Topology and smooth structure for pseudoframes

Esteban Andruchow *   Jorge Antezana †   Gustavo Corach ‡

Esteban Andruchow
Instituto de Ciencias, UNGS, San Miguel, Argentina and IAM-CONICET
e-mail: eandruch@ungs.edu.ar

Jorge Antezana
Depto. de Matemática, FCE-UNLP, La Plata, Argentina and IAM-CONICET
e-mail: antezana@mate.unlp.edu.ar

Gustavo Corach
Depto. de Matemática, FI-UBA, Buenos Aires, Argentina and IAM-CONICET
e-mail: gcorach@fi.uba.ar

Abstract

Given a closed subspace $\mathcal{S}$ of a Hilbert space $\mathcal{H}$, we study the sets $\mathcal{F}_\mathcal{S}$ of pseudo-frames, $\mathcal{CF}_\mathcal{S}$ of commutative pseudo-frames and $\mathfrak{X}_\mathcal{S}$ of dual frames for $\mathcal{S}$, via the (well known) one to one correspondence which assigns a pair of operators (F, H) to a frame pair $(\{f_n\}_{n\in\mathbb{N}}, \{h_n\}_{n\in\mathbb{N}})$,

$$F : \ell^2 \to \mathcal{H}, \quad F\big(\{c_n\}_{n\in\mathbb{N}}\big) = \sum_n c_n f_n,$$

and

$$H : \ell^2 \to \mathcal{H}, \quad H = (\{c_n\}_{n\in\mathbb{N}}) = \sum_n c_n h_n.$$

We prove that, with this identification, the sets $\mathcal{F}_\mathcal{S}$, $\mathcal{CF}_\mathcal{S}$ and $\mathfrak{X}_\mathcal{S}$ are complemented submanifolds of $\mathcal{B}(\ell^2, \mathcal{H}) \times \mathcal{B}(\ell^2, \mathcal{H})$. We examine in more detail $\mathfrak{X}_\mathcal{S}$, which carries a locally transitive action from the general linear group $GL(\ell^2)$. For instance, we characterize the homotopy theory of $\mathfrak{X}_\mathcal{S}$ and we prove that $\mathfrak{X}_\mathcal{S}$ is a strong deformation retract both of $\mathcal{F}_\mathcal{S}$ and $\mathcal{CF}_\mathcal{S}$; therefore these sets share many of their topological properties.

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1 Introduction

Let $\mathcal{H}$ be a Hilbert space, $\mathcal{B}(\ell^2, \mathcal{H})$ the Banach space of bounded (linear) operators acting from ℓ^2 to $\mathcal{H}$, and $\mathcal{S} \subset \mathcal{H}$ a closed subspace of $\mathcal{H}$. In this paper we study the set

$$\mathcal{F}_{\mathcal{S}} = \{(F, H) \in \mathcal{B}(\ell^2, \mathcal{H}) \times \mathcal{B}(\ell^2, \mathcal{H}) : FH^*|_{\mathcal{S}} = id_{\mathcal{S}}\}. \tag{1}$$

If Q is an idempotent with $R(Q) = \mathcal{S}$, then

$$\mathcal{F}_{\mathcal{S}} = \{(F, H) \in \mathcal{B}(\ell^2, \mathcal{H}) \times \mathcal{B}(\ell^2, \mathcal{H}) : FH^*Q = Q\}.$$

This set parametrizes the set of what S. Li and H. Ogawa call pseudoframes for the subspace $\mathcal{S}$ [21]. Let us recall their definition: A pair of sequences $\{f_n\}_{n\in\mathbb{N}}, \{h_n\}_{n\in\mathbb{N}}$ of vectors in $\mathcal{H}$ form a *pseudoframe for the subspace* $\mathcal{S}$ if

1. $\{f_n\}_{n\in\mathbb{N}}$ and $\{h_n\}_{n\in\mathbb{N}}$ are Bessel sequences of $\mathcal{H}$ (see definition in section 2).
2. For any $f \in \mathcal{S}$,
$$f = \sum_{n=1}^{\infty} \langle\, f,\, h_n \,\rangle\, f_n.$$

As these authors remark, pseudoframes for $\mathcal{S}$ are in one to one correspondence with pairs of bounded operators F, H, defined by

$$H^* : \mathcal{H} \to \ell^2, \quad H^*x = (\langle\, x, h_n \,\rangle)_n,$$

and

$$F : \ell^2 \to \mathcal{H}, \quad F\big(\{c_n\}_{n\in\mathbb{N}}\big) = \sum_n c_n f_n.$$

such that $FH^*|_{\mathcal{S}} = id_{\mathcal{S}}$.

In fact, Li and Ogawa require less, namely that $\{f_n\}_{n\in\mathbb{N}}$ be a Bessel sequence with respect to $\mathcal{S}$, which means that the condition $\sum_n |\langle\, f, f_n \,\rangle|^2 \leq c\|f\|^2$ holds only for $f \in \mathcal{S}$. This weaker condition gives naturally representations using unbounded operators. We shall consider this setting elsewhere. Therefore we may call the present version *bounded* pseudoframes for $\mathcal{S}$.

We shall use this one to one correspondence between bounded pseudoframes for the subspace $\mathcal{S}$ and the set $\mathcal{F}_{\mathcal{S}} \subset \mathcal{B}(\ell^2, \mathcal{H}) \times \mathcal{B}(\ell^2, \mathcal{H})$ to endow the former with a topological structure, namely the one given by the operator norm in the space $\mathcal{B}(\ell^2, \mathcal{H}) \times \mathcal{B}(\ell^2, \mathcal{H})$. This point of view is not new. There are many papers dealing with frames by examining their analysis and synthesis operators. Among them, let us mention [12], [20], [21], [15] and [10]. What is gained by this abstract presentation of frames, is that different special sets of frames, as the ones considered below, are regarded as spaces in a common ambient Banach space. Questions about the local structure of these spaces can be examined, for instance, if any given pair of frames in one of these spaces can be joined by continuous ot a smooth path inside the space.

As it turns out, $\mathcal{F}_{\mathcal{S}}$ is a smooth complemented submanifold of the Banach space $\mathcal{B}(\ell^2, \mathcal{H}) \times \mathcal{B}(\ell^2, \mathcal{H})$. We study its properties, as well as those of the following subsets:

1. The set of $\mathcal{CF}_{\mathcal{S}} \subset \mathcal{F} : \mathcal{S}$ of *commutative pseudoframes* [21], which consists of pairs (F, H) in $\mathcal{F}_{\mathcal{S}}$ which also verify that $PFH^* = P$ (where P is the orthogonal projection onto $\mathcal{S}$). This set parametrizes the set of pairs of sequences $(\{f_n\}_{n\in\mathbb{N}}, \{h_n\}_{n\in\mathbb{N}})$ such that
$$f = \sum_{n=1}^{\infty} \langle\, f, h_n \,\rangle\, f_n = \sum_{n=1}^{\infty} \langle\, f, f_n \,\rangle\, h_n,$$
for all $f \in \mathcal{S}$. This set is shown to be a complemented submanifold of $\mathcal{F}_{\mathcal{S}}$.

2. The set $\mathfrak{X}_{\mathcal{S}} \subset \mathcal{F}_{\mathcal{S}}$ of *dual frames* for $\mathcal{S}$ [14], [7], [21]. Recall that dual frames for $\mathcal{S}$ are pairs of sequences $(\{f_n\}_{n\in\mathbb{N}}, \{h_n\}_{n\in\mathbb{N}})$ which verify that
$$Pf = \sum_{n=1}^{\infty} \langle\, f, h_n \,\rangle\, f_n,$$
for all $f \in \mathcal{H}$. As operators, they are parametrized by
$$\mathfrak{X}_{\mathcal{S}} = \{(F, H) \in \mathcal{F}_{\mathcal{S}} : R(F) = R(H) = \mathcal{S}\}.$$
The set $\mathfrak{X}_{\mathcal{S}}$ is also a complemented submanifold of $\mathcal{F}_{\mathcal{S}}$, and a strong deformation retract both of of $\mathcal{F}_{\mathcal{S}}$ and $\mathcal{CF}_{\mathcal{S}}$, a fact which allows to examine many of the topological properties of $\mathcal{F}_{\mathcal{S}}$ and $\mathcal{CF}_{\mathcal{S}}$ in $\mathfrak{X}_{\mathcal{S}}$. This space is easier to handle than $\mathcal{F}_s$ because it carries a locally transitive action of the general linear group $GL(\ell^2)$. We characterize the orbits (=connected components of) $\mathfrak{X}_{\mathcal{S}}$ under this action, and analize their homotopy groups.

Apparently, $\mathfrak{X}_{\mathcal{S}} \subsetneq \mathcal{CF}_{\mathcal{S}} \subsetneq \mathcal{F}_{\mathcal{S}}$.

The contents of the paper are the following. Section 2 contains preliminaries and notations. In section 3 we prove the existence of a smooth structure for $\mathcal{F}_{\mathcal{S}}$ (Proposition 3.4) and for $\mathcal{CF}_{\mathcal{S}}$ (Proposition 3.7). In section 4 we introduce $\mathfrak{X}_{\mathcal{S}}$ and state its basic properties: the smooth structure for $\mathfrak{X}_{\mathcal{S}}$ (Proposition 4.2) and the existence of strong deformation retractions from $\mathcal{F}_{\mathcal{S}}$ and $\mathcal{CF}_{\mathcal{S}}$ onto $\mathfrak{X}_{\mathcal{S}}$ (Proposition 4.3). Section 5 examines the finer topological features of $\mathfrak{X}_{\mathcal{S}}$ by means of the action of $GL(\ell^2)$: existence of continuous local cross sections for the action (Proposition 5.1), characterization of the connected components of $\mathfrak{X}_{\mathcal{S}}$ (Proposition 5.3), description the homotopy structure of this space (Corollary 5.8).

2 Preliminaries

Given two separable Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, the space of bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$ is denoted by $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$. For an operator $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, we denote by $R(A)$ the range or image of A, $N(A)$ the null space of A, A^* the adjoint of A, $\|A\|$ the usual operator norm of A and, if $R(A)$ is closed, $A^\dagger$ the Moore-Penrose pseudoinverse of A. The space $\mathcal{B}(\mathcal{H}_2, \mathcal{H}_2)$ is denoted $\mathcal{B}(\mathcal{H}_2)$, $GL(\mathcal{H}_2)$ denotes the group of invertible operators on $\mathcal{H}_2$, and $\mathcal{B}(\mathcal{H}_2)^+$ denotes the cone of positive (semi-definite) operators of $\mathcal{B}(\mathcal{H}_2)$. If $\mathcal{H} = \mathcal{W} \oplus \mathcal{M}^\perp$ then the projection Q onto $\mathcal{W}$ defined by this decomposition is denoted by $P_{\mathcal{W}||\mathcal{M}^\perp}$. Observe that $P^*_{\mathcal{W}||\mathcal{M}^\perp} = P_{\mathcal{M}||\mathcal{W}^\perp}$. In the case of orthogonal projections, i.e. $\mathcal{W} = \mathcal{M}$, we write $P_{\mathcal{W}}$ instead of $P_{\mathcal{W}||\mathcal{W}^\perp}$.

Frames

We introduce some basic facts about frames in Hilbert spaces. For complete descriptions of frame theory and applications, the reader is referred to the paper by Duffin and Schaeffer [13], the review by Heil and Walnut [17] or the books by Young [27], Daubechies [12] and Christensen [6].

A sequence $\{f_n\}_{n\in\mathbb{N}}$ of elements of a Hilbert space $\mathcal{H}$ is called a *Bessel sequence* if there exists a positive number B such that

$$\sum_{n\in\mathbb{N}} |\langle f, f_n \rangle|^2 \leq B\|f\|^2 \quad \forall f \in \mathcal{H}.$$

If the inequality holds for all f in a certain subspace $\mathcal{S}$, then $\{f_n\}_{n\in\mathbb{N}}$ is called a Bessel sequnce for $\mathcal{S}$.

A *frame* for a closed subspace $\mathcal{W}$ of $\mathcal{H}$ is a sequence $\{f_n\}_{n\in\mathbb{N}}$ such that each f_n belongs to $\mathcal{W}$ and there exist numbers $A, B > 0$ such that, for every $f \in \mathcal{W}$,

$$A\|f\|^2 \leq \sum_{n\in\mathbb{N}} |\langle f, f_n \rangle|^2 \leq B\|f\|^2. \tag{2}$$

The optimal constants A, B for equation (2) are called the *frame bounds* for $\mathcal{F}$. $\mathcal{F}$ is a *Parseval frame* if $A = B = 1$. Note that, as each $f_n \in \mathcal{W}$, for every $f \in \mathcal{W}^\perp$ it holds that $\langle f, f_n \rangle = 0$. This shows that every frame for $\mathcal{W}$ is in particular a Bessel sequence in $\mathcal{H}$.

Any Bessel sequence $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ defines a bounded linear operator $T : \ell^2 \to \mathcal{H}$ by $Te_n = f_n$, where $\{e_n\}_{n\in\mathbb{N}}$ denotes the "canonical" basis of ℓ^2. This operator is called the *synthesis operator* of $\mathcal{F}$, $T^* \in \mathcal{B}(\mathcal{H}, \ell^2)$ is called the *analysis operator* of $\mathcal{F}$, and $S = TT^*$ is called the *frame operator* of $\mathcal{F}$. It is easy to see that $T^*f = \sum_{n\in\mathbb{N}} \langle f, f_n\rangle e_n$ and therefore

$$Sf = \sum_{n\in\mathbb{N}} \langle f, f_n \rangle f_n \qquad \forall f \in \mathcal{H}. \tag{3}$$

Observe that, in the case of a frame for a (closed) subspace $\mathcal{W}$, from (2) we obtain the operator inequality $A \cdot P_{\mathcal{W}} \leq S \leq B \cdot P_{\mathcal{W}}$. Hence, $S|_{\mathcal{W}}$ is invertible in $\mathcal{B}(\mathcal{W})$ and $R(T) = \mathcal{W}$. The dimension of $N(T)$ is called sometimes the *excess* of $\mathcal{F}$. A Riesz basis for a closed subspace $\mathcal{W}$ is a frame for this subspace with excess equal to zero. (see Balan et. al. [3])

3 Differentiable structure of $\mathcal{F}_\mathcal{S}$

Our goal in this section is to prove that $\mathcal{F}_\mathcal{S}$ is a smooth submanifold of $\mathcal{B}(\ell^2, \mathcal{H}) \times \mathcal{B}(\ell^2, \mathcal{H})$. To do this we shall use an auxiliary set $\mathcal{AF}_\mathcal{S}$ which contains $\mathcal{F}_\mathcal{S}$. Namely,

$$\mathcal{AF}_\mathcal{S} = \{(F, H) \in \mathcal{B}(\ell^2, \mathcal{H}) \times \mathcal{B}(\ell^2, \mathcal{H}) : PFH^*P = P\}$$

Throughout $P = P_{\mathcal{S}}$ denotes the orthogonal projection from $\mathcal{H}$ onto $\mathcal{S}$. Clearly $\mathcal{F}_{\mathcal{S}} \subset \mathcal{AF}_{\mathcal{S}}$. In this section we shall use the sesqui-linear map Π_A^B, for $A, B \in \mathcal{B}(\ell^2, \mathcal{H})$ fixed:

$$\Pi_A^B : \mathcal{B}(\ell^2, \mathcal{H}) \times \mathcal{B}(\ell^2, \mathcal{H}) \to \mathcal{B}(\mathcal{H}), \;\; \Pi_A^B(X, Y) = AX^*YB^*.$$

Let $\mathcal{O}_{\mathcal{S}} = \{(X, Y) \in \mathcal{B}(\ell^2, \mathcal{H}) \times \mathcal{B}(\ell^2, \mathcal{H}) : PXY^*P \text{ is invertible in } \mathcal{S}\}$. Clearly $\mathcal{O}_{\mathcal{S}}$ is open in $\mathcal{B}(\ell^2, \mathcal{H}) \times \mathcal{B}(\ell^2, \mathcal{H})$.

Lemma 3.1. *The set $\mathcal{AF}_{\mathcal{S}}$ is a C^∞ submanifold of $\mathcal{B}(\ell^2, \mathcal{H}) \times \mathcal{B}(\ell^2, \mathcal{H})$.*

Proof. Consider the map $\Pi_P^P|_{\mathcal{O}_{\mathcal{S}}} : \mathcal{O}_{\mathcal{S}} \to GL(\mathcal{S})$, where we identify $P\mathcal{B}(\mathcal{H})P$ with $\mathcal{B}(\mathcal{S})$. We claim that $\Pi_P^P|_{\mathcal{O}_{\mathcal{S}}}$ is a submersion. In that case, we have that $\mathcal{AF}_{\mathcal{S}} = \Pi_P^P|_{\mathcal{O}_{\mathcal{S}}}^{-1}(\{P\})$ is a submanifold of $\mathcal{O}_{\mathcal{S}}$, which is open in $\mathcal{B}(\ell^2, \mathcal{H}) \times \mathcal{B}(\ell^2, \mathcal{H})$, and the proof follows. Apparently $\Pi_P^P|_{\mathcal{O}_{\mathcal{S}}}$ is onto. On the other hand, the differential of Π_P^P at (F, H) is

$$d(\Pi_P^P)_{(F,H)}(X, Y) = P(XH^* + FY^*)P, \;\; X, Y \in \mathcal{B}(\ell^2, \mathcal{H}).$$

Let us exhibit a (bounded linear) cross section for $d(\Pi_P^P)_{(F,H)}$. For $Z \in (TGl(\mathcal{S})_{PFH^*P} = \mathcal{B}(\mathcal{S}))$, let $S = (PFH^*P)^\dagger$ and put

$$\sigma_{(F,H)}(Z) = (\frac{1}{2}ZSF, \frac{1}{2}Z^*S^*F).$$

Note that

$$d(\Pi_P^P)_{(F,H)}(\sigma_{(F,H)}(Z)) = \frac{1}{2}P(ZSFH^* + FH^*SZ)P = PZP = Z.$$

It follows that Π_P^P is a submersion. ■

Remark 3.2. From the proof of the above lemma it can be deduced that the tangent space of $\mathcal{AF}_{\mathcal{S}}$ at a pair (F, H) is given by

$$(T_{\mathcal{AF}_{\mathcal{S}}})_{(F,H)} = \ker d(\Pi_P^P)_{(F,H)} = \{(X, Y) \in \mathcal{B}(\ell^2, \mathcal{H}) \times \mathcal{B}(\ell^2, \mathcal{H}) : P(XH^* + FY^*)P = 0\}. \quad (4)$$

Remark 3.3. The set $\mathcal{AF}_{\mathcal{S}}$ can be interpreted as a parametrization of the pairs of Bessel sequences $(\{f_n\}_{n\in\mathbb{N}}, \{h_n\}_{n\in\mathbb{N}})$ such that

$$\langle f, g \rangle = \sum_{n=1}^{\infty} \langle f, h_n \rangle \langle f_n, g \rangle \,,$$

for every $f, g \in \mathcal{S}$.

Proposition 3.4. *The set $\mathcal{F}_{\mathcal{S}}$ is a C^∞ submanifold of $\mathcal{B}(\ell^2, \mathcal{H}) \times \mathcal{B}(\ell^2, \mathcal{H})$.*

Proof. Let $R_P = \{X \in \mathcal{B}(\mathcal{H}) : PX = P$ and $(I-P)X(I-P) = 0\}$. Note that R_P is an affine and complemented submanifold of $\mathcal{B}(\mathcal{H})$. Indeed, $R_P = P + \{Y \in \mathcal{B}(\mathcal{H}) : PY = 0, (I-P)Y(I-P) = 0\}$, and this linear subset of $\mathcal{B}(\mathcal{H})$ is complemented in $\mathcal{B}(\mathcal{H})$. Note that if $X \in R_P$, $X = P + (I-P)XP$ and in particular $XP = X$. Let

$$\Pi_P : \mathcal{AF}_{\mathcal{S}} \to R_P,\ \Pi_P(F,H) = FH^*P.$$

Clearly it is well defined and onto. Since $\mathcal{F}_{\mathcal{S}} \subset \mathcal{AF}_{\mathcal{S}}$, it follows that $\mathcal{F}_{\mathcal{S}} = \Pi_P^{-1}(\{P\})$. Let us show that Π_P is also a submersion. Its differential at a point $(F,H) \in \mathcal{AF}_{\mathcal{S}}$ is given by

$$d(\Pi_P)_{(F,H)}(X,Y) = (XH^* + FY^*)P.$$

Consider the linear map

$$\tau_{(F,H)}(Z) = (ZPF, 0).$$

First note that if Z is tangent to R_P, i.e. $PZ = 0$ and $(I-P)Z(I-P) = 0$. This implies that $ZP = Z$ and also that the pair $(ZPF, 0)$ is tangent to $\mathcal{AF}_{\mathcal{S}}$ at (F,H). Indeed, this follows by equation (4) and the fact that $P(ZPFH^* + 0)P = PZP = 0$. Next,

$$d(\Pi_P)_{(F,H)}(\tau_{(F,H)}(Z)) = (ZPFH^* + 0)P = ZP = Z.$$

Therefore Π_P is a submersion, which concludes the proof. ■

Corollary 3.5. *$\mathcal{F}_{\mathcal{S}}$ is a submanifold of $\mathcal{AF}_{\mathcal{S}}$ and the tangent space of $\mathcal{F}_{\mathcal{S}}$ at (F,H) is given by:*

$$(T\mathcal{F}_{\mathcal{S}})_{(F,H)} = \ker d(\Pi_P)_{(F,H)} = \{(X,Y) \in \mathcal{B}(\ell^2,\mathcal{H}) \times \mathcal{B}(\ell^2,\mathcal{H}) : (XH^* + FY^*)P = 0\}. \quad (5)$$

We emphasize the fact that both $\mathcal{AF}_{\mathcal{S}}$ and $\mathcal{F}_{\mathcal{S}}$ are *complemented* submanifolds of $\mathcal{B}(\ell^2,\mathcal{H}) \times \mathcal{B}(\ell^2,\mathcal{H})$. In other words, the linear subspaces $(T\mathcal{F}_{\mathcal{S}})_{(F,H)}$ are complemented in $\mathcal{B}(\ell^2,\mathcal{H}) \times \mathcal{B}(\ell^2,\mathcal{H})$. Let us exhibit a natural supplement for them, or more precisely, a linear idempotent $Q_{(F,H)}$ acting on $\mathcal{B}(\ell^2,\mathcal{H}) \times \mathcal{B}(\ell^2,\mathcal{H})$ whose range is $(T\mathcal{F}_{\mathcal{S}})_{(F,H)}$. This is done by composing the linear maps τ and σ of the preceding results with $d(\Pi_P)_{(F,H)}$. This gives, after tedious but straightforward computations, the linear map

$$Q_{(F,H)}(X,Y) = (X + (\frac{1}{2}P - I)(XH^* + FY^*)PF, Y - \frac{1}{2}P(XH^* + FY^*)PH) \quad (6)$$

for $(F,H) \in \mathcal{F}_{\mathcal{S}}$.

Proposition 3.6. *The map $Q_{(F,H)}$ is a linear idempotent with range equal to $(T\mathcal{F}_{\mathcal{S}})_{(F,H)}$.*

Proof. First note that $Q_{(F,H)}$ acts as the identity map on $(T\mathcal{F}_{\mathcal{S}})_{(F,H)}$: if $(X,Y) \in (T\mathcal{F}_{\mathcal{S}})_{(F,H)}$ then $(XH^* + FY^*)P = 0$, and therefore $Q_{(F,H)}(X,Y) = (X,Y)$.
Next note that $R(Q_{(F,H)}) \subset (T\mathcal{F}_{\mathcal{S}})_{(F,H)}$. Recall the basic relation $FH^*P = P$, and abbreviate $Z = (XH^* + FY^*)P$. If $(\widetilde{X}, \widetilde{Y}) = Q_{(F,H)}(X,Y)$, then

$$(\widetilde{X}H^* + F\widetilde{Y}^*)P = XH^*P + (\frac{1}{2} - I)(XH^* + FY^*)PFH^*P + FY^*P - \frac{1}{2}FH^*P(XH^* + FY^*)P$$

$$= ZP + (\frac{1}{2} - I)ZP + \frac{1}{2}PZP = 0.$$

Finally, we show that $Q^2_{(F,H)} = Q_{(F,H)}$. Write again $(\widetilde{X}, \widetilde{Y}) = Q_{(F,H)}(X,Y)$; then by the first fact above, $(\widetilde{X}H^* + F\widetilde{Y}^*)P = 0$, and therefore $Q_{(F,H)}(\widetilde{X}, \widetilde{Y}) = (\widetilde{X}, \widetilde{Y})$, which proves the result. ■

3.1 Commutative Pseudoframes

A pseudoframe $(\{f_n\}_{n\in\mathbb{N}}, \{h_n\}_{n\in\mathbb{N}})$ for $\mathcal{S}$ is called *commutative* [21] if the roles of $\{f_n\}_{n\in\mathbb{N}}$ and $\{h_n\}_{n\in\mathbb{N}}$ are interchangeable (see [21]). In operator theoretic terms, a pseudoframe (F,H) is commutative if

$$FH^*P = PFH^* = P$$

Let us denote by $\mathcal{CF}_\mathcal{S}$ the set of such pairs:

$$\mathcal{CF}_\mathcal{S} = \{(F,H) \in \mathcal{F}_\mathcal{S} : PFH^* = P\}.$$

More generally, given an oblique projection Q such that $R(Q) = \mathcal{S}$ we can consider the set

$$\mathcal{CF}_Q = \{(F,H) \in \mathcal{F}_\mathcal{S} : PFH^* = Q\}.$$

Let $\mathcal{Q}_\mathcal{S}$ be the set of oblique projections whose range is $\mathcal{S}$. This set is an affine subspace of $\mathcal{B}(\mathcal{H})$ with tangent space $\mathcal{Q}^0_\mathcal{S} := \{Z \in B(\mathcal{H}) : \ PZ = Z\, ,\, PZP = 0\}$. If we consider the map $\Phi : \mathcal{F}_\mathcal{S} \to \mathcal{Q}_\mathcal{S}$ defined by $\Phi(F,H) = PFH^*$, then it holds that $\mathcal{CF}_Q = \Phi^{-1}(Q)$.

Proposition 3.7. *The set $\mathcal{CF}_Q$ is a smooth submanifold of $\mathcal{F}_\mathcal{S}$ (and therefore also of $\mathcal{B}(\ell^2,\mathcal{H}) \times \mathcal{B}(\ell^2,\mathcal{H})$) and satisfies the following properties:*

1. *Every $\mathcal{CF}_Q$ is diffeomorphic to $\mathcal{CF}_\mathcal{S}$.*
2. *The family consisting of the connected components of all the submanifolds $\mathcal{CF}_Q$ form a foliation of $\mathcal{F}_\mathcal{S}$.*

Proof. Fix $Q \in \mathcal{Q}_\mathcal{S}$ and let $G_Q = Q + (I - P)$. Straightforward but simple computations show that the set $\{G_Q\}_{Q\in\mathcal{Q}_\mathcal{S}}$ is a multiplicative group of $GL(\mathcal{H})$ and the action $(F,H) \mapsto (F, G^*H)$ defines a diffeomorphism between $\mathcal{CF}_\mathcal{S}$ and $\mathcal{CF}_Q$. This proves that the sets $\mathcal{CF}_Q$ are diffeomorphic. On the other hand, since $\mathcal{CF}_Q = \Phi^{-1}(Q)$, they are disjoint. So, in order to complete the proof, it is enough to prove that $\Phi : \mathcal{F}_\mathcal{S} \to \mathcal{Q}_\mathcal{S}$ is a C^∞ surjective submersion. Clearly, this map is smooth and it is not difficult to see that it is also surjective. To prove that it is a submersion we are going to prove that the differential $d\Phi$ admits local cross sections. Fix $(F_0,H_0) \in \mathcal{F}_\mathcal{S}$ and let $Q_0 = F_0H_0^*$. Then

$$d\Phi_{(F_0,H_0)} : T(\mathcal{F}_\mathcal{S})_{(F_0,H_0)} \to T(\mathcal{Q}_\mathcal{S})_{Q_0}, \quad d\Phi_{(F_0,H_0)}(X,Y) = P(XH_0^* + F_0Y^*)$$

Define $\sigma(Z) = (ZPF_0, Z^*PH_0)$, for $Z \in \mathcal{Q}^0_\mathcal{S}$. Firstly note that $\sigma(Z) \in T(\mathcal{F}_\mathcal{S})_{(F_0,H_0)}$. Recall that a pair (X,Y) lies in the tangent space $T(\mathcal{F}_\mathcal{S})_{(F_0,H_0)}$ if $(XH_0^* + F_0Y^*)P = 0$, in our case:

$$(ZPF_0H_0^* + F_0H_0^*PZ)P = ZP + PZP = ZP = PZP = 0.$$

Next note that $d\Phi_{(F_0,H_0)}(\sigma(Z)) = Z$ for $Z \in \mathcal{Q}^0_{\mathcal{S}}$. Indeed

$$d\Phi_{(F_0,H_0)}(\sigma(Z)) = PZPF_0H_0^* + PF_0H_0^*PZ = PZ = Z.$$

Therefore Π_P is a C^∞ submersion. ■

4 Dual frames.

Inside $\mathcal{F}_{\mathcal{S}}$ there is another distinguished subset:

$$\mathfrak{X}_{\mathcal{S}} := \{(F,\ H) \in \mathcal{B}(\ell^2, \mathcal{H}) \times \mathcal{B}(\ell^2, \mathcal{H}) :\ FH^* = P,\ R(F) = R(H) = \mathcal{S}\}.$$

Note that $\mathfrak{X}_{\mathcal{S}}$ is non empty: if $V : \ell^2 \to \mathcal{H}$ is a partial isometry with final space S (for instance, take $V = PU$ with U a unitary operator), then $(V, V) \in \mathfrak{X}_{\mathcal{S}}$.

This set is the operator version of the set of all dual frames $(\{f_n\}_{n\in\mathbb{N}}, \{h_n\}_{n\in\mathbb{N}})$ for the subspace $\mathcal{S}$, i.e. , pairs of frames such that for every $f \in \mathcal{H}$:

$$Pf = \sum_{n=1}^{\infty} \langle\, f, h_n \,\rangle\, f_n.$$

In [2], the next alternative characterizations of the set $\mathfrak{X}_{\mathcal{S}}$ were proved:

Proposition 4.1. *If $F, H \in \mathcal{B}(\ell^2, \mathcal{H})$, then the following statements are equivalent:*

1. *$(F, H) \in \mathfrak{X}_{\mathcal{S}}$;*
2. *$FH^*F = F$, $H^*FH^* = H^*$ and $FH^* = P$;*
3. *$R(F) = \mathcal{S}$, $H = (QF^\dagger P)^*$ where Q is an oblique projection such that $N(Q) = N(F)$;*
4. *$FH^* = P$, $PF = F$ and $PH = H$.*

From the geometrical point of view, we have the next result:

Proposition 4.2. *The set $\mathfrak{X}_{\mathcal{S}}$ is a smooth submanifold of $\mathcal{F}_{\mathcal{S}}$.*

Since $\mathfrak{X}_{\mathcal{S}}$ is a submanifold of $\mathcal{B}(\ell^2, \mathcal{H}) \times \mathcal{B}(\ell^2, \mathcal{H})$ (see [11] and [2] for more details), the proof of this proposition is a consequence of the following result which will be used in the sequel to study the topological properties of the set $\mathcal{F}_{\mathcal{S}}$.

Proposition 4.3. *There is a smooth strong deformation retraction from $\mathcal{F}_{\mathcal{S}}$ onto $\mathfrak{X}_{\mathcal{S}}$. Moreover, when restricted to $\mathcal{CF}_{\mathcal{S}}$, it is a strong deformation retraction from this space.*

Proof. Define $\Gamma : [0, 1] \times \mathcal{F}_{\mathcal{S}} \to \mathcal{F}_{\mathcal{S}}$ by

$$\Gamma\big(t, (F, H)\big) = \big(\, PF + (1-t)(1-P)F\,,\ PH + (1-t)(1-P)H\,\big). \tag{7}$$

Straightforward computations show that $\Gamma\big(t,(F,H)\big) \in \mathcal{F}_{\mathcal{S}}$ for every $t \in [0,1]$, and, if $(F,H) \in \mathfrak{X}_{\mathcal{S}}$, then $\Gamma(t,(F,H)) = (F,H)$ for every $t \in [0,1]$. So, it only remains to prove that for every $(F,H) \in \mathcal{F}_{\mathcal{S}}$ it holds that $\Gamma(1,(F,H)) \in \mathfrak{X}_{\mathcal{S}}$. Let $(F,H) \in \mathcal{F}_{\mathcal{S}}$ and define $(F_1,H_1) = \Gamma(1,(F,H)) = (PF\,,\,PH)$. Then

$$F_1H_1^* = P\,FH^*P = P.$$

On the other hand, as P is selfadjoint, we get $F_1H_1^* = H_1F_1^* = P$. This implies that $\mathcal{S} \subseteq R(F_1)$ and $\mathcal{S} \subseteq R(H_1)$. As the other inclusions are trivial because $F_1 = PF$ and $H_1 = PH$, it holds that $(F_1,H_1) \in \mathfrak{X}_{\mathcal{S}}$.

If the pair (F,H) belongs to $\mathcal{CF}_{\mathcal{S}}$, then

$$P\Gamma(t,(F,H)) = PF(H^*P + (1-t)H^*(1-P) = PFH^*P + (1-t)PFH^*(1-P) = P,$$

because in $\mathcal{CF}_{\mathcal{S}}$, $PFH^* = P$. Then $\Gamma(t,(F,H)) \in \mathcal{CF}_{\mathcal{S}}$ ■

In particular, any element (F,H) in $\mathcal{F}_{\mathcal{S}}$ (resp. $\mathcal{CF}_{\mathcal{S}}$) can be joined in $\mathcal{F}_{\mathcal{S}}$ (resp. $\mathcal{CF}_{\mathcal{S}}$) to $(PF,PH) \in \mathfrak{X}_{\mathcal{S}}$ with the path $\Gamma(t,(F,H))$. Therefore the connected components of $\mathcal{F}_{\mathcal{S}}$ and $\mathcal{CF}_{\mathcal{S}}$ are parametrized by the components of $\mathfrak{X}_{\mathcal{S}}$. We characterize these in the next section.

5 Topological structure of $\mathfrak{X}_{\mathcal{S}}$

In this section we shall study some topological properties of the set $\mathfrak{X}_{\mathcal{S}}$. By Proposition 4.3, many of these properties will hold for $\mathcal{F}_{\mathcal{S}}$. With this aim, we consider the action $\rho : GL(\ell^2) \times \mathfrak{X}_{\mathcal{S}} \to \mathfrak{X}_{\mathcal{S}}$ defined by

$$\rho(G,(F,H)) = G \cdot (F,H) = (F\,G, H\,(G^{-1})^*).$$

This action of $GL(\ell^2)$ on $\mathfrak{X}_{\mathcal{S}}$ induces a partition of $\mathfrak{X}_{\mathcal{S}}$ in orbits: the orbit of (F,H) is $\mathcal{O}_{(F,H)} = \{G \cdot (F,H) : \ G \in GL(\ell^2)\}$. Denote by $\mathcal{C}_{(F,H)}$ the connected component of (F,H) in $\mathfrak{X}_{\mathcal{S}}$. Since $GL(\ell^2)$ is connected, $\mathcal{O}_{(F,H)}$ is automatically contained in $\mathcal{C}_{(F,H)}$. The following result establishes the existence of continuous local cross section for the action and, as a consequence, the equality $\mathcal{O}_{(F,H)} = \mathcal{C}_{(F,H)}$.

Proposition 5.1. *For every $(F,H) \in \mathfrak{X}_{\mathcal{S}}$ there is an open neighborhood $\mathcal{U}$ of (F,H) in $\mathfrak{X}_{\mathcal{S}}$ and a continuous map $\sigma : \mathcal{U} \to GL(\ell^2)$ such that*

$$\rho\big(\sigma(\tilde{F},\tilde{H}),(F,H)\big) = (\tilde{F},\tilde{H})$$

for all $(\tilde{F},\tilde{H}) \in \mathcal{U}$.

Proof. Fix $(F,H) \in \mathfrak{X}_{\mathcal{S}}$, and let $\sigma_0 : B(\ell^2,\mathcal{H}) \times B(\ell^2,\mathcal{H}) \to L(\ell^2)$ be defined by

$$\sigma_0\big((\tilde{F},\tilde{H})\big) = H^*\tilde{F} + \big(1 - H^*F\big)\big(1 - \tilde{H}^*\tilde{F}\big).$$

As H^*F is a projection, $\sigma_0((F,H)) = I$ and σ_0 is continuous, there is an open neighborhood $\mathcal{U}_0$ of (F,H) in $B(\ell^2,\mathcal{H})\times B(\ell^2,\mathcal{H})$ such that $\sigma_0|_{\mathcal{U}_0}$ takes values in $GL(\ell^2)$. Let $\mathcal{U} = \mathcal{U}_0 \cap \mathfrak{X}_{\mathcal{S}}$ and let σ be the restriction of σ_0 to $\mathcal{U}$. Note that

$$\begin{aligned} F\big(\sigma(\tilde{F},\tilde{H})\big) &= F\Big(H^*\tilde{F} + \big(1 - H^*F\big)\big(1 - \tilde{H}^*\tilde{F}\big)\Big) \\ &= Q\tilde{F} + (F - FH^*F)\big(1 - \tilde{H}^*\tilde{F}\big) \\ &= \tilde{F}, \end{aligned}$$

Analogously,

$$\begin{aligned} \tilde{H}\big(\sigma(\tilde{F},\tilde{H})\big)^* &= \tilde{H}\Big(\tilde{F}^*H + \big(1 - \tilde{F}^*\tilde{H}\big)\big(1 - F^*H\big)\Big) \\ &= Q^*H + \big(\tilde{H} - \tilde{H}\tilde{F}^*\tilde{H}\big)\big(1 - F^*H\big) \\ &= H. \end{aligned}$$

So, $\rho\big(\sigma(\tilde{F},\tilde{H}),(F,H)\big) = (\tilde{F},\tilde{H})$ for all $(\tilde{F},\tilde{H}) \in \mathcal{U}$. ■

Corollary 5.2. *The orbits are open, and therefore closed, subsets of $\mathfrak{X}_{\mathcal{S}}$. In particular, for every $(F,H) \in \mathfrak{X}_{\mathcal{S}}$ it holds $\mathcal{O}_{(F,\,H)} = \mathcal{C}_{(F,\,H)}$.*

Given $(F,H) \in \mathfrak{X}_{\mathcal{S}}$, note that $\dim N(F) = \dim N(H)$ and this quantity is invariant in $\mathcal{O}_{(F,\,H)}$. Our next objectives are to characterize $\mathcal{C}_{(F,\,H)}$ in terms of $\dim N(F)$ and to characterize the homotopy groups of each connected component.

Proposition 5.3. *Given $(F,H) \in \mathfrak{X}_{\mathcal{S}}$, then*

$$\mathcal{C}_{(F,\,H)} = \{(S,T) \in \mathfrak{X}_{\mathcal{S}} :\ \dim N(S) = \dim N(F)\}$$

To prove this result, we need the following technical lemmas:

Lemma 5.4. *Let V and W two partial isometries of $B(\ell^2,\mathcal{H})$ such that $VV^* = WW^*$ and $\dim N(V) = \dim N(W)$. Then, there is an unitary operator $U \in L(\ell^2)$ such that $VU = W$.*

Proof. Let U_0 be a partial isometry with initial subspace $N(W)$ and final subspace $N(V)$. Then, $U = V^*W + U_0$ is an unitary operator that satisfies the identity $VU = W$. ■

Lemma 5.5. *Let $(F,H) \in \mathfrak{X}_{\mathcal{S}}$ and define $A = F^*F + (1 - F^*H)(1 - H^*F)$. Then, A is a positive invertible operator such that $V = HA^{1/2} = FA^{-1/2}$ is a partial isometry with the following properties:*

$$VV^* = P \quad \text{and} \quad \dim N(V) = \dim N(F)\,. \tag{8}$$

where P denotes the orthogonal projection from $\mathcal{H}$ onto $\mathcal{S}$.

Proof. Straightforward computations show that $A^{-1} = H^*H + P_{N(F)}$. On the other hand, by Proposition 4.1 and using that $FH^* = HF^* = P$, we get $H^* - H^*FH^* = H - HF^*H = 0$ and, therefore

$$HA = HF^*F + (H - HF^*H)(H - HH^*F) = PF = F\,, \tag{9}$$

and

$$\begin{aligned} VV^* &= (HA^{1/2})(HA^{1/2})^* = HAH^* \\ &= HF^*FH^* - (H - HF^*H)(H^* - H^*FH^*) \\ &= P\,. \end{aligned} \tag{10}$$

Equation (9) implies that $HA^{1/2} = FA^{-1/2}$, whereas equation (10) implies that $V = HA^{1/2} = FA^{-1/2}$ is a partial isometry. Finally, by construction, V satisfies that

$$VV^* = P \quad \text{and} \quad \dim N(V) = \dim N(F)\,,$$

■

Proof of Proposition 5.3: As $\mathcal{C}_{(F,\,H)} = \mathcal{O}_{(F,\,H)}$ and the action preserves the dimension of the kernels of F and H, it is enough to prove that, given a pair (S,T) such that $\dim N(S) = \dim N(F)$, then $(S,T) \in \mathcal{O}_{(F,\,H)}$. So, take $(S,T) \in \mathfrak{X}_{\mathcal{S}}$ with $\dim N(S) = \dim N(F)$. By Lemma 5.5, there are positive invertible operators $A, B \in B(\ell^2)$ so that $V = FA^{-1/2} = HA^{1/2}$ and $W = SB^{-1/2} = TB^{1/2}$ are partial isometries which satisfy

$$VV^* = WW^* = P \quad \text{and} \quad \dim N(V) = \dim N(F) = \dim N(S) = \dim N(W)\,.$$

Then, by Lemma 5.4, there is an unitary operator U such that $VU = W$. Therefore, the operator $C = A^{-1/2}UB^{1/2}$ satisfies that $C \cdot (F,H) = (S,T)$, which concludes the proof. ■

The characterization of the homotopy groups of each connected component is based in the following proposition. For the topological definitions see [22].

Proposition 5.6. *Let $\mathcal{C}_n$ be the connected component of $\mathfrak{X}_{\mathcal{S}}$ consisting of all the pairs (F,H) with nulity (of F and H) equal to $n \in \mathbb{N} \cup \{\infty\}$. Let $\mathcal{J}_n$ be the subset of $\mathcal{C}_n$ of pairs that have the form (V,V), where V is a partial isometry with nullity equal to n. Then there is a smooth strong deformation retraction from $\mathcal{C}_n$ onto $\mathcal{J}_n$.*

Proof. By Lemma 5.5, given $(F,H) \in \mathcal{C}_n$, $A_{(F,\,H)} = F^*F + (1 - F^*H)(1 - H^*F)$ is a positive invertible operator of $B(\ell^2)$ such that $A^{-1/2}_{(F,\,H)} \cdot (F,H) = (FA^{-1/2}_{(F,\,H)}, HA^{1/2}_{(F,\,H)}) \in \mathcal{J}_n$. Also note that, given $(V,V) \in \mathcal{J}_n$, then $A_{(V,\,V)} = I$. So, if we define $\Gamma : [0,1] \times \mathcal{C}_n \to \mathcal{C}_n$ by

$$\Gamma\big(t, (F,H)\big) = A^{-t/2}_{(F,\,H)} \cdot (F,H) = \big(F^*F + (1 - F^*H)(1 - H^*F)\big)^{-t/2} \cdot (F,H),$$

Γ is a strong deformation retraction from $\mathcal{C}_n$ onto $\mathcal{J}_n$. ■

In particular, the above result states that the homotopy type of $\mathcal{C}_n$ is the same as the homotopy type of the set $\mathcal{I}_n$ of partial isometries with fixed final space P and nullspace of dimension n. The sets $\mathcal{J}_n$ were studied in [1]. More precisely, the space studied there is the set of partial isometries with fixed initial space. Apparently one can pass from one set to the other by taking adjoints, an operation which preserves topological and differentiable properties. Among these, let us cite a few:

1. The action of the unitary group $\mathcal{U}(\mathcal{H})$ given by right multiplication is transitive in $\mathcal{J}_n$, and defines, for a fixed $V \in \mathcal{J}_n$, a locally trivial fibration map

$$\pi_V : \mathcal{U}(\mathcal{H}) \to \mathcal{J}_n, \quad \pi_V(U) = VU.$$

   The fibre of this bundle is the subgroup $\mathcal{J}_V = \{U \in \mathcal{U}(\mathcal{H}) : VU = V\}$, which consists of unitaries whose matrices in terms of the orthogonal projection $P_V = V^*V$ are of the form

$$\begin{pmatrix} I & 0 \\ 0 & U_{22} \end{pmatrix}.$$

   It is apparent that it is homeomorphic to the unitary group $\mathcal{U}(P_V(\mathcal{H})^{\perp})$.

2. The sets $\mathcal{J}_n$ are C^∞ submanifolds of $L(\mathcal{H})$.

We may use this fibration map to compute the homotopy groups of $\mathcal{J}_n$ (and therefore of $\mathcal{C}_n$):

Proposition 5.7. *If $n = \infty$, then $\mathcal{J}_n$ is contractible. If $n < \infty$, then for all $k \geq 1$, $\pi_k(\mathcal{J}_n) \simeq \pi_{k-1}(\mathcal{U}(n))$, where $\mathcal{U}(n)$ denotes the unitary group on $\mathbb{C}^n$.*

Proof. The fibration map π_V induces the homotopy exact sequence

$$\cdots \longrightarrow \pi_k(\mathcal{J}_V) \longrightarrow \pi_k(\mathcal{U}(\mathcal{H})) \longrightarrow \pi_k(\mathcal{J}_n) \longrightarrow \pi_{k-1}(\mathcal{J}_V) \longrightarrow \pi_{k-1}(\mathcal{U}(\mathcal{H})) \longrightarrow \cdots$$

By Kuiper's theorem (see [18]), $\mathcal{U}(\mathcal{H})$ is contractible, as well as $\mathcal{J}_V$ in the $n = \infty$ case. Thus in this case $\pi_k(\mathcal{J}_n) = 0$ for all k, and $\mathcal{J}_n$ is therefore contractible , being a differentiable manifold, by Palais' theorem (see [26]). If $n < \infty$, the result follows by observing that $\mathcal{J}_V \simeq \mathcal{U}(n)$. ■

Corollary 5.8. *If $n = \infty$, then $\mathcal{C}_n$ is contractible. If $n < \infty$, then for every $k \geq 1$, $\pi_k(\mathcal{C}_n) \simeq \pi_{k-1}(\mathcal{U}(n))$.*

The next theorem summarizes the previous results about $\mathfrak{X}_{\mathcal{S}}$.

Theorem 5.9. *The following facts hold*

1. *Given $(F, H) \in \mathfrak{X}_{\mathcal{S}}$ then:*

$$\mathcal{C}_{(F,\,H)} = \big\{(S,T) \in \mathfrak{X}_{\mathcal{S}} : \ \dim N(S) = \dim N(F)\big\}.$$

2. *There is a smooth strong deformation retraction from $\mathcal{C}_n$ onto $\mathcal{J}_n$.*

3. If $n \in \mathbb{N}$, then $\pi_k(\mathcal{C}_n) = \pi_{k-1}(\mathcal{U}(n))$, and $\mathcal{C}_\infty$ is contractible. In particular, the different connected components of $\mathfrak{X}_\mathcal{S}$ are not homeomorphic.

Following Li and Ogawa [20], [21] and Christensen and Eldar [7], we shall consider now the fibers of the map $(F, H) \to F$. We start, as usual, with a decomposition $\mathcal{H} = \mathcal{S} \oplus \mathcal{S}^\perp$. By definition of the set $\mathfrak{X}_\mathcal{S}$, for every $(F, H) \in \mathfrak{X}_\mathcal{S}$ the range of the operator $F \in \mathcal{B}(\ell^2, \mathcal{H})$ is $\mathcal{S}$.

Denote by $\mathcal{CR}_\mathcal{S}$ the set

$$\big\{F \in \mathcal{B}(\ell^2, \mathcal{H}) : \ R(F) = \mathcal{S}\big\}.$$

This set has an interesting geometrical structure as a discrete union of homogeneous spaces of $GL(\ell^2) \times GL(\mathcal{H})$ (see [9]). If $\mathcal{S} = \mathcal{H}$ we write $\mathcal{E}$ instead of $\mathcal{CR}_\mathcal{H}$.

Theorem 5.10. *The map $pr_1 : \mathfrak{X}_\mathcal{S} \to \mathcal{CR}_\mathcal{S}$ defined by $pr_1((F, H)) = F$ is onto, has affine fibers, and admits a global continuous cross section.*

Proof. The surjectivity of pr_1 follows directly from Proposition 4.1. Moreover, this result parametrizes the fiber $pr_1^{-1}(F) = \{(F, H) \in \mathfrak{X}_\mathcal{S}\}$ for every $F \in \mathcal{CR}_\mathcal{S}$ as

$$pr_1^{-1}(F) = \big\{(F, QF^\dagger P) : \ Q \in L(\ell^2), \ N(Q) = N(F), \ Q^2 = Q\big\}.$$

Observe that, since the set of all projections with a fixed kernel is an affine manifold, the same holds for $pr_1^{-1}(F)$. In particular, $pr_1^{-1}(F)$ is contractible. To end the proof it suffices to show that the Moore-Penrose map $F \to F^\dagger$ is continuous when it is restricted to sets like $\mathcal{CR}_\mathcal{S}$. This result easily follows from the next lemma. ■

Lemma 5.11. *If $F \in \mathcal{CR}_\mathcal{S}$ then $FF^* + P_{\mathcal{S}^\perp} \in GL(\mathcal{H})$ and $F^\dagger = F^*\big((FF^*) + P_{\mathcal{S}^\perp}\big)^{-1}$.*

Proof. One first checks that the inverse of $FF^* + P_{\mathcal{S}^\perp}$ is $(FF^*)^\dagger + P_{\mathcal{S}^\perp}$. Then, a simple computation, using elementary properties of †, shows that $F^*\big((FF^*) + P_{\mathcal{S}^\perp}\big)^{-1} = F^\dagger$. ■

Corollary 5.12. *$\mathfrak{X}_\mathcal{S}$ is homotopy equivalent to $\mathcal{CR}_\mathcal{S}$.*

Proof. The fiber bundle

$$pr_1^{-1}(F) \longrightarrow \mathfrak{X}_\mathcal{S} \longrightarrow \mathcal{CR}_\mathcal{S}$$

has a global cross section and contractible fibers. Therefore, as in the proof of Proposition 5.7, we get that $\mathfrak{X}_\mathcal{S} \simeq \mathcal{CR}_\mathcal{S}$. ■

Remark 5.13. It should be mentioned that the results of this section generalize those about the connected components of the set $\mathcal{E} = \{T \in B(\mathcal{H}) : \ R(T) = \mathcal{H}\}$ studied in [10]. Recall that $\mathcal{E}$ is an open subset of $B(\mathcal{H})$ with a natural action

$$GL(\mathcal{H}) \times \mathcal{E} \longrightarrow \mathcal{E}$$

defined by $(G, T) \to TG^{-1}$. The orbits by this action are exactly the connected components of $\mathcal{E}$. Moreover, the components are determined by the nullity of T: $T, T' \in \mathcal{E}$ belong to the same component if and only if $\dim N(T) = \dim N(T')$. Fix an orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ of $\mathcal{H}$ and define the unilateral shift $S \in B(\mathcal{H})$ by $Se_n = e_{n+1}$ ($n \in \mathbb{N}$). Then $S^* \in \mathcal{E}$ and all "finite" connected components of $\mathcal{E}$ have the form $GL(\mathcal{H}) \cdot S^{*n}$. The map $\mathcal{E} \to \mathcal{E}$ defined by $T \to (TT^*)^{-1/2}T$ is a retraction from $\mathcal{E}$ onto the subset $\mathcal{E}_0 = \{T \in B(\mathcal{H}) :\ TT^* = I\}$. It is well known that $\mathcal{E}$ corresponds naturally to the set of all frames on $\mathcal{H}$ and, under this correspondence, $\mathcal{E}_0$ is mapped onto the set of all Parseval frames.

Remark 5.14. As remarked above, any operator whose range equals $\mathcal{S}$, is the first coordinate of a pair in $\mathfrak{X}_{\mathcal{S}}$. Note that any $F : \ell^2 \to \mathcal{H}$ such that $\mathcal{S} \subset R(F)$ is the first coordinate of a pair in $\mathcal{F}_{\mathcal{S}}$. Indeed, consider $F|_{F^{-1}(\mathcal{S})} : F^{-1}(\mathcal{S}) \to \mathcal{S} \subset \mathcal{H}$, and let $F_0 = F|_{F^{-1}(\mathcal{S})} \oplus 0_{F^{-1}(\mathcal{S})^\perp}$ whose range equals $\mathcal{S}$, then $(F, F_0^{\dagger\,*})$ belongs to $\mathcal{F}_{\mathcal{S}}$.

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