A note on perturbations of Fusion Frames

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Abstract

In this work, we consider some relationships between a closed range operator T and a fusion frame $\mathcal{W} = (W_i, w_i)_{i\in I}$ for a Hilbert space $\mathcal{H}$ that provides that the sequence $(\overline{T(W_i)}, v_i)_{i\in I}$ is a fusion frame sequence for $\mathcal{H}$, if we consider a suitable family of weights $\{v_i\}_{i\in I}$. This (sufficient) condition generalizes some previous work in the subject.

Keywords: Fusion frames, Linear perturbation of Fusion Frames, Angle between subspaces
2000 MSC: 42C15, 15A60

1. Introduction

Let $\mathcal{H}$ be a separable Hilbert space. Let $\{W_i\}_{i\in I}$ be a family of closed subspaces of $\mathcal{H}$ and $\{w_i\}_{i\in I}$ be a family of positive weights. The pair $\mathcal{W} = (W_i, w_i)_{i\in I}$ is a **fusion frame** for $\mathcal{H}$ if there exists $A, B > 0$ which satisfy that

$$A\|f\|^2 \le \sum_{i\in I} w_i^2 \|P_{W_i} f\|^2 \le B\|f\|^2 \text{ for every } f \in \mathcal{H} \tag{1}$$

where by P_W we denote the orthogonal projection onto W. We say that $\mathcal{W} = (W_i, w_i)_{i\in I}$ is a **fusion frame sequence** for $\mathcal{H}$ if it is a fusion frame for a closed proper subspace of $\mathcal{H}$.

Fusion frames were introduced by P. Casazza and G. Kutyniok in [3], under the name of *frame of subspaces*. They were motivated by the problem of finding a way to join together local frames in order to get global frames. These authors were also studying methods to decompose a frame into a family of frame sequences. Both problems lead them to define the concept of frame of subspaces, which clearly generalizes classical vector frames. Namely, given a frame $\{f_i\}_{i\in I}$ for $\mathcal{H}$, then $(\text{span}\{f_i\}, \|f_i\|)_{i\in I}$ constitutes a fusion frame (of one-dimensional subspaces) for $\mathcal{H}$.

During the last decade, fusion frame theory has been a fast-growing area of research, driven by several applications such as sensor networks, neurology, coding theory, among others.

As well as for vector frames, there are some bounded operators associated to a fusion frame. First, we set the Hilbert space $\mathcal{K}_{\mathcal{W}} := \bigoplus_{i\in I} W_i$ (endowed with the ℓ^2 norm) and we define the **synthesis operator** $T_{\mathcal{W}} : \mathcal{K}_{\mathcal{W}} \to \mathcal{H}$ and the **analysis operator** $T^*_{\mathcal{W}} : \mathcal{H} \to \mathcal{K}_{\mathcal{W}}$ of $(W_i, w_i)_{i\in I}$ given by:

$$T_{\mathcal{W}}(g) = \sum_{i\in I} w_i\, g_i \text{ and } T^*_{\mathcal{W}}(f) = \{w_i\, P_{W_i} f\}_{i\in I}.$$

It is clear that some properties and problems in classical vector frames need to be treated in a completely different way if we are in the context of fusion frames. For example, problems of design of frames, such as the existence and construction of finite frames with prescribed norms need to be attacked with different tools when they are posed for fusion frames (see [2, 4, 13]).

Also, the equivalence between frames and epimorphisms useful in abstract frame theory, is not longer true for fusion frames (see for example [14]). Related to this, there is another interesting issue to study in

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the fusion frame setting: linear perturbation of fusion frames. That is, the goal is to find conditions on a bounded linear operator T in $\mathcal{H}$ and a fusion frame $\mathcal{W} = (W_i, w_i)_{i\in I}$ in order that the perturbed sequence $(\overline{T(W_i)}, w_i)_{i\in I}$ is a fusion frame for $T(\mathcal{H})$. In [3], [10], [8] and [14] it is shown that for an invertible operator T it is always true that $(\overline{T(W_i)}, w_i)_{i\in I}$ is a fusion frame for $\mathcal{H}$. Later, in [9] and [11] the authors proved that the perturbed sequence is a fusion frame sequence (or fusion frame if T is surjective), by adding some extra conditions (such as $T^*T(W_i) \subset W_i$, for example). As it is noticed by the authors in [10], these hypothesis can be described in a geometrical way by means of the *gap* between $\overline{T^*T(W_i)}$ (or $\overline{P_{R(T^*)}(W_i)}$) and W_i. The gap between two subspaces M and N, denoted by $\delta(M,N)$ is the supremum among the distances $d(x,N)$, where x is in M and $\|x\| = 1$ (see [12] for results and properties of the gap). Namely, in [10], the authors show that if the gaps $\delta(\overline{T^*T(W_i)}, W_i)$ are uniformely bounded above by a constant $c < 1$ then the sequence $(\overline{T(W_i)}, w_i)_{i\in I}$ is a fusion frame sequence for $\mathcal{H}$. In the same article, the authors presents alternative sufficient conditions which are related to the previous one, replacing the Grammian operator T^*T by the orthogonal projection onto the range of T^* (see [10, Cor 2.14]).

In this note, we follow the techniques and tools used in [14] to establish a sufficient condition that slightly generalizes those found in [10] using the notion of angle between subspaces.

2. Preliminaries

We shall consider different notions that provides some information about the geometrical position between two subspaces. They are the Friedrichs and Dixmier angles (angle and minimal angle respectively) and the gap defined by Kato. They will play a central role in the study of perturbation of fusion frames.

Definition 2.1. *Given two closed subspaces M and N of a Hilbert space $\mathcal{H}$, the* ***angle*** *between M and N is the angle in $[0, \pi/2]$ whose cosine is defined by*

$$c\,(\,M,\,N\,) = \sup\{\,|\,\langle x, y\rangle\,| :\ x \in M \ominus N,\ y \in N \ominus M \text{ and } \|x\| = \|y\| = 1\}.$$

We get the ***minimal angle*** *between M and N as the angle in $[0, \pi/2]$ whose cosine is defined by*

$$c_0\,(\,M,\,N\,) = \sup\{\,|\,\langle x, y\rangle\,| :\ x \in M,\ y \in N \text{ and } \|x\| = \|y\| = 1\}.$$

We list below some of the properties that satisfy these constants:

Proposition 2.2. *Let M and N be closed subspaces of $\mathcal{H}$. Then*

1. $c\,(\,M,\,N\,) = c\,(\,N,\,M\,) = c\left(\,M^\perp,\,N^\perp\,\right)$
2. $c\,(\,M,\,N\,) = c\,(\,M,\,N \ominus M\,) = c_0\,(\,M,\,N \ominus M\,)$
3. $c\,(\,M,\,N\,) = \|P_M P_N P_{(M\cap N)^\perp}\|$
4. $c_0\,(\,M,\,N\,) = \|P_M P_N\|$

The minimal angle is also related with the **gap** between closed subspaces of $\mathcal{H}$:

Definition 2.3. *Let M and N be closed subspaces of $\mathcal{H}$. The gap between M and N is defined as*

$$\delta(M,N) = \sup_{x\in M,\,\|x\|=1} \operatorname{dist}(x,N) = \|P_{N^\perp}P_M\| = c_0\left(\,N^\perp,\,M\,\right).$$

The following result shows the closed connection between angles and closed range operators.

Proposition 2.4. *Let $A, B \in L(\mathcal{H})$ be closed range operators. Then, AB has closed range if and only if* $c\,(\,R(B), N(A)\,) < 1$.

For more details and properties we refer the reader to the work by Deutsch [7].

For an operator $T \in L(\mathcal{H})$ we define its *reduced minimum modulus* by

$$\gamma(T) := \inf\{\|Tx\| \mid \|x\| = 1\ ,\ x \in N(T)^\perp\}. \tag{2}$$

It is well known that $\gamma(T) = \gamma(T^*) = \gamma(T^*T)^{1/2} = \|T^\dagger\|^{-1}$, where $T^\dagger$ is the Moore-Penrose pseudoinverse. Also, it can be shown that an operator T has closed range if and only if $\gamma(T) > 0$.

The following proposition will be useful for our results. It relates the notion of angle and reduced minimum modulus of a closed range operator:

Proposition 2.5. *(Prop. 4.14, [1]) Let $T \in L(\mathcal{H})$ a closed range operator in $\mathcal{H}$, W a closed subspace of $\mathcal{H}$ and P_M its orthogonal projection. Suppose that $c := c\,(\,N(T),\, W\,) < 1$. Then,*

$$\gamma(T)(1-c^2)^{1/2} \leq \gamma(TP_W) \leq \|T\|(1-c^2)^{1/2}. \tag{3}$$

Our next goal is to obtain inequalities that relate the gap $\delta(\overline{A(W)}, W)$ and the cosine of the Friedrichs' angle between $N(A)$ and W, where A is a closed range semidefinite positive operator and W is a closed subspace of $\mathcal{H}$.

Theorem 2.6. *Let $A \in L(\mathcal{H})$ be a semidefinite positive and closed range operator, W be a closed subspace of $\mathcal{H}$ such that $c\,(\,N(A),\, W\,) < 1$. Then*

$$c\,(\,N(A),\, W\,) \leq c\left(\,A^{-1}(W^\perp),\, W\,\right) \leq \left[1 - \frac{\gamma(A^{1/2})^4}{\|A\|^2}\left(1 - c\,(N(A), W)^2\right)\right]^{1/2}.$$

Proof. Notice that $A^{-1}(W^\perp) \cap W = N(A) \cap W$. Hence, since $N(A) \subset A^{-1}(W^\perp)$ we have that $N(A) \ominus (N(A) \cap W) \subset A^{-1}(W^\perp) \ominus (A^{-1}(W^\perp) \cap W)$ so the first inequality follows by Definition 2.1.

On the other hand, notice that $A^{1/2}P_W$ has closed range, since $N(A^{1/2}) = N(A)$ and $c\,(\,N(A),\, W\,) < 1$ by hypothesis. Therefore, by Prop. 2.5,

$$\gamma(P_W A P_W) = \gamma(A^{1/2}P_W)^2 \geq \gamma(A^{1/2})^2\left(1 - c^2\,(N(A), W)\right). \tag{4}$$

We also have that, by Prop. 2.5,

$$\gamma(P_W A P_W) \leq \|P_W A\|\left(1 - c\left(A^{-1}(W^\perp), W\right)^2\right)^{1/2} \leq \|A\|\left(1 - c\left(A^{-1}(W^\perp), W\right)^2\right)^{1/2} \tag{5}$$

since $N(P_W A) = A^{-1}(W^\perp)$.

Combining eqs. (4) and (5) we deduce that

$$c\left(\,A^{-1}(W^\perp),\, W\,\right) \leq \left[1 - \frac{\gamma(A^{1/2})^4}{\|A\|^2}\left(1 - c\,(N(A), W)^2\right)\right]^{1/2}.$$

□

Corollary 2.7. *Let $A \in L(\mathcal{H})^+$ be a closed range operator and $\{W_i\}_{i\in I}$ be a sequence of closed subspaces of $\mathcal{H}$. Then, the following are equivalent:*

1. $\inf_{i\in I} \gamma(AP_{W_i}) > 0$.
2. $\sup_{i\in I} c\,(\,N(A),\, W_i\,) < 1$.
3. $\sup_{i\in I} \delta\left(\overline{A(W_i)}, W_i\right) < 1$.

Proof. The equivalence between 1. and 2. follows from Prop. 2.5. In order to prove 2. ⇔ 3. we claim that any of them implies that $\delta\left(\overline{A(W_i)}, W_i\right) = c\left(\,A^{-1}(W_i^\perp),\, W_i\,\right)$, for all i in I. Then, is clear that the rest of the proof follows by Thm. 2.6.

Suppose first that $c\,(\,N(A),\, W_i\,) < 1$. Therefore $A(W_i) = \overline{A(W_i)}$, which implies

$$\begin{aligned}
\delta\left(\overline{A(W_i)}, W_i\right) &= \|P_{W_i^\perp} P_{A(W_i)}\| = \|P_{W_i^\perp} P_{A(W_i)} P_{(W_i^\perp \cap A(W_i))^\perp}\| \\
&= c\left(\,W_i^\perp,\, A(W_i)\,\right) = c\left(\,W_i,\, A^{-1}(W_i^\perp)\,\right) \\
&= c\left(\,A^{-1}(W_i^\perp),\, W_i\,\right).
\end{aligned}$$

On the other hand, if $\delta\left(\overline{A(W_i)}, W_i\right) < 1$ then $\overline{A(W_i)} \cap W_i^{\perp} = \{0\}$, therefore

$$\begin{aligned}\delta\left(\overline{A(W_i)}, W_i\right) = \|P_{W_i^{\perp}} P_{A(W_i)}\| &= c_0\left(\overline{A(W_i)},\, W_i^{\perp}\right)\\ &= c\left(\overline{A(W_i)},\, W_i^{\perp}\right)\\ &= c\left(W_i,\, A^{-1}(W_i^{\perp})\right).\end{aligned}$$

□

3. Fusion frame perturbation

In this section we apply the results stated previously to operator perturbations of fusion frames to obtain a sufficient condition that generalizes the results stated in [10] (see Cor. 2.10 and 2.12 and Thm. 3.4). We show that these conditions can be reduced to check a compatibility relationship between $N(T)$ and the family $\{W_i\}_{i\in I}$.

We need the following theorem, stated in [14], which can be seen as a result concerning operator perturbation of orthonormal basis of subspaces. In order to make the paper clear and self-contained, we state it in a general form, using perturbations by closed range operators instead of surjective operators. It is worth to mention that the proof follows the same lines as the proof of the cited result. Notice that the family of weights of the perturbed sequence can vary.

Theorem 3.1. *[14] Let $\mathcal{E} = \{E_i\}_{i\in I}$ be an orthonormal basis of subspaces for a Hilbert space $\mathcal{K}$ and let $T \in L(\mathcal{K}, \mathcal{H})$ be a closed range operator. Suppose that $0 < c := \inf_{i\in I} \frac{\gamma(TP_{E_i})^2}{\|TP_{E_i}\|^2}$.*

Let $0 < A \le B < \infty$ such that, $\frac{A}{B} \le c$ for all $i \in I$.

Let $w = \{w_i\}_{i\in I} \in \ell^\infty_+(I)$ such that $\frac{\|TP_{E_i}\|^2}{B} \le w_i^2 \le \frac{\gamma(TP_{E_i})^2}{A}$ for all $i \in I$.

Then $T(\mathcal{E}) := (T(E_i), w_i)_{i\in I}$ is a fusion frame sequence for $\mathcal{H}$. Moreover, the optimal frame bounds for the sequence satisfy the inequalities

$$\frac{\gamma(T)^2}{B} \le A_{T(\mathcal{E})} \le \frac{\gamma(T)^2}{A} \quad \text{and} \quad \frac{\|T\|^2}{B} \le B_{T(\mathcal{E})} \le \frac{\|T\|^2}{A}.$$

From this result it can be easily deduced the following:

Theorem 3.2. *Let $\mathcal{W} = (W_i, w_i)_{i\in I}$ be a fusion frame for $\mathcal{H}$ and $T \in L(\mathcal{H})$ a closed range operator such that*

$$0 < c := \inf_{i\in I} \frac{\gamma(TP_{W_i})^2}{\|TP_{W_i}\|^2}. \tag{6}$$

Let $0 < A \le B < \infty$ such that $\frac{A}{B} \le c$ and suppose that the family of weights $v = \{v_i\}_{i\in I} \in \ell^\infty_+(I)$ satisfy

$$w_i^2 \frac{\|TP_{W_i}\|^2}{B} \le v_i^2 \le w_i^2 \frac{\gamma(TP_{W_i})^2}{A} \quad \text{for all } i \in I,$$

then $(T(W_i), v_i)_{i\in I}$ is a fusion frame sequence for $\mathcal{K}$.

Proof. Denote by E_j to the isometric copy of W_j in $\mathcal{K}_\mathcal{W} = \oplus_{i\in I} W_i$, hence $\{E_j\}_{j\in I}$ is an orthonormal basis of subspaces of $\mathcal{K}_\mathcal{W}$. Consider the composition of the synthesis operator $T_\mathcal{W} : \mathcal{K}_\mathcal{W} \to \mathcal{H}$ with T:

$$F(g) := TT_\mathcal{W}(g) = \sum_{i\in I} w_i\, T(g_i), g \in \mathcal{K}_\mathcal{W}.$$

Notice that $F \in L(\mathcal{K}_\mathcal{W}, \mathcal{H})$ is a closed range operator, since T is closed range and $TT_\mathcal{W}$ is surjective. It is clear that

$$\|FP_{E_i}\| = w_i \|TP_{W_i}\| \text{for all } i \in I. \tag{7}$$

On the other side, we claim that $\gamma(FP_{E_i}) = w_i\gamma(TP_{W_i}), i \in I$. Indeed, from the equality

$$N(FP_{E_i})^\perp = ((N(F) \cap E_i) \oplus E_i^\perp)^\perp = E_i \cap (N(F) \cap E_i)^\perp$$

it follows that every unitary vector $x \in N(FP_{E_i})^\perp$ has the form $x = (0, \ldots, x_i, \ldots, 0, \ldots)$ with $||x_i|| = 1$ and $x_i \in (N(T) \cap W_i)^\perp$.

Then, if $x \in N(FP_{E_i})^\perp$ with $\|x\| = 1$, $\|FP_{E_i}x\| = \|w_iTx_i\| = w_i\|TP_{W_i}x_i\|$, therefore

$$\gamma(FP_{E_i}) \geq w_i\gamma(TP_{W_i}). \tag{8}$$

The reverse inequality can be proven in a similar way: we take $x \in (N(T)\cap W_i)^\perp$ and the corresponding $\tilde{x} = (0, \ldots, x, \ldots, 0, \ldots)$, then $\tilde{x} \in (N(F) \cap E_i)^\perp$. Then we have that

$$\|TP_{W_i}x\| = 1/w_i\|F\tilde{x}\| = 1/w_i\|FP_{E_i}\tilde{x}\| \geq 1/w_i\gamma(FP_{E_i}). \tag{9}$$

From (8) and (9) we have the equality

$$\gamma(FP_{E_i}) = w_i\gamma(TP_{W_i}). \tag{10}$$

The result follows by Thm.3.1, applied to $\mathcal{E} = \{E_i\}_{i\in I}$ and $F \in L(\mathcal{K}_\mathcal{W}, \mathcal{H})$. □

Notice that in both results the equality $T(E_i) = \overline{T(E_i)}$ (resp. $T(W_i) = \overline{T(W_i)}$) is guaranteed for every $i \in I$ since $\gamma(TP_{E_i}) > 0$, $\forall i \in I$ (resp. $\gamma(TP_{W_i}) > 0$, $\forall i \in I$).

Remark 3.3. Suppose that $\mathcal{W} = (W_i, w_i)_{i\in I}$ is a fusion frame for $\mathcal{H}$ and $T \in L(\mathcal{H})$ is a closed range operator. Then, using Corollary 2.7 and the fact that $\|TP_{W_i}\| \leq \|T\|$, $\forall i \in I$, we have that any of the following conditions

1. $0 < \inf_{i\in I} \gamma(TP_{W_i})$
2. $\sup_{i\in I} c\,(\,N(T),\, W_i\,) < 1$.
3. $\sup_{i\in I} \delta\left(\overline{T^*T(W_i)}, W_i\right) < 1$.

implies Eq. (6). In particular, Theorem 3.2 generalizes all the sufficient conditions proved in [10].

It is easy to construct examples in which Eq. (6) holds but $\inf_{i\in\mathbb{N}} \gamma(TP_{W_i}) = 0$. Notice that in such cases Thm. 3.2 implies, by considering a different family of weights (which tends to 0), that the perturbed sequence is a fusion frame sequence.

The following example shows that Eq. (6) is not neccesary to produce a fusion frame sequence via an operator perturbation of a fusion frame for $\mathcal{H}$:

Example 3.4. Let $\{e_i\}_{i\in\mathbb{N}}$ be an orthonormal basis for $\mathcal{H}$. Let $\{\theta_k\}_{k=0}^\infty$ be a sequence of positive numbers such that $\theta_0 < \frac{\pi}{4}$ and $\lim_{k\to\infty} \theta_k = 0$. For $k = 0, 1, 2, \ldots$, let

$$E_k = \text{span}\{e_{3k+1}\ ,\ \sin(\theta_k)e_{3k+2} + \cos(\theta_k)e_{3(k+1)}\}$$

and

$$F_k = \text{span}\{e_{3k+1},\ \frac{1}{\sqrt{2}}e_{3k+2} + \frac{1}{\sqrt{2}}e_{3(k+1)}\}.$$

Then, the family $\{W_i\}_{i\in\mathbb{N}}$ of closed subspaces of $\mathcal{H}$ given by $W_{2k+1} = E_k$ and $W_{2(k+1)} = F_k$ determines a fusion frame for $\mathcal{H}$ by considering the weights $w_i = 1$, $\forall i \in \mathbb{N}$. Indeed, this claim follows by the fact that the union of the orthonormal basis that generates each E_k and F_k is a frame of vectors for $\mathcal{H}$. To be more precise, for each k, the family of vectors

$$\begin{aligned}
f_{1,k} &= e_{3k+1}\\
f_{2,k} &= \sin(\theta_k)e_{3k+2} + \cos(\theta_k)e_{3(k+1)}\\
f_{3,k} &= e_{3k+1}\\
f_{4,k} &= \frac{1}{\sqrt{2}}e_{3k+2} + \frac{1}{\sqrt{2}}e_{3(k+1)}
\end{aligned}$$

is a frame for $\text{span}\{e_{3k+1}\ ,\ e_{3k+2}\ ,\ e_{3(k+1)}\}$, with frame bounds $A_k = 1 - \sqrt{\frac{1}{2} + \cos(\theta_k)\sin(\theta_k)}$ and $B_k = 2$.

Since the lower frame bounds are uniformly bounded by $0 < 1 - \sqrt{\frac{1}{2} + \cos(\theta_0)\sin(\theta_0)}$ and the subspaces generated by these frame sequences are mutually orthogonal, the claim follows from Thm. 3.2 in [3].

Let $T \in L(\mathcal{H})$ be the orthonormal projection onto $\mathcal{K} = \overline{\text{span}}\{e_{3k+1}\ , e_{3k+2}\ , k = 0, 1, \ldots\}$. Since $T(E_k) = T(F_k) = \text{span}\{e_{3k+1}\ , e_{3k+2}\}$ for every k, we deduce that $(T(W_i), w_i)_{i\in\mathbb{N}}$ is a fusion frame sequence for $\mathcal{H}$. On the other side, it is not difficult to show that

$$\gamma(TP_{W_i}) = \begin{cases} sin^2(\theta_k), & \text{for } i = 2k+1 \\ \frac{1}{2}, & \text{for } i = 2(k+1) \end{cases}$$

and $\|TP_{W_i}\| = 1$ for every $i \in \mathbb{N}$. Therefore, $\inf_{i\in\mathbb{N}} \frac{\gamma(TP_{W_i})}{\|TP_{W_i}\|} = 0$, so Eq.(6) is not a necessary condition for linear perturbation of fusion frames.

Aknowledgments

This work was partially supported by CONICET (PIP 0435/10) and Universidad Nacional de La Plata (UNLP 11 X681). The authors would like to thank Professor D. Stojanoff for his valuable comments related to the content of this note.

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