since $c(\mathcal{W}, N(V)) < 1$ it follows by Proposition 2.3 that $VP_{\mathcal{W}}$ has closed range, so $\gamma(VP_{\mathcal{W}}) > 0$. Then, $\{f_n\}_{n\in\mathbb{N}}$ is a frame for $\mathcal{E}$, with frame bounds $0 < A\gamma(VP_{\mathcal{W}})^2 < B\|V\|^2$. □

Given a sequence $\{f_n\}_{n\in\mathbb{N}}$ in $\mathcal{E}$, if $N(T)\cap N(V)=\{0\}$ it is easy to see that $\{f_n\}_{n\in\mathbb{N}}$ is a frame for $\mathcal{E}$ if and only if $\{g_n\}_{n\in\mathbb{N}}$ is a frame for $T^\# T(N(V))^{[\perp]}$, where g_n is the (unique) (T,V)-interpolant to f_n (see Proposition 3.9 below for a proof of this assertion). However, the following example shows that, if $N(T)\cap N(V)\neq\{0\}$, given a frame $\{f_n\}_{n\in\mathbb{N}}$ for $\mathcal{E}$ it is easy to construct $g_n\in sp(T,V,f_n)$ (for every $n\in\mathbb{N}$) such that $\{g_n\}_{n\in\mathbb{N}}$ is not a frame for $|\mathcal{H}|$.

Example 1. Suppose that $N(T)\cap N(V)\neq\{0\}$. Given a frame $\{f_n\}_{n\in\mathbb{N}}$ for $\mathcal{E}$, we construct a sequence $\{g_n\}_{n\in\mathbb{N}}$ such that $g_n\in sp(T,V,f_n)$ for every $n\in\mathbb{N}$ but $\{g_n\}_{n\in\mathbb{N}}$ is not a frame for $|\mathcal{H}|$.

Observe that if $\{f_n\}_{n\in\mathbb{N}}$ is a frame with frame bounds $0<A<B$ then $\|f_n\|^2\leq B$. Given $u\in N(T)\cap N(V)$ with $\|u\|=1$, define

$$Z_n(x)=\begin{cases} n\alpha u & \text{if } x=\alpha V^\dagger f_n,\ \alpha\in\mathbb{C};\\ 0 & \text{if } x\perp V^\dagger f_n.\end{cases}$$

Then, $Z_n\in L(\mathcal{H})$ and satisfies $N(V)\subseteq N(Z_n)$ and $R(Z_n)\subseteq N(T)\cap N(V)$. Furthermore, by Lemma 3.7, $Q_n=Q_0+Z_n$ is a projection with $R(Q_n)=N(V)$ and $N(Q_n)\subseteq T^\# T(N(V))^{[\perp]}$. Therefore, $g_n=(I-Q_n)V^\dagger f_n\in sp(T,V,f_n)$ for every $n\in\mathbb{N}$.

But observe that $\{g_n\}_{n\in\mathbb{N}}$ can not be a frame for $|\mathcal{H}|$ because $\|g_n\|\to+\infty$ as $n\to\infty$. Indeed, it is easy to see that

$$\|g_n\|\geq\|Z_nV^\dagger f_n\|-\|(I-Q)V^\dagger f_n\|\geq n-\|I-Q\|\|V^\dagger\|B^{1/2}\to+\infty \quad \text{as } n\to\infty.$$

The next result shows that, given a frame $\{f_n\}_{n\in\mathbb{N}}$ for $\mathcal{E}$, if $N(T)\cap N(V)\neq\{0\}$ there are several ways to obtain frames of splines for any complement of $N(V)$ contained in $T^\# T(N(V))^{[\perp]}$, associated to $\{f_n\}_{n\in\mathbb{N}}$.

Proposition 3.9. *Given a sequence $\{f_n\}_{n\in\mathbb{N}}$ in $\mathcal{E}$, consider $g_n=(I-Q)V^\dagger f_n\in sp(T,V,f_n)$, $n\in\mathbb{N}$, where $Q\in L(\mathcal{H})$ is any fixed projection such that $R(Q)=N(V)$ and $N(Q)\subseteq T^\# T(N(V))^{[\perp]}$. Then,*

1. *$\{f_n\}_{n\in\mathbb{N}}$ is a frame for $\mathcal{E}$ if and only if $\{g_n\}_{n\in\mathbb{N}}$ is a frame for $N(Q)$.*
2. *$\{f_n\}_{n\in\mathbb{N}}$ is a Riesz basis of $\mathcal{E}$ if and only if $\{g_n\}_{n\in\mathbb{N}}$ is a Riesz basis of $N(Q)$.*
3. *$\{f_n\}_{n\in\mathbb{N}}$ is a (Schauder) basis of $\mathcal{E}$ if and only if $\{g_n\}_{n\in\mathbb{N}}$ is a (Schauder) basis of $N(Q)$.*

Proof. Observe that, if $W=(I-Q)V^\dagger$, then $R(W)=R(I-Q)=N(Q)$ is closed. Then, $\gamma(W)>0$.
1. Suppose that $\{f_n\}_{n\in\mathbb{N}}$ is a frame for $\mathcal{E}$. Notice that $\sum_{n=1}^\infty|\langle x,g_n\rangle|^2=\sum_{n=1}^\infty|\langle x,Wf_n\rangle|^2=\sum_{n=1}^\infty|\langle W^*x,f_n\rangle|^2$ for every $x\in\mathcal{H}$. So, if $0<A<B$ are frame bounds for $\{f_n\}_{n\in\mathbb{N}}$ then

$$A\gamma(W)^2\|x\|^2=A\gamma(W^*)^2\|x\|^2\leq A\|W^*x\|^2\leq\sum_{n=1}^\infty|\langle x,g_n\rangle|^2\leq B\|W^*x\|^2\leq B\|W\|^2\|x\|^2,$$

for every $x\in N(W^*)^\perp=N(Q)$. Therefore, $\{g_n\}_{n\in\mathbb{N}}$ is a frame for $N(Q)$. The other implication is a consequence of Proposition 3.8.
2. Suppose that $\{f_n\}_{n\in\mathbb{N}}$ is a Riesz basis of $\mathcal{E}$. Then it is also a frame for $\mathcal{E}$ and, by item *1*, the sequence $\{g_n\}_{n\in\mathbb{N}}$ is a frame for $N(Q)$. Furthermore, if there exists a sequence $(\alpha_k)_{k\in\mathbb{N}}$ such that $\sum_{k=1}^\infty\alpha_k g_k=0$, then applying V to both sides of the equation we obtain that $\sum_{k=1}^\infty\alpha_k f_k=0$. So, $\alpha_k=0$ for every $k\in\mathbb{N}$ because $\{f_n\}_{n\in\mathbb{N}}$ is a Riesz basis of $\mathcal{E}$. Therefore, $\{g_n\}_{n\in\mathbb{N}}$ is a Riesz basis of $N(Q)$. The other implication follows in the same way.
3. It is analogous to the proof of [4, Chapter III, Proposition 1.1]. □

4 Indefinite smoothing splines

Let $\mathcal{H}$ and $\mathcal{K}$ be Krein spaces with fundamental symmetries $J_\mathcal{H}$ and $J_\mathcal{K}$, respectively, and consider a Hilbert space $\mathcal{E}$. Given surjective operators $T\in L(\mathcal{H},\mathcal{K})$ and $V\in L(\mathcal{H},\mathcal{E})$, consider the following generalization of the smoothing problem [4]:

Problem 3. Given $\rho > 0$ and fixed $z_0 \in \mathcal{E}$, find $x_0 \in \mathcal{H}$ such that

$$[\,Tx_0, Tx_0\,]_{\mathcal{K}} + \rho \|Vx_0 - z_0\|^2_{\mathcal{E}} = \min_{x \in \mathcal{H}} \left([\,Tx, Tx\,]_{\mathcal{K}} + \rho \|Vx - z_0\|^2_{\mathcal{E}} \right). \tag{4.1}$$

Definition. Any element $x_0 \in \mathcal{H}$ satisfying Eq. (4.1) is called a (T, V, ρ)-*smoothing spline* to $z_0 \in \mathcal{E}$. The set of (T, V, ρ)-smoothing splines to z_0 is denoted by $sm(T, V, \rho, z_0)$.

To study this problem consider the indefinite metric defined on $\mathcal{K} \times \mathcal{E}$ by:

$$[\,(y, z), (y', z')\,]_{\mathcal{K}\times\mathcal{E}} = [\,y, y'\,]_{\mathcal{K}} + \rho \,\langle\, z, z'\,\rangle_{\mathcal{E}}\,, \tag{4.2}$$

where $y, y' \in \mathcal{K}$ and $z, z' \in \mathcal{E}$. Notice that, $\mathcal{K} \times \mathcal{E}$ is a Krein space with the indefinite metric defined above. In fact, considering the inner product $\langle\ ,\ \rangle_\rho$ in $\mathcal{E}$ given by $\langle\, z, z'\,\rangle_\rho = \rho \,\langle\, z, z'\,\rangle$ for every $z, z' \in \mathcal{E}$, the operator $J \in L(\mathcal{K} \times \mathcal{E})$ defined as

$$J(y, z) = (J_{\mathcal{K}} y, z), \qquad y \in \mathcal{K},\ z \in \mathcal{E},$$

is the fundamental symmetry associated to $\mathcal{K} \times \mathcal{E}$. Consider, also, the operator $L : \mathcal{H} \to \mathcal{K} \times \mathcal{E}$ defined by

$$Lx = (Tx, Vx), \qquad x \in \mathcal{H}.$$

The proof of the next lemma is analogous to the one given in [4, Chapter III, Lemma 2.1] for the Hilbert space case.

Lemma 4.1. *If $T(N(V))$ is a closed subspace of $\mathcal{K}$ then $R(L)$ is a closed subspace of $\mathcal{K} \times \mathcal{E}$.*

Proof. Given $y \in \mathcal{K}$ and $z \in \mathcal{E}$, suppose that $\{x_n\}_{n\geq 1} \subseteq \mathcal{H}$ is such that $Tx_n \to y$ and $Vx_n \to z$. If $v_n = V^\dagger V x_n \in \mathcal{H}$, then $v_n \to V^\dagger z \in \mathcal{H}$ and $u_n = x_n - v_n \in N(V)$. Therefore, $Vv_n = Vx_n \to z$ and $Tu_n \to y - TV^\dagger z$.

Since $T(N(V))$ is a closed subspace of $\mathcal{K}$, the operator $W = T|_{N(V)} : N(V) \to \mathcal{K}$ has closed range. Thus, $\{x_n\}_{n\geq 1}$ converges to some $x \in \mathcal{H}$ because $x_n = v_n + u_n = v_n + W^\dagger T u_n$. Furthermore, $Tx = y$ and $Vx = z$. Therefore, $R(L)$ is a closed subspace of $\mathcal{K} \times \mathcal{E}$. □

Therefore, assuming that $T(N(V))$ is a closed subspace of $\mathcal{K}$, observe that Problem 3 can be restated as the following indefinite least squares problem:

Problem 4. Given $\rho > 0$ and a fixed $z_0 \in \mathcal{E}$, find $x_0 \in \mathcal{H}$ such that

$$[\,Lx_0 - (0, z_0), Lx_0 - (0, z_0)\,]_{\mathcal{K}\times\mathcal{E}} = \min_{x \in \mathcal{H}} [\,Lx - (0, z_0), Lx - (0, z_0)\,]_{\mathcal{K}\times\mathcal{E}}. \tag{4.3}$$

The next result characterizes the solutions of the indefinite smoothing problem.

Lemma 4.2. *Given $z_0 \in \mathcal{E}$, $x_0 \in \mathcal{H}$ is a solution of Problem 4 if and only if $R(L)$ is J-nonnegative and x_0 is a solution of the equation:*

$$(T^\# T + \rho V^\# V)x = \rho V^\# z_0.$$

Proof. Following the same arguments as in Lemma 3.1, it is easy to see that $x_0 \in \mathcal{H}$ satisfies Eq. (4.3) if and only if $R(L)$ is J-nonnegative and

$$[\,Lx_0 - (0, z_0), Ly\,] = 0, \qquad \text{for every } y \in \mathcal{H}.$$

or equivalently, $L^\#(Lx_0 - (0, z_0)) = 0$. Since $L^\# \in L(\mathcal{K} \times \mathcal{E}, \mathcal{H})$ is given by $L^\#(y, z) = T^\# y + \rho V^\# z$, $(y, z) \in \mathcal{K} \times \mathcal{E}$, it follows that $(T^\# T + \rho V^\# V)x_0 = \rho V^\# z_0$. □

Proposition 4.3. *Suppose that $T(N(V))$ is a closed subspace of $\mathcal{K}$. Then, Problem 4 admits a solution for every $z \in \mathcal{E}$ if and only if $R(L)$ is a (closed) J-uniformly positive subspace of $\mathcal{K} \times \mathcal{E}$.*

Proof. Suppose that, for every $z \in \mathcal{E}$ there exists $u \in \mathcal{H}$ such that

$$[\,Lu - (0,z), Lu - (0,z)\,] = \min_{x\in\mathcal{H}} [\,Lx - (0,z), Lx - (0,z)\,].$$

Applying Lemma 4.2, it follows that $R(L)$ is J-nonnegative. Given $(y,z) \in \mathcal{K}\times\mathcal{E}$, consider $w = u' + T^{\dagger}y$, where $u' \in \mathcal{H}$ satisfies

$$[\,Lu' - (0, z - VT^{\dagger}y), Lu' - (0, z - VT^{\dagger}y)\,] = \min_{x\in\mathcal{H}} [\,Lx - (0, z - VT^{\dagger}y), Lx - (0, z - VT^{\dagger}y)\,].$$

Then, for every $x \in \mathcal{H}$,

$$\begin{aligned}
[\,Lw - (y,z), Lw - (y,z)\,] &= [\,Lu' + (y, VT^{\dagger}y) - (y,z), Lu' + (y, VT^{\dagger}y) - (y,z)\,] \\
&= [\,Lu' - (0, z - VT^{\dagger}y), Lu' - (0, z - VT^{\dagger}y)\,] \\
&\le [\,L(x - T^{\dagger}y) - (0, z - VT^{\dagger}y), L(x - T^{\dagger}y) - (0, z - VT^{\dagger}y)\,] \\
&= [\,Lx - (y,z), Lx - (y,z)\,].
\end{aligned}$$

Therefore, for every $(y,z) \in \mathcal{K}\times\mathcal{E}$, there exists $w \in \mathcal{H}$ such that

$$[\,Lw - (y,z), Lw - (y,z)\,] = \min_{x\in\mathcal{H}} [\,Lx - (y,z), Lx - (y,z)\,].$$

Furthermore, for every $(y,z) \in \mathcal{K}\times\mathcal{E}$ there exists $w \in \mathcal{H}$ such that $Lw - (y,z) \in R(L)^{[\perp]}$. So, $\mathcal{K}\times\mathcal{E} = R(L) + R(L)^{[\perp]}$, i.e. $R(L)$ is a regular subspace of $\mathcal{K}\times\mathcal{E}$. Thus, $R(L)$ is a (closed) uniformly positive subspace of $\mathcal{K}\times\mathcal{E}$.

The converse implication follows from Corollary 2.2, considering a J-selfadjoint projection $Q \in L(\mathcal{K}\times\mathcal{E})$ onto $R(L)$. □

4.1 Every smoothing spline is an interpolating spline

This subsection is devoted to show that $sm(T,V,\rho,z_0) = sp(T,V,z')$ for a suitable $z' \in \mathcal{E}$.

Lemma 4.4. *If $T(N(V))$ is a regular subspace of $\mathcal{K}$ then, if Q_0 is the projection considered in Eq.* (3.3),

$$R(L) = (T(N(V))\times\{0\}) \dotplus \{(T(I-Q_0)V^{\dagger}z, z) :\ z \in \mathcal{E}\}.$$

Furthermore, this decomposition of $R(L)$ is orthogonal respect to the indefinite metric defined in (4.2).

Proof. Since $R(Q_0) = N(V)$, observe that $R(L) = L(N(V)) + L(N(Q_0))$ and $L(N(V)) = T(N(V))\times\{0\}$. In order to compute $L(N(Q_0))$, consider $W = (I-Q_0)V^{\dagger}$. It follows that $VW = I_{\mathcal{E}}$ and $WV = I - Q_0$. Therefore, $WVx = x$ for every $x \in N(Q_0)$, so if $z = Vx$ then $Wz = (I-Q_0)x = x$ and $Tx = TWz = T(I-Q_0)V^{\dagger}z$, i.e.

$$L(N(Q_0)) = \{(T(I-Q_0)V^{\dagger}z, z) :\ z \in \mathcal{E}\}.$$

Furthermore, since $T(N(Q_0)) \subseteq T(N(V))^{[\perp]}$, it follows that $L(N(V))\ [\perp]\ L(N(Q_0))$. □

Theorem 4.5. *Suppose that $T(N(V))$ is a closed subspace of $\mathcal{K}$ and $R(L)$ is a uniformly positive subspace of $\mathcal{K}\times\mathcal{E}$. Then, given $z_0 \in \mathcal{E}$, $sm(T,V,\rho,z_0) = sp(T,V,z')$, where z' is an adequate vector in $\mathcal{E}$.*

Proof. If $z_0 = 0$ then $sm(T,V,\rho,0) = N(L) = N(T)\cap N(V) = sp(T,V,0)$. On the other hand, notice that $T(N(V))\times\{0\}$ is a regular subspace of $\mathcal{K}\times\mathcal{E}$ because it is a closed subspace of $R(L)$ and $R(L)$ is uniformly positive. Then, since $T(N(V))$ is a regular subspace of $\mathcal{K}$, the projection considered in Eq. (3.3) is bounded. So, given $x \in \mathcal{H}$, it can be decomposed as

$$x = Q_0x + (I-Q_0)x = v + (I-Q_0)V^{\dagger}z,$$

where $v = Q_0x \in N(V)$ and $z = Vx \in \mathcal{E}$.

Observe that, by Lemma 4.4,

$$[\,Lx - (0,z_0), Lx - (0,z_0)\,] = [\,(Tv,0),(Tv,0)\,] + [\,(T(I-Q_0)V^{\dagger}z, z - z_0), (T(I-Q_0)V^{\dagger}z, z - z_0)\,].$$

Then, $x_0 \in sm(T,V,\rho,z_0)$ if and only if $[\,TQ_0x_0, TQ_0x_0\,] = \min_{u\in N(V)} [\,Tu, Tu\,]$ and $z_1 = Vx_0$ satisfies

$$[\,(T(I-Q_0)V^\dagger z_1, z_1-z_0), (T(I-Q_0)V^\dagger z_1, z_1-z_0)\,] = \min_{z\in\mathcal{E}}[\,(T(I-Q_0)V^\dagger z, z-z_0), (T(I-Q_0)V^\dagger z, z-z_0)\,]. \tag{4.4}$$

Notice that $\min_{u\in N(V)}[\,Tu,Tu\,]$ is attained at every $u \in N(T)\cap N(V)$, because $T(N(V))$ is uniformly positive. Therefore, $Q_0x_0 \in N(T) \cap N(V)$.

On the other hand, to characterize z_1, consider the (bounded) operator $W : \mathcal{E} \to \mathcal{K} \times \mathcal{E}$ given by

$$Wz = (T(I-Q_0)V^\dagger z, z).$$

Observe that $N(W) = \{0\}$ and $R(W)$ is closed (because it is isometrically isomorphic to the graph of the bounded operator $T(I-Q_0)V^\dagger$). Then, Eq. (4.4) is equivalent to

$$[\,Wz_1 - (0,z_0), Wz_1 - (0,z_0)\,] = \min_{z\in\mathcal{E}}[\,Wz - (0,z_0), Wz - (0,z_0)\,].$$

Since $R(W)$ is a regular subspace of $R(L)$ (see Lemma 4.4), $R(W)$ is a (closed) uniformly positive subspace of $\mathcal{K}\times\mathcal{E}$. Thus, by Corollary 2.2, z_1 satisfies the above equation if and only if $Wz_1 = P(0,z_0)$, where P is the J-selfadjoint projection onto $R(W)$. If $S : \mathcal{K}\times\mathcal{E} \to \mathcal{E}$ is defined as $S(y,z) = z$ then $SW = I_\mathcal{E}$ and $z_1 = SWz_1 = SP(0,z_0)$. So, $(I-Q_0)V^\dagger z_1 = (I-Q_0)V^\dagger SP(0,z_0)$.

Therefore, $x_0 \in sm(T,V,\rho,z_0)$ if and only if $x_0 \in (I-Q_0)V^\dagger SP(0,z_0) + N(T)\cap N(V)$, i.e.

$$sm(T,V,\rho,z_0) = sp(T,V,SP(0,z_0)).$$ □

4.2 The smoothing splines converge to the interpolating spline

In the following paragraph we show that, given $z_0 \in \mathcal{E}$, if $\{x_\rho\}_{\rho>0}$ is a net in $\mathcal{H}$ such that $x_\rho \in sm(T,V,\rho,z_0)$, $\rho > 0$, then it converges to an interpolating spline $x_0 \in sp(T,V,z_0)$ as $\rho\to\infty$. The proof of this result is analogous to [4, Chapter III, Proposition 2.2].

Proposition 4.6. *Given a fixed vector $z_0 \in \mathcal{E}$, suppose that $T(N(V))$ is a closed subspace of $\mathcal{K}$ and $R(L)$ is a uniformly positive subspace of $\mathcal{K}\times\mathcal{E}$. Let $x_\rho \in sm(T,V,\rho,z_0)$ for every $\rho > 0$. Then, there exists $x_0 \in sp(T,V,z_0)$ such that*

$$\lim_{\rho\to\infty} \|x_\rho - x_0\| = 0.$$

Proof. First, notice that if $x_\rho \in sm(T,V,\rho,z_0)$, $\rho > 0$, then $\{[\,Tx_\rho, Tx_\rho\,]\}_{\rho>0}$ is an increasing net in $\mathbb{R}$ with an upper bound, and $\|Vx_\rho - z_0\| \to 0$ as $\rho\to\infty$.

Indeed, given $\rho_1,\rho_2 > 0$, notice that $[\,Tx_{\rho_i}, Tx_{\rho_i}\,] + \rho_i\|Vx_{\rho_i} - z_0\|^2 \le [\,Tx_{\rho_j}, Tx_{\rho_j}\,] + \rho_i\|Vx_{\rho_j} - z_0\|^2$, if $i \ne j$. Then, if $\rho_1 < \rho_2$ it follows that $\|Vx_{\rho_1} - z_0\|^2 - \|Vx_{\rho_2} - z_0\|^2 \ge 0$ and

$$[\,Tx_{\rho_2}, Tx_{\rho_2}\,] - [\,Tx_{\rho_1}, Tx_{\rho_1}\,] \ge \rho_1(\|Vx_{\rho_1} - z_0\|^2 - \|Vx_{\rho_2} - z_0\|^2) \ge 0.$$

Furthermore, if $x \in sp(T,V,z_0)$, for every $\rho > 0$, $[\,Tx_\rho, Tx_\rho\,] + \rho\|Vx_\rho - z_0\|^2 \le [\,Tx, Tx\,] + \rho\|Vx - z_0\|^2 = [\,Tx, Tx\,]$. So, $[\,Tx, Tx\,] - [\,Tx_\rho, Tx_\rho\,] \ge \rho\|Vx_\rho - z_0\|^2 \ge 0$ for every $\rho > 0$, and this inequality implies that

$$\lim_{\rho\to\infty} \|Vx_\rho - z_0\| = 0.$$

The next step is to prove that $\lim_{\rho\to\infty}\|x_\rho - x_0\| = 0$, where $x_0 = V^\dagger z_0 + u$ for some $u \in N(V)$.

Let $y_\rho = P_{N(V)^\perp}x_\rho$ and observe that $y_\rho = V^\dagger V x_\rho \to V^\dagger z_0$ as $\rho\to\infty$. If $u_\rho = x_\rho - y_\rho = P_{N(V)}x_\rho \in N(V)$, we are going to show that $\{u_\rho\}_{\rho\ge1}$ converges to some $u \in N(V)$.

For this purpose, we shall consider a particular generalized inverse of the closed range operator $W = T|_{N(V)} : N(V) \to \mathcal{K}$ (see Lemma 4.1). If $Q = P_{T(N(V))//T(N(V))^{[\perp]}}$, let $W' = W^\dagger Q$. It is easy to see that $WW'W = W$, $W'WW' = W'$ and $N(W') = T(N(V))^{[\perp]}$.

Applying Theorem 4.5 $x_\rho \in sp(T,V,z_\rho)$ for a suitable $z_\rho \in \mathcal{E}$; then, it follows that $Tx_\rho \in T(N(V))^{[\perp]}$ (see Lemma 3.1). Therefore, $W'Tx_\rho = 0$ for every $\rho \ge 1$, and

$$W'Tu_\rho = -W'Ty_\rho \to -W'TV^\dagger z_0 = u \in R(W') \subseteq N(V) \quad \text{as } \rho\to\infty.$$ □

5 The indefinite mixed problem

Given Hilbert spaces $\mathcal{E}_1$ and $\mathcal{E}_2$, and Krein spaces $\mathcal{H}$ and $\mathcal{K}$ with fundamental symmetries $J_{\mathcal{H}}$ and $J_{\mathcal{K}}$ respectively, let $T \in L(\mathcal{H}, \mathcal{K})$, $V_1 \in L(\mathcal{H}, \mathcal{E}_1)$ and $V_2 \in L(\mathcal{H}, \mathcal{E}_2)$ be surjective operators. Then, consider the following problem:

Problem 5. Let $\rho > 0$. Fixed $(z_1, z_2) \in \mathcal{E}_1 \times \mathcal{E}_2$, estimate

$$\min \left([Tx, Tx]_{\mathcal{K}} + \rho \|V_2 x - z_2\|^2_{\mathcal{E}_2} \right) \quad \text{subject to } V_1 x = z_1. \tag{5.1}$$

This is a generalization to Krein spaces of the mixed problem for abstract splines in Hilbert spaces proposed by A. I. Rozhenko and V. A. Vasilenko in [19].

As in the previous section, $\mathcal{K} \times \mathcal{E}_2$ is a Krein space with the indefinite metric defined in Eq. (4.2) and its fundamental symmetry $J \in L(\mathcal{K} \times \mathcal{E}_2)$ is given by $J(y, z) = (J_{\mathcal{K}} y, z)$, $y \in \mathcal{K}$, $z \in \mathcal{E}_2$. Also, consider the operators $L \in L(\mathcal{H}, \mathcal{K} \times \mathcal{E}_2)$ given by

$$Lx = (Tx, V_2 x), \qquad x \in \mathcal{H},$$

and $L_1 = LP_{N(V_1)} \in L(\mathcal{H}, \mathcal{K} \times \mathcal{E}_2)$. Then, Problem 5 can be restated as:

Problem 6. Given $\rho > 0$ and fixed $(z_1, z_2) \in \mathcal{E}_1 \times \mathcal{E}_2$, estimate

$$\min_{x \in \mathcal{H}} [\, L_1 x - (w_1, w_2), L_1 x - (w_1, w_2) \,]_{\mathcal{K} \times \mathcal{E}_2}, \tag{5.2}$$

where $w_1 = -TV_1^{\dagger} z_1$ and $w_2 = z_2 - V_2 V_1^{\dagger} z_1$.

Remark 5.1. If $N(V_1) + N(V_2)$ is closed then it is easy to see that $R(L_1)$ is closed. In fact, $R(L_1) = R(LP_{N(V_1)})$ is closed if and only if $c(N(T) \cap N(V_2), N(V_1)) < 1$ (see Proposition 2.3). But,

$$c(N(T) \cap N(V_2), N(V_1)) \leq c(N(V_2), N(V_1)) < 1,$$

since $N(V_1) + N(V_2)$ is closed.

Lemma 5.2. *Given $(z_1, z_2) \in \mathcal{E}_1 \times \mathcal{E}_2$, $x_0 \in \mathcal{H}$ is a solution of Problem 6 if and only if $R(L_1)$ is J-nonnegative and x_0 is a solution of the equation:*

$$P^{\#}_{N(V_1)} (T^{\#} T + \rho V_2^{\#} V_2) P_{N(V_1)} x_0 = P^{\#}_{N(V_1)} (T^{\#} w_1 + \rho V_2^{\#} w_2).$$

Proof. It is analogous to the proof of Lemma 4.2. Notice that, in this case, $L_1^{\#} \in L(\mathcal{K} \times \mathcal{E}_2, \mathcal{H})$ is given by $L_1^{\#}(y, z) = P^{\#}_{N(V_1)} L^{\#}(y, z) = P^{\#}_{N(V_1)} (T^{\#} y + \rho V_2^{\#} z)$, $(y, z) \in \mathcal{K} \times \mathcal{E}_2$. □

Proposition 5.3. *Suppose that $N(V_1) + N(V_2)$ is a closed subspace of $\mathcal{H}$. Then, Problem 6 admits a solution for every $(z_1, z_2) \in \mathcal{E}_1 \times \mathcal{E}_2$ if and only if $R(L_1)$ is a (closed) J-uniformly positive subspace of $\mathcal{K} \times \mathcal{E}_2$.*

Proof. Suppose that, for every $(z_1, z_2) \in \mathcal{E}_1 \times \mathcal{E}_2$ there exists $u \in \mathcal{H}$ such that

$$[\, L_1 u - (w_1, w_2), L_1 u - (w_1, w_2) \,] = \min_{x \in \mathcal{H}} [\, L_1 x - (w_1, w_2), L_1 x - (w_1, w_2) \,],$$

where $w_1 = -TV_1^{\dagger} z_1$ and $w_2 = z_2 - V_2 V_1^{\dagger} z_1$.

Given $(y, z) \in \mathcal{K} \times \mathcal{E}_2$, let $z_1 = -V_1 T^{\dagger} y$ and $z_2 = z - V_2 T^{\dagger} y$. Consider $x_0 = u' + T^{\dagger} y$, where $u' \in \mathcal{H}$ satisfies

$$[\, L_1 u' - (w_1, w_2), L_1 u' - (w_1, w_2) \,] = \min_{x \in \mathcal{H}} [\, L_1 x - (w_1, w_2), L_1 x - (w_1, w_2) \,],$$

for this particular pair $(z_1, z_2) \in \mathcal{E}_1 \times \mathcal{E}_2$.

Observe that $L_1x_0 - (y,z) = L_1u' + (TP_{N(V_1)}T^\dagger y, V_2P_{N(V_1)}T^\dagger y) - (y,z) = L_1u' - (-TV_1^\dagger z_1, z_2 - V_2V_1^\dagger z_1) = L_1u' - (w_1,w_2)$. Then, for every $x \in \mathcal{H}$,

$$\begin{aligned}[\,L_1x_0 - (y,z), L_1x_0 - (y,z)\,] &= [\,L_1u' - (w_1,w_2), L_1u' - (w_1,w_2)\,] \\ &\leq [\,L_1(x - T^\dagger y) - (w_1,w_2), L_1(x - T^\dagger y) - (w_1,w_2)\,] \\ &= [\,L_1x - (y,z), L_1x - (y,z)\,],\end{aligned}$$

because $L_1x - (y,z) = L_1(x - T^\dagger y) - (w_1,w_2)$. Therefore, for every $(y,z) \in \mathcal{K} \times \mathcal{E}_2$, there exists $x_0 \in \mathcal{H}$ such that

$$[\,L_1x_0 - (y,z), L_1x_0 - (y,z)\,] = \min_{x\in\mathcal{H}}[\,L_1x - (y,z), L_1x - (y,z)\,].$$

Following the same arguments as in the proof of Proposition 4.3, it is easy to see that the above condition holds if and only if $R(L_1)$ is a (closed) uniformly positive subspace of $\mathcal{K} \times \mathcal{E}_2$. □

5.1 Parametrization of the set of solutions of the mixed problem

The following paragraphs follow analogous ideas to those presented in the previous section to show that every smoothing spline is an interpolating spline. Before stating the main theorem, we need the following lemma.

Consider the operator $V \in L(\mathcal{H}, \mathcal{K} \times \mathcal{E}_2)$ given by $Vx = (V_1x, V_2x)$, $x \in \mathcal{H}$, and notice that $N(V) = N(V_1) \cap N(V_2)$. Then,

Lemma 5.4. *Suppose that $T(N(V))$ is a regular subspace of $\mathcal{K}$ and $N(V_1) + N(V_2)$ is closed in $\mathcal{H}$. If $Q_0 = P_{N(V)//\mathcal{W}}$, where $\mathcal{W} = T^{\#}T(N(V))^{[\perp]} \ominus N(V)$, then:*

1. *$\mathcal{M}_1 = (I - Q_0)(N(V_1))$ and $\mathcal{M}_2 = V_2(N(V_1))$ are closed subspaces of $\mathcal{H}$ and $\mathcal{E}_2$, respectively.*
2. *$V_2|_{\mathcal{M}_1} : \mathcal{M}_1 \to \mathcal{M}_2$ is an isomorphism.*
3. *$R(L_1) = (T(N(V)) \times \{0\}) \dot{+} L(\mathcal{M}_1)$. Furthermore, $L(\mathcal{M}_1)$ is closed in $\mathcal{K}$ and the decomposition is orthogonal respect to the indefinite metric defined on $\mathcal{K} \times \mathcal{E}_2$.*

Proof. 1. First of all, notice that $\mathcal{M}_1 = R(I - Q_0) \cap N(V_1)$. Therefore, it is closed and $N(V_1) = N(V) \dot{+} \mathcal{M}_1$ because $Q_0(N(V_1)) = N(V)$. On the other hand, by Proposition 2.3, $\mathcal{M}_2 = R(V_2P_{N(V_1)})$ is closed if and only if $c(N(V_2), N(V_1)) < 1$, or equivalently, $N(V_1) + N(V_2)$ is closed. Therefore, $\mathcal{M}_2$ is closed.

2. To show that $V_2|_{\mathcal{M}_1} : \mathcal{M}_1 \to \mathcal{M}_2$ is an isomorphism, it only remains to prove that $V_2|_{\mathcal{M}_1}$ is injective. But, if $x \in \mathcal{M}_1$ and $V_2x = 0$ then $x \in N(V_2) \cap \mathcal{M}_1 = N(V) \cap R(I - Q_0) = \{0\}$.

3. Observe that $R(L_1) = L(N(V_1)) = L(N(V)) + L(\mathcal{M}_1)$ because $N(V_1) = N(V) \dot{+} \mathcal{M}_1$. Furthermore, as in Lemma 4.4, it is easy to see that $R(L_1) = L(N(V)) \dot{+} L(\mathcal{M}_1)$ and that this decomposition is orthogonal respect to the indefinite metric defined on $\mathcal{K} \times \mathcal{E}_2$. Recall that $L(\mathcal{M}_1)$ is closed if and only if $c(N(L), \mathcal{M}_1) < 1$, but

$$c(N(L), \mathcal{M}_1) = c(N(T) \cap N(V_2), \mathcal{M}_1) \leq c(N(V_2), N(V_1)) < 1,$$

because $N(V_1) + N(V_2)$ is closed. Finally, if $x \in N(V_1)$ then $Q_0x \in N(V)$ and $(I - Q_0)x \in \mathcal{M}_1$. So, $Lx = (TQ_0x, 0) + L(I - Q_0)x$. Therefore, $R(L_1) = L(N(V)) \dot{+} L(\mathcal{M}_1) = (T(N(V)) \times \{0\}) \dot{+} L(\mathcal{M}_1)$. □

The next theorem shows that that every smoothing spline is an interpolating spline.

Theorem 5.5. *Suppose that $N(V_1) + N(V_2)$ is closed in $\mathcal{K}$, $T(N(V))$ is a closed subspace of $\mathcal{K}$ and $R(L_1)$ is a (closed) uniformly positive subspace of $\mathcal{K} \times \mathcal{E}_2$. Then, if P is the J-selfadjoint projection onto $L(\mathcal{M}_1)$ and $S \in L(\mathcal{K} \times \mathcal{E}_2, \mathcal{E}_2)$ is the canonical projection onto $\mathcal{E}_2$, given $(z_1, z_2) \in \mathcal{E}_1 \times \mathcal{E}_2$, an element $x_0 \in \mathcal{H}$ is a solution of Problem 6 if and only if*

$$x_0 \in sp(T, V, (e_1, e_2)),$$

where (e_1, e_2) is a suitable vector in $\mathcal{E}_1 \times \mathcal{E}_2$.

Proof. Given $(z_1, z_2) \in \mathcal{E}_1 \times \mathcal{E}_2$, recall that if $x_0 \in \mathcal{H}$ is a solution of Problem 6 then $V_1 x_0 = z_1$. So, $P_{N(V_1)^\perp} x_0 = V_1^\dagger z_1$. Assuming that $T(N(V))$ is a regular subspace of $\mathcal{K}$, it can be decomposed as $V_1^\dagger z_1 = u_1 + v_1$, where $u_1 = Q_0 V_1^\dagger z_1 \in N(V)$ and $v_1 = (I - Q_0) V_1^\dagger z_1 \in \mathcal{W}$. Observe that the pair (w_1, w_2) considered in Eq. (5.2) satisfies

$$-w_1 = Tu_1 + Tv_1 \in T(N(V)) \dotplus T(N(V))^{[\perp]},$$

and $w_2 = z_2 - V_2 v_1$.

Furthermore, if $N(V_1) + N(V_2)$ is a closed subspace of $\mathcal{H}$, given $x \in \mathcal{H}$ there exist (unique) $u \in N(V)$ and $m \in \mathcal{M}_1$ such that $P_{N(V_1)x} = u + m$ (see Lemma 5.4). So,

$$L_1 x - (w_1, w_2) = (T(u + u_1), 0) + Lm - (-Tv_1, w_2),$$

and $Lm - (-Tv_1, w_2) = L(m + v_1) - (0, z_2) \in (T(N(V)) \times \{0\})^{[\perp]}$. Then,

$$[\, L_1 x - (w_1, w_2), L_1 x - (w_1, w_2) \,]_{\mathcal{K} \times \mathcal{E}_2} = [\, T(u+u_1), T(u+u_1) \,]_{\mathcal{K}} + [\, Lm - (-Tv_1, w_2), Lm - (-Tv_1, w_2) \,]_{\mathcal{K} \times \mathcal{E}_2}.$$

Therefore, x_0 is a solution to Problem 6 if and only if $P_{N(V_1)} x_0 = u_0 + m_0$, with $u_0 \in N(V)$ and $m_0 \in \mathcal{M}_1$ satisfying $[\, T(u_0 + u_1), T(u_0 + u_1) \,] = \min_{u \in N(V)} [\, (T(u + u_1), 0), (T(u + u_1), 0) \,]$ and

$$[\, Lm_0 - (-Tv_1, w_2), Lm_0 - (-Tv_1, w_2) \,] = \min_{m \in \mathcal{M}_1} [\, Lm - (-Tv_1, w_2), Lm - (-Tv_1, w_2) \,].$$

Notice that, if $R(L_1)$ is a (closed) uniformly positive subspace of $\mathcal{K} \times \mathcal{E}_2$, then $T(N(V))$ is a closed uniformly positive subspace of $\mathcal{K}$ and $\min_{u \in N(V)} [\, T(u + u_1), T(u + u_1) \,]_{\mathcal{K}}$ is attained at every $y \in -u_1 + N(V) \cap N(T)$.

On the other hand, consider the bounded operator $W : \mathcal{M}_2 \to \mathcal{K} \times \mathcal{M}_2$ defined by

$$Wz = (T(V_2|_{\mathcal{M}_1})^{-1} z, z).$$

Observe that W has closed range, because it is isometrically isomorphic to the graph of the bounded operator $T(V_2|_{\mathcal{M}_1})^{-1}$, and

$$\min_{m \in \mathcal{M}_1} [\, Lm - (-Tv_1, w_2), Lm - (-Tv_1, w_2) \,] = \min_{z \in \mathcal{M}_2} [\, Wz - (-Tv_1, w_2), Wz - (-Tv_1, w_2) \,].$$

Thus, following the same argument as in Theorem 4.5 and observing that $R(W) = L(\mathcal{M}_1)$ is a closed uniformly positive subspace of $\mathcal{K} \times \mathcal{E}_2$, this last problem admits a (unique) solution given by $z_0 = V_2 m_0 = SP(-Tv_1, w_2)$, where P is the J-selfadjoint projection onto $L(\mathcal{M}_1)$ and $S : \mathcal{K} \times \mathcal{E}_2 \to \mathcal{E}_2$ is defined by $S(y, z) = z$. So, $x_0 \in \mathcal{H}$ is a solution to Problem 6 if and only if

$$x_0 = V_1^\dagger z_1 + P_{N(V_1)} x_0 = u_1 + v_1 + u_0 + m_0 \in (v_1 + (V_2|_{\mathcal{M}_1})^{-1} SP(-Tv_1, w_2)) + N(T) \cap N(V).$$

Therefore, $x_0 \in \mathcal{H}$ is a solution to Problem 6 if and only if $x_0 \in sp(T, V, (e_1, e_2))$, where $e_1 = z_1 + V_1 (V_2|_{\mathcal{M}_1})^{-1} SP(-Tv_1, w_2) \in \mathcal{E}_1$ and $e_2 = V_2 V_1^\dagger z_1 + SP(-Tv_1, w_2) \in \mathcal{E}_2$. $\square$

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