

# Nonpositive Curvature in $p$ -Schatten class <sup>\*</sup>

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## Abstract

We study the geometry of the set  $\Delta_p$ , with  $1 < p < \infty$ , which consists of perturbations of the identity operator by  $p$ -Schatten class operators, which are positive and invertible as elements of  $B(H)$ . These manifolds have natural and invariant Finsler structures. In [3], we introduced the metric  $d_p$  and exposed several results about this metric space. The aim of this work is to prove that the space  $(\Delta_p, d_p)$  behaves in many senses like a nonpositive curvature metric space.<sup>1</sup>

## 1 Introduction

Let  $B(H)$  denote the algebra of bounded operators acting on a complex and separable Hilbert space  $H$ ,  $Gl(H)$  the group of invertible elements of  $B(H)$  and  $Gl(H)^+$  the set of all positive elements of  $Gl(H)$ .

If  $X \in B(H)$  is compact we denote by  $\{s_j(X)\}$  the sequence of singular values of  $X$  (decreasingly ordered). For  $1 \leq p < \infty$ , let

$$\|X\|_p = (\sum s_j(X)^p)^{1/p} = (tr |X|^p)^{1/p},$$

where  $tr$  is the usual trace functional, this defines a norm on the set

$$B_p(H) = \{X \in B(H) : \|X\|_p < \infty\},$$

called the  $p$ -Schatten class of  $B(H)$  (to simplify notation we use  $B_p$ ). A reference for this subject is [8].

The Clarkson-McCarthy inequalities on  $B_p$  (see [11]) assert that for  $2 \leq p < \infty$

$$2(\|A\|_p^p + \|B\|_p^p) \leq \|A - B\|_p^p + \|A + B\|_p^p \leq 2^{p-1}(\|A\|_p^p + \|B\|_p^p), \quad (1)$$

and for  $1 \leq p \leq 2$ , the inequality is reversed.

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In a fascinating paper [1], Ball, Carlen and Lieb defined that a Banach space  $X$  is  $q$ -uniformly convex with  $q \geq 2$ , if there exists a positive constant  $K > 0$  such that

$$2\left(\frac{1}{K}\|v\|^q + \|w\|^q\right) \leq \|v - w\|^q + \|v + w\|^q,$$

for any  $u, v \in X$  (it is easily to see that any Banach space  $q$ -uniformly convex is uniformly convex). In this paper Ball et al. proved that for  $1 < p \leq 2$ ,  $B_p$  is 2-uniformly convex (see Proposition 3.3). For the cases  $2 \leq p < \infty$ , the Clarkson-McCarthy inequalities imply the  $p$ -uniform convexity of  $B_p$ .

On  $B_p$  we define the following norm associated with  $a \in Gl(H)^+$ :

$$\|X\|_{p,a} := \|a^{-1/2} X a^{-1/2}\|_p.$$

This metric is invariant under the action by  $Gl(H, B_p) = \{x \in Gl(H) : x - 1 \in B_p\}$ ,

$$l : Gl(H, B_p) \times \Delta_p \rightarrow \Delta_p, \quad l_g(X) = gXg^*.$$

For each  $a \in \Delta_p$ , the tangent space  $T_a(\Delta_p)$  can be identified with  $B_p^{sa}$ , the selfadjoint part of  $B_p$  and the geometry induced by the norm  $\|\cdot\|_{p,a}$  has a rich geometrical structure, for example uniqueness of the short curve (or geodesic) connecting two points.

In this work we expose several results about the geometrical structure of the manifold

$$\Delta_p = \{a = 1 + X : X \in B_p\} \subset Gl(H)^+,$$

for  $1 < p < \infty$ . This study relates to previous work on differential geometry of positive operators (or positive definite matrices). Mainly a series of papers [4], [5] and [6] by Corach, Porta and Recht, where the geometry of the set of positive invertible elements of a  $C^*$  algebra was studied. Also this study is related to classical work on the geometry of positive matrices [12]. In all these works, the authors have obtained among other properties that the distance function between two short curves is a convex function, that is if  $\gamma_1, \gamma_2$  are geodesics then for all  $t \in [0, 1]$

$$d(\gamma_1(t), \gamma_2(t)) \leq (1 - t)d(\gamma_1(0), \gamma_2(0)) + td(\gamma_1(1), \gamma_2(1)).$$

The notion of convexity of the distance between short curves (with the same initial point) or the strict convexity of the power 2 of the distance are related with the notion of nonpositive curvate in metric spaces in sense of Busemann or Alexandrov respectively. For the development of the theory of nonpositively curved metric spaces, we shall consider works that have been carried out in two different directions: the works of H. Busemann and the works of A. D. Alexandrov and his collaborators. Both Busemann and Alexandrov started their works in the 1940s, and they showed that the notions of upper and lower curvature bounds make sense for a more general class of metric spaces than Riemannian manifolds, namely for “geodesic spaces”. Let us briefly describe these spaces: a complete metric space  $(X, d)$  is called a *geodesic length space*, or simply a *geodesic space*, if for any two points  $x, y \in X$ , there exists a shortest geodesic joining them, i.e. a continuous curve such that  $\gamma : [0, 1] \rightarrow X$  with  $\gamma(0) = x, \gamma(1) = y$ , and

$d(x, y) = L^d(\gamma)$ . Here,  $L^d(\gamma)$  denotes the *length* of  $\gamma$  (respect to the metric  $d$ ) and it is defined as

$$L^d(\gamma) := \sup \left\{ \sum_{i=1}^n d(\gamma(t_{i-1}), \gamma(t_i)) : 0 = t_0 < t_1 < \dots < t_n = 1, n \in \mathbb{N} \right\}.$$

A curve  $\gamma : [0, 1] \rightarrow X$  is called a *geodesic* if there exists  $\epsilon > 0$  such that

$$L^d(\gamma|_{[t, t']}) = d(\gamma(t), \gamma(t')) \quad \text{whenever } |t - t'| < \epsilon.$$

Finally, a geodesic  $\gamma : [0, 1] \rightarrow X$  is called a *shortest geodesic* if  $L^d(\gamma) = d(\gamma(0), \gamma(1))$ .

For more details on such metric spaces we refer to [9] and [2]. The ramifications of these two theories continue to grow today, especially since the rekindling of interest that was given to nonpositive curvature by M. Gromov in the 1970s. Let us briefly describe the basic underlying ideas of these works.

A geodesic space  $(X, d)$  is said to be a *Busemann nonpositive curvature space* (BNPC) if for every  $p \in X$  there exists  $\delta_p > 0$  such that for any  $x, y, z \in B(p, \delta_p)$  and any shortest geodesic  $\gamma_1, \gamma_2 : [0, 1] \rightarrow X$  with  $\gamma_1(0) = \gamma_2(0) = x \in B(p, \delta_p)$  and with endpoints  $\gamma_1(1), \gamma_2(1) \in B(p, \delta_p)$ , we have

$$d(\gamma_1(\frac{1}{2}), \gamma_2(\frac{1}{2})) \leq \frac{1}{2}d(\gamma_1(1), \gamma_2(1)).$$

A geodesic space  $(X, d)$  is said to be an *Alexandrov nonpositive curvature space* (ANPC) if for every  $p \in X$  there exists  $\rho_p > 0$  such that for any  $x, y, z \in B(p, \rho_p)$  and any shortest geodesic  $\gamma : [0, 1] \rightarrow X$  with  $\gamma(0) = x, \gamma(1) = z$ , we have for  $0 \leq t \leq 1$

$$d^2(y, \gamma(t)) \leq (1-t)d^2(y, \gamma(0)) + td^2(y, \gamma(1)) - t(1-t)L^d(\gamma)^2.$$

This paper is organized as follows: In Section 2 we introduce a metric in the tangent spaces of  $\Delta_p$ , which are  $r$ -uniformly convex with  $r = \max\{2, p\}$ . This strong geometric condition implies the uniqueness of short curves in the metric space  $(\Delta_p, d_p)$ , where  $d_p$  is the metric given by the infima of the lengths of curves joining two given points in  $\Delta_p$ , measured with the Finsler metric  $\|\cdot\|_{p,a}$ . This space was previously studied in [3].

In Section 3 we introduce the notion of  $p$ -uniform convexity of a metric space (see Definition 3.2) and we prove that the space  $\Delta_p$  is  $r$ -uniformly convex. As a consequence of this metric condition we obtain existence and uniqueness of best approximation to convex closed subsets of  $\Delta_p$  (where by a convex set  $K$  we mean a set such that any geodesic joining two elements of  $K$  is entirely contained in  $K$ ).

## 2 The geometry of $\Delta_p$

We shall need the following facts concerning the geometric structure of the submanifold  $\Delta_p$ , which we take from [3].

The first fact states that if we measure the length of a smooth curve  $\gamma(t) \subset \Delta_p$ ,  $t \in [a, b]$ , using the Finsler norm, i.e.

$$L_p(\gamma) = \int_a^b \|\dot{\gamma}(t)\|_{p, \gamma(t)} dt,$$

then the curves of the form  $\gamma_{a,b}(t) = a^{1/2}(a^{-1/2}ba^{-1/2})^t a^{1/2}$ , with  $a, b \in \Delta_p$ , have minimal length along their paths for  $t \in [0, 1]$ . Note that  $\gamma_{a,b}(0) = a$  and  $\gamma_{a,b}(1) = b$ , and thus

$$d_p(a, b) = \inf\{L_p(\gamma) : \gamma \in \Omega_{a,b}\} = L_p(\gamma_{a,b}) = \|\log(a^{-1/2}ba^{-1/2})\|_p,$$

where we denote by

$$\Omega_{a,b} = \{\alpha : [0, 1] \rightarrow \Delta_p : \alpha \text{ is a } C^1 \text{ curve, } \alpha(0) = a \text{ and } \alpha(1) = b\}$$

the set of smooth curves in  $\Delta_p$  joining  $a$  to  $b$ .

**Lemma 2.1.** [3, Proposition 5.2] For all  $X, Y \in B_p^{sa}$

$$\|Y\|_p \leq \|e^{-X/2} d \exp_X(Y) e^{-X/2}\|_p,$$

where  $d \exp_X$  denotes the differential of the exponential map at  $X$ .

We give now a different proof of the existence of minimal curves. This proof involves the uniform convexity of the tangent spaces.

For a piecewise differentiable curve  $\beta : [0, 1] \rightarrow B_p^{sa}$ , one computes the *length* of the curve  $\beta$  by

$$L(\beta) = \int_0^1 \|\dot{\beta}(t)\|_p dt.$$

Let  $\delta$  be a piecewise smooth curve  $\delta \subset \Delta_p$ . Then  $\delta = e^\beta$  for a uniquely determined piecewise smooth curve  $\beta = \log(\delta)$  such that  $\beta \subseteq B_p^{sa}$ . By the previous lemma, we get

$$\dot{\delta} = d \exp_\beta(\dot{\beta}) = \int_0^1 e^{(1-t)\beta} \dot{\beta} e^{t\beta} dt.$$

We begin comparing the lengths of the curves  $\delta$  and  $\beta$ .

**Theorem 2.2.** Let  $\delta = e^\beta \subseteq \Delta_p$  be a piecewise smooth curve. Then  $L(\beta) \leq L_p(\delta)$ .

*Proof.* Let us compute the speed of  $\delta$  using the Lemma 2.1:

$$\|\dot{\delta}\|_p \leq \|e^{-\frac{\beta}{2}} d \exp_\beta(\dot{\beta}) e^{-\frac{\beta}{2}}\|_p = \|\dot{\beta}\|_{p,\delta}.$$

□

**Corollary 2.3.** (EMI property) Let  $X, Y \in B_p^{sa}$ . Then

$$\|X - Y\|_p \leq d_p(e^X, e^Y).$$

*Proof.* Let  $\delta \in \Omega_{e^X, e^Y}$ , put  $\delta = e^\beta$  as before. Then

$$\|Y - X\|_p = \|\beta(1) - \beta(0)\|_p = \left\| \int_0^1 \dot{\beta} dt \right\|_p \leq \int_0^1 \|\dot{\beta}\|_p dt = L(\beta) \leq L_p(\delta).$$

Hence

$$\|Y - X\|_p \leq d_p(e^X, e^Y). \tag{2}$$

□

**Remark 2.4.** The hypothesis of Theorem 2.4 holds for any  $p \in (1, \infty)$ , since it is a simple consequence of the uniform convexity of these spaces.

We summarize in the following proposition the basic properties of the metric space  $\Delta_p$ .

**Proposition 2.5.** *Given  $a, b \in \Delta_p$  and  $g \in Gl(H, B_p)$  we get*

1.

$$d_p(a, b) = d_p(a^{-1}, b^{-1}).$$

2. For all  $t \in \mathbb{R}$

$$d_p(a, \gamma_{a,b}(t)) = |t|d_p(a, b).$$

3. Invariance of the metric under the action by  $Gl(H, B_p)$ :

$$d_p(a, b) = d_p(gag^*, bgb^*).$$

4. If  $X, Y \in B_p^{sa}$  commute, we have

$$\|X - Y\|_p = d_p(e^X, e^Y).$$

In particular on each line  $\mathbb{R}X \subseteq B_p^{sa}$  the exponential map preserves distances.

5. If 1 lies on the geodesic  $\gamma_{a,b}$ , then  $a$  and  $b$  commute and

$$\log(b) = -\frac{1-t}{t} \log(a).$$

where  $t = d_p(a, 1)/d_p(a, b)$ .

6. Let  $s \in [1, \infty)$ . Then the  $s$ -energy functional

$$E_s : \Omega_{a,b} \rightarrow \mathbb{R}^+, \quad E_s(\beta) := \int_0^1 \|\dot{\beta}(t)\|_{p, \beta(t)}^s dt,$$

has its global minimum  $d_p^s(a, b)$  precisely at  $\gamma_{a,b}$ .

7.  $\Delta_p$  is a complete metric space with the geodesic distance  $d_p$ .

A midpoint map for a metric space  $(X, d)$  is a map  $m : X \times X \rightarrow X$  satisfying

$$d(m(x, y), x) = \frac{1}{2}d(x, y) = d(m(x, y), y) \quad \forall x, y \in X.$$

Given  $a, b \in \Delta_p$  and  $\gamma_{a,b}$  the shortest curve joining them, we can define the following midpoint map

$$m : \Delta_p \times \Delta_p \rightarrow \Delta_p, \quad m(a, b) := \gamma_{a,b}\left(\frac{1}{2}\right).$$

**Definition 2.6.** Let  $K \subseteq \Delta_p$ . We say that  $K$  is convex if, for any  $a, b \in K$ ,  $\gamma_{a,b}(t) \in K$  for any  $t \in [0, 1]$ .

### 3 Clarkson's inequalities and uniform convexity

#### 3.1 Weak semi parallelogram law

Let  $(V, \langle \cdot, \cdot \rangle)$  be an euclidean space, i.e.  $V$  is a real vector space (finite or infinite dimensional) and  $\langle \cdot, \cdot \rangle$  is a positive definite symmetric bilinear form on  $V$ . Then  $\|v\| = \sqrt{\langle v, v \rangle}$  defines a norm on  $V$  and the parallelogram law states that for  $u, v \in V$  we have

$$\|v - w\|^2 + \|v + w\|^2 = 2(\|v\|^2 + \|w\|^2).$$

If we consider a parallelogram with vertices  $x, x_1, x_2$  and  $x_3 = x_1 + x_2 - x$ , then the parallelogram law reads

$$d(x_1, x_2)^2 + d(x, x_3)^2 = 2d(x, x_1)^2 + 2d(x, x_2)^2$$

where  $d(a, b) := \|a - b\|$ . If  $z = \frac{x_1 + x_2}{2}$  is the midpoint of  $x_1$  and  $x_2$ , then we get

$$d(x_1, x_2)^2 + 4d(x, z)^2 = 2d(x, x_1)^2 + 2d(x, x_2)^2$$

**Definition 3.1.** Let  $(X, d)$  be a metric space. We say that  $(X, d)$  satisfies the semi parallelogram law (SPL) if for  $x_1, x_2 \in X$  there exists a point  $z \in X$  such that for each  $x \in X$  we have

$$d(x_1, x_2)^2 + 4d(x, z)^2 \leq 2d(x, x_1)^2 + 2d(x, x_2)^2.$$

Note that the point  $z$  occurring in the preceding definition plays the role of a midpoint between  $x_1$  and  $x_2$ .

The above condition can be rephrased as follows in a geodesic length space (see [9]):

For any  $x \in X$  and any minimal curve  $\eta : [0, 1] \rightarrow X$  with  $\eta(0) = x_1, \eta(1) = x_2$ , we have

$$d(x, \eta(\frac{1}{2}))^2 \leq \frac{1}{2}d(x, \eta(0))^2 + \frac{1}{2}d(x, \eta(1))^2 - \frac{1}{4}d(\eta(0), \eta(1))^2. \quad (3)$$

One natural generalization of the  $p$ -uniform convexity to a metric space is the following:

**Definition 3.2.** Let  $(X, d)$  be a metric space and  $p \geq 2$ . We say that  $(X, d)$  is  $p$ -uniformly convex if for any  $x \in X$  and any minimal curve  $\eta : [0, 1] \rightarrow X$  with  $\eta(0) = x_1, \eta(1) = x_2$ , there exists a constant  $c_p > 0$  such that

$$d(x, \eta(\frac{1}{2}))^p \leq \frac{1}{2}d(x, \eta(0))^p + \frac{1}{2}d(x, \eta(1))^p - \frac{1}{4}c_p d(\eta(0), \eta(1))^p. \quad (4)$$

If  $p = 2$  and  $c_2 = 1$ , then the inequality (4) corresponds to the SPL.

At this point we can easily prove the  $r$ -uniform convexity “at the origin” of  $\Delta_p$ .

The proof of this fact requires some preliminaries. We begin with the following inequalities, proved in [1] by Ball, Carlen and Lieb.

**Proposition 3.3.** For  $A, B \in B_p(H)$  and  $1 \leq p \leq 2$  it holds that

$$\|A\|_p^2 + (p-1)\|B\|_p^2 \leq \frac{1}{2}(\|A+B\|_p^2 + \|A-B\|_p^2). \quad (5)$$

**Lemma 3.4.** *Let  $X, B \in B_p^{sa}$  and  $\gamma : [0, 1] \rightarrow \Delta_p$  be the geodesic joining  $e^B$  with  $e^{-B}$ . Then for  $1 < p < \infty$*

$$d_p(e^X, \gamma(\frac{1}{2}))^r \leq \frac{1}{2}(d_p(e^X, \gamma(0))^r + d_p(e^X, \gamma(1))^r) - \frac{1}{4}c_r d_p(\gamma(0), \gamma(1))^r, \quad (6)$$

*with  $r = \max\{p, 2\}$  and  $c_r = p - 1$  if  $r = 2$  or  $c_r = \frac{1}{2^{p-2}}$  if  $r \neq 2$ .*

*Proof.* By (1), if  $2 < p < \infty$

$$\begin{aligned} 2(\|X\|_p^p + \|B\|_p^p) &= 2(d_p(e^X, 1)^p + d_p(e^B, 1)^p) \\ &\leq \|X - B\|_p^p + \|X + B\|_p^p \\ &\leq d_p(e^X, e^B)^p + d_p(e^X, e^{-B})^p. \end{aligned}$$

Since  $d_p(e^B, 1) = \frac{1}{2}d_p(e^B, e^{-B}) = \frac{1}{2}L_p(\gamma)$  we have

$$\frac{1}{2^p}L_p(\gamma)^p \leq \frac{1}{2}(d_p(e^X, \gamma(0))^p + d_p(e^X, \gamma(1))^p) - d_p(e^X, \gamma(\frac{1}{2}))^p.$$

Now, we consider the case  $1 < p \leq 2$ . Applying the exponential map and using the EMI property in the inequality (5) we obtain

$$\begin{aligned} \|X\|_p^2 + (p-1)\|B\|_p^2 &= d_p(e^X, 1)^2 + (p-1)d_p(e^B, 1)^2 \\ &\leq \frac{1}{2}(\|X + B\|_p^2 + \|X - B\|_p^2) \\ &\leq \frac{1}{2}(d_p(e^X, e^B)^2 + d_p(e^X, e^{-B})^2). \end{aligned}$$

Since  $d_p(e^B, 1) = \frac{1}{2}d_p(e^B, e^{-B}) = \frac{1}{2}L_p(\gamma)$ , we have

$$(p-1)\frac{1}{2^2}L_p(\gamma)^2 \leq \frac{1}{2}(d_p(e^X, \gamma(0))^2 + d_p(e^X, \gamma(1))^2) - d_p(e^X, \gamma(\frac{1}{2}))^2.$$

□

The following proposition establishes the  $r$ -uniform convexity for  $\Delta_p$ .

**Theorem 3.5.** *Let  $X \in B_p^{sa}$ , and  $\gamma : [0, 1] \rightarrow \Delta_p$  be a geodesic. Then for  $1 < p < \infty$*

$$d_p(e^X, \gamma(\frac{1}{2}))^r \leq \frac{1}{2}(d_p(e^X, \gamma(0))^r + d_p(e^X, \gamma(1))^r) - \frac{1}{4}c_r d_p(\gamma(0), \gamma(1))^r, \quad (7)$$

*with  $r$  and  $c_r$  as above.*

*Proof.* Given  $a = \gamma(0), b = \gamma(1) \in \Delta_p$ , let  $m = m(a, b)$  be the midpoint of  $a$  and  $b$ . We claim that there exist  $g \in GL(H, B_p)$  and  $X \in B_p^{sa}$  with

$$l_g(a) = e^X; \quad l_g(b) = e^{-X}.$$

First, observe that  $h_1 = b^{-\frac{1}{2}}$  satisfies  $l_{h_1}(b) = 1$ . Let  $x := l_{h_1}(a)$  and we define  $h_2 : x^{-\frac{1}{4}}$  and  $g := h_2 h_1$ .

Then

$$l_g(a) = h_2 h_1 a h_1 h_2 = l_{h_2}(x) = x^{\frac{1}{2}}.$$

$$l_g(b) = h_2 h_1 b h_1 h_2 = l_{h_2}(1) = x^{-\frac{1}{2}}.$$

Now the claim above follows with  $x = e^{2X}$ .

By the invariance of the distance under the action of  $GL(H, B_p)$  and the claim, it suffices to verify the inequality for pairs  $a, b$  with  $b = a^{-1}$ . This case follows from the previous lemma.  $\square$

Following [7], one can define two different notions of convexity for metric spaces.

**Definition 3.6.** Let  $(X, d)$  be a metric space admitting a midpoint map.  $(X, d)$  is called

1. *ball convex* if for all  $x, y, z \in X$

$$d(m(x, y), z) \leq \max\{d(x, z), d(y, z)\}. \quad (8)$$

for any midpoint  $m$ . It is called *strictly ball convex* if the inequality is strict whenever  $x \neq y$ ;

2. *distance convex* if for all  $x, y, z \in X$

$$d(m(x, y), z) \leq \frac{1}{2}[d(x, z) + d(y, z)], \quad (9)$$

for any midpoint map  $m$ .

Note that the condition (9) implies condition (8), and also that strict ball convexity implies the uniqueness of a midpoint map.

Now, we give the definition of uniform ball convexity of metric spaces, see [7]:

**Definition 3.7.** Let  $(X, d)$  be a metric space admitting a midpoint map.  $(X, d)$  is called *uniformly ball convex* if for all  $\epsilon > 0$  there exists a  $\rho(\epsilon) > 0$  such that for all  $x, y, z \in X$  satisfying  $d(x, y) > \epsilon \max\{d(x, z), d(y, z)\}$ , it holds that

$$d(m(x, y), z) \leq (1 - \rho(\epsilon)) \max\{d(x, z), d(y, z)\}$$

for the (unique) midpoint map  $m$ .

**Corollary 3.8.** The metric space  $(\Delta_p, d_p)$  is strictly ball convex for  $1 < p < \infty$ .

*Proof.* Let  $a, b, c \in \Delta_p$  with  $a \neq b$ , then

$$\begin{aligned} d_p(c, \gamma_{a,b}(\frac{1}{2}))^r &\leq \frac{1}{2}(d_p(c, a)^r + d_p(c, b)^r) - \frac{1}{4}c_r d_p(a, b)^r \\ &< (\max\{d_p(c, a), d_p(c, b)\})^r. \end{aligned}$$

$\square$



**Corollary 3.9.** *If  $a, b, c \in \Delta_p$  ( $1 < p < \infty$ ) are three arbitrary points, then we have*

$$2^r(L_A)^r \leq 2^{r-1}B^r + 2^{r-1}C^r - c_r A^r$$

*with  $A = d_p(b, c), B = d_p(c, a), C = d_p(a, b)$  and  $L_A$  the length of the geodesic joining  $a$  to  $m(b, c)$ .*

Another interesting consequence of Theorem 3.5 is the uniform convexity of  $(\Delta_p, d_p)$ .

**Corollary 3.10.** *For  $1 < p < \infty$ , the metric space  $(\Delta_p, d_p)$  is uniformly ball convex.*

One can compute the modulus of uniform convexity explicitly in these cases.

Consider  $\epsilon > 0$ , set  $d_p(a, b), d_p(a, c) \leq s$  and  $d_p(b, c) \geq \epsilon s$ . Then,

$$d_p(a, m(b, c))^r \leq s^r - \frac{1}{4}c_r d_p(b, c)^r \leq s^r - \frac{1}{4}c_r(\epsilon s)^r = [1 - c_r(\frac{\epsilon}{2^{2/r}})^r]s^r.$$

Then an admissible value for the modulus of uniform convexity is  $\rho_{\Delta_p}(\epsilon) = 1 - [1 - c_r(\frac{\epsilon}{2^{2/r}})^r]^{1/r}$ .

Note that the formula of the modulus is similar with the one obtained by Clarkson for the space  $L_p$ .

**Proposition 3.11.** *Let  $X \in B_p^{sa}$ ,  $\gamma : [0, 1] \rightarrow \Delta_p$  a geodesic and  $1 < p \leq 2$ . Then for all  $t \in [0, 1]$*

$$d_p(e^X, \gamma(t))^r \leq (1-t)d_p(e^X, \gamma(0))^r + td_p(e^X, \gamma(1))^r - t(1-t)c_r d_p(\gamma(0), \gamma(1))^r. \quad (10)$$

*Proof.* Let us denote  $W_2(t) = t(1-t)$ . Given any geodesic  $\gamma : [0, 1] \rightarrow \Delta_p$ , it suffices to prove the previous inequality for all dyadic  $t \in [0, 1]$ . It obviously holds for  $t = 0$  and  $t = 1$ . Assume that it holds for all  $t = k2^{-n}$  with  $k = 0, 1, \dots, 2^n$ . We want to prove that (10) also holds for all  $t = k2^{-(n+1)}$  with  $k = 0, 1, \dots, 2^{n+1}$ . For  $k$  even this is clear. Fix  $t = k2^{-(n+1)}$  with  $k$  odd; and put  $\Delta t = 2^{-(n+1)}$ . Then by (7)

$$d_p(e^X, \gamma(t))^r \leq \frac{1}{2}(d_p(e^X, \gamma(t - \Delta t))^r + d_p(e^X, \gamma(t + \Delta t))^r) - \frac{1}{4}c_r d_p(\gamma(t - \Delta t), \gamma(t + \Delta t))^r. \quad (11)$$

By the assumption for multiples of  $2^{-n}$

$$\begin{aligned} d_p(e^X, \gamma(t \pm \Delta t))^r &\leq (1-t \mp \Delta t)d_p(e^X, \gamma(0))^r + (t \pm \Delta t)d_p(e^X, \gamma(1))^r \\ &\quad - W_2(t \pm \Delta t)c_r d_p(\gamma(0), \gamma(1))^r. \end{aligned}$$

Thus, by (11)

$$d_p(e^X, \gamma(t))^r \leq (1-t)d_p(e^X, \gamma(0))^r + td_p(e^X, \gamma(1))^r - [g(t, \Delta t)]c_r d_p(\gamma(0), \gamma(1))^r,$$

where  $g(t, \Delta t) = (\Delta t)^2 + \frac{1}{2}W_2(t - \Delta t) + \frac{1}{2}W_2(t + \Delta t)$ .

Since

$$W_2(t) = (\Delta t)^2 + \frac{1}{2}W_2(t - \Delta t) + \frac{1}{2}W_2(t + \Delta t) = g(t, \Delta t), \quad (12)$$

then

$$d_p(e^X, \gamma(t))^r \leq (1-t)d_p(e^X, \gamma(0))^r + td_p(e^X, \gamma(1))^r - W_2(t)c_r d_p(\gamma(0), \gamma(1))^r.$$

□

**Corollary 3.12.** *Let  $\gamma, \eta : [0, 1] \rightarrow \Delta_p$ ,  $1 < p \leq 2$  and  $t \in [0, 1]$ , then*

$$\begin{aligned} d_p(\eta(t), \gamma(t))^r &\leq (1-t)^2 d_p(\eta(0), \gamma(0))^r + t^2 d_p(\eta(1), \gamma(1))^r - t(1-t) c_r(L_p(\eta)^r + L_p(\gamma)^r) \\ &\quad + t(1-t)[d_p(\eta(0), \gamma(1))^r + d_p(\eta(1), \gamma(0))^r]. \end{aligned}$$

*Proof.* Applying (10) twice, we obtain that

$$\begin{aligned} d_p(\eta(t), \gamma(t))^r &\leq (1-t) d_p(\eta(0), \gamma(t))^r + t d_p(\eta(1), \gamma(t))^r - t(1-t) c_r L_p(\eta)^r \\ &\leq (1-t)[(1-t) d_p(\eta(0), \gamma(0))^r + t d_p(\eta(0), \gamma(1))^r - t(1-t) c_r L_p(\gamma)^r] \\ &\quad + t[(1-t) d_p(\eta(1), \gamma(0))^r + t d_p(\eta(1), \gamma(1))^r - t(1-t) c_r L_p(\gamma)^r] \\ &\quad - t(1-t) c_r L_p(\eta)^r \\ &= (1-t)^2 d_p(\eta(0), \gamma(0))^r + t^2 d_p(\eta(1), \gamma(1))^r - t(1-t) c_r(L_p(\eta)^r + L_p(\gamma)^r) \\ &\quad + t(1-t)[d_p(\eta(0), \gamma(1))^r + d_p(\eta(1), \gamma(0))^r]. \end{aligned}$$

□

In particular if  $p = r = 2$ , the metric space  $(\Delta_2, d_2)$  is an Alexandrov nonpositive curvature space and we get

$$d_2(\eta(t), \gamma(t)) < (1-t) d_2(\eta(0), \gamma(0)) + t d_2(\eta(1), \gamma(1)),$$

or equivalently,  $d_2$  is strictly convex on geodesics.

Now, we try to extend this result for  $p > 1$  with  $p \neq 2$ . Recently, Larotonda [10] using the theory of dissipative operators and the theory of entire functions, derived several operator inequalities for unitarily invariant norms. Among them, if  $X, Y \in B_p^{sa}$

$$\|\log(e^{\frac{-t}{2}X} e^{tY} e^{\frac{-t}{2}X})\|_p \leq t \|\log(e^{\frac{-X}{2}} e^Y e^{\frac{-X}{2}})\|_p. \quad (13)$$

This inequality establishes the convexity of the geodesic distance  $d_p$  in the Finsler manifold  $\Delta_p$ , that is

**Proposition 3.13.** *Let  $a, b, c, d \in \Delta_p$ , then*

$$d_p(\gamma_{a,b}(t), \gamma_{c,d}(t)) \leq t d_p(a, c) + (1-t) d_p(b, d). \quad (14)$$

*Proof.* Consider the geodesic rectangle with vertices  $a, b, c, d$ . Let  $\gamma_{c,b}$  be the short curve joining  $c$  to  $b$  in  $\Sigma_\Phi$ , and consider the triangle with sides  $c, b, d$ , and the geodesic triangle with sides  $b, a, c$ . Note that  $\gamma_{c,b}(t) = \gamma_{b,c}(1-t)$  and the same holds for  $\gamma_{a,b}$ . Then, by the triangle inequality

$$d_p(\gamma_{a,b}(t), \gamma_{c,d}(t)) \leq d_p(\gamma_{a,b}(t), \gamma_{c,b}(t)) + d_p(\gamma_{c,b}(t), \gamma_{c,d}(t)),$$

and by (13)

$$d_p(\gamma_{c,b}(t), \gamma_{c,d}(t)) \leq t d_p(b, d).$$

Also

$$d_p(\gamma_{b,c}(1-t), \gamma_{b,a}(1-t)) \leq (1-t) d_p(a, b).$$

Adding these two inequalities yields the convexity of  $d_p$ . □

### 3.2 Best approximation

Given a subset  $K \subseteq \Delta_p$  and an element  $a \in \Delta_p$ , put

$$d_p(a, K) = \inf\{d_p(a, k) : k \in K\}.$$

In Theorem 3.14, we shall prove that, as in a Hilbert space, one can define a metric projection onto convex closed subsets of  $\Delta_p$ . In other words given  $K$  a convex closed subset of  $\Delta_p$  and  $a \in \Delta_p$ , there is a unique  $k_0 \in K$  such that the length of the geodesic joining  $a$  with  $k_0$  is the distance between  $a$  and  $K$ . That is, there is a unique solution to the minimization problem

$$\begin{cases} k_0 \in K \\ d_p(a, k_0) \leq d_p(a, k) \forall k \in K. \end{cases} \quad (15)$$

**Theorem 3.14.** (*Best approximation*) Let  $K \subseteq \Delta_p$  be a convex closed set,  $1 < p < \infty$  and  $a \in \Delta_p$ . Then the problem (15) has a unique solution. In other words, there is a unique  $\pi_K(a) = k_0 \in K$  such that  $d_p(a, k_0) = d_p(a, K)$ . In addition, if  $\tilde{a}$  belongs to the geodesic segment  $[a, \pi_K(a)]$ , then  $\pi_K(\tilde{a}) = \pi_K(a)$ .

*Proof.* Let  $\{k_n\}_{n \in \mathbb{N}}$  be a sequence in  $K$ , such that  $d_p(a, k_n) \rightarrow d_p(a, K)$ . By Theorem 3.5 we obtain that

$$\begin{aligned} \frac{1}{4}c_r d_p(k_n, k_m)^r &\leq \frac{1}{2}(d_p(k_n, a)^r + d_p(a, k_m)^r) - d_p(a, k_{n,m})^r \\ &\leq \frac{1}{2}(d_p(k_n, a)^r + d_p(a, k_m)^r) - d_p(a, K)^r, \end{aligned} \quad (16)$$

where  $k_{n,m} = \gamma(\frac{1}{2}) \in K$ , with  $\gamma$  the geodesic joining  $k_n$  and  $k_m$ .

This implies that  $\{k_n\}_{n \in \mathbb{N}}$  is a Cauchy sequence in  $K$ , hence convergent to some  $k_0 \in K$ . Since  $K$  is closed,  $k_0 \in K$ . By the continuity of the distance we have

$$d_p(k_0, a) = \lim_{n \rightarrow \infty} d_p(k_n, a) = d_p(a, K).$$

For the uniqueness part, let  $k_1, k_2 \in K$  such that

$$d_p(k_1, a) = d_p(a, K) = d_p(k_2, a).$$

Replacing  $k_n$  and  $k_m$  by  $k_1$  and  $k_2$  respectively in (16), we obtain

$$\begin{aligned} d_p(a, k_{1,2})^r &\leq \frac{1}{2}(d_p(k_1, a)^r + d_p(a, k_2)^r) - \frac{1}{4}c_r d_p(k_1, k_2)^r \\ &= d_p(a, K)^r - \frac{1}{4}c_r d_p(k_1, k_2)^r, \end{aligned}$$

since  $k_{1,2} \in K$ , the above inequality proves that  $d_p(k_1, k_2) = 0$ . □

**Definition 3.15.** Let  $K \subseteq \Delta_p$  be a convex closed set,  $1 < p < \infty$  and  $a \in \Delta_p$ . By Theorem 3.14 there is exactly one point  $\pi_K(a) \in K$  such that

$$d_p(a, \pi_K(a)) = d_p(a, K).$$

Then  $\pi_K(a)$  is called the projection of  $a$  to  $K$ . The map  $\pi_K : \Delta_p \rightarrow K$  is called the projection map to  $K$ .

**Remark 3.16.** 1. We remark that we have proved the existence of a unique projection without assuming any kind of compactness for  $K$ .

2. Clearly  $\pi_K^2 = \pi_K$ .

**Theorem 3.17.** Let  $K \subseteq \Delta_p$  be a convex closed set,  $1 < p < \infty$  and  $\pi_K$  the projection map onto  $K$ . Then  $\pi_K$  is continuous.

*Proof.* Let the sequence  $\{c_n\}$  converge to  $c$  in  $\Delta_p$ . For simplicity, denote  $\pi_K(c_n)$  by  $u_n$ . Now  $\{u_n\}$  is a Cauchy sequence in  $K$ , otherwise there are positive numbers  $\epsilon$  and subsequences  $\{u_{n_k}\}$  and  $\{u_{m_k}\}$  such that  $n_k < m_k$  and  $d_p(u_{n_k}, u_{m_k}) \geq \epsilon$  for all  $k$ . Put  $a_k = u_{n_k}, b_k = u_{m_k}$  and  $M_k = \max\{d_p(c, a_k), d_p(c, b_k)\}$ .

Note that  $M_k \rightarrow d_p(c, K)$  as  $k \rightarrow \infty$ . Now  $d_p(c, a_k) \leq M_k, d_p(c, b_k) \leq M_k$  and  $d_p(a_k, b_k) \geq (\frac{\epsilon}{M_k})M_k$ . This implies

$$d_p(c, m(a_k, b_k)) \leq M_k \left( 1 - \rho_{\Delta_p} \left( \frac{\epsilon}{M_k} \right) \right) \leq M_k \left( 1 - \rho_{\Delta_p} \left( \frac{d(a_k, b_k)}{M_k} \right) \right).$$

Also  $\rho_{\Delta_p}(\frac{\epsilon}{M_k}) \leq 1 - \frac{d_p(c, K)}{M_k}$ , letting  $k \rightarrow \infty$ , one has  $\delta_{\Delta_p}(\frac{\epsilon}{M_k}) \rightarrow 0$  and  $\epsilon$  can not be positive. Thus  $\{\pi_K(c_n)\}$  is a Cauchy sequence in  $K$  and therefore converges to a point  $z$  in  $K$ , as  $d_p(c, z) = d_p(c, K)$ , then  $z = \pi_K(c)$ .  $\square$

Another useful property of the  $\Delta_p$  spaces with  $1 < p \leq 2$  is the following Pythagoras type inequality.

**Corollary 3.18.** Under the same conditions stated above, we have that for all  $k \in K$  with  $K$  a closed and convex set and  $t \in (0, 1]$

$$d_p(a, \pi_K(a))^2 + (1-t)(p-1)d_p(\pi_K(a), k)^2 \leq d_p(a, k)^2, \quad (17)$$

in particular

$$d_p(a, \pi_K(a))^2 + (p-1)d_p(\pi_K(a), k)^2 \leq d_p(a, k)^2. \quad (18)$$

*Proof.* Let  $\gamma : [0, 1] \rightarrow \Delta_p$  be the geodesic joining  $\gamma(0) = \pi_K(a)$  and  $\gamma(1) = k$ , then  $\gamma(t) \in K$  by the convexity of  $K$ . Hence, by Proposition 3.11

$$\begin{aligned} d_p(a, \pi_K(a))^2 &\leq d_p(a, \gamma(t))^2 \\ &\leq (1-t)d_p(a, \pi_K(a))^2 + td_p(a, k)^2 - t(1-t)(p-1)d_p(\pi_K(a), k)^2, \end{aligned}$$

and therefore

$$td_p(a, \pi_K(a))^2 \leq td_p(a, k)^2 - t(1-t)(p-1)d_p(\pi_K(a), k)^2.$$

Now if  $t \in (0, 1]$ , this is the desired inequality.  $\square$

We shall prove now that the inequality (18) characterizes solutions of the minimization problem.

**Theorem 3.19.** Let  $K \subseteq \Delta_p$  be a convex closed subset and  $q \in \Delta_p$  with  $1 < p \leq 2$ . Suppose that  $q_0 \in K$  verifies (18), then  $q_0$  is the unique solution of (15).

*Proof.* For all  $k \in K$  and  $t \in (0, 1]$  we have

$$d(q, q_0)^2 + (p - 1)d(q_0, k)^2 \leq d(q, k)^2.$$

Then  $d(q, q_0) \leq d(q, k)$ . For the uniqueness part, let  $q_0, q_1 \in K$  satisfying (18), then

$$(p - 1)d(q_1, q_0)^2 \leq d(q, q_1)^2 - d(q, q_0)^2 = d(q, K)^2 - d(q, K)^2.$$

□

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