Metric properties of projections in semi-Hilbertian spaces

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To our teacher Mischa Cotlar, in memoriam

Abstract. Several results on norms of projections on a Hilbert space $\mathcal{H}$ are extended for the operator seminorm defined by a positive semidefinite operator $A \in L(\mathcal{H})^+$.

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1. Introduction

In this paper, $\mathcal{H}$ denotes a Hilbert space, $L(\mathcal{H})$ is the algebra of bounded linear operators on $\mathcal{H}$ and $\mathcal{Q}$ is the subset of $L(\mathcal{H})$ of all projections (i.e. idempotents). Given a closed subspace $\mathcal{S}$ of $\mathcal{H}$, $\mathcal{Q}_{\mathcal{S}}$ denotes the subset of $\mathcal{Q}$ of all projections with image $\mathcal{S}$. The topology and differential geometry of $\mathcal{Q}$ and $\mathcal{P} = \{P \in \mathcal{Q} : P^* = P\}$ have been studied in detail in many places [3], [12], [13], [14], [27], [28], [29], [34], [35] and [39]. This paper is devoted to the study of several metrical properties of $\mathcal{Q}$ and $\mathcal{Q}_{\mathcal{S}}$ when an additional seminorm is consider on $\mathcal{H}$. Let $P_{\mathcal{S}} \in \mathcal{Q}_{\mathcal{S}}$ denote the unique Hemitian projection with image $\mathcal{S}$. The following properties are well known:

(I) For all $0 \neq Q \in \mathcal{Q}$ it holds $\|Q\| = 1$ if and only if $Q^* = Q$;
(II) For every non trivial $Q \in \mathcal{Q}$ it holds $\|Q\| = \|I - Q\|$;
(III) Given closed subspaces $\mathcal{S}$ and $\mathcal{T}$ of $\mathcal{H}$ it holds $\|P_{\mathcal{S}} - P_{\mathcal{T}}\| \leq \|Q_{\mathcal{S}} - Q_{\mathcal{T}}\|$ for every $Q_{\mathcal{S}} \in \mathcal{Q}_{\mathcal{S}}$ and $Q_{\mathcal{T}} \in \mathcal{Q}_{\mathcal{T}}$;
(IV) For all closed subspaces $\mathcal{S}$ and $\mathcal{T}$ of $\mathcal{H}$ it holds $\|P_{\mathcal{S}} - P_{\mathcal{T}}\| \leq 1$. Equality holds if and only if $P_{\mathcal{S}}$ and $P_{\mathcal{T}}$ commute;

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(V) For all closed subspaces $\mathcal{S}$ and $\mathcal{T}$ of $\mathcal{H}$ it holds $\|P_{\mathcal{S}} - P_{\mathcal{T}}\| = \max\{\ \|P_{\mathcal{S}}(I - P_{\mathcal{T}})\|,\ \|P_{\mathcal{T}}(I - P_{\mathcal{S}})\|\ \}$;

(VI) For every $Q \in \mathcal{Q}$ it holds $\|Q\| = \dfrac{1}{\sin\theta}$ if $\theta \in [0, \pi/2]$ is the angle such that $\cos\theta = \sup\{|\langle\xi, \eta\rangle| : \xi \in R(Q), \eta \in N(Q) \text{ and } \|\xi\| = \|\eta\| = 1\}$.

Here $R(Q)$ is the image of the projection Q and $N(Q)$ is its nullspace. Proofs of properties (I), (II) and (IV) can be found in textbooks like [8] and [23]. A proof of property (V) can be found in the book by Akhiezer and Glazman [1]. Property (III) is due to T. Kato [[23], Th. 6.35, p. 58] (see also M. Mbektha [[30], 1.10]). Property (VI) is due to V. Ljance [26]. Proofs of it can be found in the monograph of Gokhberg and Krein [20] and in the papers by V. Ptak [32], J. Steinberg [37], D. Buckholtz [6] and I. Ipsen and C. Meyer [22] (for finite dimensional spaces).

The main goal of this paper is to study these properties if we consider an additional seminorm $\|\ .\ \|_A$, defined by a positive semidefinited operator $A \in L(\mathcal{H})$ by $\|\xi\|_A^2 = \langle A\xi, \xi\rangle$, $\xi \in \mathcal{H}$, and we replace the operator norm in formulas (I) to (VI) by the quantity

$$\|T\|_A = \sup\{\|T\xi\|_A : \ \|\xi\|_A = 1\}.$$

Of course, many difficulties arise. For instance, it may happen that $\|T\|_A = +\infty$ for some $T \in L(\mathcal{H})$. Besides, there is no obvious choice for an adjoint operation defined by A. In order to describe our results, we need to introduce a certain relationship between positive operators and closed subspaces called compatibility in the recent literature. We say that a positive semidefinite operator A on $\mathcal{H}$ and a closed subspace $\mathcal{S}$ of $\mathcal{H}$ are **compatible** if there exists a projection $Q \in \mathcal{Q}_{\mathcal{S}}$ such that AQ is Hermitian (or symmetric). This means that $\langle Q\xi, \eta\rangle_A = \langle \xi, Q\eta\rangle_A$ for all $\xi, \eta \in \mathcal{H}$ where $\langle \xi, \eta\rangle_A = \langle A\xi, \eta\rangle$. In this case, it can be proved that $\mathcal{H} = \mathcal{S} \dot{+} (A\mathcal{S})^{\perp}$ and the projection $P_{A,\mathcal{S}}$ onto $\mathcal{S}$ with nullspace $(A\mathcal{S})^{\perp} \ominus \mathcal{S} \cap N(A)$ satisfies $AP_{A,\mathcal{S}} = P_{A,\mathcal{S}}^{*}A$. This operator, $P_{A,\mathcal{S}}$, has similar, but not identical, metric properties like the orthogonal projection $P_{\mathcal{S}}$. For example, if the pair $(A, \mathcal{S})$ is compatible then for every $\xi \in \mathcal{H}$ it holds that $\|(I - P_{A,\mathcal{S}})\xi\|_A = d_A(\xi, \mathcal{S}) = \inf\{\|\xi - \eta\|_A : \ \eta \in \mathcal{S}\}$. See [11] for its proof. Under convenient hypothesis of compatibility we are able to prove properties analogous to (I)-(VI) for the operator seminorm $\|\ .\ \|_A$ and a convenient adjoint operation.

The subject of operators which are symmetric under a certain inner product is quite old. Papers by M.G. Krein [24] in 1937 and W. T. Reid [33] in 1951, with references to earlier works, studied many spectral properties of the so-called **symmetrizable** operators. Later, P. Lax [25] and J. Dieudonné [15] studied conditions for the symmetrizability of operators. In more recent times, Z. Sebestyén [36], B.A. Barnes [4], S. Hassi, Z. Sebestyén and H. de Snoo [21] and P. Cojuhari and A. Gheondea [7] have found many interesting results and applications of various notions of symmetrizability.

The contents of the paper are the following. In section 2 we collect some facts about Moore-Penrose pseudoinverses, compatibility between positive operators and closed subspaces, and a brief description of a result by R. G. Douglas [17]

which plays a relevant role in this paper. Douglas theorem (sometimes called **range inclusion theorem**) gives necessary and sufficient conditions for the existence and uniqueness of solution for equations of the type $AX = TA$, with an additional condition on the range of X.
In section 3 we explore the existence of A-adjoints for projections. If a projection Q admits an A-adjoint then we define $Q^\sharp$ as the unique solution of the problem

$$AX = Q^*A, \quad R(X) \subseteq \overline{R(A)}.$$

Properties of $Q^\sharp$ are described.
Sections 4 and 5 contain the main results of the paper, i.e., the extension of properties (I) to (VI) above, as follows

(I') every projection Q such that $AQ = Q^*A \neq 0$ satisfies $\|Q\|_A = 1$;

(II') equality $\|Q\|_A = \|I-Q\|_A$ holds for any projection Q such that $R(Q) \cap \overline{R(A)} \neq \{0\}$ and $R(I-Q) \cap \overline{R(A)} \neq \{0\}$;

(III') if $(A, \mathcal{S})$, $(A, \mathcal{T})$ are compatible pairs then for every $Q_\mathcal{S} \in \mathcal{Q}_\mathcal{S}$ and $Q_\mathcal{T} \in \mathcal{Q}_\mathcal{T}$ which admit adjoint respect to $\langle\ ,\ \rangle_A$ it holds

$$\|P_{A,\mathcal{S}} - P_{A,\mathcal{T}}\|_A \leq \|Q_\mathcal{S} - Q_\mathcal{T}\|_A;$$

(III") if $\mathcal{S} = \mathcal{S}_1 + \mathcal{S}_2$ and $\mathcal{T} = \mathcal{T}_1 + \mathcal{T}_2$, where $\mathcal{S}_1, \mathcal{T}_1 \subseteq \overline{R(A)}$ and $\mathcal{S}_2, \mathcal{T}_2 \subseteq N(A)$ and the pairs $(A, \mathcal{S}_1)$ and $(A, \mathcal{T}_1)$ are compatible then, for every $Q_\mathcal{S} \in \mathcal{Q}_\mathcal{S} \cap L^A(\mathcal{H})$ and $Q_\mathcal{T} \in \mathcal{Q}_\mathcal{T} \cap L^A(\mathcal{H})$ it holds

$$\|P_{A,\mathcal{S}} - P_{A,\mathcal{T}}\|_A \leq \|Q_\mathcal{S} - Q_\mathcal{T}\|_A;$$

(IV') if A is compatible with the closed subspaces $\mathcal{S}$ and $\mathcal{T}$ then $\|P_{A,\mathcal{S}} - P_{A,\mathcal{T}}\|_A \leq 1$ and equality holds if $P^\sharp_{A,\mathcal{S}}$ commutes with $P^\sharp_{A,\mathcal{T}}$;

(V') if A is compatible with the closed subspaces $\mathcal{S}$ and $\mathcal{T}$ then $\|P_{A,\mathcal{S}} - P_{A,\mathcal{T}}\|_A = \max\{\ \|P_{A,\mathcal{S}}(I - P_{A,\mathcal{T}})\|_A,\ \|P_{A,\mathcal{T}}(I - P_{A,\mathcal{S}})\|_A\ \}$;

(VI') if $(A, \mathcal{S})$ and $(A, \mathcal{T})$ are compatible pairs and $\mathcal{S} \cap \overline{R(A)} \neq \{0\}$ then it holds $\|Q_{\mathcal{S}//\mathcal{T}}\|_A = \dfrac{1}{\sin\theta_A}$, where $\theta_A \in [0, \pi/2]$ is the angle such that $\cos\theta_A = \sup\{|\langle \xi, \eta\rangle_A| : \xi \in \mathcal{S}, \eta \in \mathcal{T}$ and $\|\xi\|_A = \|\eta\|_A = 1\}$.

2. Preliminaries

Throughout $\mathcal{H}$ denotes a complex Hilbert space. $L(\mathcal{H})$ is the space of bounded linear operators on $\mathcal{H}$, $L(\mathcal{H})^+$ denotes the cone of all positive operators of $L(\mathcal{H})$, i.e., $L(\mathcal{H})^+ = \{A \in L(\mathcal{H}) : \langle A\eta, \eta\rangle \geq 0$ for all $\eta \in \mathcal{H}\}$, $Gl(\mathcal{H})$ is the group of invertible operators of $L(\mathcal{H})$ and $Gl(\mathcal{H})^+ = Gl(\mathcal{H}) \cap L(\mathcal{H})^+$. For every $T \in L(\mathcal{H})$, its range is denoted by $R(T)$, its nullspace by $N(T)$ and its adjoint by T^*. $\mathcal{S}$ and $\mathcal{T}$ are closed subspaces of $\mathcal{H}$ and $\mathcal{S} \ominus \mathcal{T} = \mathcal{S} \cap \mathcal{T}^\perp$. In this paper, given closed subspaces $\mathcal{S}, \mathcal{T}$ of $\mathcal{H}$, by $L(\mathcal{S}, \mathcal{T})$ we denote the subspace $\{T \in L(\mathcal{H}) :\ T(\mathcal{S}^\perp) = \{0\}$ and $T(\mathcal{S}) \subseteq \mathcal{T}\}$. If $\mathcal{H}$ is decomposed as a direct sum $\mathcal{H} = \mathcal{S} \dot{+} \mathcal{T}$, where $\mathcal{S}$ and $\mathcal{T}$ are closed subsapces of $\mathcal{H}$, then the unique projection with range $\mathcal{S}$ and nullspace $\mathcal{T}$ is denoted by $Q_{\mathcal{S}//\mathcal{T}}$.

2.1. Moore-Penrose pseudoinverse

Recall that given $T \in L(\mathcal{H})$, the Moore-Penrose inverse of T, denoted by $T^\dagger$, is defined as the unique linear extension of $\tilde{T}^{-1}$ to $\mathcal{D}(T^\dagger) := R(T) + R(T)^\perp$ with $N(T^\dagger) = R(T)^\perp$, where $\tilde{T}$ is the isomorphism $T|_{N(T)^\perp} : N(T)^\perp \longrightarrow R(T)$. It holds that $T^\dagger$ is the unique solution of the four "Moore-Penrose equations":

$$TXT = T, \quad XTX = X, \quad XT = P_{N(T)^\perp} \quad \text{and} \quad TX = P_{\overline{R(T)}}|_{\mathcal{D}(T^\dagger)}.$$

$T^\dagger$ has closed graph and $T^\dagger$ is bounded if and only if $R(T)$ is closed. Proofs of these facts can be found in many places, e.g. the books [31], [5] and [18]. Even though $T^\dagger$ is, in general, unbounded, for every $B \in L(\mathcal{H})$ such that $R(B) \subseteq \mathcal{D}(T^\dagger)$ it holds that $T^\dagger B$ is bounded. In fact, let $(\xi_n)_{n\in\mathbb{N}} \subset \mathcal{H}$ such that $\xi_n \underset{n\to\infty}{\longrightarrow} \xi$. Now, as $B\xi_n, B\xi \in \mathcal{D}(T^\dagger)$, $B\xi_n \underset{n\to\infty}{\longrightarrow} B\xi$ and $T^\dagger$ has closed graph then $T^\dagger B\xi_n \underset{n\to\infty}{\longrightarrow} T^\dagger B\xi$, i.e., $T^\dagger B \in L(\mathcal{H})$. In the next proposition we collect without proof some properties of $T^\dagger$ that we will need in this work.

Proposition 2.1. *Let $T \in L(\mathcal{H})$.*

1. *If $T = T^*$ then $(T^\dagger)^* = T^\dagger$.*
2. *If $T \in L(\mathcal{H})^+$ then $T^\dagger = (T^{1/2})^\dagger (T^{1/2})^\dagger|_{\mathcal{D}(T^\dagger)}$.*

A bounded linear densely defined operator T can be uniquely extended to $L(\mathcal{H})$; its unique extension will be denoted by $\overline{T}$. Clearly, $\|\overline{T}\| = \|T\|$. In the next proposition we enunciate some elementary properties of $\overline{T}$.

Proposition 2.2. *Let T and R be bounded densely defined linear operators. Then:*

1. *If $T = R^*R$ then $\overline{T} = \overline{R}^*\overline{R}$.*
2. $\overline{T^*} = \overline{T}^* = T^*$.

2.2. A-selfadjoint projections and compatibility

Any $A \in L(\mathcal{H})^+$ defines a positive semidefinite sesquilinear form:

$$\langle \ , \ \rangle_A : \mathcal{H} \times \mathcal{H} \to \mathbb{C}, \ \langle \xi, \eta \rangle_A = \langle A\xi, \eta \rangle.$$

By $\| \ . \ \|_A$ we denote the seminorm induced by $\langle \ , \ \rangle_A$, i.e., $\|\xi\|_A = \langle \xi, \xi \rangle_A^{1/2}$. Observe that $\|\xi\|_A = 0$ if and only if $\xi \in N(A)$. Then $\| \ . \ \|_A$ is a norm if and only if A is an injective operator. Moreover, $\langle \ , \ \rangle_A$, induces a seminorm on a subset of $L(\mathcal{H})$. Namely, given $T \in L(\mathcal{H})$, if there exists a constant $c > 0$ such that $\|T\omega\|_A \leq c\|\omega\|_A$ for every $\omega \in \overline{R(A)}$ it holds

$$\|T\|_A = \sup_{\substack{\omega \in \overline{R(A)} \\ \omega \neq 0}} \frac{\|T\omega\|_A}{\|\omega\|_A} < \infty.$$

It is straightforward that

$$\|T\|_A = \sup\{|\langle T\xi, \eta \rangle_A| : \xi, \eta \in \mathcal{H} \text{ and } \|\xi\|_A \leq 1 \ \|\eta\|_A \leq 1\}.$$

From now on we will denote

$$L^A(\mathcal{H}) = \{T \in L(\mathcal{H}) : \ \|T\|_A < \infty\}.$$

It can be seen that $L^A(\mathcal{H})$ is not a subalgebra of $L(\mathcal{H})$. In [4] it is proved that if $A \in L(\mathcal{H})^+$ is injective then $T \in L^A(\mathcal{H})$ if and only if $A^{1/2}TA^{-1/2}$ is bounded. In the next proposition we extend this result for a non necessary injective operator $A \in L(\mathcal{H})^+$.

Proposition 2.3. *Let $A \in L(\mathcal{H})^+$ and $T \in L(\mathcal{H})$. Then the following conditions are equivalent:*

1. *$A^{1/2}T(A^{1/2})^\dagger$ is a bounded linear operator.*
2. *There exists $c > 0$ such that $\|T\omega\|_A \leq c\|\omega\|_A$ for every $\omega \in \overline{R(A)}$.*
3. *$R(A^{1/2}T^*A^{1/2}) \subseteq R(A)$.*

Moreover, if one of these conditions holds then

$$\|T\|_A = \|A^{1/2}T(A^{1/2})^\dagger\|.$$

Proof.
1⇒2 Let $A^{1/2}T(A^{1/2})^\dagger$ be a bounded linear operator. Then, for every $\xi \in \overline{R(A)}$ we have that

$$\begin{aligned} \|T\xi\|_A &= \|TP_{\overline{R(A)}}\xi\|_A = \|A^{1/2}T(A^{1/2})^\dagger A^{1/2}\xi\| \\ &\leq \|A^{1/2}T(A^{1/2})^\dagger\|\|A^{1/2}\xi\| \\ &= \|A^{1/2}T(A^{1/2})^\dagger\|\|\xi\|_A, \end{aligned}$$

i.e., item 2. holds. Moreover, $\|T\|_A \leq \|A^{1/2}T(A^{1/2})^\dagger\|$.
2⇒1 Let $c > 0$ such that $\|T\omega\|_A \leq c\|\omega\|_A$ for every $\omega \in \overline{R(A)}$. Then, for every $\xi \in \mathcal{D}((A^{1/2})^\dagger)$ it holds that

$$\|A^{1/2}T(A^{1/2})^\dagger\xi\| = \|T(A^{1/2})^\dagger\xi\|_A \leq \|T\|_A\|(A^{1/2})^\dagger\xi\|_A \leq \|T\|_A\|\xi\|.$$

Therefore, $A^{1/2}T(A^{1/2})^\dagger$ is bounded and $\|A^{1/2}T(A^{1/2})^\dagger\| \leq \|T\|_A$.
2⇔3 It is clear that $\|T\xi\|_A \leq c\|\xi\|_A$ for every $\xi \in \overline{R(A)}$ if and only if $\|A^{1/2}T\xi\| \leq c\|A^{1/2}\xi\|$ for every $\xi \in R(A^{1/2})$, i.e. if and only if $\|A^{1/2}TA^{1/2}\eta\| \leq c\|A\eta\|$ for every $\eta \in \mathcal{H}$. Now, by Douglas theorem, this is equivalent to $R(A^{1/2}T^*A^{1/2}) \subseteq R(A)$. □

By Proposition 2.3, if $A \in L(\mathcal{H})^+$ has closed range then $L^A(\mathcal{H}) = L(\mathcal{H})$ because $(A^{1/2})^\dagger$ is bounded. But, as the next example shows, if A has not closed range then $L^A(\mathcal{H}) \subsetneq L(\mathcal{H})$.

Example 1. Let $A \in L(\mathcal{H})^+$ with non closed range and let $\mu \in R(A^{1/2}) \setminus R(A)$. Then, there exists $\eta \in \overline{R(A)} \setminus R(A^{1/2})$ such that $\mu = A^{1/2}\eta$. Now, let $\xi \in R(A^{1/2})$ and $\mathcal{S}$ a closed subspace of $\mathcal{H}$ such that $\mathcal{H} = span\{\xi\} + span\{\eta\} + \mathcal{S}$. Then, define $T : \mathcal{H} \to \mathcal{H}$ by $T\xi = \eta, T\eta = \eta$ and $T(\mathcal{S}) = \{0\}$. Thus, $T \in L(\mathcal{H})$. Moreover, $T \in \mathcal{Q}$. Then, $T^* \in \mathcal{Q}$ but $T^* \notin L^A(\mathcal{H})$. In fact, $\mu = A^{1/2}\eta = A^{1/2}T\xi \in R(A^{1/2}TA^{1/2})$ and $\mu \notin R(A)$. So, $R(A^{1/2}TA^{1/2}) \not\subseteq R(A)$, i.e., $T^* \notin L^A(\mathcal{H})$ by Proposition 2.3.

A bounded linear operator $W \in L(\mathcal{H})$ is called an **A-adjoint** of $T \in L(\mathcal{H})$ if

$$\langle T\xi, \eta \rangle_A = \langle \xi, W\eta \rangle_A \quad \text{for every} \quad \xi, \eta \in \mathcal{H},$$

or, which is equivalent, if W satisfies the equation $AW = T^*A$. The operator T is said **A-selfadjoint** if $AT = T^*A$. The existence of an A-adjoint operator is not guaranteed. This kind of equations can be studied applying the next theorem of R. G. Douglas (for its proof see [17] or [19]).

Theorem (Douglas). Let $A, B \in L(\mathcal{H})$. The following conditions are equivalent:

1. $R(B) \subseteq R(A)$.
2. There exists a positive number λ such that $BB^* \leq \lambda AA^*$.
3. There exists $C \in L(\mathcal{H})$ such that $AC = B$.

If one of these conditions holds there exists an unique operator $D \in L(\mathcal{H})$ such that $AD = B$ and $R(D) \subseteq \overline{R(A^*)}$. Furthermore, $N(D) = N(B)$. Such D is called the **reduced solution** or **Douglas solution** of $AX = B$.

Note that if the equation $AX = B$ has solution then $A^\dagger B$ is the reduced solution. Indeed, since $R(B) \subseteq R(A) \subseteq \mathcal{D}(A^\dagger)$, $A^\dagger B \in L(\mathcal{H})$. Moreover, $AA^\dagger B = P_{\overline{R(A)}}|_{\mathcal{D}(A^\dagger)}B = B$ and $R(A^\dagger B) \subseteq \overline{R(A)}$.
By Douglas theorem, $T \in L(\mathcal{H})$ admits an A-adjoint operator if and only if $R(T^*A) \subseteq R(A)$. We shall denote by $L_A(\mathcal{H})$ the subalgebra of $L(\mathcal{H})$ consisting of such operators, i.e,

$$L_A(\mathcal{H}) = \{T \in L(\mathcal{H}) : \ R(T^*A) \subseteq R(A)\}.$$

Again, by Douglas theorem, it is easy to see that

$$L_{A^{1/2}}(\mathcal{H}) = \{T \in L(\mathcal{H}) : \ \exists\, c > 0 \ \ \|T\xi\|_A \leq c\|\xi\|_A \ \ \forall\, \xi \in \mathcal{H}\}.$$

The inclusions $L_A(\mathcal{H}) \subseteq L_{A^{1/2}}(\mathcal{H}) \subseteq L^A(\mathcal{H})$ hold. The first of them was proved in Theorem 5.1 of [21], the second one is trivial. Observe that these inclusions assure that $\|T\|_A$ is finite for every T which admits an A-adjoint. If $T \in L_A(\mathcal{H})$ then there exists a distinguished A-adjoint operator of T, namely, the reduced solution of equation $AX = T^*A$. We denote this operator by $T^\sharp$. Therefore $T^\sharp = A^\dagger T^*A$ and its main properties are

$$AT^\sharp = T^*A, \ \ R(T^\sharp) \subseteq \overline{R(A)} \ \ \text{and} \ \ N(T^\sharp) = N(T^*A).$$

Observe that if W is an A-adjoint of T then $T^\sharp = P_{\overline{R(A)}}W$. In [2] we have studied some properties of the $\sharp$ operation which are relevant for studying A-partial isometries, i.e. operator which behave as partial isometries with respect to $\langle\ ,\ \rangle_A$. We add now a few properties.

Proposition 2.4. *Let $A \in L(\mathcal{H})^+$ and $T \in L_A(\mathcal{H})$. Then*

1. $\|T\|_A = \|T^\sharp\|_A = \|T^\sharp T\|_A^{1/2}$.
2. $\|W\|_A = \|T^\sharp\|_A$ *for every $W \in L(\mathcal{H})$ which is an A-adjoint of T.*
3. *If $W \in L_A(\mathcal{H})$ then $\|TW\|_A = \|WT\|_A$.*

4. $\|T^\sharp\| \le \|W\|$ *for every* $W \in L(\mathcal{H})$ *which is an* A*-adjoint of* T*. Nevertheless,* $T^\sharp$ *is not in general the unique* A*-adjoint of* T *that realizes the minimal norm.*

Proof.
1. It is easy to check that $\overline{A^{1/2}T(A^{1/2})^\dagger}^* = \overline{A^{1/2}(A^\dagger T^* A)(A^{1/2})^\dagger}$. Then

$$\begin{aligned}\|T\|_A &= \|A^{1/2}T(A^{1/2})^\dagger\| = \|\overline{A^{1/2}T(A^{1/2})^\dagger}\| = \|\overline{A^{1/2}T(A^{1/2})^\dagger}^*\| \\ &= \|\overline{A^{1/2}(A^\dagger T^* A)(A^{1/2})^\dagger}\| = \|A^{1/2}(A^\dagger T^* A)(A^{1/2})^\dagger\| \\ &= \|A^{1/2}T^\sharp(A^{1/2})^\dagger\| = \|T^\sharp\|_A.\end{aligned}$$

On the other hand,

$$\begin{aligned}\|T^\sharp T\|_A &= \|A^{1/2}T^\sharp T(A^{1/2})^\dagger\| = \|A^{1/2}A^\dagger T^* AT(A^{1/2})^\dagger\| \\ &= \|(A^{1/2})^\dagger T^* AT(A^{1/2})^\dagger\| = \|\overline{(A^{1/2})^\dagger T^* AT(A^{1/2})^\dagger}\| \\ &= \|\overline{(A^{1/2}T(A^{1/2})^\dagger)}^* \ \overline{(A^{1/2}T(A^{1/2})^\dagger)}\| = \|\overline{A^{1/2}T(A^{1/2})^\dagger}\|^2 \\ &= \|A^{1/2}T(A^{1/2})^\dagger\|^2 = \|T\|_A^2.\end{aligned}$$

2. If $W \in L(\mathcal{H})$ is an A-adjoint operator of T then $W = T^\sharp + Z$, where Z is a solution of the homogeneous equation $AX = 0$. Then $\|W\|_A = \|A^{1/2}W(A^{1/2})^\dagger\| = \|A^{1/2}(T^\sharp + Z)(A^{1/2})^\dagger\| = \|A^{1/2}T^\sharp(A^{1/2})^\dagger\| = \|T^\sharp\|_A$.
3. Note that

$$\begin{aligned}\|TW\|_A &= \|(TW)^\sharp\|_A = \|W^\sharp T^\sharp\|_A = \|A^{1/2}W^\sharp T^\sharp(A^{1/2})^\dagger\| \\ &= \|A^{1/2}W^\sharp(A^{1/2})^\dagger A^{1/2}T^\sharp(A^{1/2})^\dagger\| \\ &= \|A^{1/2}T^\sharp(A^{1/2})^\dagger A^{1/2}W^\sharp(A^{1/2})^\dagger\| \\ &= \|T^\sharp W^\sharp\|_A = \|(WT)^\sharp\|_A \\ &= \|WT\|_A.\end{aligned}$$

4. Let $W \in L(\mathcal{H})$ be an A-adjoint operator of T. Then $W = T^\sharp + Z$, where $AZ = 0$. Let $\xi \in \mathcal{H}$ with $\|\xi\| = 1$. Since $R(T^\sharp) \subseteq \overline{R(A)}$ and $R(Z) \subseteq N(A)$ we get $\|W\xi\|^2 = \|T^\sharp\xi\|^2 + \|Z\xi\|^2$. Then $\|T^\sharp\xi\|^2 \le \|W\xi\|^2$ and so $\|T^\sharp\| \le \|W\|$. Now, let $A = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \in M_2(\mathbb{R})^+$ and $T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \in M_2(\mathbb{R})$. It is easy to check that T admits A-adjoint operators and that $T^\sharp = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Furthermore, observe that the identity matrix I is an A-adjoint of T, $\|T^\sharp\| = \|I\| = 1$ and $T^\sharp \neq I$. □

Given $A \in L(\mathcal{H})^+$ and a closed subspace $\mathcal{S}$, we denote by $\mathcal{P}(A, \mathcal{S})$ the set of A-selfadjoint projections with fixed range $\mathcal{S}$:

$$\mathcal{P}(A, \mathcal{S}) = \{Q \in \mathcal{Q}_{\mathcal{S}} : \ AQ = Q^*A\}.$$

Fixed $A \in L(\mathcal{H})^+$ the set $\mathcal{P}(A, \mathcal{S})$ can be empty, or have one element (for example if $A \in Gl(\mathcal{H})^+$) or have infinite elements. If $\mathcal{P}(A, \mathcal{S}) \neq \emptyset$ then the pair $(A, \mathcal{S})$ is said to be **compatible**. For a fuller treatment on the theory of compatibility see [9], [10] and [12]. If the pair $(A, \mathcal{S})$ is compatible, the unique element in $\mathcal{P}(A, \mathcal{S})$ with nullspace $(A\mathcal{S})^\perp \ominus \mathcal{N}$, where $\mathcal{N} = N(A) \cap \mathcal{S}$, is denoted by $P_{A,\mathcal{S}}$. This element has minimal norm in $P(A, \mathcal{S})$. Nevertheless, $P_{A,\mathcal{S}}$ is not in general the unique $Q \in \mathcal{P}(A, \mathcal{S})$ that realizes the minimal norm. See [9] Theorem 3.5 for its proof. The next proposition provides a parametrization of $\mathcal{P}(A, \mathcal{S})$ and it expresses the element $P_{A,\mathcal{S}}$ as the solution of certain Douglas-type equations.

Proposition 2.5. *Let $A \in L(\mathcal{H})^+$ such that the pair $(A, \mathcal{S})$ is compatible. If Q is the reduced solution of the equation $(P_\mathcal{S} A P_\mathcal{S})X = P_\mathcal{S} A$ then*

1. $Q = P_{A,\mathcal{S}\ominus\mathcal{N}}$.
2. $P_{A,\mathcal{S}} = P_{A,\mathcal{S}\ominus\mathcal{N}} + P_\mathcal{N}$.
3. *$\mathcal{P}(A, \mathcal{S})$ is an affine manifold that can be parametrized as $\mathcal{P}(A, \mathcal{S}) = P_{A,\mathcal{S}} + L(\mathcal{S}^\perp, \mathcal{N})$. In particular, if $\mathcal{N} = \{0\}$ then $\mathcal{P}(A, \mathcal{S}) = \{P_{A,\mathcal{S}}\}$.*

3. The A-adjoint operation $^\sharp$ on projections

In this paper, we are mainly interested in how the A-adjoint operation $^\sharp$ acts on A-adjointable projections. We first notice that there is no obvious notion of self-adjointness: an operator T such that $AT = T^*A$ could be named A-Hermitian, but also an operator $T \in L_A(\mathcal{H})$ such that $T^\sharp = T$. We discuss this problem focusing in the set of projections. For this, we consider the following subsets of $\mathcal{Q}$:

$$\mathcal{W} = \{Q \in \mathcal{Q} \cap L_A(\mathcal{H}) : \ Q^\sharp = Q\}$$

$$\mathcal{X} = \{Q \in \mathcal{Q} \cap L_A(\mathcal{H}) : \ AQ = Q^*A\}$$

$$\mathcal{Y} = \{Q \in \mathcal{Q} \cap L_A(\mathcal{H}) : \ (Q^\sharp)^2 = Q^\sharp\}$$

$$\mathcal{Z} = \mathcal{Q} \cap L_A(\mathcal{H}).$$

Proposition 3.1. *The next inclusions hold: $\mathcal{W} \subsetneq \mathcal{X} \subsetneq \mathcal{Y} = \mathcal{Z}$.*

Proof. Let $Q \in \mathcal{W}$ then $Q^\sharp = Q$. Thus, $Q^*A = AQ^\sharp = AQ$ and so $Q \in \mathcal{X}$. On the other hand, consider $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in M_2(\mathbb{C})^+$ and $Q = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$. Then it is easy to check that $Q \in \mathcal{X}$, but $Q \notin \mathcal{W}$. It is immediate that $\mathcal{X} \subseteq \mathcal{Z}$. In order to see that this is a strict inclusion consider $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \in M_2(\mathbb{C})^+$ and $Q = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$. Since A is invertible then $R(Q^*A) \subseteq R(A)$, i.e., $Q \in \mathcal{Z}$, but $Q \notin \mathcal{X}$. Finally, let $Q \in \mathcal{Z}$, i.e, $Q^2 = Q$ and there exists $Q^\sharp$. Let see that $(Q^\sharp)^2 = Q^\sharp$. Indeed, $(Q^\sharp)^2 = A^\dagger Q^* A A^\dagger Q^* A = A^\dagger Q^* P_{\overline{R(A)}}|_{\mathcal{D}(A^\dagger)} Q^* A = A^\dagger (Q^*)^2 A = A^\dagger Q^* A = Q^\sharp$. i.e., $Q \in \mathcal{Y}$. The other inclusion is trivial. □

Proposition 3.2. *If $Q \in \mathcal{P}(A, \mathcal{S})$ then:*

1. $Q^\sharp = Q^\sharp Q = P_{\overline{R(A)}} Q = P_{\overline{R(A)}} P_{A,\mathcal{S}}$ *is a projection.*
2. $I - Q^\sharp \in \mathcal{P}(A, N(P_\mathcal{S} A))$.

Proof.
1. It is sufficient to prove that $Q^\sharp Q$ is the reduced solution of the equation $AX = Q^*A$. In fact, $AQ^\sharp Q = Q^*AQ = (Q^*)^2 A = Q^*A$ and $R(Q^\sharp Q) \subseteq R(Q^\sharp) \subseteq \overline{R(A)}$. Therefore, $Q^\sharp Q = Q^\sharp$. In order to see that $Q^\sharp = P_{\overline{R(A)}} P_{A,\mathcal{S}}$, observe that, by Proposition 2.5, we get $Q = P_{A,\mathcal{S}} + Z$, where $Z \in L(\mathcal{S}^\perp, \mathcal{N})$. Therefore, $Q^\sharp = A^\dagger Q^* A = P_{\overline{R(A)}} Q = P_{\overline{R(A)}} (P_{A,\mathcal{S}} + Z) = P_{\overline{R(A)}} P_{A,\mathcal{S}}$.
2. If $Q \in \mathcal{P}(A, \mathcal{S})$ then $Q^\sharp$ is also an A-selfadjoint projection. On the other hand, $R(I - Q^\sharp) = N(Q^\sharp) = N(Q^*A) = R(AQ)^\perp = R(AP_\mathcal{S})^\perp = N(P_\mathcal{S} A)$. Then $I - Q^\sharp \in \mathcal{P}(A, N(P_\mathcal{S} A))$. □

Remarks 3.3. Considering the subsets defined before, it is clear that if the pair $(A, \mathcal{S})$ is compatible then $\mathcal{P}(A, \mathcal{S}) \subseteq \mathcal{X}$. On the other hand, $\mathcal{P}(A, \mathcal{S}) \cap \mathcal{W} \neq \emptyset$ if and only if $\mathcal{S} \subseteq \overline{R(A)}$ and the pair $(A, \mathcal{S})$ is compatible. In fact, if there exists $Q \in \mathcal{P}(A, \mathcal{S}) \cap \mathcal{W}$ then $Q^\sharp = Q$ and so $\mathcal{S} = R(Q) = R(Q^\sharp) \subseteq \overline{R(A)}$. Conversely, if $\mathcal{S} \subseteq \overline{R(A)}$ and $(A, \mathcal{S})$ is compatible then $P^\sharp_{A,\mathcal{S}} = P_{\overline{R(A)}} P_{A,\mathcal{S}} = P_{A,\mathcal{S}}$, i.e. $P_{A,\mathcal{S}} \in \mathcal{P}(A, \mathcal{S}) \cap \mathcal{W}$.

4. Identities on the seminorm of projections

In this section we generalize several identities on the norm of projections when the seminorm induced by $A \in L(\mathcal{H})^+$ is considered. We start by establishing an useful relationship between orthogonal projections and A-selfadjoint projections.

Proposition 4.1. *Let $A \in L(\mathcal{H})^+$ and $Q \in L(\mathcal{H})$ such that $\mathcal{S} = R(Q)$ is a closed subspace of $\overline{R(A)}$.*

1. *If $Q \in \mathcal{Q}_\mathcal{S} \cap L^A(\mathcal{H})$ then $\overline{A^{1/2} Q (A^{1/2})^\dagger}$ is a projection.*
2. *The following conditions are equivalent:*
   (a) $Q \in \mathcal{P}(A, \mathcal{S})$.
   (b) $Q \in L_A(\mathcal{H})$ *and* $\overline{A^{1/2} Q (A^{1/2})^\dagger}$ *is an orthogonal projection.*

   If one of these conditions holds then $\|Q\|_A = \|\overline{A^{1/2} Q (A^{1/2})^\dagger}\| = 1$.

Proof.
1. Since $Q \in \mathcal{Q}_\mathcal{S}$ and $\mathcal{S} \subseteq \overline{R(A)}$ then $A^{1/2} Q (A^{1/2})^\dagger$ is a projection. Futhermore, as $Q \in L^A(\mathcal{H})$, by Proposition 2.3, it holds that $A^{1/2} Q (A^{1/2})^\dagger$ is bounded. Therefore $\overline{A^{1/2} Q (A^{1/2})^\dagger}$ is a projection of $L(\mathcal{H})$.
2. Let $Q \in \mathcal{P}(A, \mathcal{S})$. By item 1. it holds that $\overline{A^{1/2} Q (A^{1/2})^\dagger}$ is a projection. In order to see that $(\overline{A^{1/2} Q (A^{1/2})^\dagger})^* = \overline{A^{1/2} Q (A^{1/2})^\dagger}$, observe that $(\overline{A^{1/2} Q (A^{1/2})^\dagger})^* = (A^{1/2} Q (A^{1/2})^\dagger)^* \supset (A^{1/2})^\dagger Q^* A^{1/2}$. Furthermore, since $\mathcal{D}((A^{1/2})^\dagger Q^* A^{1/2}) = \mathcal{H}$, we obtain that $(\overline{A^{1/2} Q (A^{1/2})^\dagger})^* = (A^{1/2})^\dagger Q^* A^{1/2} = \overline{(A^{1/2})^\dagger Q^* A^{1/2}|_{\mathcal{D}((A^{1/2})^\dagger)}} = \overline{A^{1/2} Q (A^{1/2})^\dagger}$ where the last equality holds since $AQ = Q^*A$.

Conversely, let $\overline{A^{1/2}Q(A^{1/2})^{\dagger}}$ be an orthogonal projection. First, let see that Q is a projection. Since, $\overline{A^{1/2}Q(A^{1/2})^{\dagger}}$ is a projection, then $A^{1/2}Q(A^{1/2})^{\dagger}$ is also a projection. Thus, $A^{1/2}Q(A^{1/2})^{\dagger} = (A^{1/2}Q(A^{1/2})^{\dagger})^2 = A^{1/2}Q^2(A^{1/2})^{\dagger}$. Then, $Q(A^{1/2})^{\dagger} = Q^2(A^{1/2})^{\dagger}$, i.e., $(Q^2 - Q)(A^{1/2})^{\dagger} = 0$. Hence, $\overline{R(A)} \subseteq N(Q^2 - Q)$, or which is the same $R((Q^*)^2 - Q^*) \subseteq N(A)$. Thus, $R(((Q^*)^2 - Q^*)A) \subseteq N(A)$. On the other hand, since $R(Q^*A) \subseteq R(A)$, it is easy to prove that $R((Q^*)^2A) \subseteq R(A)$. So, $R(((Q^*)^2 - Q^*)A) \subseteq R(A)$.Then, $((Q^*)^2 - Q^*)A = 0$, i.e., $AQ^2 = AQ$ and so $Q^2 = Q$. It only remains to show that Q is A-selfadjoint. Now, as $\overline{A^{1/2}Q(A^{1/2})^{\dagger}}$ is selfadjoint, it holds $\overline{A^{1/2}Q(A^{1/2})^{\dagger}} = (\overline{A^{1/2}Q(A^{1/2})^{\dagger}})^* = (A^{1/2}Q(A^{1/2})^{\dagger})^* = (A^{1/2})^{\dagger}Q^*A^{1/2}$. Hence, $A^{1/2}Q(A^{1/2})^{\dagger} = (A^{1/2})^{\dagger}Q^*A^{1/2}|_{\mathcal{D}((A^{1/2})^{\dagger})}$ and as a consequence, $AQP_{\overline{R(A)}} = P_{\overline{R(A)}}|_{\mathcal{D}((A^{1/2})^{\dagger})}Q^*A = Q^*A$. Now, taking adjoints we get $Q^*A = AQ$. Hence $Q \in \mathcal{P}(A, \mathcal{S})$.
The equality $\|Q\|_A = \|\overline{A^{1/2}Q(A^{1/2})^{\dagger}}\|$ follows by Proposition 2.3. □

For the seminorm $\| \ \|_A$, it is not true, in general, that $1 \leq \|Q\|_A$ for every $Q \in \mathcal{Q}_{\mathcal{S}}$. See example 2.

Proposition 4.2. *Let $A \in L(\mathcal{H})^+$. If $\mathcal{S} \cap \overline{R(A)} \neq \{0\}$ then $1 \leq \|Q\|_A$ for every $Q \in \mathcal{Q}_{\mathcal{S}}$.*

Proof. If $Q \notin L^A(\mathcal{H})$ then the assertion is trivial. Now, suppose $Q \in L^A(\mathcal{H})$. Let $0 \neq \xi \in \mathcal{S} \cap \overline{R(A)}$ and $\eta = A^{1/2}\xi$. Then, we get $\dfrac{\|A^{1/2}Q(A^{1/2})^{\dagger}\eta\|}{\|\eta\|} = \dfrac{\|A^{1/2}Q\xi\|}{\|A^{1/2}\xi\|} = \dfrac{\|A^{1/2}\xi\|}{\|A^{1/2}\xi\|} = 1$. Therefore, $\|Q\|_A = \|A^{1/2}Q(A^{1/2})^{\dagger}\| \geq 1$. □

In what follows, given A in $L(\mathcal{H})^+$ we shall say that a projection Q is **trivial for** A if $AQ = 0$. In that case, $\|Q\|_A = 0$ and $\|I - Q\|_A = 1$. Note that if $Q \in \mathcal{P}(A, \mathcal{S})$ then $\|Q\|_A$ is finite. Moreover, in the next proposition we show that if $Q \in \mathcal{P}(A, \mathcal{S})$ is non-trivial for A then $\|Q\|_A = 1$.

Proposition 4.3. *Let $A \in L(\mathcal{H})^+$. If $Q \in \mathcal{Q}_{\mathcal{S}}$ is non-trivial for A then the following conditions are equivalent:*

1. $Q \in \mathcal{P}(A, \mathcal{S})$.
2. $\|Q\|_A = 1$ *and* $Q \in L_A(\mathcal{H})$.
3. Q *is A-selfadjoint.*
4. $\langle Q\xi, \xi\rangle_A \geq 0$ *for all* $\xi \in \mathcal{H}$.

Proof.
$1 \Rightarrow 2$. If $Q \in \mathcal{P}(A, \mathcal{S})$ then, by Proposition 3.2, $Q^{\sharp}Q$ is a projection. In addition, $R(Q^{\sharp}Q) \subseteq \overline{R(A)}$. Then applying Proposition 4.1 we deduce that $\overline{A^{1/2}Q^{\sharp}Q(A^{1/2})^{\dagger}}$ is an orthogonal projection. Moreover, since Q is non-trivial, $R(Q) \not\subseteq N(A)$ and so $\overline{A^{1/2}Q^{\sharp}Q(A^{1/2})^{\dagger}} \neq 0$. Thus, $\|Q\|_A^2 = \|Q^{\sharp}Q\|_A = \|A^{1/2}Q^{\sharp}Q(A^{1/2})^{\dagger}\|^2 = \|\overline{A^{1/2}Q^{\sharp}Q(A^{1/2})^{\dagger}}\|^2 = 1$.

2 ⇒ 1. As $R(Q^*A) \subseteq R(A)$ then $Q^\sharp$ is a projection whose range is contained in $\overline{R(A)}$. Then, $(A^{1/2}Q^\sharp(A^{1/2})^\dagger)^2 = A^{1/2}Q^\sharp(A^{1/2})^\dagger$ and so $\overline{A^{1/2}Q^\sharp(A^{1/2})^\dagger}$ is a projection. In addition, as $1 = \|Q\|_A = \|Q^\sharp\|_A = \|\overline{A^{1/2}Q^\sharp(A^{1/2})^\dagger}\|$, it follows that $\overline{A^{1/2}Q^\sharp(A^{1/2})^\dagger}$ is an orthogonal projection. On the other hand, since $Q^\sharp = A^\dagger Q^* A$ we get that $\overline{A^{1/2}Q^\sharp(A^{1/2})^\dagger} = \overline{(A^{1/2})^\dagger Q^* A^{1/2}|_{\mathcal{D}((A^{1/2})^\dagger)}}$ is an orthogonal projection. Hence, it holds $\overline{(A^{1/2})^\dagger Q^* A^{1/2}|_{\mathcal{D}((A^{1/2})^\dagger)}} = (\overline{(A^{1/2})^\dagger Q^* A^{1/2}|_{\mathcal{D}((A^{1/2})^\dagger)}})^*$ and $((A^{1/2})^\dagger Q^* A^{1/2}|_{\mathcal{D}((A^{1/2})^\dagger)})^* \supset A^{1/2}Q(A^{1/2})^\dagger$. As a consequence, we have that $\overline{(A^{1/2})^\dagger Q^* A^{1/2}|_{\mathcal{D}((A^{1/2})^\dagger)}} = \overline{A^{1/2}Q(A^{1/2})^\dagger}$ and so $\overline{A^{1/2}Q(A^{1/2})^\dagger}$ is an orthogonal projection. Thus $\overline{A^{1/2}Q(A^{1/2})^\dagger} = (A^{1/2}Q(A^{1/2})^\dagger)^* \supset (A^{1/2})^\dagger Q^* A^{1/2}$. Moreover, since $\mathcal{D}((A^{1/2})^\dagger Q^* A^{1/2}) = \mathcal{H}$ then $\overline{A^{1/2}Q(A^{1/2})^\dagger} = (A^{1/2})^\dagger Q^* A^{1/2}$. In particular, $A^{1/2}Q(A^{1/2})^\dagger = (A^{1/2})^\dagger Q^* A^{1/2}|_{\mathcal{D}((A^{1/2})^\dagger)}$. So $AQ(A^{1/2})^\dagger = Q^* A^{1/2}|_{\mathcal{D}((A^{1/2})^\dagger)}$ and then $AQ = Q^*A$. Thus $Q \in \mathcal{P}(A, \mathcal{S})$.
1 ⇒ 3. It is clear.
3 ⇒ 4. If $AQ = Q^*A$ then $AQ = AQ^2 = Q^*AQ \in L(\mathcal{H})^+$ which means that $\langle Q\xi, \xi\rangle_A = \langle AQ\xi, \xi\rangle \geq 0$ for all $\xi \in \mathcal{H}$.
4 ⇒ 1. If $\langle Q\xi, \xi\rangle_A = \langle AQ\xi, \xi\rangle \geq 0$ for every $\xi \in \mathcal{H}$ then $AQ = (AQ)^*$. Therefore Q is A-selfadjoint and since $Q^2 = Q$ then $Q \in \mathcal{P}(A, \mathcal{S})$. □

Corollary 4.4. *Let $A \in L(\mathcal{H})^+$ and $(A, \mathcal{S})$ be a compatible pair. If $\mathcal{S} \cap \overline{R(A)} \neq \{0\}$ then, for every $Q_\mathcal{S} \in \mathcal{Q}_\mathcal{S}$ it holds*

$$\|P_{A,\mathcal{S}}\|_A \leq \|Q_\mathcal{S}\|_A. \tag{4.1}$$

Proof. Note that $\|P_{A,\mathcal{S}}\|_A = 1$. Therefore, the assertion follows from Proposition 4.2. □

In [[23], Th. 6.35, p. 58] T. Kato proved that $\|P_\mathcal{S} - P_\mathcal{T}\| \leq \|Q_1 - Q_2\|$ for every $Q_1 \in \mathcal{Q}_\mathcal{S}$ and $Q_2 \in \mathcal{Q}_\mathcal{T}$ (see also M. Mbekhta [[30], 1.10]) . We shall generalize this property for A-selfadjoint projections and the seminorm induced by $A \in L(\mathcal{H})^+$ in three different manners. In Proposition 4.5 the inequality is proved for every $Q_\mathcal{S}$, $Q_\mathcal{T} \in L_A(\mathcal{H})$. In order to obtain this inequality for every $Q_\mathcal{S}$, $Q_\mathcal{T} \in Q$ new hypotheses on the subspaces $\mathcal{S}$ and $\mathcal{T}$ are required (Proposition 4.6, Corollary 4.7). The proof of the next proposition follows the same lines that the proof of [30], Proposition 1.10.

Proposition 4.5. *Let $A \in L(\mathcal{H})^+$ and $(A, \mathcal{S})$, $(A, \mathcal{T})$ be compatible pairs. Then, for every $Q_\mathcal{S} \in \mathcal{Q}_\mathcal{S} \cap L_A(\mathcal{H})$ and $Q_\mathcal{T} \in \mathcal{Q}_\mathcal{T} \cap L_A(\mathcal{H})$ it holds*

$$\|P_{A,\mathcal{S}} - P_{A,\mathcal{T}}\|_A \leq \|Q_\mathcal{S} - Q_\mathcal{T}\|_A.$$

Proof. First observe that $Q_\mathcal{S} P_{A,\mathcal{S}} = P_{A,\mathcal{S}}$, $P_{A,\mathcal{S}} Q_\mathcal{S} = Q_\mathcal{S}$, $Q_\mathcal{T} P_{A,\mathcal{T}} = P_{A,\mathcal{T}}$ and $P_{A,\mathcal{T}} Q_\mathcal{T} = Q_\mathcal{T}$. From this it holds that

$$(I - Q_\mathcal{S})(P_{A,\mathcal{S}} - P_{A,\mathcal{T}}) = (Q_\mathcal{S} - Q_\mathcal{T})P_{A,\mathcal{T}},$$

$$(P_{A,\mathcal{S}} - P_{A,\mathcal{T}})Q_\mathcal{S} = (I - P_{A,\mathcal{T}})(Q_\mathcal{S} - Q_\mathcal{T})$$

and as consequence $((P_{A,\mathcal{S}} - P_{A,\mathcal{T}})Q_{\mathcal{S}})^\sharp = ((I - P_{A,\mathcal{T}})(Q_{\mathcal{S}} - Q_{\mathcal{T}}))^\sharp$. On the other hand, simple computations show that $((I - P_{A,\mathcal{T}})(Q_{\mathcal{S}} - Q_{\mathcal{T}}))^\sharp = (Q_{\mathcal{S}}^\sharp - Q_{\mathcal{T}}^\sharp)(I - P_{A,\mathcal{T}})$ and $((P_{A,\mathcal{S}} - P_{A,\mathcal{T}})Q_{\mathcal{S}})^\sharp = Q_{\mathcal{S}}^\sharp(P_{A,\mathcal{S}} - P_{A,\mathcal{T}})$.
Now, if $\xi \in \mathcal{H}$, then it is easy to check that

$$\|\xi\|_A^2 + \|(Q_{\mathcal{S}} - Q_{\mathcal{S}}^\sharp)\xi\|_A^2 = \|(I - Q_{\mathcal{S}})\xi\|_A^2 + \|Q_{\mathcal{S}}^\sharp \xi\|_A^2.$$

Therefore, if $\eta \in \overline{R(A)}$ and we define $\xi = (P_{A,\mathcal{S}} - P_{A,\mathcal{T}})\eta$:

$$\begin{aligned}
\|(P_{A,\mathcal{S}} - P_{A,\mathcal{T}})\eta\|_A^2 &\leq \|(P_{A,\mathcal{S}} - P_{A,\mathcal{T}})\eta\|_A^2 + \|(Q_{\mathcal{S}} - Q_{\mathcal{S}}^\sharp)(P_{A,\mathcal{S}} - P_{A,\mathcal{T}})\eta\|_A^2 \\
&= \|(I - Q_{\mathcal{S}})(P_{A,\mathcal{S}} - P_{A,\mathcal{T}})\eta\|_A^2 + \|Q_{\mathcal{S}}^\sharp(P_{A,\mathcal{S}} - P_{A,\mathcal{T}})\eta\|_A^2 \\
&= \|(Q_{\mathcal{S}} - Q_{\mathcal{T}})P_{A,\mathcal{T}}\eta\|_A^2 + \|(Q_{\mathcal{S}}^\sharp - Q_{\mathcal{T}}^\sharp)(I - P_{A,\mathcal{T}})\eta\|_A^2 \\
&\leq \|Q_{\mathcal{S}} - Q_{\mathcal{T}}\|_A^2(\|P_{A,\mathcal{T}}\eta\|_A^2 + \|(I - P_{A,\mathcal{T}})\eta\|_A^2) \\
&= \|Q_{\mathcal{S}} - Q_{\mathcal{T}}\|_A^2\|\eta\|_A^2.
\end{aligned}$$

So, $\|P_{A,\mathcal{S}} - P_{A,\mathcal{T}}\|_A \leq \|Q_{\mathcal{S}} - Q_{\mathcal{T}}\|_A$. □

Proposition 4.6. *Let $A \in L(\mathcal{H})^+$ and $\mathcal{S}, \mathcal{T} \subseteq \overline{R(A)}$. If the pairs $(A, \mathcal{S})$ and $(A, \mathcal{T})$ are compatible then, for every $Q_{\mathcal{S}} \in \mathcal{Q}_{\mathcal{S}} \cap L^A(\mathcal{H})$ and $Q_{\mathcal{T}} \in \mathcal{Q}_{\mathcal{T}} \cap L^A(\mathcal{H})$ it holds*

$$\|P_{A,\mathcal{S}} - P_{A,\mathcal{T}}\|_A \leq \|Q_{\mathcal{S}} - Q_{\mathcal{T}}\|_A. \tag{4.2}$$

Proof. Since the subspaces $\mathcal{S}, \mathcal{T} \subseteq \overline{R(A)}$, it holds that $Q_1 = A^{1/2}Q_{\mathcal{S}}(A^{1/2})^\dagger$ and $Q_2 = A^{1/2}Q_{\mathcal{T}}(A^{1/2})^\dagger$ are projections with the same range as $A^{1/2}P_{A,\mathcal{S}}(A^{1/2})^\dagger$ and $A^{1/2}P_{A,\mathcal{T}}(A^{1/2})^\dagger$, respectively. On the other hand, by Proposition 4.1, it holds that $\overline{A^{1/2}P_{A,\mathcal{S}}(A^{1/2})^\dagger}$ and $\overline{A^{1/2}P_{A,\mathcal{T}}(A^{1/2})^\dagger}$ are orthogonal projections. Therefore,

$$\begin{aligned}
\|P_{A,\mathcal{S}} - P_{A,\mathcal{T}}\|_A &= \|A^{1/2}(P_{A,\mathcal{S}} - P_{A,\mathcal{T}})(A^{1/2})^\dagger\| \\
&= \|\overline{A^{1/2}P_{A,\mathcal{S}}(A^{1/2})^\dagger} - \overline{A^{1/2}P_{A,\mathcal{T}}(A^{1/2})^\dagger}\| \\
&\leq \|\overline{A^{1/2}Q_{\mathcal{S}}(A^{1/2})^\dagger} - \overline{A^{1/2}Q_{\mathcal{T}}(A^{1/2})^\dagger}\| \\
&= \|A^{1/2}Q_{\mathcal{S}}(A^{1/2})^\dagger - A^{1/2}Q_{\mathcal{T}}(A^{1/2})^\dagger\| \\
&= \|Q_{\mathcal{S}} - Q_{\mathcal{T}}\|_A
\end{aligned}$$

where the inequality holds by [[23], p. 58]. □

Corollary 4.7. *Let $A \in L(\mathcal{H})^+$ and $\mathcal{S}, \mathcal{T} \subseteq \mathcal{H}$ such that $\mathcal{S} = \mathcal{S}_1 + \mathcal{S}_2$ and $\mathcal{T} = \mathcal{T}_1 + \mathcal{T}_2$, where $\mathcal{S}_1, \mathcal{T}_1 \subseteq \overline{R(A)}$ and $\mathcal{S}_2, \mathcal{T}_2 \subseteq N(A)$. If the pairs $(A, \mathcal{S}_1)$ and $(A, \mathcal{T}_1)$ are compatible then, for every $Q_{\mathcal{S}} \in \mathcal{Q}_{\mathcal{S}} \cap L^A(\mathcal{H})$ and $Q_{\mathcal{T}} \in \mathcal{Q}_{\mathcal{T}} \cap L^A(\mathcal{H})$ it holds*

$$\|P_{A,\mathcal{S}} - P_{A,\mathcal{T}}\|_A \leq \|Q_{\mathcal{S}} - Q_{\mathcal{T}}\|_A.$$

Proof. Observe that $\mathcal{S}_1$ and $\mathcal{S}_2$ are orthogonal subspaces then every projection $Q_{\mathcal{S}}$ can be decomposed as $Q_{\mathcal{S}_1} + Q_{\mathcal{S}_2}$ where $Q_{\mathcal{S}_1} = P_{\mathcal{S}_1}Q_{\mathcal{S}}$ and $Q_{\mathcal{S}_2} = P_{\mathcal{S}_2}Q_{\mathcal{S}}$.

Furthermore, since $\mathcal{S}_2 \subseteq N(A)$ then $P_{A,\mathcal{S}} = P_{A,\mathcal{S}_1} + P_{\mathcal{S}_2}$. Then,

$$\begin{aligned}
\|P_{A,\mathcal{S}} - P_{A,\mathcal{T}}\|_A &= \|A^{1/2}(P_{A,\mathcal{S}_1} - P_{A,\mathcal{T}_1})(A^{1/2})^\dagger\| \\
&= \|\overline{A^{1/2}P_{A,\mathcal{S}_1}(A^{1/2})^\dagger} - \overline{A^{1/2}P_{A,\mathcal{T}_1}(A^{1/2})^\dagger}\| \\
&\leq \|\overline{A^{1/2}Q_{\mathcal{S}_1}(A^{1/2})^\dagger} - \overline{A^{1/2}Q_{\mathcal{T}_1}(A^{1/2})^\dagger}\| \\
&= \|A^{1/2}Q_{\mathcal{S}_1}(A^{1/2})^\dagger - A^{1/2}Q_{\mathcal{T}_1}(A^{1/2})^\dagger\| \\
&= \|A^{1/2}(Q_{\mathcal{S}_1} + Q_{\mathcal{S}_2})(A^{1/2})^\dagger - A^{1/2}(Q_{\mathcal{T}_1} + Q_{\mathcal{T}_2})(A^{1/2})^\dagger\| \\
&= \|Q_{\mathcal{S}} - Q_{\mathcal{T}}\|_A
\end{aligned}$$

□

As the next example shows, a naive extension of Kato's theorem is false. Our results 4.5, 4.6 and 4.7 offer different additional hypothesis which guarantee the conclusion.

Example 2. Consider $\mathcal{H} = \mathbb{R}^2$, $\mathcal{S} = span\{(1,1)\}$, $\mathcal{T} = span\{(-1,2)\}$ and $A = \begin{pmatrix} 2 & 1 \\ 1 & 1/2 \end{pmatrix} \in L(\mathbb{R}^2)^+$. Therefore $R(A) = span\{(2,1)\}$ and $\mathcal{S}$ does not verify the condition of Corollary 4.7. Moreover, $\mathcal{Q}_{\mathcal{T}} = \left\{ \begin{pmatrix} -\xi & -1/2(\xi+1) \\ 2\xi & \xi+1 \end{pmatrix}, \xi \in \mathbb{R} \right\}$ and $\mathcal{Q}_{\mathcal{S}} = \left\{ \begin{pmatrix} 1/2(1+\xi) & 1/2(1-\xi) \\ 1/2(1+\xi) & 1/2(1-\xi) \end{pmatrix}, \xi \in \mathbb{R} \right\}$. It is easy to check that $P_{A,\mathcal{S}} = \begin{pmatrix} 2/3 & 1/3 \\ 2/3 & 1/3 \end{pmatrix}$ and $P_{A,\mathcal{T}} = \begin{pmatrix} 1/5 & -2/5 \\ -2/5 & 4/5 \end{pmatrix}$. Now, if we take $Q_{\mathcal{S}} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ and $Q_{\mathcal{T}} = \begin{pmatrix} 0 & -1/2 \\ 0 & 1 \end{pmatrix}$ then $Q_{\mathcal{S}}$ does not admit an A-adjoint operator, $\|P_{A,\mathcal{S}} - P_{A,\mathcal{T}}\|_A = 1$ and $\|Q_{\mathcal{S}}\|_A = \|Q_{\mathcal{S}} - Q_{\mathcal{T}}\|_A = 0.6$.

The following lemma shows that in Corollary 4.4, Proposition 4.5, Corollary 4.7 and Proposition 4.10, the elements $P_{A,\mathcal{S}}$ and $P_{A,\mathcal{T}}$ can be replaced for any element of $\mathcal{P}(A,\mathcal{S})$ and $P(A,\mathcal{T})$ respectively.

Lemma 4.8. *Let $A \in L(\mathcal{H})^+$. If $(A,\mathcal{S})$ and $(A,\mathcal{T})$ are compatible pairs then*

$$\|Q_1 - Q_2\|_A = \|P_{A,\mathcal{S}} - P_{A,\mathcal{T}}\|_A$$

for every $Q_1 \in \mathcal{P}(A,\mathcal{S})$ and $Q_2 \in \mathcal{P}(A,\mathcal{T})$.

Proof. By Propositions 2.4 and 3.2 it holds that $\|Q_1 - Q_2\|_A = \|Q_1^\sharp - Q_2^\sharp\|_A = \|P_{\overline{R(A)}}P_{A,\mathcal{S}} - P_{\overline{R(A)}}P_{A,\mathcal{T}}\|_A = \|P_{A,\mathcal{S}} - P_{A,\mathcal{T}}\|_A$. □

Given a non trivial projection Q in $L(\mathcal{H})$, i.e., one which is different from 0 and I, it holds $\|Q\| = \|I-Q\|$. In [38] different proofs of this fact are collected. In the next proposition we generalize this identity for the seminorm induced by $A \in L(\mathcal{H})^+$. The proof we present is similar to the one due to Krainer presented in [38].

Proposition 4.9. *Let $A \in L(\mathcal{H})^+$. Therefore, for every $Q \in \mathcal{Q}_{\mathcal{S}}$ such that $R(Q) \cap \overline{R(A)} \neq \{0\}$ and $R(I-Q) \cap \overline{R(A)} \neq \{0\}$ it holds*

$$\|Q\|_A = \|I-Q\|_A.$$

Proof. Observe that by Proposition 4.2, the conditions $R(Q) \cap \overline{R(A)} \neq \{0\}$ and $R(I-Q) \cap \overline{R(A)} \neq \{0\}$ imply that $\|Q\|_A \geq 1$ and $\|I-Q\|_A \geq 1$. Let $\xi \in \mathcal{H}$ such that $\|\xi\|_A = 1$. Define $\eta = Q\xi$ and $\mu = (I-Q)\xi$. Then $\xi = \eta + \mu$. Let see that $\|Q\xi\|_A \leq \|I-Q\|_A$. If $\eta \in N(A)$ then $\|Q\xi\|_A = 0$ and so the inequality holds. If $\mu \in N(A)$ then $\|Q\xi\|_A = 1$ and so the inequality holds. Consider $\eta, \mu \notin N(A)$ and define $\omega = \tilde{\eta} + \tilde{\mu}$ where $\tilde{\eta} = \frac{\|\mu\|_A}{\|\eta\|_A}\eta$ and $\tilde{\mu} = \frac{\|\eta\|_A}{\|\mu\|_A}\mu$. Then $\|\omega\|_A^2 = \|\tilde{\eta}\|_A^2 + \|\tilde{\mu}\|_A^2 + 2Re\,\langle\tilde{\eta},\tilde{\mu}\rangle_A = \|\eta\|_A^2 + \|\mu\|_A^2 + 2Re\,\langle\eta,\mu\rangle_A = \|\xi\|_A^2 = 1$. Therefore, $\|Q\xi\|_A = \|\eta\|_A = \|\tilde{\mu}\|_A = \|(I-Q)\omega\|_A \leq \|I-Q\|_A$. Thus, $\|Q\|_A \leq \|I-Q\|_A$. The other inequality holds by symmetry. □

The conditions $R(Q) \cap \overline{R(A)} \neq \{0\}$ and $R(I-Q) \cap \overline{R(A)} \neq \{0\}$ in the above Proposition are necessary. Indeed, if $Q = P_{N(A)}$ then $I - Q = P_{\overline{R(A)}}$ and so $\|Q\|_A = 0$ and $\|I-Q\|_A = 1$.
In [1] § 34, properties (IV) and (V) enunciated in the introduction are proved. They where first proved by M. G. Krein, M. A. Krasnoselski and B. Sz.-Nagy. We extend now these facts for A-selfadjoint projections and the operator seminorm induced by A, with convenient compatibility hypothesis.

Proposition 4.10. *Let $A \in L(\mathcal{H})^+$ such that the pairs $(A, \mathcal{S})$ and $(A, \mathcal{T})$ are compatible. Then:*

(a) $\|P_{A,\mathcal{S}} - P_{A,\mathcal{T}}\|_A \leq 1$;
(b) *If $P^\sharp_{A,\mathcal{S}}$ and $P^\sharp_{A,\mathcal{T}}$ commute then* $\|P_{A,\mathcal{S}} - P_{A,\mathcal{T}}\|_A = 1$;
(c) $\|P_{A,\mathcal{S}} - P_{A,\mathcal{T}}\|_A = max\,\{\, \|P_{A,\mathcal{S}}(I - P_{A,\mathcal{T}})\|_A,\ \|P_{A,\mathcal{T}}(I - P_{A,\mathcal{S}})\|_A \,\}$.

Proof. By Proposition 3.1, the element $P^\sharp_{A,\mathcal{S}}$ is an A-selfadjoint projection. Furthermore, $R(P^\sharp_{A,\mathcal{S}}) \subseteq \overline{R(A)}$. Therefore, by Proposition 4.1, we get that $P_1 = \overline{A^{1/2}P^\sharp_{A,\mathcal{S}}(A^{1/2})^\dagger}$ is an orthogonal projection. Analogously, $P_2 = \overline{A^{1/2}P^\sharp_{A,\mathcal{T}}(A^{1/2})^\dagger}$ is an orthogonal projection. By the above remarks,

$$\begin{aligned}
\|P_{A,\mathcal{S}} - P_{A,\mathcal{T}}\|_A &= \|P^\sharp_{A,\mathcal{S}} - P^\sharp_{A,\mathcal{T}}\|_A \\
&= \|A^{1/2}(P^\sharp_{A,\mathcal{S}} - P^\sharp_{A,\mathcal{T}})(A^{1/2})^\dagger\| \\
&= \|\overline{A^{1/2}P^\sharp_{A,\mathcal{S}}(A^{1/2})^\dagger} - \overline{A^{1/2}P^\sharp_{A,\mathcal{T}}(A^{1/2})^\dagger}\| \\
&= \|P_1 - P_2\|
\end{aligned}$$

and so, by (IV), $\|P_{A,\mathcal{S}} - P_{A,\mathcal{T}}\|_A \leq 1$; this proves (a).
It is easy to check that if $P^\sharp_{A,\mathcal{S}}$ and $P^\sharp_{A\mathcal{T}}$ commute then P_1 and P_2 commute. Therefore, applying (IV), $\|P_{A,\mathcal{S}} - P_{A,\mathcal{T}}\|_A = \|P_1 - P_2\| = 1$, which proves (b).

For the proof of (c) observe that

$$\begin{aligned}\|P_{A,\mathcal{S}}(I-P_{A,\mathcal{T}})\|_A &= \|(I-P_{A,\mathcal{T}})^\sharp P^\sharp_{A,\mathcal{S}}\|_A = \|(P_{\overline{R(A)}} - P^\sharp_{A,\mathcal{T}})P^\sharp_{A,\mathcal{S}}\|_A \\ &= \|(I-P^\sharp_{A,\mathcal{T}})P^\sharp_{A,\mathcal{S}}\|_A = \|A^{1/2}(I-P^\sharp_{A,\mathcal{T}})P^\sharp_{A,\mathcal{S}}(A^{1/2})^\dagger\| \\ &= \|\overline{A^{1/2}(I-P^\sharp_{A,\mathcal{T}})P^\sharp_{A,\mathcal{S}}(A^{1/2})^\dagger}\| = \|(I-P_2)P_1\| \\ &= \|P_1(I-P_2)\|.\end{aligned}$$

Analogously, $\|P_{A,\mathcal{T}}(I-P_{A,\mathcal{S}})\|_A = \|P_2(I-P_1)\|$. On the other hand, $\|P_{A,\mathcal{S}} - P_{A,\mathcal{T}}\|_A = \|P_1 - P_2\|$, by the proof of (b). Then the assertion follows applying (V). □

5. Angles and seminorm of projections

In [26], V. Ljance proved that if $\mathcal{H}$ is decomposed as $\mathcal{H} = \mathcal{S} \dotplus \mathcal{T}$ then the norm of the projection $Q_{\mathcal{S}//\mathcal{T}}$ equals $1/\sin\theta$, where $\theta \in [0, \pi/2]$ is the angle between the subspaces $\mathcal{S}$ and $\mathcal{T}$ introduced by Dixmier in [16]. Proof of this theorem can be found in the papers by Ptak [32], Steinberg [37], Buckholtz [6] and Ipsen and Meyer [22] (for finite dimensional spaces).
As a final result, we extend Ljance's theorem for the A-seminorm, with a convenient definition of angle between subspaces depending on the semi-inner product $\langle\ ,\ \rangle_A$. First, recall that given two closed subspaces $\mathcal{S}$ and $\mathcal{T}$ of $\mathcal{H}$ the Dixmier's angle between them is the angle $\theta(\mathcal{S},\mathcal{T}) \in [0, \frac{\pi}{2}]$ whose cosine is defined by

$$\cos\theta(\mathcal{S},\mathcal{T}) = \sup\{|\langle\xi,\eta\rangle| : \xi\in\mathcal{S}, \eta\in\mathcal{T} \text{ and } \|\xi\|\le 1\ \|\eta\|\le 1\}.$$

Note that, even though if $\mathcal{S}$ and $\mathcal{T}$ are not closed subspaces then the angle between them can be also defined as above. Moreover, it holds $\cos\theta(\mathcal{S},\mathcal{T}) = \cos\theta(\overline{\mathcal{S}},\overline{\mathcal{T}})$.

Definition 5.1. Let $A \in L(\mathcal{H})^+$. The **A-angle** between two closed subspaces $\mathcal{S}$ and $\mathcal{T}$ is the angle $\theta_A(\mathcal{S},\mathcal{T}) \in [0, \frac{\pi}{2}]$ whose cosine is defined by

$$\cos\theta_A(\mathcal{S},\mathcal{T}) = \sup\{|\langle\xi,\eta\rangle_A| : \xi\in\mathcal{S}, \eta\in\mathcal{T} \text{ and } \|\xi\|_A\le 1\ \|\eta\|_A\le 1\}.$$

Observe that $0 \le \cos\theta_A(\mathcal{S},\mathcal{T}) \le 1$. Furthermore, it holds that $\cos\theta_A(\mathcal{S},\mathcal{T}) = \cos\theta(A^{1/2}(\mathcal{S}), A^{1/2}(\mathcal{T}))$.

Proposition 5.2. *Let $A \in L(\mathcal{H})^+$. If $(A,\mathcal{S})$ and $(A,\mathcal{T})$ are compatible pairs then* $\cos\theta_A(\mathcal{S},\mathcal{T}) = \|P_{A,\mathcal{S}}P_{A,\mathcal{T}}\|_A$.

Proof.

$$\begin{aligned}\cos\theta_A(\mathcal{S},\mathcal{T}) &= \sup\{|\langle\xi,\eta\rangle_A| : \xi\in\mathcal{S}, \eta\in\mathcal{T} \text{ and } \|\xi\|_A\le 1\ \|\eta\|_A\le 1\} \\ &= \sup\{|\langle P_{A,\mathcal{S}}\xi, P_{A,\mathcal{T}}\eta\rangle_A| : \xi,\eta\in\mathcal{H} \text{ and } \|\xi\|_A\le 1\ \|\eta\|_A\le 1\} \\ &= \sup\{|\langle \xi, P_{A,\mathcal{S}}P_{A,\mathcal{T}}\eta\rangle_A| : \xi,\eta\in\mathcal{H} \text{ and } \|\xi\|_A\le q\ \|\eta\|_A\le 1\} \\ &= \|P_{A,\mathcal{S}}P_{A,\mathcal{T}}\|_A.\end{aligned}$$

□

Proposition 5.3. *Let $A \in L(\mathcal{H})^+$. If $(A, \mathcal{S})$ and $(A, \mathcal{T})$ are compatible pairs and $\mathcal{S} \cap \overline{R(A)} \neq \{0\}$ then for $Q = Q_{\mathcal{S}//\mathcal{T}}$ it holds*

$$\|Q\|_A = (1 - \|P_{A,\mathcal{T}} P_{A,\mathcal{S}}\|_A^2)^{-1/2}.$$

Proof. Let $\xi \in \mathcal{H}$. Then $\xi = P_{A,\mathcal{T}}\xi + (I - P_{A,\mathcal{T}})\xi$, so $Q\xi = Q(I - P_{A,\mathcal{T}})\xi$ and $\|(I - P_{A,\mathcal{T}})\xi\|_A \leq \|\xi\|_A$. Therefore, as $R(I - P_{A,\mathcal{T}}) = N(P_{A,\mathcal{T}}) = \mathcal{T}^{\perp_A} \ominus \mathcal{N}$, where $\mathcal{N} = \mathcal{T} \cap N(A)$ then $\|Q\|_A = \|Q|_{\mathcal{T}^{\perp_A} \ominus \mathcal{N}}\|_A$. Now, consider $\xi \in (\mathcal{T}^{\perp_A} \ominus \mathcal{N}) \cap \overline{R(A)}$. Thus $P_{A,\mathcal{T}} Q\xi = P_{A,\mathcal{T}}\xi + P_{A,\mathcal{T}}(Q\xi - \xi) = Q\xi - \xi$ and as a consequence $\|Q\xi\|_A^2 = \|\xi\|_A^2 + \|Q\xi - \xi\|_A^2 = \|\xi\|_A^2 + \|P_{A,\mathcal{T}} P_{A,\mathcal{S}} Q\xi\|_A^2$. Note that, with out loss of generality, we can consider $Q\xi \in \overline{R(A)}$. Then we get that $1 = \frac{\|\xi\|_A^2}{\|Q\xi\|_A^2} + \frac{\|P_{A,\mathcal{T}} P_{A,\mathcal{S}} Q\xi\|_A^2}{\|Q\xi\|_A^2}$ and from this

$$\left(1 - \frac{\|P_{A,\mathcal{T}} P_{A,\mathcal{S}} Q\xi\|_A^2}{\|Q\xi\|_A^2}\right)^{-1/2} = \frac{\|Q\xi\|_A}{\|\xi\|_A}.$$

Now, since $\|Q\|_A = \|Q|_{\mathcal{T}^{\perp_A} \ominus \mathcal{N}}\|_A$ and $\|P_{A,\mathcal{T}} P_{A,\mathcal{S}}\|_A = \|P_{A,\mathcal{T}} P_{A,\mathcal{S}}|_{\mathcal{S}}\|_A$ the assertion follows. □

Corollary 5.4. *Let $A \in L(\mathcal{H})^+$. If $(A, \mathcal{S})$ and $(A, \mathcal{T})$ are compatible pairs and $\mathcal{S} \cap \overline{R(A)} \neq \{0\}$ then for every $Q_{\mathcal{S}//\mathcal{T}}$ it holds*

$$\|Q_{\mathcal{S}//\mathcal{T}}\|_A = \frac{1}{\sin \theta_A(\mathcal{T}, \mathcal{S})}.$$

The following example shows that the condition $\mathcal{S} \cap \overline{R(A)} \neq \{0\}$ in Proposition 5.3 is not superfluous.

Example 3. Let $\mathcal{H} = \mathbb{R}^2$, $A = \begin{pmatrix} 2 & 1 \\ 1 & 1/2 \end{pmatrix} \in L(\mathbb{R}^2)^+$ and $Q = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$. Then $\mathcal{S} = R(Q) = span\{(1,1)\}$ and $\mathcal{T} = N(Q) = span\{(1,0)\}$. Furthermore, $P_{A,\mathcal{T}} = \begin{pmatrix} 2/3 & 1/3 \\ 2/3 & 1/3 \end{pmatrix}$ and $P_{A,\mathcal{S}} = \begin{pmatrix} 1 & 1/2 \\ 0 & 0 \end{pmatrix}$. Now, $\|P_{A,\mathcal{T}} P_{A,\mathcal{S}}\|_A = 1$ and $\|Q\|_A = 0.6$.

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