Indefinite least squares problems and oblique projections

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Abstract

The existence and uniqueness of solutions of some indefinite least squares problems are studied using the notion of compatibility between closed subspaces and selfadjoint operators. Also, the set of such solutions is related to a set of weighted generalized inverses of a closed range operator.

1 Introduction

A classical problem studied in signal processing is the least mean square estimation of stochastic processes. Under the assumption that the processes are stationary, this problem has been studied since the 1940's by A. N. Kolmogorov [1] and by N. Wiener [2], who introduced this idea in the engineering areas of control theory and signal processing. Later, at the end of 1950's, R. E. Kalman [3] extended this theory to non-stationary processes and, since then, the so called Kalman filter became one of the most useful tools in control applications. After Kalman's work, similar theories on optimal control have been developed, known as linear quadratic Gaussian (LQG) control. In the LQG control theory, the statistical properties of the underlying stochastic processes are assumed to be known, an important limitation in some engineering applications. At the beginning of 1980's, the $\mathcal{H}^\infty$ control theory emerged as a useful alternative to solve this limitation in practical problems, see [4]. It became a very active area, which has been studied from different approaches, some of them, related to problems in indefinite metric spaces. In particular, the introduction of Krein spaces proposed by B. Hassibi, A. H. Sayed and T. Kailath in $\mathcal{H}^\infty$ estimation and control techniques made possible to adapt traditional tools of the LQG control theory to $\mathcal{H}^\infty$ control (see [5] for a complete exposition on this subject). Moreover, the introduction of Krein spaces provided an explanation of some important issues in adaptive filter theory, see [6] and [7].

Some of these (finite dimensional) problems can be adequately stated as indefinite least squares problems. Given a matrix $A \in \mathbb{R}^{m\times n}$ with $m \geq n$ and a vector $y \in \mathbb{R}^m$, an indefinite least squares solution of the equation $Ax = y$ is a vector $u \in \mathbb{R}^n$ such that

$$(Au - y)^t J(Au - y) = \min_{x\in\mathbb{R}^n} (Ax - y)^t J(Ax - y),$$

where $J = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$ is a signature matrix with $p + q = m$. The description of the indefinite least squares solutions of $Ax = y$ is related to a linear estimation tecnique in Krein spaces, studied by S. Chandrasekaran et al. [8] and A. H. Sayed et al. [9].

Analogously, given Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, a closed range operator $C \in L(\mathcal{H}, \mathcal{K})$, a selfadjoint operator $B \in L(\mathcal{K})$ and a vector $y \in \mathcal{K}$, we say that a vector $u \in \mathcal{H}$ is a B-least squares solution (B-LSS) of the equation $Cx = y$ if it satisfies

$$\langle B(Cu - y), Cu - y \rangle = \min_{x\in\mathcal{H}} \langle B(Cx - y), Cx - y \rangle . \tag{1}$$

In particular, if $B = I$ the above equation defines the classical least squares problem, i.e. find $u \in \mathcal{H}$ such that

$$\|Cu - y\|^2 = \min_{x\in\mathcal{H}} \|Cx - y\|^2.$$

The last problem always admits a (unique) solution of minimal norm, given by $u = C^\dagger y$, where $C^\dagger$ is the Moore-Penrose inverse of C. On the other hand, if B is a signature operator, i.e. $B = B^* = B^{-1}$, Eq. (1) is an optimization problem in Krein spaces similar to those considered before in finite dimension.

When the sesquilinear form $\langle x, y \rangle_B = \langle Bx, y \rangle$ induced by an arbitary selfadjoint operator $B \in L(\mathcal{K})$ is considered, nor existence neither uniqueness of B-LSS are in general guaranteed, even in the finite-dimensional case. The existence of B-LSS of $Cx = y$ turns out to be intimately related to the existence of B-selfadjoint projections with the same range as the operator C.

The existence of selfadjoint projections respect to an indefinite metric has been studied in the classical literature on indefinite metric spaces [10], [11], and recently in [12], [13], and [14]. In [13] the notion of compatibility is introduced. Given a closed subspace $\mathcal{S}$ of $\mathcal{K}$, the pair $(B, \mathcal{S})$ is *compatible* if there exists an (oblique) projection Q with range $\mathcal{S}$, which is selfadjoint with respect to the sesquilinear form induced by B.

In [15], using this idea and assuming that B is a (semidefinite) positive operator, the problem of finding B-LSS has been studied. The present work can be seen as an extension of [15] to selfadjoint operators.

The paper is organized as follows. In Section 3 we show that a necessary condition for the existence of B-LSS is that the range of C, hereafter denoted by $R(C)$, is a B-nonnegative subspace (i.e. $\langle Bx, x \rangle \geq 0$ for every $x \in R(C)$). Under this hypothesis, given $y \in \mathcal{K}$, the B-LSS of the equation $Cx = y$ coincide with the solutions of the "normal equation" associated to the problem:

$$C^* B(Cx - y) = 0.$$

Using this fact, we prove that equation $Cx = y$ admits B-LSS for every $y \in \mathcal{K}$ if and only if the pair $(B, R(C))$ is compatible. Furthermore, if $(B, R(C))$ is compatible and $y \in \mathcal{K} \setminus R(C)$, we show that $u \in \mathcal{H}$ is a B-LSS of $Cx = y$ if and only if $Cu = Qy$ for some B-selfadjoint projection Q with $R(Q) = R(C)$.

We also prove that, fixed a vector $y \in \mathcal{K}$, the set of solutions of the normal equation is an affine manifold, parallel to the nullspace of BC. Finally, a minimization problem among the B-LSS of the equation $Cx = y$ is presented. If $A \in L(\mathcal{H})$ is a selfadjoint operator which satisfies certain compatibility condition, we look for those $w \in \mathcal{H}$ which are B-LSS of $Cx = y$ and satisfy

$$\langle w, w \rangle_A \leq \langle u, u \rangle_A \qquad \text{for every } B\text{-LSS } u \in \mathcal{H} \text{ of } Cx = y.$$

A vector $w \in \mathcal{H}$ satisfying the above conditions is called an AB-LSS of the equation $Cx = y$. It is shown that $w \in \mathcal{H}$ is an AB-LSS of $Cx = y$ if and only if $w = (I - Q)C^\dagger Py$, where P and Q are appropriate B-selfadjoint and A-selfadjoint projections, respectively. In this case, the operator $D = (I - Q)C^\dagger P \in L(\mathcal{K}, \mathcal{H})$ can be seen as a "weighted inverse" of C because it is a solution of

$$CXC = C, \quad XCX = X, \quad A(XC) = (XC)^* A, \quad B(CX) = (CX)^* B. \tag{2}$$

The solutions of the above equations have been studied, in the finite dimensional case, by X. Sheng and G. Chen [16]. Similarly, X. Mary [17] studied the existence of generalized inverses in Krein spaces. The solutions of (2) can also be seen as an extension of the weighted inverses considered, for positive weights, by Eldén [18] (in finite dimensional spaces) and by G. Corach and A. Maestripieri [15] (in infinite dimensional Hilbert spaces). See also the book by A. Ben-Israel and T. N. E. Greville [19].

The purpose of Section 4 is to characterize, under certain compatibility conditions, the set of weighted inverses of a fixed closed range operator C, and, assuming the definiteness of certain subspaces, relate them with the set of AB-LSS of $Cx = y$.

Section 5 is devoted to study the solutions of the normal equation without the restriction of $R(C)$ being B-definite. In this case, there no longer exist B-LSS of the equation $Cx = y$. However, the solutions of the normal equation $C^* B(Cx - y) = 0$ can be related, under certain decomposability condition of $R(C)$, to the solutions of a min-max problem.

2 Preliminaries

Along this work $\mathcal{H}$ and $\mathcal{K}$ denote complex (separable) Hilbert spaces, $L(\mathcal{H}, \mathcal{K})$ is the algebra of bounded linear operators from $\mathcal{H}$ into $\mathcal{K}$, $L(\mathcal{H}) = L(\mathcal{H}, \mathcal{H})$ and $CR(\mathcal{H}, \mathcal{K})$ is the set of (bounded linear) closed range operators. If $T \in L(\mathcal{H}, \mathcal{K})$ then $T^* \in L(\mathcal{K}, \mathcal{H})$ denotes the adjoint operator of T, $R(T)$ stands for the range of T and $N(T)$ for its nullspace.

Consider the following subsets of $L(\mathcal{H})$: let $L(\mathcal{H})^+$ be the cone of (semidefinite) positive operators, $L(\mathcal{H})^s$ the (real) vector space of selfadjoint operators, $GL(\mathcal{H})^s$ the group of invertible selfadjoint operators and denote by $\mathcal{Q}$ the set of (oblique) projections, i.e. $\mathcal{Q} = \{Q \in L(\mathcal{H}) \,:\, Q^2 = Q\}$.

If $\mathcal{S}$ and $\mathcal{T}$ are two (closed) subspaces of $\mathcal{H}$, denote by $\mathcal{S} \dotplus \mathcal{T}$ the direct sum of $\mathcal{S}$ and $\mathcal{T}$, $\mathcal{S} \oplus \mathcal{T}$ the (direct) orthogonal sum of them and $\mathcal{S} \ominus \mathcal{T} := \mathcal{S} \cap (\mathcal{S} \cap \mathcal{T})^\perp$. If $\mathcal{H} = \mathcal{S} \dotplus \mathcal{T}$, the oblique projection onto $\mathcal{S}$ along $\mathcal{T}$, $P_{\mathcal{S}//\mathcal{T}}$, is the projection with $R(P_{\mathcal{S}//\mathcal{T}}) = \mathcal{S}$ and $N(P_{\mathcal{S}//\mathcal{T}}) = \mathcal{T}$. In particular, $P_{\mathcal{S}} := P_{\mathcal{S}//\mathcal{S}^\perp}$ is the orthogonal projection onto $\mathcal{S}$. If $C \in CR(\mathcal{H}, \mathcal{K})$, $C^\dagger$ denotes the Moore-Penrose pseudoinverse of C.

Given $B \in L(\mathcal{H})^s$ consider the sesquilinear form in $\mathcal{H} \times \mathcal{H}$ defined by $\langle x, y \rangle_B := \langle Bx, y \rangle$, for $x, y \in \mathcal{H}$. If $\mathcal{S}$ is a closed subspace of $\mathcal{H}$, the B-*orthogonal companion* to $\mathcal{S}$ is given by

$$\mathcal{S}^{\perp_B} := \{x \in \mathcal{H} \,:\, \langle x, s \rangle_B = 0 \text{ for every } s \in \mathcal{S}\}.$$

It holds that $\mathcal{S}^{\perp_B} = B^{-1}(\mathcal{S}^\perp) = B(\mathcal{S})^\perp$. Given two closed subspaces $\mathcal{S}$ and $\mathcal{T}$ of $\mathcal{H}$, we say that $\mathcal{S}$ is B-*orthogonal* to $\mathcal{T}$ if $\mathcal{T} \subseteq \mathcal{S}^{\perp_B}$ and denote it by $\mathcal{S} \perp_B \mathcal{T}$. Define $\mathcal{N} := \mathcal{S} \cap \mathcal{S}^{\perp_B}$, in general $\mathcal{N} \neq \{0\}$.

A vector $x \in \mathcal{H}$ is B-*positive* if $\langle x, x \rangle_B > 0$. A subspace $\mathcal{S}$ of $\mathcal{H}$ is B-positive if every $x \in \mathcal{S}$, $x \neq 0$, is a B-positive vector. B-nonnegative, B-neutral, B-negative and B-nonpositive vectors (and subspaces) are defined analogously.

Definition. *Given $B \in L(\mathcal{H})^s$, a closed subspace $\mathcal{S}$ of $\mathcal{H}$ is said to be* B-decomposable *if it can be represented as the B-orthogonal direct sum of a B-neutral subspace $\mathcal{S}_0$, a B-positive subspace $\mathcal{S}_+$ and a B-negative subspace $\mathcal{S}_-$, i.e.*

$$\mathcal{S} = \mathcal{S}_0 \dotplus \mathcal{S}_+ \dotplus \mathcal{S}_-.$$

It is important to notice that not every subspace $\mathcal{S}$ of $\mathcal{H}$ is B-decomposable, see [10, Example 1.33]. Observe that if $\mathcal{S}$ is B-decomposable then $\mathcal{S}_0 = \mathcal{N}$, see [10] for a complete exposition on this subject.

An operator $T \in L(\mathcal{H})$ is B-*selfadjoint* if $\langle Tx, y \rangle_B = \langle x, Ty \rangle_B$ for every $x, y \in \mathcal{H}$. It is easy to see that T satisfies this condition if and only if $BT = T^*B$.

Definition. *Let $B \in L(\mathcal{H})^s$ and $\mathcal{S}$ be a closed subspace of $\mathcal{H}$. The pair $(B, \mathcal{S})$ is* compatible *if there exists a B-selfadjoint projection with range $\mathcal{S}$, i.e. if the set*

$$\mathcal{P}(B, \mathcal{S}) := \{Q \in \mathcal{Q} \,:\, R(Q) = \mathcal{S},\ BQ = Q^*B\}$$

is not empty.

Notice that a projection Q is B-selfadjoint if and only if its nullspace satisfies the inclusion $N(Q) \subseteq R(Q)^{\perp_B}$, see [13, Lemma 3.2]. Then, $(B, \mathcal{S})$ is compatible if and only if

$$\mathcal{H} = \mathcal{S} + B(\mathcal{S})^\perp.$$

Given a compatible pair $(B, \mathcal{S})$, $\mathcal{N} = \mathcal{S} \cap N(B)$ and the Hilbert space $\mathcal{H}$ can be decomposed as $\mathcal{H} = \mathcal{S} \dotplus (B(\mathcal{S})^\perp \ominus \mathcal{N})$, so the following oblique projection is well defined:

$$P_{B,\mathcal{S}} := P_{\mathcal{S}//B(\mathcal{S})^\perp \ominus \mathcal{N}}. \tag{3}$$

Observe that $P_{B,\mathcal{S}} \in \mathcal{P}(B, \mathcal{S})$ because $R(P_{B,\mathcal{S}}) = \mathcal{S}$ and $N(P_{B,\mathcal{S}}) \subseteq B(\mathcal{S})^\perp$. In what follows, we state several results about the set $\mathcal{P}(B, \mathcal{S})$, which will be needed later.

Theorem 1. *Let $B \in L(\mathcal{H})^s$ and $\mathcal{S}$ be a closed subspace of $\mathcal{H}$ such that $(B, \mathcal{S})$ is compatible. Then $\mathcal{P}(B, \mathcal{S})$ is an affine manifold that can be parametrized as*

$$\mathcal{P}(B, \mathcal{S}) = P_{B,\mathcal{S}} + L(\mathcal{S}^\perp, \mathcal{N}),$$

where $L(\mathcal{S}^\perp, \mathcal{N})$ is viewed as a subspace of $L(\mathcal{H})$. Moreover, $P_{B,\mathcal{S}}$ has minimal norm in $\mathcal{P}(B, \mathcal{S})$.

Proof. *Proof. See [13, Theorem 3.5].* □

Proposition 1. *Let $B \in L(\mathcal{H})^s$ and $\mathcal{S}$ be a closed subspace of $\mathcal{H}$ such that $(B, \mathcal{S})$ is compatible. If $Q \in \mathcal{P}(B, \mathcal{S})$ then*

$$Q = P_{B,\mathcal{S}\ominus\mathcal{N}} + P_{\mathcal{N}//\mathcal{S}\ominus\mathcal{N}+N(Q)}.$$

Proof. *Proof. It is a particular case of [20, Proposition 3.5].* □

Proposition 2. *Let $B \in L(\mathcal{H})^s$ and $\mathcal{S}$ be a closed subspace of $\mathcal{H}$ such that $(B, \mathcal{S})$ is compatible. If $x \in \mathcal{H}$ then $(I - P_{B,\mathcal{S}})x$ is the unique minimal norm element in the set*

$$\{(I - Q)x : Q \in \mathcal{P}(B, \mathcal{S})\}. \tag{4}$$

Proof. *Proof. It is analogous to the proof of [21, Theorem 3.2].* □

Proposition 3. *Let $B \in L(\mathcal{H})^s$ and $\mathcal{S}$ be a closed subspace of $\mathcal{H}$. Then, $(B, \mathcal{S})$ is compatible if and only if there exists a (unique) orthogonal decomposition $\mathcal{S} \ominus N(B) = \mathcal{S}_+ \oplus \mathcal{S}_-$, where $\mathcal{S}_+$ is a (closed) B-positive subspace, $\mathcal{S}_-$ is a (closed) B-negative subspace, $(B, \mathcal{S}_\pm)$ is compatible and $\mathcal{S}_+ \perp_B \mathcal{S}_-$.*

Proof. *Proof. See [14, Theorem 5.1 and Proposition 5.2].* □

Using the decomposition of $\mathcal{S}$ provided by Proposition 3, there is a min-max interpretation of the values of $(I - Q)x$, for every $Q \in \mathcal{P}(B, \mathcal{S})$.

Proposition 4. *Let $B \in L(\mathcal{H})^s$ and $\mathcal{S}$ be a closed subspace of $\mathcal{H}$ such that $(B, \mathcal{S})$ is compatible. Then, for every $Q \in \mathcal{P}(B, \mathcal{S})$ and $x \in \mathcal{H}$,*

$$\langle (I-Q)x, (I-Q)x \rangle_B = \min_{s\in\mathcal{S}_+} \max_{t\in\mathcal{S}_-} \langle x-(s+t), x-(s+t) \rangle_B = \max_{t\in\mathcal{S}_-} \min_{s\in\mathcal{S}_+} \langle x-(s+t), x-(s+t) \rangle_B\,, \tag{5}$$

where $\mathcal{S} \ominus N(B) = \mathcal{S}_+ \dot{+} \mathcal{S}_-$ is the decomposition given in Proposition 3.

Proof. *Proof. See [22, Corollary 3.13].* □

3 Indefinite Least Squares Problems

Given an operator $C \in CR(\mathcal{H}, \mathcal{K})$ and a vector $y \in \mathcal{K}$, a weighted least squares solution of the equation $Cx = y$ is a vector $u \in \mathcal{H}$ such that

$$\|Cu - y\|_A = \min_{x\in\mathcal{H}} \|Cx - y\|_A, \tag{6}$$

where $A \in L(\mathcal{K})^+$ and $\| \, \|_A$ is the seminorm on $\mathcal{K}$ defined by $\|x\|_A = \|A^{1/2}x\| = \langle Ax, x \rangle^{1/2}$ (see [15]). Notice that (6) is equivalent to

$$\langle A(Cu - y), Cu - y \rangle = \min_{x\in\mathcal{H}} \langle A(Cx - y), Cx - y \rangle\,.$$

In this section, we study the same problem considering selfadjoint weights. Given $C \in CR(\mathcal{H}, \mathcal{K})$, $y \in \mathcal{K}$ and an operator $B \in L(\mathcal{K})^s$, we are interested in characterizing, if there is any, those vectors $u \in \mathcal{H}$ such that

$$\langle B(Cu - y), Cu - y \rangle = \min_{x\in\mathcal{H}} \langle B(Cx - y), Cx - y \rangle\,.$$

We are going to establish necessary and sufficient conditions for the existence of vectors $u \in \mathcal{H}$ satisfying the above equation; and, in this case, we will provide a parametrization of the set of solutions.

This kind of problems has been previously studied, for instance, by A. H. Sayed et al. [9]. Given two invertible Hermitian matrices Π and W, a column vector y, and an arbitrary matrix T of appropriate dimensions, they studied the following minimization problem: characterize those vectors z_0 such that

$$z_0^*\Pi^{-1}z_0 + (y - Tz_0)^*W^{-1}(y - Tz_0) = \min_z \left[z^*\Pi^{-1}z + (y - Tz)^*W^{-1}(y - Tz)\right].$$

Observe that, if $\mathcal{H}_1$ and $\mathcal{H}_2$ are finite-dimensional Hilbert spaces, $\Pi \in GL(\mathcal{H}_1)^s$, $W \in GL(\mathcal{H}_2)^s$, $y \in \mathcal{H}_2$ and $T \in L(\mathcal{H}_1, \mathcal{H}_2)$, the above problem can be restated as: characterize those vectors $z_0 \in \mathcal{H}_1$ such that

$$\langle B(w - Cz_0), w - Cz_0 \rangle = \min_{z \in \mathcal{H}_1} \langle B(w - Cz), w - Cz \rangle ,$$

where $w = \begin{pmatrix} 0 \\ y \end{pmatrix} \in \mathcal{H}_1 \oplus \mathcal{H}_2$, $B = \begin{pmatrix} \Pi^{-1} & 0 \\ 0 & W^{-1} \end{pmatrix} \in GL(\mathcal{H}_1 \oplus \mathcal{H}_2)^s$ and $C = \begin{pmatrix} I \\ T \end{pmatrix} \in L(\mathcal{H}_1, \mathcal{H}_1 \oplus \mathcal{H}_2)$.

Definition. *Given $C \in CR(\mathcal{H}, \mathcal{K})$, $B \in L(\mathcal{K})^s$ and $y \in \mathcal{K}$, an element $u \in \mathcal{H}$ is a B-least squares solution (B-LSS) of the equation $Cx = y$ if*

$$\langle Cu - y, Cu - y \rangle_B = \min_{x \in \mathcal{H}} \langle Cx - y, Cx - y \rangle_B . \tag{7}$$

The next lemma shows necessary and sufficient conditions for the existence of B-LSS of the equation $Cx = y$. Similar results have been presented in [15, Remark 4.3] for positive operators and in [11, Theorem 8.4] for indefinite metric spaces.

Lemma 1. *Let $C \in CR(\mathcal{H}, \mathcal{K})$, $B \in L(\mathcal{K})^s$ and $y \in \mathcal{K}$. Then, $u \in \mathcal{H}$ is a B-LSS of the equation $Cx = y$ if and only if $R(C)$ is B-nonnegative and $y - Cu \in R(BC)^\perp$.*

Proof. *Proof. Let $u \in \mathcal{H}$ be a B-LSS of $Cx = y$. If $x \in \mathcal{H}$ and $\alpha \in \mathbb{R}$, then*

$$\begin{aligned} \langle Cu - y, Cu - y \rangle_B &\leq \langle Cu + \alpha Cx - y, Cu + \alpha Cx - y \rangle_B = \\ &= \langle Cu - y, Cu - y \rangle_B + 2\alpha \operatorname{Re} \langle Cu - y, Cx \rangle_B + \alpha^2 \langle Cx, Cx \rangle_B . \end{aligned}$$

Therefore, $2\alpha \operatorname{Re} \langle Cu - y, Cx \rangle_B + \alpha^2 \langle Cx, Cx \rangle_B \geq 0$ for every $\alpha \in \mathbb{R}$, and a standard argument shows that $\operatorname{Re} \langle Cu - y, Cx \rangle_B = 0$. In the same way, considering $\beta = i\alpha$, $\alpha \in \mathbb{R}$, it follows that $\operatorname{Im} \langle Cu - y, Cx \rangle_B = 0$. Then, $\langle Cu - y, Cx \rangle_B = 0$ and $\langle Cx, Cx \rangle_B \geq 0$ for every $x \in \mathcal{H}$.

Conversely, suppose that $R(C)$ is B-nonnegative and there exists $u \in \mathcal{H}$ such that $y - Cu \in R(BC)^\perp$. Then, for every $x \in \mathcal{H}$,

$$\langle y - Cx, y - Cx \rangle_B = \langle y - Cu, y - Cu \rangle_B + \langle C(u - x), C(u - x) \rangle_B \geq \langle y - Cu, y - Cu \rangle_B .$$

Therefore, u is a B-LSS of $Cx = y$. □

Corollary 1. *Let $C \in CR(\mathcal{H}, \mathcal{K})$, $B \in L(\mathcal{K})^s$ and $y \in \mathcal{K}$. If $u, v \in \mathcal{H}$ are two B-LSS of the equation $Cx = y$, then $C(u - v) \in \mathcal{N} = R(C) \cap R(BC)^\perp$.*

Proof. *Proof. If u and v are B-LSS of $Cx = y$ then, by Lemma 1, $y - Cu, y - Cv \in R(BC)^\perp$. Therefore, $C(u - v) = (y - Cv) - (y - Cu) \in R(C) \cap R(BC)^\perp = \mathcal{N}$.* □

3.1 The normal equation

Let $C \in CR(\mathcal{H}, \mathcal{K})$, $B \in L(\mathcal{K})^s$ and $y \in \mathcal{K}$. Given $u \in \mathcal{H}$, $y - Cu \in R(BC)^\perp$ if and only if $\langle Cu - y, Cx \rangle_B = 0$ for every $x \in \mathcal{H}$, or equivalently, $\langle C^*B(Cu - y), x \rangle = 0$ for every $x \in \mathcal{H}$. Thus, $y - Cu \in R(BC)^\perp$ if and only if u is a solution of the *normal equation*

$$C^*B(Cx - y) = 0.$$

Therefore, we have proved the following proposition.

Proposition 5. *Let $C \in CR(\mathcal{H}, \mathcal{K})$ and $B \in L(\mathcal{K})^s$. Given $y \in \mathcal{K}$, the normal equation $C^*B(Cx - y) = 0$ admits a solution $u \in \mathcal{H}$ if and only if $y - Cu \in R(BC)^\perp$.*

Proposition 5 shows that there is a solution of the normal equation $C^*B(Cx - y) = 0$ for every $y \in \mathcal{K}$ if and only if $\mathcal{K} = R(C) + R(BC)^\perp$, and the last condition is equivalent to the compatibility of the pair $(B, R(C))$ (see the Preliminaries). This is exactly what the next result states.

Corollary 2. *Let $C \in CR(\mathcal{H}, \mathcal{K})$ and $B \in L(\mathcal{K})^s$. Then, there exists a solution $u \in \mathcal{H}$ of the normal equation $C^*B(Cx - y) = 0$ for every $y \in \mathcal{K}$ if and only if the pair $(B, R(C))$ is compatible.*

Proposition 6. *Let $C \in CR(\mathcal{H}, \mathcal{K})$ and $B \in L(\mathcal{K})^s$. Given $y \in \mathcal{K}$, suppose that the normal equation $C^*B(Cx - y) = 0$ admits a solution $u_0 \in \mathcal{H}$. Then, the set of solutions of $C^*B(Cx - y) = 0$ coincides with $u_0 + N(C^*BC)$.*

Proof. *Proof. If $z \in N(C^*BC)$, observe that $u = u_0 + z$ satisfies $C^*B(Cu - y) = C^*B(Cu_0 - y) = 0$. Conversely, if $u_1 \neq u_0$ is another solution of the normal equation $C^*B(Cx - y) = 0$, then $u_1 - u_0 \in N(C^*BC)$. Therefore, $u_1 \in u_0 + N(C^*BC)$.* □

Proposition 7. *Given $C \in CR(\mathcal{H}, \mathcal{K})$ and $B \in L(\mathcal{K})^s$, suppose that $(B, R(C))$ is compatible. If $y \in \mathcal{K} \setminus R(C)$, then $u \in \mathcal{H}$ is a solution of $C^*B(Cx - y) = 0$ if and only if there exists $Q \in \mathcal{P}(B, R(C))$ such that $Cu = Qy$.*

Proof. *Proof. For $y \in \mathcal{K}$ and $u \in \mathcal{H}$ suppose that there exists $Q \in \mathcal{P}(B, R(C))$ such that $Cu = Qy$. Then, $y - Cu = (I - Q)y \in N(Q) \subseteq R(BC)^\perp$ (see Preliminaries), and by Proposition 5, u is a solution of $C^*B(Cx - y) = 0$.*

Conversely, for $y \in \mathcal{K} \setminus R(C)$, let u be a solution of $C^*B(Cx - y) = 0$. By Proposition 5, $y = Cu + z$ with $z \in R(BC)^\perp$, $z \notin R(C)$. Since $z \in R(BC)^\perp \setminus R(C)$ and $\mathcal{K} = R(C) + R(BC)^\perp$, it is easy to see that there exists a closed subspace $\mathcal{S}$ of $R(BC)^\perp$ such that $z \in \mathcal{S}$ and $\mathcal{H} = R(C) \dotplus \mathcal{S}$. Therefore, $Q = P_{R(C)//\mathcal{S}} \in \mathcal{P}(B, R(C))$ and

$$Qy = Q(Cu + z) = Cu.$$

□

Remark 1. *With the hypothesis of Proposition 7,*

(a) *It follows from the proof of Proposition 7 that, given $y \in \mathcal{K}$, if $Cu = Qy$ for some $Q \in \mathcal{P}(B, R(C))$, then u is a solution of the normal equation.*

(b) *The converse is no longer true when $y \in R(C)$. In this case, $Qy = y$ for every $Q \in \mathcal{P}(B, R(C))$ and the set of solutions of $Cx = y$ is $C^\dagger y + N(C)$ but the set of solutions of the normal equation $C^*B(Cx - y) = 0$ can be parametrized as $C^\dagger y + N(BC)$ (see Proposition 9).*

Corollary 3. *Given $C \in CR(\mathcal{H}, \mathcal{K})$ and $B \in L(\mathcal{K})^s$ such that $(B, R(C))$ is compatible, suppose that $\mathcal{N} = \{0\}$. Then, if $y \in \mathcal{K}$, $u \in \mathcal{H}$ is a solution of $C^*B(Cx - y) = 0$ if and only if $Cu = P_{B,R(C)}y$.*

Proof. *Proof. If $(B, R(C))$ is compatible and $\mathcal{N} = \{0\}$ then $\mathcal{P}(B, R(C)) = \{P_{B,R(C)}\}$ and $N(BC) = N(C)$. Therefore, the corollary follows from Proposition 7 and the above remark.* □

As it was stated in Lemma 1, a necessary condition for the existence of B-LSS of the equation $Cx = y$ is that $R(C)$ is B-nonnegative. The following theorem describes the B-least squares problem assuming this fact.

Theorem 2. *Given $B \in L(\mathcal{K})^s$, let $C \in CR(\mathcal{H}, \mathcal{K})$ such that $R(C)$ is B-nonnegative. Then, the following conditions hold:*

1. *Given $y \in \mathcal{K}$, $u \in \mathcal{H}$ is a B-LSS of the equation $Cx = y$ if and only if u is a solution of the normal equation $C^*B(Cx - y) = 0$.*

2. *There exists a B-LSS of the equation $Cx = y$ for every $y \in \mathcal{K}$ if and only if the pair $(B, R(C))$ is compatible. In this case, if $y \in \mathcal{K} \setminus R(C)$, $u \in \mathcal{H}$ is a B-LSS of $Cx = y$ if and only if there exists $Q \in \mathcal{P}(B, R(C))$ such that $Cu = Qy$.*

Proof. *Proof. It is a consequence of Lemma 1, Propositions 5 and 7, and Corollary 2.* □

The following result shows that, if Q is a B-selfadjoint projection with B-nonnegative range, it behaves as a classical orthogonal projection, in the sense that, fixed $x \in \mathcal{H}$, Qx minimizes the "B-distance" to the subspace $R(Q)$.

Corollary 4. *Let $Q \in L(\mathcal{H})$ be a B-selfadjoint projection and consider $x \in \mathcal{H}$. If $R(Q)$ is B-nonnegative then $\langle (I-Q)x, (I-Q)x \rangle_B = \min_{s \in R(Q)} \langle x-s, x-s \rangle_B$. Moreover, if $R(Q)$ is B-positive, Qx is the unique vector in $R(Q)$ which attains the minimum.*

Proof. *Proof. It is a consequence of the above theorem, considering the B-LSS of the equation $Pz = x$, where $P \in L(\mathcal{H})$ is the orthogonal projection onto $R(Q)$.* □

Remark 2. *Given $C \in CR(\mathcal{H}, \mathcal{K})$ and $B \in L(\mathcal{K})^s$, if $R(C)$ is a B-nonpositive subspace of $\mathcal{K}$, vectors $u \in \mathcal{H}$ satisfying*

$$\langle B(Cu-y), Cu-y \rangle = \max_{x \in \mathcal{H}} \langle B(Cx-y), Cx-y \rangle \quad \text{for every } x \in \mathcal{H},$$

can be characterized following the same ideas as in the B-least squares problem. In fact, similar results to Theorem 2 and Corollary 4 can be established mutatis mutandis.

3.2 The set of B-Least Squares Solutions

Let $C \in CR(\mathcal{H}, \mathcal{K})$, $B \in L(\mathcal{K})^s$ and suppose that $(B, R(C))$ is compatible. Given $y \in \mathcal{K}$, it is possible to characterize, if there is any, every solution of the normal equation $C^*B(Cx-y) = 0$. For this purpose we define a particular solution of the normal equation, based on the minimal norm element $P_{B,R(C)} \in \mathcal{P}(B, R(C))$.

Definition. *Let $C \in CR(\mathcal{H}, \mathcal{K})$, $B \in L(\mathcal{K})^s$ and $y \in \mathcal{K}$. Suppose that the pair $(B, R(C))$ is compatible. Then,*

$$u_y := C^\dagger P_{B,R(C)} y \tag{8}$$

is the minimal solution *of the equation $C^*B(Cx-y) = 0$.*

Observe that $Cu_y = P_{B,R(C)} y$ so that u_y is a solution of $C^*B(Cx-y) = 0$ (see Remark 1). The following result characterizes the minimal solution of the equation $C^*B(Cx-y) = 0$.

Proposition 8. *Let $C \in CR(\mathcal{H}, \mathcal{K})$ and $B \in L(\mathcal{K})^s$ such that $(B, R(C))$ is compatible. Given $y \in \mathcal{K}$, $u_y \in \mathcal{H}$ is the unique solution of $C^*B(Cx-y) = 0$ in $N(C)^\perp$ which satisfies*

$$\|y - Cu_y\| = \min\{\|y - Cu\| : u \text{ is a solution of } C^*B(Cx-y) = 0\}. \tag{9}$$

Proof. *Proof. If $y \in R(C)$ it is easy to see that u_y satisfies Eq. (9). Let $y \in \mathcal{K} \setminus R(C)$. If u_0 is a solution of $C^*B(Cx-y) = 0$ then there exists $Q \in \mathcal{P}(B, R(C))$ such that $Cu_0 = Qy$ and, by Proposition 2,*

$$\|y - Cu_0\| = \|(I-Q)y\| \geq \|(I - P_{B,R(C)})y\| = \|y - Cu_y\|.$$

Then, u_0 satisfies $\|y - Cu_0\| = \min\{\|y - Cu\| : u \text{ is a solution of } C^*B(Cx-y) = 0\}$ if and only if $Qy = P_{B,R(C)} y$, i.e. $Cu_0 = Cu_y$, or equivalently, $u_0 \in u_y + N(C)$. □

Given $y \in \mathcal{K}$, the next proposition presents a parametrization of the set of solutions of the normal equation $C^*B(Cx-y) = 0$ in terms of the minimal solution of $C^*B(Cx-y) = 0$. For this purpose observe that, if $(B, R(C))$ is compatible, then $C^{-1}(\mathcal{N}) = N(C^*BC) = N(BC)$.

Proposition 9. *Let $C \in CR(\mathcal{H}, \mathcal{K})$, $B \in L(\mathcal{K})^s$ and $y \in \mathcal{K}$. If $(B, R(C))$ is compatible then the set of solutions of $C^*B(Cx-y) = 0$ is the affine manifold*

$$u_y + N(BC).$$

Proof. *Proof. If $(B, R(C))$ is compatible then $N(C^*BC) = N(BC)$ and, given $y \in \mathcal{K}$, $u_y = C^\dagger P_{B,R(C)} y$ is a solution of the normal equation $C^*B(Cx-y) = 0$. Then, by Proposition 6, the set of solutions of $C^*B(Cx-y) = 0$ coincides with $u_y + N(BC)$.* □

Corollary 5. *Let $C \in CR(\mathcal{H},\mathcal{K})$, $B \in L(\mathcal{K})^s$ and $y \in \mathcal{K}$. Suppose that $(B, R(C))$ is compatible and that $R(C)$ is B-nonnegative. Then, the affine manifold*

$$u_y + N(BC),$$

is the set of B-LSS of the equation $Cx = y$.

Proof. *Proof. It is a consequence of Theorem 2 and Proposition 9.* □

Corollary 6. *Let $C \in CR(\mathcal{H},\mathcal{K})$, $B \in L(\mathcal{K})^s$ and $y \in \mathcal{K}$. Suppose that $(B, R(C))$ is compatible and $R(C)$ is B-nonnegative. Then, the following conditions are equivalent:*

1. *$\mathcal{N} = \{0\}$;*
2. *$R(C)$ is B-positive;*
3. *u_y is the unique B-LSS in $N(C)^\perp$ of $Cx = y$.*

Proof. *Proof. Since $C^*BC \in L(\mathcal{H})^+$, $Cx \in \mathcal{N}$ if and only if $\langle Cx, Cx \rangle_B = 0$. Therefore, item (i) is equivalent to item (ii). On the other hand, item (i) is equivalent to item (iii) by Corollary 5.* □

In [23], given $B \in L(\mathcal{H})^s$, a (closed) B-nonnegative subspace $\mathcal{S}$ of $\mathcal{H}$ and $y \in \mathcal{H}$, B. Hassibi et al. studied the problem of finding vectors $u \in \mathcal{S}$ such that

$$\langle u - y, u - y \rangle_B = \min_{s\in\mathcal{S}} \langle s - y, s - y \rangle_B = \min_{x\in\mathcal{H}} \langle P_{\mathcal{S}}x - y, P_{\mathcal{S}}x - y \rangle_B , \tag{10}$$

where $P_{\mathcal{S}}$ is the orthogonal projection onto $\mathcal{S}$. They were particularly interested in cases where there is a unique solution of the problem. By Lemma 1, $u \in \mathcal{S}$ satisfies Eq. (10) if and only if $y - u \in R(BP_{\mathcal{S}})^\perp$. It is easy to see that this condition holds if and only if

$$P_{\mathcal{S}}BP_{\mathcal{S}}u = P_{\mathcal{S}}By. \tag{11}$$

When $\mathcal{H}$ is finite dimensional, if for some $y_0 \in \mathcal{H}$ there exists a unique solution $u \in \mathcal{S}$ of (11), then the operator $P_{\mathcal{S}}BP_{\mathcal{S}}|_{\mathcal{S}}$ is injective. Therefore, $P_{\mathcal{S}}BP_{\mathcal{S}}|_{\mathcal{S}}$ is invertible, and there exists a unique solution of (11) for every $y \in \mathcal{H}$.

If $\mathcal{H}$ is an infinite dimensional Hilbert space this may be not true, since $P_{\mathcal{S}}BP_{\mathcal{S}}|_{\mathcal{S}}$ may be injective but not invertible. In fact, in this case, there is a solution of Eq. (11) for every $y \in \mathcal{H}$ if and only if the equation

$$(P_{\mathcal{S}}BP_{\mathcal{S}})X = P_{\mathcal{S}}B$$

admits a solution in $L(\mathcal{H})$, or equivalently, the pair $(B, \mathcal{S})$ is compatible; see [13, Proposition 3.3].

3.3 Minimizing in the set of B-LSS

Recall that, given $y \in \mathcal{K}$, the classical least squares problem (in Hilbert spaces) associated to the equation $Cx = y$ always admits a (unique) solution of minimal norm, namely, $u = C^\dagger y$.

In the following paragraphs we are going to study a minimization problem in the set of B-LSS of $Cx = y$. Let $C \in CR(\mathcal{H},\mathcal{K})$ and $B_2 \in L(\mathcal{K})^s$ such that $(B_2, R(C))$ is compatible and $R(C)$ is B_2-nonnegative. Fixed $y \in \mathcal{K}$, consider the set $u_y + N(B_2C)$ of B_2-LSS of the equation $Cx = y$. Given $B_1 \in L(\mathcal{H})^s$, we look for those B_2-LSS $w \in \mathcal{H}$ of $Cx = y$ such that

$$\langle w, w \rangle_{B_1} \le \langle u, u \rangle_{B_1} , \quad \text{for every } u \in u_y + N(B_2C).$$

In the following, denote by $\mathcal{N}_2 = R(C) \cap R(B_2C)^\perp$. Observe that, if $(B_2, R(C))$ is compatible then $C^{-1}(\mathcal{N}_2) = N(C^*B_2C) = N(B_2C)$.

Definition. *An element $w \in \mathcal{H}$ is a B_1B_2-least squares solution (hereafter B_1B_2-LSS) of $Cx = y$ if w is a B_2-LSS of $Cx = y$ and*

$$\langle w, w \rangle_{B_1} \le \langle u, u \rangle_{B_1} ,$$

for every B_2-LSS u of $Cx = y$.

Theorem 3. *Let $C \in CR(\mathcal{H},\mathcal{K})$, $B_1 \in L(\mathcal{H})^s$ and $B_2 \in L(\mathcal{K})^s$ such that $(B_2, R(C))$ is compatible and $R(C)$ is B_2-nonnegative. Then, there exists a B_1B_2-LSS of the equation $Cx = y$ for every $y \in \mathcal{K}$ if and only if $(B_1, N(B_2C))$ is compatible and $N(B_2C)$ is B_1-nonnegative. Moreover if $y \in \mathcal{K} \setminus R(B_2C)^\perp$, w is a B_1B_2-LSS of $Cx = y$ if and only if there exists $Q \in \mathcal{P}(B_1, N(B_2C))$ and $P \in \mathcal{P}(B_2, R(C))$ such that*

$$w = (I - Q)C^\dagger Py.$$

Proof. *Proof. Suppose that, for every $y \in \mathcal{K}$, there exists a B_1B_2-LSS $w_y \in \mathcal{H}$ of the equation $Cx = y$, i.e.*

$$\langle w_y, w_y \rangle_{B_1} = \min_{u \in u_y + N(B_2C)} \langle u, u \rangle_{B_1}, \tag{12}$$

where $u_y = C^\dagger P_{B_2,R(C)}y$ is the minimal B_2-LSS of the equation $Cx = y$. Let $z_y \in N(B_2C)$ such that $w_y = u_y + z_y = u_y + Ez_y$, where $E = P_{N(B_2C)}$ is the orthogonal projection onto $N(B_2C)$. Then,

$$\langle u_y + Ez_y, u_y + Ez_y \rangle_{B_1} = \langle w_y, w_y \rangle_{B_1} = \min_{z \in N(B_2C)} \langle u_y + z, u_y + z \rangle_{B_1} = \min_{x \in \mathcal{H}} \langle u_y + Ex, u_y + Ex \rangle_{B_1}.$$

Therefore, z_y is a B_1-LSS of the equation $Ex = -u_y$. Hence, by Lemma 1, $R(E) = N(B_2C)$ is B_1-nonnegative.

The compatibility of $(B_2, R(C))$ implies that $\{u_y : y \in \mathcal{K}\} = N(C)^\perp$, therefore the equation $Ex = z$ admits a B_1-LSS for every $z \in N(C)^\perp$. Also, the equation $Ex = z$ admits an exact solution (which is also a B_1-LSS) for every $z \in N(C)$ because $N(C) \subseteq R(E)$. Thus, $Ex = z$ admits a B_1-LSS for every $z \in \mathcal{K}$ and, applying Theorem 2, it follows that $(B_1, N(B_2C))$ is compatible.

Observe that if $y \notin R(B_2C)^\perp$ then $u_y \notin R(E) = N(B_2C)$: in fact, $u_y \in N(B_2C)$ if and only if $B_2P_{B_2,R(C)}y = 0$ and $N(B_2P_{B_2,R(C)}) = R(B_2P_{B_2,R(C)})^\perp = R(B_2C)^\perp$. Therefore, by Theorem 2, if $y \notin R(B_2C)^\perp$ then there exists $Q \in \mathcal{P}(B_1, N(B_2C))$ such that $Ez_y = -Qu_y$. So,

$$w_y = u_y + z_y = u_y + Ez_y = (I - Q)u_y = (I - Q)C^\dagger P_{B_2,R(C)}y.$$

Conversely, suppose that $(B_1, N(B_2C))$ is compatible, $N(B_2C)$ is B_1-nonnegative and let $Q \in \mathcal{P}(B_1, N(B_2C))$. If $P \in \mathcal{P}(B_2, R(C))$, observe that $(I - Q)C^\dagger P = (I - Q)C^\dagger P_{B_2,R(C)}$. Indeed, if $P \in \mathcal{P}(B_2, R(C))$, there exists $Z \in L(R(C)^\perp, \mathcal{N}_2)$ such that $P = P_{B_2,R(C)} + Z$ (see Theorem 1). Then, $(I - Q)C^\dagger P = (I - Q)C^\dagger P_{B_2,R(C)}$ because $(I - Q)C^\dagger Z = 0$.

Given $y \in \mathcal{K}$, consider $w = (I - Q)C^\dagger P_{B_2,R(C)}y$. Then, $w \in u_y + N(B_2C)$ and therefore, it is a B_2-LSS of $Cx = y$ (see Corollary 5). On the other hand, given any B_2-LSS u of $Cx = y$, there exists $z \in N(B_2C)$ such that $u = u_y + z = u_y + Qz$ and

$$\begin{aligned}
\langle u, u \rangle_{B_1} &= \langle (I - Q)u_y + Q(u_y + z), (I - Q)u_y + Q(u_y + z) \rangle_{B_1} = \\
&= \langle w, w \rangle_{B_1} + 2\,\mathrm{Re}\,\langle (I - Q)u_y, Q(u_y + z) \rangle_{B_1} + \langle Q(u_y + z), Q(u_y + z) \rangle_{B_1} = \\
&= \langle w, w \rangle_{B_1} + \langle Q(u_y + z), Q(u_y + z) \rangle_{B_1} \geq \langle w, w \rangle_{B_1},
\end{aligned}$$

because $R(Q) \perp_{B_1} N(Q)$ and $R(Q) = N(B_2C)$ is B_1-nonnegative. Thus, w is a B_1B_2-LSS of $Cx = y$. □

It follows from the proof of Theorem 3 that, given $y \in \mathcal{K} \setminus R(B_2C)^\perp$, the set of B_1B_2-LSS of the equation $Cx = y$ is

$$\{(I - Q)C^\dagger P_{B_2,R(C)}y : \; Q \in \mathcal{P}(B_1, N(B_2C))\}. \tag{13}$$

On the other hand, if $y \in R(B_2C)^\perp$ then $u_y \in N(B_2C)$. So, given $y \in R(B_2C)^\perp$, the problem of finding a B_1B_2-LSS of $Cx = y$ translates into finding a vector $u \in N(B_2C)$ such that $\langle u, u \rangle_{B_1} = 0$ (see Eq. (12)). Hence, if $y \in R(B_2C)^\perp$, the set of B_1B_2-LSS is $\mathcal{N}_1 = N(B_1) \cap N(B_2C)$.

Corollary 7. *Let $C \in CR(\mathcal{H},\mathcal{K})$, $B_1 \in L(\mathcal{H})^s$ and $B_2 \in L(\mathcal{K})^s$ such that $(B_2, R(C))$ is compatible and $R(C)$ is B_2-nonnegative. Suppose that $(B_1, N(B_2C))$ is compatible, $N(B_2C)$ is B_1-nonnegative and $\mathcal{N}_1 = \{0\}$. If $y \in \mathcal{K}$ then, w is a B_1B_2-LSS of $Cx = y$ if and only if*

$$w = (I - P_{B_1,N(B_2C)})C^\dagger P_{B_2,R(C)}\, y.$$

As mentioned before, if $y \in R(B_2C)^\perp$, Eq. (13) describes a proper subset of the set of B_1B_2-LSS of $Cx = y$. However, it contains the minimal norm B_1B_2-LSS of $Cx = y$.

Proposition 10. *Let $C \in CR(\mathcal{H}, \mathcal{K})$, $B_1 \in L(\mathcal{H})^s$ and $B_2 \in L(\mathcal{K})^s$ such that $(B_2, R(C))$ is compatible and $R(C)$ is a B_2-nonnegative subspace of $\mathcal{K}$. Suppose that $(B_1, N(B_2C))$ is also compatible and $N(B_2C)$ is a B_1-nonnegative subspace of $\mathcal{H}$. If $y \in \mathcal{K}$ then $v_y = (I - P_{B_1,N(B_2C)})C^\dagger P_{B_2,R(C)}y$ is the unique minimal norm element of the set of B_1B_2-LSS of $Cx = y$.*

Proof. *Proof. If $y \in \mathcal{K} \setminus R(B_2C)^\perp$ and v is a B_1B_2-LSS of $Cx = y$, there exists $Q \in \mathcal{P}(B_1, N(B_2C))$ such that $v = (I - Q)u_y$ and, by Proposition 2,*

$$\|v\| = \|(I - Q)u_y\| \geq \|(I - P_{B_1,N(B_2C)})u_y\| = \|v_y\|.$$

Therefore, $v_y = (I - P_{B_1,N(B_2C)})C^\dagger P_{B_2,R(C)}y$ is the unique minimal norm element of the set of B_1B_2-LSS of $Cx = y$.

On the other hand, if $y \in R(B_2C)^\perp$ then $u_y \in N(B_2C)$. Therefore, v_y is the minimal norm element of the set of B_1B_2-LSS of $Cx = y$ because $v_y = 0$. □

4 Weighted generalized inverses

Given $C \in CR(\mathcal{H}, \mathcal{K})$ and $B_2 \in L(\mathcal{K})^s$, suppose that $(B_2, R(C))$ is compatible and $R(C)$ is B_2-positive. Then, by Corollary 6, $\mathcal{N}_2 = \{0\}$ so that $N(B_2C) = N(C)$. Consider $B_1 \in L(\mathcal{H})^s$ such that $(B_1, N(C))$ is compatible and $N(C)$ is B_1-nonnegative. Therefore, by Theorem 3, $w \in \mathcal{H}$ is a B_1B_2-LSS of the equation $Cx = y$ if and only if

$$w = (I - Q)C^\dagger P_{B_2,R(C)}y,$$

where $Q \in \mathcal{P}(B_1, N(C))$. This leads us to study the set

$$\{(I - Q)C^\dagger P : \ Q \in \mathcal{P}(B_1, N(C)) \text{ and } P \in \mathcal{P}(B_2, R(C))\}.$$

Let $Q \in \mathcal{P}(B_1, N(C))$, $P \in \mathcal{P}(B_2, R(C))$ and consider $D = (I - Q)C^\dagger P$. Then, using that $C^\dagger \ CC^\dagger P = P$ and $(I - Q)C^\dagger C = (I - Q)$ it is easy to see that

$$CD = P \text{ and } DC = I - Q.$$

Therefore, D is a weighted generalized inverse of C in the following sense:

Definition. *Given $C \in CR(\mathcal{H}, \mathcal{K})$ and weights $B_1 \in L(\mathcal{H})^s$ and $B_2 \in L(\mathcal{K})^s$, $D \in L(\mathcal{K}, \mathcal{H})$ is a* weighted generalized inverse *of C if D is a solution of*

$$CXC = C, \quad XCX = X, \quad B_1(XC) = (XC)^* B_1, \quad B_2(CX) = (CX)^* B_2. \tag{14}$$

The above equations can be seen as an extension of the previous definitions given for positive weights by Eldén [18] (in finite dimensional spaces) and by G. Corach and A. Maestripieri [15] (in infinite dimensional Hilbert spaces).

This section is devoted to present conditions for the existence of weighted generalized inverses (respect to selfadjoint weights B_1 and B_2) of a closed range operator C and to characterize those inverses in terms of the Moore-Penrose inverse of C and the set of B_i-selfadjoint projections ($i = 1, 2$). The proof of Theorem 4 is omitted since it is analogous to the proofs given in [15, Section 3] for positive weights.

Theorem 4. *Given $C \in CR(\mathcal{H}, \mathcal{K})$, $B_1 \in L(\mathcal{H})^s$ and $B_2 \in L(\mathcal{K})^s$ there exists $D \in L(\mathcal{K}, \mathcal{H})$ such that D is a solution of (14) if and only if $(B_1, N(C))$ and $(B_2, R(C))$ are compatible pairs. In this case,*

$$GI(C, B_1, B_2) = \{(I - Q)C^\dagger P : \ Q \in \mathcal{P}(B_1, N(C)) \text{ and } P \in \mathcal{P}(B_2, R(C))\}$$

is the set of all bounded linear solutions of (14).

In order to characterize the set of B_1B_2-LSS of $Cx = y$ we need two technical lemmas.

Lemma 2. *Let $C \in CR(\mathcal{H},\mathcal{K})$ and consider a closed subspace $\mathcal{S} \subseteq R(C)$. Then, $P_{\mathcal{S}}C \in CR(\mathcal{H},\mathcal{K})$ and*

$$(P_{\mathcal{S}}C)^\dagger = P_{N(P_{\mathcal{S}}C)^\perp}C^\dagger.$$

Proof. *Proof. Let $\tilde{C} = P_{\mathcal{S}}C$. Observe that $R(\tilde{C}) = \mathcal{S}$. Consider $T = P_{N(\tilde{C})^\perp}C^\dagger$. Then,*

$$\tilde{C}T = \tilde{C}P_{N(\tilde{C})^\perp}C^\dagger = \tilde{C}C^\dagger = P_{\mathcal{S}}CC^\dagger = P_{\mathcal{S}}P_{R(C)} = P_{\mathcal{S}} = P_{R(\tilde{C})}.$$

Therefore, $\tilde{C}T\tilde{C} = \tilde{C}$. Also, $(T\tilde{C})^2 = T(\tilde{C}T\tilde{C}) = T\tilde{C}$, i.e. $T\tilde{C}$ is a projection. Since $N(\tilde{C}) \subseteq N(T\tilde{C}) \subseteq N(\tilde{C}T\tilde{C}) = N(\tilde{C})$ and $R(T\tilde{C}) \subseteq R(T) \subseteq N(\tilde{C})^\perp$, then $T\tilde{C} = P_{N(\tilde{C})^\perp}$. Therefore, $T\tilde{C}T = T$. Hence, $(P_{\mathcal{S}}C)^\dagger = T = P_{N(P_{\mathcal{S}}C)^\perp}C^\dagger$. □

Lemma 3. *Let $C \in CR(\mathcal{H},\mathcal{K})$, $B_1 \in L(\mathcal{H})^s$ and $B_2 \in L(\mathcal{K})^s$. If $Q \in \mathcal{P}(B_1, N(B_2C))$ then*

$$(I-Q)C^\dagger = (I-Q)\tilde{C}^\dagger,$$

where $\tilde{C} = P_{R(C)\ominus\mathcal{N}_2}C$.

Proof. *Proof. If $\tilde{C} = P_{R(C)\ominus\mathcal{N}_2}C$ then, by Lemma 2, $\tilde{C}^\dagger = P_{N(P_{R(C)\ominus\mathcal{N}_2}C)^\perp}C^\dagger = P_{N(B_2C)^\perp}C^\dagger$ because*

$$N(P_{R(C)\ominus\mathcal{N}_2}C) = C^{-1}(\mathcal{N}_2) = N(B_2C).$$

Furthermore, if $Q \in \mathcal{P}(B_1, N(B_2C))$ then $(I-Q)P_{N(B_2C)} = 0$. So, $(I-Q)\tilde{C}^\dagger = (I-Q)P_{N(B_2C)^\perp}C^\dagger = (I-Q)C^\dagger$. □

Proposition 11. *Let $C \in CR(\mathcal{H},\mathcal{K})$, $B_1 \in L(\mathcal{H})^s$ and $B_2 \in L(\mathcal{K})^s$ such that $(B_2, R(C))$ and $(B_1, N(B_2C))$ are compatible, $R(C)$ is B_2-nonnegative and $N(B_2C)$ is B_1-nonnegative. Given $y \in \mathcal{K} \setminus R(B_2C)^\perp$, the set of B_1B_2-LSS of the equation $Cx = y$ is given by*

$$\{Ty : T \in GI(P_{R(C)\ominus\mathcal{N}_2}C, B_1, B_2)\}.$$

Proof. *Proof. By Theorem 3 we know that given $y \in \mathcal{K} \setminus R(B_2C)^\perp$, the set of B_1B_2-LSS of $Cx = y$ is*

$$\{(I-Q)C^\dagger Py : \ Q \in \mathcal{P}(B_1, N(B_2C)), P \in \mathcal{P}(B_2, R(C))\}.$$

Given $P \in \mathcal{P}(B_2, R(C))$, let $E = P_{\mathcal{N}_2//R(C)\ominus\mathcal{N}_2+N(P)}$. If $\tilde{C} = P_{R(C)\ominus\mathcal{N}_2}C$ then $R(\tilde{C}) = R(C)\ominus\mathcal{N}_2$ and, by Proposition 1, it follows that $P = P_{B_2,R(\tilde{C})} + E$. On the other hand, for any $Q \in \mathcal{P}(B_1, N(B_2C))$, $(I-Q)C^\dagger E = 0$, because $R(C^\dagger E) \subseteq N(B_2C)$. Then the set of B_1B_2-LSS of $Cx = y$ can be written as

$$\{(I-Q)C^\dagger P_{B_2,R(\tilde{C})}y : \ Q \in \mathcal{P}(B_1, N(B_2C))\}.$$

Applying Lemma 3, and the fact that $\mathcal{P}(B_2, R(\tilde{C})) = \{P_{B_2,R(\tilde{C})}\}$, it is easy to see that this set coincides with $\{Ty : T \in GI(\tilde{C}, B_1, B_2)\}$. □

Finally, we characterize the set of solutions of the normal equation $C^*B(Cx-y) = 0$ by means of an operator solution of the equations $CXC = C$ and $BCX = (CX)^*B$.

Proposition 12. *Let $C \in CR(\mathcal{H},\mathcal{K})$ and $B \in L(\mathcal{K})^s$ such that the pair $(B, R(C))$ is compatible. Given $y \in \mathcal{K} \setminus R(C)$, $u \in \mathcal{H}$ is a solution of the normal equation $C^*B(Cx-y) = 0$ if and only if there exists a solution $D \in L(\mathcal{K},\mathcal{H})$ of*

$$CXC = C, \quad BCX = (CX)^*B,$$

such that $Dy = u$.

Proof. *Proof. Given $y \in \mathcal{K} \setminus R(C)$, suppose that $u = Dy$, with $D \in L(\mathcal{K},\mathcal{H})$ satisfying $CDC = C$ and $BCD = (CD)^*B$. It is easy to see that $Q = CD$ is a B-selfadjoint projection. Furthermore, $R(Q) \subseteq R(C) = R(CDC) \subseteq R(Q)$ i.e. $Q \in \mathcal{P}(B, R(C))$. Then $Cu = CDy = Qy$ with $Q \in \mathcal{P}(B, R(C))$ and, by Proposition 7, u is a solution of the normal equation.*

Conversely, if $u \in \mathcal{H}$ is a solution of $C^*B(Cx-y) = 0$, there exists $Q \in \mathcal{P}(B, R(C))$ such that $Cu = Qy$. Then, $u = C^\dagger Qy + z$ where $z \in N(C)$. Consider an operator $T \in L(\mathcal{K},\mathcal{H})$ with $R(T) \subseteq N(C)$ such that $Ty = z$, and define $D = C^\dagger Q + T$. Since $CD = C(C^\dagger Q + T) = Q$ it is easy to see that D is a solution of

$$CXC = C, \quad BCX = (CX)^*B,$$

and $Dy = C^\dagger Qy + Ty = u$. □

5 A min-max solution of the equation $Cx = y$

In Section 3 we established a relationship between the B-LSS of $Cx = y$ and the solutions of the normal equation $C^*B(Cx - y) = 0$. More precisely, supposing that $R(C)$ is a B-nonnegative subspace of $\mathcal{K}$, we proved that $u \in \mathcal{H}$ is a B-LSS of $Cx = y$ if and only if it is a solution of the normal equation $C^*B(Cx - y) = 0$. We also showed that the normal equation $C^*B(Cx - y) = 0$ admits a solution for every $y \in \mathcal{K}$ if and only if the pair $(B, R(C))$ is compatible, and that the solutions of the normal equation are related to the elements of $\mathcal{P}(B, R(C))$.

If $R(C)$ is B-indefinite, there no longer exist B-LSS of the equation $Cx = y$ (see Lemma 1). However, the solutions of the normal equation $C^*B(Cx - y) = 0$ can be related to the solutions of a min-max problem.

Given $C \in CR(\mathcal{H}, \mathcal{K})$, $B \in L(\mathcal{K})^s$ and $y \in \mathcal{K} \setminus R(C)$, suppose that $(B, R(C))$ is compatible and $u \in \mathcal{H}$ is a solution of $C^*B(Cx - y) = 0$. Then, by Proposition 7, there exists $Q \in \mathcal{P}(B, R(C))$ such that $Cu = Qy$ and, by Proposition 4,

$$\langle\, y - Cu, y - Cu \,\rangle_B = \langle\, (I - Q)y, (I - Q)y \,\rangle_B = \min_{s \in \mathcal{S}_+} \max_{t \in \mathcal{S}_-} \langle\, y - (s + t), y - (s + t) \,\rangle_B \,, \tag{15}$$

where $R(C) \ominus \mathcal{N} = R(C) \ominus N(B) = \mathcal{S}_+ \oplus \mathcal{S}_-$ is the decomposition given in Proposition 3. Observe that this representation depends on the existence of a B-orthogonal decomposition of $R(C)$.

Definition. *Let $C \in CR(\mathcal{H}, \mathcal{K})$ and $B \in L(\mathcal{K})^s$ such that $R(C)$ is B-decomposable. Given $y \in \mathcal{K}$, a vector $u \in \mathcal{H}$ is a B-min-max solution (B-MMS) of the equation $Cx = y$ if*

$$\langle\, y - Cu, y - Cu \,\rangle_B = \min_{s \in \mathcal{S}_+} \max_{t \in \mathcal{S}_-} \langle\, y - (s + t), y - (s + t) \,\rangle_B = \max_{t \in \mathcal{S}_-} \min_{s \in \mathcal{S}_+} \langle\, y - (s + t), y - (s + t) \,\rangle_B \,, \tag{16}$$

where $R(C) = \mathcal{N} \dot{+} \mathcal{S}_+ \dot{+} \mathcal{S}_-$ is a decomposition as in Definition 2.

Remark 3. *Given a decomposition $R(C) = \mathcal{N} \dot{+} \mathcal{S}_+ \dot{+} \mathcal{S}_-$ as above, observe that $\mathcal{N} \dot{+} \mathcal{S}_+$ is a closed subspace and*

$$\max_{t \in \mathcal{S}_-} \min_{s \in \mathcal{S}_+} \langle\, y - (s + t), y - (s + t) \,\rangle_B = \max_{t \in \mathcal{S}_-} \min_{x \in \mathcal{S}_+ + \mathcal{N}} \langle\, y - (x + t), y - (x + t) \,\rangle_B \,. \tag{17}$$

Indeed, for a fixed $t \in \mathcal{S}_-$, $\min\limits_{x \in \mathcal{S}_+ + \mathcal{N}} \langle\, y - (x + t), y - (x + t) \,\rangle_B \leq \min\limits_{s \in \mathcal{S}_+} \langle\, y - (s + t), y - (s + t) \,\rangle_B$ because $\mathcal{S}_+ \subseteq \mathcal{S}_+ + \mathcal{N}$. On the other hand, if $w \in \mathcal{S}_+ + \mathcal{N}$ satisfies

$$\langle\, y - (w + t), y - (w + t) \,\rangle_B = \min_{x \in \mathcal{S}_+ + \mathcal{N}} \langle\, y - (x + t), y - (x + t) \,\rangle_B \,,$$

then, by Lemma 1, $y - (w + t)$ is B-orthogonal to $\mathcal{S}_+ + \mathcal{N}$, that is $\langle\, y - (w + t), x \,\rangle_B = 0$ for every $x \in \mathcal{S}_+ + \mathcal{N}$. Suppose that $w = s_0 + n_0$, with $s_0 \in \mathcal{S}_+$ and $n_0 \in \mathcal{N}$. Hence, $\langle\, (y - t) - w, n_0 \,\rangle_B = 0$ and $\langle\, n_0, n_0 \,\rangle_B = 0$ because $n_0 \in \mathcal{N}$. Therefore,

$$\langle\, y - (w + t), y - (w + t) \,\rangle_B = \langle\, (y - t) - s_0, (y - t) - s_0 \,\rangle_B \geq \min_{s \in \mathcal{S}_+} \langle\, y - (t + s), y - (t + s) \,\rangle_B \,.$$

So, considering the maximum over the vectors $t \in \mathcal{S}_-$, Eq. (17) follows.

Also, notice that the decomposition $R(C) = \mathcal{N} \dot{+} \mathcal{S}_+ \dot{+} \mathcal{S}_-$ in Definition 2 is not necessarily unique. However, the following result shows that the B-MMS definition is independent of the selected decomposition. Furthermore, it characterizes the B-MMS of the equation $Cx = y$. Along the following paragraphs, $\mathcal{M}$ denotes the set of B-neutral vectors in $\mathcal{K}$.

Theorem 5. *Let $C \in CR(\mathcal{H}, \mathcal{K})$ and $B \in L(\mathcal{K})^s$ such that $R(C)$ is B-decomposable. Given $y \in \mathcal{K}$, $u \in \mathcal{H}$ is a B-MMS of $Cx = y$ if and only if $u \in u_0 + C^{-1}(\mathcal{M})$, where $u_0 \in \mathcal{H}$ is a solution of the normal equation $C^*B(Cx - y) = 0$.*

Proof. *Proof.* *Suppose that* $u \in \mathcal{H}$ *is a B-MMS of* $Cx = y$*, Then,*

$$\langle\, y - Cu, y - Cu \,\rangle_B = \max_{t \in \mathcal{S}_-} \min_{s \in \mathcal{S}_+} \langle\, y - (s+t), y - (s+t) \,\rangle_B \,,$$

where $R(C) = \mathcal{N} \dot{+} \mathcal{S}_+ \dot{+} \mathcal{S}_-$ *is a decomposition as in Definition 2. Fixed* $t \in \mathcal{S}_-$*, by Lemma 1, it follows that* $\min_{s \in \mathcal{S}_+} \langle\, (y-t) - s, (y-t) - s \,\rangle_B$ *is attained at* $s_0(t) \in \mathcal{S}_+$ *if and only if* $\langle\, y - t - s_0(t), x \,\rangle_B = 0$ *for every* $x \in \mathcal{S}_+$*. Then,* $\langle\, y - s_0(t), x \,\rangle_B = \langle\, y - t - s_0(t), x \,\rangle_B = 0$ *for every* $x \in \mathcal{S}_+$ *because* $\mathcal{S}_-$ *is B-orthogonal to* $\mathcal{S}_+$*. Therefore,* $s_0(t) = s_0$ *is the unique vector in* $\mathcal{S}_+$ *which satisfies*

$$\langle\, (y-t) - s_0, (y-t) - s_0 \,\rangle_B = \min_{s \in \mathcal{S}_+} \langle\, (y-t) - s, (y-t) - s \,\rangle_B \,.$$

Hence, $\langle\, y - Cu, y - Cu \,\rangle_B = \max_{t \in \mathcal{S}_-} \langle\, (y - s_0) - t, (y - s_0) - t \,\rangle_B$*.*

Analogously, by Remark 2, this maximum is attained at $t_0 \in \mathcal{S}_-$ *if and only if* $\langle\, y - s_0 - t_0, t \,\rangle_B = 0$ *for every* $t \in \mathcal{S}_-$*. Moreover, by Remark 3,* $\langle\, y - s_0 - t_0, x \,\rangle_B = 0$ *for every* $x \in \mathcal{S}_- + \mathcal{N}$*.*

Let $u_0 \in \mathcal{H}$ *such that* $Cu_0 = s_0 + t_0$*. Since* $R(C) = \mathcal{N} \dot{+} \mathcal{S}_+ \dot{+} \mathcal{S}_-$*, it is easy to see that* $\langle\, y - Cu_0, z \,\rangle_B = 0$ *for every* $z \in R(C)$*. Hence,* u_0 *is a solution of the normal equation* $C^*B(Cx - y) = 0$ *and*

$$\langle\, y - Cu, y - Cu \,\rangle_B = \langle\, y - Cu_0, y - Cu_0 \,\rangle_B \,.$$

Since $\langle\, y - Cu_0, z \,\rangle_B = 0$ *for every* $z \in R(C)$*, then*

$$\begin{aligned}
\langle\, y - Cu_0, y - Cu_0 \,\rangle_B &= \langle\, y - Cu, y - Cu \,\rangle_B = \\
&= \langle\, (y - Cu_0) + (Cu_0 - Cu), (y - Cu_0) + (Cu_0 - Cu) \,\rangle_B = \\
&= \langle\, y - Cu_0, y - Cu_0 \,\rangle_B + \langle\, Cu_0 - Cu, Cu_0 - Cu \,\rangle_B \,,
\end{aligned}$$

and the above equation holds if and only if $\langle\, Cu_0 - Cu, Cu_0 - Cu \,\rangle_B = 0$*. Therefore,* $u \in u_0 + C^{-1}(\mathcal{M})$*.*

Conversely, consider a solution $u_0 \in \mathcal{H}$ *of the normal equation* $C^*B(Cx - y) = 0$ *and let* $u \in u_0 + C^{-1}(\mathcal{M})$*. Then,* $\langle\, Cu - y, Cu - y \,\rangle_B = \langle\, Cu_0 - y, Cu_0 - y \,\rangle_B$*.*

Suppose that $R(C) = \mathcal{N} \dot{+} \mathcal{S}_+ \dot{+} \mathcal{S}_-$ *is a decomposition as in Definition 2, and let* $x_0 \in \mathcal{S}_+ + \mathcal{N}$ *and* $t_0 \in \mathcal{S}_-$ *such that* $Cu_0 = x_0 + t_0$*. Since* $\langle\, y - Cu_0, x \,\rangle_B = 0$ *for every* $x \in R(C)$ *and* $\mathcal{S}_+ + \mathcal{N}$ *is B-orthogonal to* $\mathcal{S}_-$*, it is easy to see that* $\langle\, y - x_0 - t, x \,\rangle_B = \langle\, y - x - t_0, t \,\rangle_B = 0$ *for every* $x \in \mathcal{S}_+ + \mathcal{N}$ *and* $t \in \mathcal{S}_-$*. Then, considering the equation* $Px = y - t_0$ *(where* P *is the orthogonal projection onto* $\mathcal{S}_+ + \mathcal{N}$*) it follows by Lemma 1 that*

$$\begin{aligned}
\langle\, y - Cu, y - Cu \,\rangle_B &= \langle\, y - x_0 - t_0, y - x_0 - t_0 \,\rangle_B = \min_{x \in \mathcal{H}} \langle\, y - t_0 - Px, y - t_0 - Px \,\rangle_B = \\
&= \min_{s \in \mathcal{S}_+ + \mathcal{N}} \langle\, y - t_0 - s, y - t_0 - s \,\rangle_B = \min_{s \in \mathcal{S}_+} \langle\, y - t_0 - s, y - t_0 - s \,\rangle_B = \\
&= \min_{s \in \mathcal{S}_+ + \mathcal{N}} \max_{t \in \mathcal{S}_-} \langle\, s + t - y, s + t - y \,\rangle_B \,.
\end{aligned}$$

□

Corollary 8. *Let* $C \in CR(\mathcal{H}, \mathcal{K})$ *and* $B \in L(\mathcal{K})^s$ *such that* $R(C)$ *is B-decomposable. Given* $y \in \mathcal{K}$*, if* $(B, R(C))$ *is compatible then,* u *is a B-MMS of* $Cx = y$ *if and only if* $Cu \in P_{B,R(C)}y + \mathcal{M}$*.*

Proof. *Proof. Suppose that* $(B, R(C))$ *is compatible. If* u *is a B-MMS of* $Cx = y$ *then* $u = u_0 + z$*, where* u_0 *is a solution of the normal equation* $C^*B(Cx - y) = 0$ *and* $z \in C^{-1}(\mathcal{M})$*. By Corollary 5,* $u_0 = C^\dagger P_{B,R(C)}y + n$ *with* $n \in N(BC) = C^{-1}(\mathcal{N})$*. Hence,* $Cu = Cu_0 + Cz = P_{B,R(C)}y + C(n + z) \in P_{B,R(C)}y + \mathcal{M}$*.*

Conversely, if $Cu = P_{B,R(C)}y + z$ *and* $z \in \mathcal{M}$ *then* $u = C^\dagger P_{B,R(C)}y + (C^\dagger z + P_{N(C)}u) \in u_y + C^{-1}(\mathcal{M})$*, where* u_y *is the minimal solution of the normal equation* $C^*B(Cx - y) = 0$*. Then, by Theorem 5,* u *is a B-MMS of the equation* $Cx = y$*.* □

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