Curves of projections and operator inequalities

Esteban Andruchow and Gustavo Corach

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Abstract

Given two orthogonal projections P and Q in a complex Hilbert space such that

$$R(P) \cap N(Q) = N(P) \cap R(Q) = \{0\},$$

there exists a unique selfadjoint operator $X_{P,Q}$, which is P-codiagonal, has norm at most $\pi/2$ and satisfies that the curve

$$\delta(t) = e^{itX_{P,Q}} P e^{-itX_{P,Q}}$$

joins $\delta(0) = P$ and $\delta(1) = Q$, and has minimal length among all piecewise smooth curves of projections joining P and Q. We use this fact to obtain operator inequalities in particular examples. Namely, given projections P, Q as above, and a path $P(t)$, $t \in [a, b]$, joining them, then one has

$$\int_a^b \| \frac{d}{dt} P(t) \| dt \geq \|X_{P,Q}\|,$$

where the right hand term is the length of δ.

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1 Introduction

Let $\mathcal{H}$ be a Hilbert space, $\mathcal{B}(\mathcal{H})$ the algebra of bounded linear operators in $\mathcal{H}$, $\mathcal{U}(\mathcal{H})$ the group of unitary operators and $\mathcal{P}(\mathcal{H})$ the set of orthogonal projections, i.e.,

$$\mathcal{P}(\mathcal{H}) = \{P \in \mathcal{B}(\mathcal{H}) : P^2 = P = P^*\}.$$

By identifying a closed subspace $\mathcal{M}$ of $\mathcal{H}$ with $P_{\mathcal{M}}$, the orthogonal projection onto $\mathcal{M}$, sometimes $\mathcal{P}(\mathcal{H})$ is called the Grassmann manifold of $\mathcal{H}$. The set $\mathcal{P}(\mathcal{H})$ has a rich geometrical structure: each component of $\mathcal{P}(\mathcal{H})$ is a homogeneous space of $\mathcal{U}(\mathcal{H})$ and a closed and complemented submanifold of $\mathcal{B}_h(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) : T^* = T\}$, with a natural connection and explicit geodesics. Moreover, there is a natural length functional ℓ for curves in $\mathcal{P}(\mathcal{H})$. It turns out that, under precise conditions, the geodesic δ joining P and Q is a global minimum for ℓ. Since the geodesics and their lengths can be explicitely characterized, finding another curve γ in $\mathcal{P}(\mathcal{H})$ with end-points P and Q yields the (operator) inequality $\ell(\delta) \leq \ell(\gamma)$. The main goal of this paper is to show, in three different examples, some operator inequalities which follow this scheme. In order to make the exposition relatively self-contained, we collect in Section 2 several facts concerning sums, differences and products of projections, in Section 3 several results on idempotents $E \in \mathcal{B}(\mathcal{H})$ and their relationship with the orthogonal projections onto the range $R(E)$ and the nullspace $N(E)$; in Section 4 we briefly describe the geometry of $\mathcal{P}(\mathcal{H})$ and its connected components; finally, Section 5 contains the inequalities mentioned above, corresponding to the following examples:

1. For $\mathcal{H} = L^2(\mathbb{R}^n)$, $I, J \subset \mathbb{R}^n$ Lebesgue measurable sets with finite positive measure and $P = P_I$, $Q = Q_J$, where $P_I f = \chi_I f$ and $Q_J f = (P_J \hat{f})\check{}$, where χ denotes the characteristic function and $\hat{}, \check{}$ denote the L^2-Fourier-Plancherel transform and anti-transform.

2. For $\mathcal{H} = L^2(\mathbb{T})$, $\mathbb{T}$ the 1-torus, we consider the closed subspaces $\mathcal{H}_\varphi = \varphi H^2(\mathbb{T})$, $\mathcal{H}_\psi = \psi H^2(\mathbb{T})$, where $\varphi, \psi : \mathbb{T} \to \mathbb{T}$ are continuous maps and $H^2(\mathbb{T})$ is the Hardy space; we study existence and minimality of geodesics joining $P = P_{\mathcal{H}_\varphi}$, $Q = P_{\mathcal{H}_\psi}$.

3. For each idempotent $E \in \mathcal{B}(\mathcal{H})$, i.e., $E^2 = E$, we consider the orthogonal projections $P_{R(E)}$ and $P_{R(E^*)} = P_{N(E)^\perp}$, and study the existence and minimality of geodesics joining them.

2 On pairs of projections

In this section, we collect several known results about two projections $P, Q \in \mathcal{P}(\mathcal{H})$, the products sums and differences $PQ, P(1-Q), P+Q, P-Q, P+Q-1$ and other operations between them. The proofs can be found in the book by Havin and Jöricke [33] (Chapter 3, Section 1), the expository papers by Deutsch [22], Galantai [30] and Böttcher and Spitkovsky [12], and in other papers which will be mentioned. Many of the proofs, if not all, rest on the following three facts:

- Kato's identities [36], [37] (p.33): for any $P, Q \in \mathcal{P}(\mathcal{H})$ it holds

  $$(P-Q)^2 + (P+Q-1)^2 = 1,$$

  $$\|(P-Q)\xi\|^2 + \|(P+Q-1)\xi\|^2 = \|\xi\|^2,$$

  for all $\xi \in \mathcal{H}$.

  The proof is straightforward.

- Krein, Krasnoselski, Milman identity [45]: for any $P, Q \in \mathcal{P}(\mathcal{H})$

  $$\|P - Q\| = \max\{\|P(1-Q)||, \|(1-P)Q\|\}.$$

  The shortest proof we know is due to S. Izumino and Y. Watatani (see [38], Appendix). First, they notice that if $A, B \in \mathcal{B}(\mathcal{H})$ are positive operators such that $AB = 0$ then $\|A + B\| = \max\{\|A\|, \|B\|\}$, and, then, they apply this to $A = (1-Q)P(1-Q), B = Q(1-P)Q$ and observe that $\|P - Q\|^2 = \|A + B\|$.

- If $\mathcal{M}, \mathcal{N}$ are closed subspaces of $\mathcal{H}$, then $\mathcal{M} + \mathcal{N}$ is closed if and only if $\mathcal{M}^\perp + \mathcal{N}^\perp$ is closed.

  This result holds in the more general context of Banach spaces. For a proof, see Kato's book [37] (Theorem 4.8, p.221).

Proposition 2.1. *If $P, Q \in \mathcal{P}(\mathcal{H})$, then the following conditions are equivalent :*

1. *$R(P)+R(Q)$ is closed.*
2. *There exists $\epsilon > 0$ such that the intersection of the spectrum $\sigma(PQ)$ with the real interval $(1-\epsilon, 1)$ is empty.*
3. $\|PQ - P_{R(P)\cap R(Q)}\| < 1$.
4. *$N(P)+N(Q)$ is closed .*
5. *$R((1-P)Q)$ is closed.*
6. *$R(1-PQ)$ is closed.*
7. *$R(P-Q)$ is closed.*
8. *$R(P+Q)$ is closed.*

If one of these conditions holds, then

$$R(P+Q) = R(P) + R(Q).$$

The proof of these equivalences can be found in [22], [43] and [26]; however, some of them have been known by Krein, Krasnoselski and Milman [45], Dixmier [24], Kato [36] and Ljance [47]. For other equivalent properties, see the papers by Bouldin [13] and Izumino [34].

Proposition 2.2. *If $P, Q \in \mathcal{P}(\mathcal{H})$, then the following conditions are equivalent:*

1. $\|PQ\| < 1$.
2. $c_0(R(P), R(Q)) := \sup\{|\langle \mu, \nu\rangle| : \mu \in R(P), \nu \in R(Q), \|\mu\| = \|\nu\| = 1\} < 1$.
3. *There exists $K > 0$ such that $\|(1-P)\nu\| \geq K\|\nu\|$ for all $\nu \in R(Q)$.*
4. *$1-PQ$ is invertible.*
5. *There exists $K > 0$ such that $\|\xi\| \leq K\left(\|(1-P)\xi\| + \|(1-Q)\xi\|\right)$ for all $\xi \in \mathcal{H}$.*
6. *$R(P)\cap R(Q) = \{0\}$ and $R(P)+R(Q)$ is closed.*
7. $R(1-P+1-Q) = \mathcal{H}$.
8. *$1-P+1-Q$ is invertible.*
9. $N(P)+N(Q) = \mathcal{H}$.

For the proof, see Havin and Jöricke [33] for the equivalences $1. - 6.$. The other equivalences folllow from the equivalence between closedness of $\mathcal{M}+\mathcal{N}$ and $\mathcal{M}^\perp + \mathcal{N}^\perp$ mentioned above, plus the fact that $R(P)\cap R(Q) = \{0\}$ if and only if $N(P)+N(Q)$ is dense in $\mathcal{H}$.

The quantity $c_0(R(P), R(Q))$ is the cosine of the so called Dixmier angle between $R(P)$ and $R(Q)$; indeed, $c_0(R(P), R(Q)) = \|PQ\|$. A more subtle notion, due to Friedrichs [29] is

$$c(\mathcal{M}, \mathcal{N}) := \sup\{|\langle \mu, \nu\rangle| : \mu \in \mathcal{M}\ominus\mathcal{N}, \nu \in \mathcal{N}\ominus\mathcal{M}, \|\mu\| = \|\nu\| = 1\}.$$

It holds that $c(\mathcal{M},\mathcal{N}) = \|P_{\mathcal{M}}P_{\mathcal{N}} - P_{\mathcal{M}\cap\mathcal{N}})\|$. Of course, Dixmier's angle is much easier to compute than Friedrichs', but the relevant fact that $c(\mathcal{M},\mathcal{N}) = c(\mathcal{M}^\perp,\mathcal{N}^\perp)$ does not hold for c_0, in general. We refer the reader to Deutsch [22], [23] for a complete discussion on these notions; see also the papers by Galántai [30] and Knyazev, Jujunashvili and Argentati [42] and Jujunashvili's thesis [35].

Corollary 2.3. *If $P, Q \in \mathcal{P}(\mathcal{H})$, the following conditions are equivalent:*

1. *$P + Q$ is invertible.*
2. *$\|(1 - P)(1 - Q)\| < 1$.*
3. *$P + Q - PQ = 1 - (1 - P)(1 - Q)$ is invertible.*

For the proof it suffices to apply Proposition 2.2 to $1 - P, 1 - Q$.

Corollary 2.4. *If $P, Q \in \mathcal{P}(\mathcal{H})$, then the following conditions are equivalent:*

1. *$P - Q$ is invertible.*
2. *$P + Q$ and $1 - PQ$ are invertible.*
3. *$\|PQ\| < 1$ and $\|(1 - P)(1 - Q)\| < 1$.*
4. *$R(P) + R(Q) = \mathcal{H}$, and the sum is direct.*
5. *$N(P) + N(Q) = \mathcal{H}$, and the sum is direct.*
6. *$P : R(Q) \to R(P)$ is bijective.*
7. *$1 - PQ$ and $P + Q - PQ$ are invertible.*
8. *$\|P + Q - 1\| < 1$.*

The proof is contained in Buckholtz [15].

The last results of this section are also simple consequences of Kato's identities and the Krein-Krasnoselski-Milman formula.

Proposition 2.5. *For $P, Q \in \mathcal{P}(\mathcal{H})$ it holds $R(P) \cap R(Q) = \{0\}$ and $N(P) \cap N(Q) = \{0\}$ if and only if $\|(P - Q)\xi\| < \|\xi\|$ for all $\xi \neq 0$.*

This is a result by Maeda [48], related to the so called *position p*, (a.k.a. *generic position*) of two subspaces. In a breakthrough paper, Dixmier [24] defined a pair of subspaces to be in position p if

$$M \cap N = M \cap N^\perp = M^\perp \cap N = M^\perp \cap N^\perp = \{0\}.$$

Maeda's result deals with a weaker assumption called *position p'*, meaning

$$M \cap N^\perp = M^\perp \cap N = \{0\}.$$

We also present a result which we shall need later and which is also a consequence of Kato's identities.

Proposition 2.6. *If $P, Q \in \mathcal{P}(\mathcal{H})$, then $R(P) + R(Q)$ is dense and $R(P) \cap R(Q) = \{0\}$ if and only if $\|(P + Q - 1)\xi\| < \|\xi\|$ for all $\xi \in \mathcal{H} \setminus \{0\}$.*

Proof. According to the mentioned identities, $\|(P + Q - 1)\xi\| < \|\xi\|$ for all $\xi \neq 0$ if and only if $\|(P - Q)\xi\| > 0$ for all $\xi \neq 0$, i.e. , $N(P - Q) = \{0\}$. But it holds in general that $N(P - Q) = R(P) \cap R(Q) \oplus N(P) \cap N(Q)$. □

Proposition 2.7. *For $P, Q \in \mathcal{P}(\mathcal{H})$, one and only one of the following conditions holds:*

1. $\|PQ\| < 1$.
2. $\|PQ\| = 1$ *and* $\|PQ\xi\| < \|\xi\|$ *for all* $\xi \neq 0$.
3. *There exists* $\xi \neq 0$ *such that* $\|PQ\xi\| = \|\xi\|$.

This alternative can be equivalently stated as:

1. $1 - P + 1 - Q$ *is invertible.*
2. $1 - P + 1 - Q$ *is injective but not invertible.*
3. $1 - P + 1 - Q$ *is not injective.*

3 On idempotents

Let us denote by

$$\mathcal{Q}(\mathcal{H}) = \{E \in \mathcal{B}(\mathcal{H}) : E^2 = E\}$$

the set of idempotent operators in $\mathcal{H}$. In this section we study the map

$$\mathbf{\Upsilon} : \mathcal{Q}(\mathcal{H}) \to \mathcal{P}(\mathcal{H}) \times \mathcal{P}(\mathcal{H}), \quad \mathbf{\Upsilon}(E) = (P_{R(E)}, P_{N(E)}),$$

its image and its inverse. On one side we show formulas on how to obtain $E \in \mathcal{Q}(\mathcal{H})$ from $P_{R(E)}$ and $P_{N(E)}$; on the other side we show formulas that express $P_{R(E)}, P_{N(E)}$ in terms of E and E^*. We start with some easy results about idempotents.

Lemma 3.1. *Given idempotents E, F it holds that*

1. $N(E) = R(1 - E)$.
2. $R(E) \subset R(F)$ *if and only if* $FE = E$.
3. $R(E) = R(F)$ *if and only if* $FE = E$, $EF = F$.
4. $N(E) \subset N(F)$ *if and only if* $FE = F$.
5. $N(E) = N(F)$ *if and only if* $FE = F$, $EF = E$.

Corollary 3.2. *If $E \in \mathcal{Q}(\mathcal{H})$ then*

1. $EP_{R(E)} = P_{R(E)}$, $P_{R(E)}E = E$.
2. $P_{N(E)}E = E + P_{N(E)} - 1$, $EP_{N(E)} = 0$.

3. $E^*P_{R(E)} = E^*$, $P_{R(E)}E^* = P_{R(E)}$.

4. $E^*P_{N(E)} = E^* + P_{N(E)} - 1$, $P_{N(E)}E^* = 0$.

Proof. Straightforward. For 2., notice that $P_{N(E)}(1-E) = 1-E$, because $R(1-E) = N(E)$. □

The next result is essentially due to Buckholtz [15].

Corollary 3.3. *If $E \in \mathcal{Q}(\mathcal{H})$ then*

1. $(P_{R(E)} - P_{N(E)})(E + E^* - 1) = (E + E^* - 1)(P_{R(E)} - P_{N(E)}) = 1$.
2. $P_{R(E)} = E(E + E^* - 1)^{-1}$.
3. $P_{N(E)} = (E - 1)(E + E^* - 1)^{-1}$.
4. $P_{R(E)} + P_{N(E)} = (2E - 1)(E + E^* - 1)^{-1}$.

As a consequence, we get:

Corollary 3.4. *If $E \in \mathcal{Q}(\mathcal{H})$ then*

1. $E = (1 - P_{N(E)}P_{R(E)})^{-1}(1 - P_{N(E)})$ (Greville [31], Ptak [53] p.347).
2. $E = P_{R(E)}(P_{R(E)} + P_{N(E)} - P_{N(E)}P_{R(E)})^{-1}$ (Greville [31]).
3. $E = P_{R(E)}(P_{R(E)} + P_{N(E)})^{-1}$ (Ando [2]).
4. $E = P_{R(E)}(P_{R(E)} - P_{N(E)})^{-1}$ (Buckholtz [15]).
5. $E = (1 - P_{R(E)}P_{N(E)})^{-1}P_{R(E)}(1 - P_{R(E)}P_{N(E)})$ (Afriat [1]).
6. $E^* = (1 - P_{N(E)})(P_{R(E)} - P_{N(E)})^{-1}$.

Proof. Straightforward. Observe that the invertibility of $1 - P_{R(E)}P_{N(E)}$ follows, because $\|P_{R(E)}P_{N(E)}\| < 1$ since $P_{R(E)} - P_{N(E)}$ is invertible (see the Corollary 2.4 in the previous section). □

Remark 3.5. Concerning the Dixmier angle c_0, for $M = R(E)$, $N = N(E)$ it holds

$$c_0(M, N) = c_0(M^\perp, N^\perp) = \|P_M P_N\| = \|P_{M^\perp} P_{N^\perp}\| = \|P_M + P_N - 1\| = (1 - \|E\|^{-2})^{1/2}.$$

We refer the reader to Buckholtz [15] and to a nice paper by Ando [2], which contains many new results and expressions for E in terms of M and N, and of $\|E\|$ and relatives.

Remark 3.6. It is worth mentioning that in Arias et al. [10] (Theorem 4.1) some of the formulas in Corollary 3.4 have been extended as follows: if $A, B \in \mathcal{B}(\mathcal{H})$ are positive operators such that $R(A) = R(E)$ and $R(B) = N(E)$ for $E \in \mathcal{Q}(\mathcal{H})$, then

$$E = A(A \pm B)^{-1}.$$

More generally, for $S, T \in \mathcal{B}(\mathcal{H})$ with $R(S) = R(E)$ and $R(T) = N(E)$, it holds

$$E = SS^*(SS^* \pm TT^*)^{-1}.$$

We include here a discussion on the Moore-Penrose inverse of an idempotent E. Recall that the Moore-Penrose inverse of a closed range linear bounded operator T is the unique solution $X = T^\dagger$ of the system

$$\begin{cases} TXT = T \\ XTX = X \\ (XT)^* = XT \\ (TX)^* = TX \end{cases}.$$

See [21] for a nice treatment of this subject.

The next formula for the Moore-Penrose inverse of $E \in \mathcal{Q}(\mathcal{H})$ is due to Penrose [51]; without noticing his result, Greville [31] found the result again, in both cases for matrices. For operators in Hilbert space, see [18].

Proposition 3.7. *If $E \in \mathcal{Q}(\mathcal{H})$ then*

$$E^\dagger = P_{N(E)^\perp} P_{R(E)} = (1 - P_{N(E)})P_{R(E)} = P_{R(E)} - P_{N(E)}P_{R(E)} = (1 - P_{N(E)}P_{R(E)})P_{R(E)}$$
$$= (P_{R(E)} - P_{N(E)})P_{R(E)}.$$

Remark 3.8. The definition of the Moore-Penrose inverse $\dagger$ can be extended to operators $T \in \mathcal{B}(\mathcal{H})$ with non closed range. In particular, it can be shown that for $T = PQ$, $T^\dagger$ is a (generally unbounded) idempotent $\tilde{E}$ with dense domain and $R(\tilde{E}), N(\tilde{E})$ closed subspaces with trivial intersection and $R(\tilde{E}) + N(\tilde{E})$ dense in $\mathcal{H}$. Denoting this set by $\tilde{\mathcal{Q}}(\mathcal{H})$, it can be seen that

$$\dagger : \mathcal{P}(\mathcal{H}) \cdot \mathcal{P}(\mathcal{H}) \to \tilde{\mathcal{Q}}(\mathcal{H})$$

is a bijection (see [17] for details).

Returning to the beginning of the section, we can resume some information about the map

$$\boldsymbol{\Upsilon} : \mathcal{Q}(\mathcal{H}) \to \mathcal{P}(\mathcal{H}) \times \mathcal{P}(\mathcal{H}) \ , \ \boldsymbol{\Upsilon}(E) = (P_{R(E)}, P_{N(E)}).$$

By Buckholtz' result (Corollary 2.4), the image of $\boldsymbol{\Upsilon}$ is

$$\{(P, Q) \in \mathcal{P}(\mathcal{H}) \times \mathcal{P}(\mathcal{H}) : P - Q \text{ is invertible}\},$$

and on this set the inverse $\boldsymbol{\Upsilon}^{-1}$ can be expressed according to any of the formulas stated in Corollary 3.4. As a sample

$$\boldsymbol{\Upsilon}^{-1}(P, Q) = P(P \pm Q)^{-1} = (P - Q)^{-1}(1 - Q).$$

The set

$$\mathcal{D} = \{P - Q : P, Q \in \mathcal{P}(\mathcal{H})\}$$

plays a relevant role in any geometrical study of the space of projections. This set was first characterized by Davis [20]. For a more recent treatment on the geometric relevance of $\mathcal{D}$, see [3]. Denote by $\mathcal{G}(\mathcal{H})$ the group of invertible operators in $\mathcal{H}$. Buckholtz' results show that $\mathcal{D} \cap \mathcal{G}(\mathcal{H})$ is the image of $\delta \circ \boldsymbol{\Upsilon}$, where

$$\delta : \mathcal{P}(\mathcal{H}) \times \mathcal{P}(\mathcal{H}) \to \mathcal{D} \ , \ \delta(P, Q) = P - Q.$$

Notice that $\delta \circ \boldsymbol{\Upsilon} : \mathcal{Q}(\mathcal{H}) \to \mathcal{D} \cap \mathcal{G}(\mathcal{H})$ is not bijective, because $\delta \circ \boldsymbol{\Upsilon}(E^*) = \delta \circ \boldsymbol{\Upsilon}(E)$. Moreover, $\delta \circ \boldsymbol{\Upsilon}(E) = \delta \circ \boldsymbol{\Upsilon}(F)$ if and only if $E + E^* = F + F^*$, i.e., $Re(E) = Re(F)$.

3.1 On 2×2 matrix decompositions

This short subsection is devoted to collect several 2×2 matrix representations of an idempotent E and its associates E^*, EE^*, $|E| = (E^*E)^{1/2}$, $P_{R(E)}$, and so on. First recall that every $P \in \mathcal{P}(\mathcal{H})$ induces a representation of $\mathcal{B}(\mathcal{H})$ as a C^*-algebra of 2×2 operator matrices. For any $T \in \mathcal{B}(\mathcal{H})$ the identity $T = PTP + PT(1-P) + (1-P)TP + (1-P)T(1-P)$ can be seen as a matrix

$$M_T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$$

where $T_{11} = PTP \in \mathcal{B}(R(P))$, $T_{12} = PT(1-P) \in \mathcal{B}(N(P), R(P))$, $T_{21} = (1-P)TP \in \mathcal{B}(R(P), N(P))$ and $T_{22} = (1-P)T(1-P) \in \mathcal{B}(N(P))$. This is a C^*-algebra representation so $M_{T_1T_2} = M_{T_1}M_{T_2}$ and $M_{T^*} = M_T^*$. We shall identify $T = M_T$. In particular, every idempotent $E \in \mathcal{B}(\mathcal{H})$ can be represented as

$$E = \begin{pmatrix} 1 & B \\ 0 & 0 \end{pmatrix} \tag{1}$$

for some $B \in \mathcal{B}(R(E)^\perp, R(E))$, if $P = P_{R(E)}$ and we write $1 = 1_{R(E)} = P$, the unit of $\mathcal{B}(R(P))$. Observe the E is determined by $R(E)$ and the operator $B : N(P) \to R(E)$. So we try to understand properties of E in terms of B.

Notice that we use thoroughly the rule $\varphi(TT^*)T = T\varphi(T^*T)$ which holds for any Borelian function φ defined in $\mathbb{R}^+$.

$$E = \begin{pmatrix} 1 & B \\ 0 & 0 \end{pmatrix},\ E^* = \begin{pmatrix} 1 & 0 \\ B^* & 0 \end{pmatrix},\ EE^* = \begin{pmatrix} 1+BB^* & 0 \\ 0 & 0 \end{pmatrix}, E + E^* - 1 = \begin{pmatrix} 1 & B \\ B^* & -1 \end{pmatrix},$$

$$(E + E^* - 1)^2 = \begin{pmatrix} 1+BB^* & 0 \\ 0 & 1+B^*B \end{pmatrix},$$

$$(E+E^*-1)^{-1} = (E+E^*-1)^{-2}(E+E^*-1) = \begin{pmatrix} (1+BB^*)^{-1} & (1+BB^*)^{-1}B \\ (1+B^*B)^{-1}B^* & -(1+B^*B)^{-1} \end{pmatrix} \tag{2}$$

$$P_{R(E)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$P_{N(E)} = (E-1)(E+E^*-1)^{-1} = \begin{pmatrix} (1+BB^*)^{-1}BB^* & (1+BB^*)^{-1}B \\ (1+B^*B)^{-1}B^* & (1+B^*B)^{-1} \end{pmatrix} \tag{3}$$

$$P_{R(E)}P_{N(E)}P_{R(E)} = \begin{pmatrix} (1+BB^*)^{-1}BB^* & 0 \\ 0 & 0 \end{pmatrix},\ |E^*| = (EE^*)^{1/2} = \begin{pmatrix} (1+BB^*)^{1/2} & 0 \\ 0 & 0 \end{pmatrix},$$

$$|E+E^*-1| = \begin{pmatrix} (1+BB^*)^{1/2} & 0 \\ 0 & (1+B^*B)^{1/2} \end{pmatrix},$$

and

$$E^\dagger = (E+E^*-1)^{-1}P_{R(E)} = \begin{pmatrix} (1+BB^*)^{-1} & 0 \\ B^*(1+BB^*)^{-1} & 0 \end{pmatrix}. \tag{4}$$

4 Geometry of $\mathcal{P}(\mathcal{H})$

In this section we survey several results which, put together, offer a quite complete description of $\mathcal{P}(\mathcal{H})$ as a discrete union of components, each of one with a structure of differentiable submanifold and of homogeneous space of the unitary group $\mathcal{U}(\mathcal{H})$ of $\mathcal{H}$. There is a natural linear connection whose geodesics have minimal length with respect to a Finsler metric, which is also natural. Our main references are Porta and Recht [52], Wilkins [55], Brown [14], Chung [16], Corach, Porta and Recht [19] and Andruchow [3], [4].

We start with the action of $\mathcal{U}(\mathcal{H})$ on $\mathcal{P}(\mathcal{H})$, defined by $U \cdot P = UPU^*$, for $U \in \mathcal{U}(\mathcal{H})$, $P \in \mathcal{P}(\mathcal{H})$. The action is locally transitive: if P, Q are close, then there exists U such that $U \cdot P = Q$. In fact, this was known at least by Sz. Nagy [49]:

Lemma 4.1. *If $P, Q \in \mathcal{P}(\mathcal{H})$ and $\|P - Q\| < 1$, there exists $U = U(P, Q) \in \mathcal{U}(\mathcal{H})$ such that $UPU^* = Q$.*

Corollary 4.2. *The orbits of the action coincide with the connected components of $\mathcal{P}(\mathcal{H})$.*

This is a consequence of the fact that the unitary group is connected.

We fix $P_0 \in \mathcal{P}(\mathcal{H})$ and denote by $\mathcal{P}_0$ the connected component of P_0. As in the previous section, we represent every $A \in \mathcal{B}(\mathcal{H})$ as a 2×2 matrix in terms of P_0,

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

and write

$$A_d = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}, \quad A_c = \begin{pmatrix} 0 & a_{12} \\ a_{21} & 0 \end{pmatrix}.$$

A_d will be called the P_0-diagonal part of A, and A_c the codiagonal part. Observe that A_d commutes with P_0. If $\mathcal{B}_h(\mathcal{H}) = \{A \in \mathcal{B}(\mathcal{H}) : A^* = A\}$, denote

$$\mathcal{D}_{P_0} = \{A_d : A \in \mathcal{B}_h(\mathcal{H})\}, \quad \mathcal{C}_{P_0} = \{A_c : A \in \mathcal{B}_h(\mathcal{H})\}.$$

Observe that $\mathcal{B}_h(\mathcal{H}) = \mathcal{D}_{P_0} \oplus \mathcal{C}_{P_0}$. Consider the map

$$\phi : \mathcal{B}_h(\mathcal{H}) \to \mathcal{B}_h(\mathcal{H}), \quad \phi(X) = X_d + e^{\tilde{X}_c} P_0 e^{-\tilde{X}_c},$$

where

$$\tilde{X}_c = \begin{pmatrix} 0 & -x_{12} \\ x_{12}^* & 0 \end{pmatrix} \quad \text{if} \quad X_c = \begin{pmatrix} 0 & x_{12} \\ x_{12}^* & 0 \end{pmatrix}.$$

ϕ is well defined, because $e^{\tilde{X}_c} \in \mathcal{U}(\mathcal{H})$, and then $\phi(X) \in \mathcal{B}_h(\mathcal{H})$.

Lemma 4.3. *The map ϕ is differentiable, and its differential $d\phi_{P_0}$ at P_0 is the identity.*

By the inverse mapping theorem, ϕ is a local diffeomorphism, which maps a neighbourhood of 0 in $\mathcal{B}_h(\mathcal{H})$ onto a neighbourhood of P_0 in the same space. When restricted to $\mathcal{C}_{P_0}$, it takes values in an open neighbourhood of P_0 in $\mathcal{P}_0$. As a consequence:

Proposition 4.4. *$\mathcal{P}_0$ is a submanifold of $\mathcal{B}_h(\mathcal{H})$. The map*

$$\pi_{P_0} : \mathcal{U}(\mathcal{H}) \to \mathcal{P}_0, \quad \pi_{P_0}(U) = UP_0U^*$$

is a smooth submersion. The tangent space $(T\mathcal{P}_0)_{P_0}$ of $\mathcal{P}_0$ is identified with $\mathcal{C}_{P_0}$.

There is a natural linear connection in $\mathcal{P}(\mathcal{H})$, which is a particular case of a reductive structure for an homogeneous space. For a smooth curve $\rho : [0,1] \to \mathcal{P}(\mathcal{H})$, a *co-diagonal lifitng* (or *horizontal lifting*) for ρ is a smooth curve $U : [0,1] \to \mathcal{U}(\mathcal{H})$ such that

$$\begin{cases} U(t)\rho(0)U^*(t) = \rho(t) \\ iU^*(t)\dot{U}(t) \in \mathcal{C}_{\rho(t)} \end{cases},$$

for all $t \in [0,1]$. It was shown in [19] that the (unique) co-diagonal lifting of ρ satisfying $U(0) = 1$ is the solution of the problem

$$\begin{cases} \dot{U} = [\dot{\rho}, \rho]U \\ U(0) = 1 \end{cases}.$$

Given a tangent vector $X \in \mathcal{C}_{P_0}$ at P_0, and a smooth curve $\rho : [0,1] \to \mathcal{P}_0$ with $\rho(0) = P_0$, the *parallel transport* of a tangent field X along ρ is $U(t)XU^*(t)$, where U is the horizontal lifting of ρ with $U(0) = 1$. A *geodesic* of $\mathcal{P}(\mathcal{H})$ is a curve δ, such that the field $\dot{\delta}$ equals the parallel transport of $\dot{\delta}(0)$ along δ. The geodesics can be explicitely computed:

Proposition 4.5. *If $P_0 \in \mathcal{P}_0$ and $X_0 \in \mathcal{C}_{P_0}$, then the unique geodesic $\delta : [0,1] \to \mathcal{P}_0$ of the above connection, with $\delta(0) = P_0$ and initial velocity $\dot{\delta}(0) = X_0$ is given by*

$$\delta(t) = e^{t\tilde{X}_0} P_0 e^{-\tilde{X}_0}.$$

In Riemannian geometry, one expects that, at least locally, geodesics have minimal length. This may not be the case when one deals with non Riemannian manifolds. This is the case of $\mathcal{P}(\mathcal{H})$. If $\mathcal{H}$ is finite dimensional, $\mathcal{P}(\mathcal{H})$ can be endowed with a Riemannian metric, by considering the Frobenius norm at every tangent space: $|X| = Tr(X^*X)^{1/2}$. For infinite dimensional $\mathcal{H}$, this norm is not available, and the natural choice is the usual operator norm. This norm is highly non-smooth. We define thus the length functional ℓ in $\mathcal{P}(\mathcal{H})$ as

$$\ell(\rho) = \int_0^1 \|\frac{d}{dt}\rho(t)\| dt,$$

for $\rho : [0,1] \to \mathcal{P}(\mathcal{H})$ a smooth curve. Combining results of Porta and Recht [52] and [3] one has

Theorem 4.6. *Let $P, Q \in \mathcal{P}(\mathcal{H})$, then the following are equivalent:*

1. *P and Q can be be joined by a geodesic of $\mathcal{P}(\mathcal{H})$.*
2. *P and Q can be be joined by a geodesic of $\mathcal{P}(\mathcal{H})$, which is a global minimum of ℓ.*
3. $\dim R(P) \cap N(Q) = \dim R(Q) \cap N(P)$.

Moreover, there exists a **unique** *geodesic which is a minimum for ℓ if and only if*

$$R(P) \cap N(Q) = R(Q) \cap N(P) = \{0\}. \tag{5}$$

It is a remarkable fact that some of these results hold for the rectifiable metric in $\mathcal{P}(\mathcal{H})$, which does not take into account the differentiable structure of $\mathcal{P}(\mathcal{H})$. This theory was developed by Brown [14].

Condition (5) above, implies the existence of a selfadjoint operator $X_{P,Q}$ ($= -i\tilde{X}_0$ in the above notation) such that

$$\delta_{P,Q}(t) = e^{itX_{P,Q}}Pe^{-itX_{P,Q}}, \tag{6}$$

is the unique minimal geodesic joining P and Q. The exponent $X_{P,Q}$ is a selfadjoint operator, which is P (and Q)-codiagonal, i.e., its matrix in terms of the decomposition $\mathcal{H} = R(P) \oplus N(P)$ is codiagonal, with $\|X_{P,Q}\| \le \pi/2$. This curve $\delta_{P,Q}$ has minimal length (equal to $\|X_{P,Q}\|$) among all possible piecewise differentiable curves of projections joining P and Q. The norm of $X_{P,Q}$ is related to the usual distance between P and Q (see for instance [9]):

$$\|P - Q\| = \sin(\|X_{P,Q}\|),$$

including the case $\|P - Q\| = 1$, when $\|X_{P,Q}\| = \pi/2$.

5 Operator inequalities from short paths of projections

Let $P, Q \in \mathcal{P}(\mathcal{H})$ which satisfy condition (5). If $P(t),\ t \in [a, b]$ is a piecewise smooth curve in $\mathcal{P}(\mathcal{H})$, then its length is not smaller than the length of $\delta_{P,Q}$,

$$\ell(P(t)) = \int_a^b \|\frac{d}{dt}P(t)\|dt \ge \|X_{P,Q}\| = \sin^{-1}(\|P - Q\|).$$

As we shall see in the examples, the integral on the left hand side is often the norm of a commutator, thus the above generic inequality takes the form of the lower bound for the norm of a commutator.

The reader is invited to produce examples of his interest, and try this method to obtain a new inequalities. We shall consider three families of examples, which will be discussed below:

Example 5.1. Let $\mathcal{H} = L^2(\mathbb{R}^n)$, $I, J \subset \mathbb{R}^n$ measurable sets of finite Lebesgue measure, and the projections $P = P_I$ and $Q = Q_J$ given by

$$P_I f = \chi_I f \quad \text{and} \quad Q_J f = \left(\chi_J \hat{f}\right)\check{}\,,$$

where χ_L denotes the characteristic function of the set L, and $\hat{}, \check{}$ denote the Fourier-Plancherel transform and anti-transform, respectively. Equivalently, denoting by $U_{\mathcal{F}}$ the Fourier-Plancherel transformation regarded as a unitary operator acting in $L^2(\mathbb{R}^n)$ and by M_φ multiplication by φ, then

$$P_I = M_{\chi_I} \ \text{ and } Q_J = U_{\mathcal{F}}^* P_J U_{\mathcal{F}}.$$

This pair of projections, and specifically the norm of $\|P_I Q_J\|$ are central in mathematical formulation of the uncertainty principle (see [28]).

Example 5.2. Let $\mathcal{H} = L^2(\mathbb{T})$, $\mathbb{T}$ the 1-torus, $\varphi, \psi : \mathbb{T} \to \mathbb{T}$ continuous functions, and $\mathcal{H}_\varphi = \varphi H^2(\mathbb{T})$, $\mathcal{H}_\psi = \psi H^2(\mathbb{T})$, where $H^2(\mathbb{T})$ is the Hardy space. Put

$$P = P_{\mathcal{H}_\varphi} \text{ and } Q = P_{\mathcal{H}_\psi},$$

the orthogonal projections onto $\mathcal{H}_\varphi$ and $\mathcal{H}_\psi$, respectively. If we denote by P_+ the projection onto $H^2(\mathbb{T})$, then the fact that φ and ψ are unimodular means that the multiplication operators M_φ and M_ψ are unitary operators in $\mathcal{H}$, and thus

$$P = M_\varphi P_+ M_{\bar\varphi} \text{ and } Q = M_\psi P_+ M_{\bar\psi}.$$

Example 5.3. Let E be an idempotent operator. Consider $P = P_{R(E)}$ and $Q = P_{N(E)^\perp} = P_{R(E^*)}$. Note that

$$R(P) \cap N(Q) = R(E) \cap N(E) = \{0\} \quad \text{and} \quad N(P) \cap R(Q) = N(E^*) \cap R(E^*) = \{0\}.$$

Thus there exists a unique geodesic joining the ranges of E and E^*.

5.1 The first example.

The facts exposed here are either known in the literature (see the excellent survey article [28] by Folland and Sitaram), or were obtained in the paper [8]. Condition (5) is well established in Examples 5.1:

Lemma 5.4. *Let P and Q as in example 5.1. Then condition (5) holds.*

Proof. Lenard proved in [46] (see also [28]) that the only common eigenvectors of P_I and Q_J are those of $N(P_I) \cap N(Q_J)$, which has infinite dimension. □

It is also known that $P_I Q_J$ is a nuclear operator (thus compact) [28], and that $\|P_I - Q_J\| = 1$. This can be derived from the fact that $P_I Q_J$ is compact: in the Calkin algebra, the classes $[P_I] \neq 0$ and $[Q_J] \neq 0$ are projections such that $[P_I Q_J] = 0$, thus $\|[P_I] - [Q_J]\| = 1$. Then

$$1 = \|[P_I - Q_J]\| \leq \|P_I - Q_J\| \leq 1.$$

Then $\|X_{P_I,Q_J}\| = \pi/2$

The co-diagonal exponent X_{P_I,Q_J} has interesting features when $I = J$.

If we pick $I = J$ (with $|I| < \infty$), and denote by $X_I = X_{P_I,Q_I}$, there are two unitary operators intertwining P_I and Q_I. Namely, the Fourier transform $U_{\mathcal{F}}$ and the exponential e^{iX_I},

$$U_{\mathcal{F}}^* P_I U_{\mathcal{F}} = Q_I = e^{iX_I} P_I e^{-iX_I}.$$

Let $H = H^*$ be the natural logarithm of the Fourier transform, $e^{iH} = U_{\mathcal{F}}$. Namely, denoting by E_1, E_{-1}, E_i and E_{-i} the eigenprojections of $U_{\mathcal{F}}$,

$$H = -\pi E_{-1} + \frac{\pi}{2} E_i - \frac{\pi}{2} E_{-i}.$$

One obtains a smooth path joining P_I and Q_I:

$$\varphi(t) = e^{-itH} P_I e^{itH}.$$

Indeed, apparently $\varphi(1) = Q_I$.

Theorem 5.5. *For any Lebesgue measurable set $I \subset \mathbb{R}^n$ with $|I| < \infty$, one has*

$$\|[H, P_I]\| = \|[H, Q_I]\| \geq \pi/2.$$

Proof. The geodesic δ_I with exponent X_I is the shortest curve in $\mathcal{P}(\mathcal{H})$ joining P_I and Q_I. Its length is $\pi/2$. Then

$$\pi/2 \leq \ell(\varphi) = \int_0^1 \|\dot{\varphi}(t)\| dt = \int_0^1 \|e^{itH}[H, P_I]e^{-itH}\| dt = \|[H, P_I]\|.$$

Note that

$$U_{\mathcal{F}}^*[H, P_I]U_{\mathcal{F}} = [H, U_{\mathcal{F}}^* P_I U_{\mathcal{F}}] = [H, Q_I]$$

because $U_{\mathcal{F}}$ and H commute. □

Remark 5.6. We may write H in terms of $U_{\mathcal{F}}$ using the formulas

$$E_{-1} = \frac{1}{4}(1 - U_{\mathcal{F}} + U_{\mathcal{F}}^2 - U_{\mathcal{F}}^3),\ E_i = \frac{1}{4}(1 - iU_{\mathcal{F}} - U_{\mathcal{F}}^2 + iU_{\mathcal{F}}^3),\ E_{-i} = \frac{1}{4}(1 + iU_{\mathcal{F}} - U_{\mathcal{F}}^2 - iU_{\mathcal{F}}^3),$$

and thus

$$H = \frac{\pi}{4}\{-1 + (1+i)U_{\mathcal{F}} - U_{\mathcal{F}}^2 + (1+i)U_{\mathcal{F}}^3\}.$$

Then

$$[H, P_I] = \frac{\pi}{4}\{(1+i)[U_{\mathcal{F}}, P_I] - [U_{\mathcal{F}}^2, P_I] + (1+i)[U_{\mathcal{F}}^3, P_I]\}.$$

The inequality in Theorem 5.5 can be written

$$\|(1+i)[U_{\mathcal{F}}, P_I] - [U_{\mathcal{F}}^2, P_I] + (1+i)[U_{\mathcal{F}}^3, P_I]\| \geq 2.$$

5.2 The second example

The facts presented in this section were obtained in [5]. We begin by analizing condition (5).

Lemma 5.7. *Let $P = P_{\mathcal{H}_\varphi}$ and $Q = P_{\mathcal{H}_\psi}$ with φ and ψ as in example 5.2. Then condition (5) holds if and only if*

$$w(\varphi) = w(\psi),$$

where $w(f)$ stands for the winding number of f.

Proof. We give a sketch of the proof. It relies on basic facts on Toeplitz operators (see for instance [25]). If $h \in L^\infty(\mathbb{T})$, denote by T_h the Toeplitz operator with symbol h. First note that the restriction of the multiplication operator

$$M_\psi|_{N(T_{\bar{\varphi}\psi})} : N(T_{\bar{\varphi}\psi}) \to \mathcal{H}_\varphi^\perp \cap \mathcal{H}_\psi$$

is an isomorphism, and similarly $N(T_{\varphi\bar{\psi}})$ is isomorphic to $\mathcal{H}_\varphi \cap \mathcal{H}_\psi^\perp$. Thus condition (5) is equivalent to both $T_{\bar{\varphi}\psi}$ and $T_{\varphi\bar{\psi}}$ having trivial nullspace.

Since $\bar{\varphi}\psi$ is invertible in $C(\mathbb{T})$, $T_{\bar{\varphi}\psi}$ is a Fredholm operator. Its index is

$$w(\bar{\varphi}\psi) = w(\psi) - w(\varphi).$$

If the winding numbers coincide, the index is zero and thus $T_{\bar{\varphi}\psi}$ is invertible, and in particular $N(T_{\bar{\varphi}\psi})$ is trivial. The other nullspace is trivial analogously.

Conversely, if both nullspaces are trivial, the index of $T_{\bar{\varphi}\psi}$ is trivial, and thus $T_{\bar{\varphi}\psi}$ (being a Toeplitz operator) is in fact invertible. □

In what follows, we assume that $w(\varphi) = w(\psi)$. Let us denote by $X_{\varphi,\psi} = X_{P_{\mathcal{H}_\varphi}, P_{\mathcal{H}_\psi}}$.

In order to compute the norm of $X_{\varphi,\psi}$, it will be useful to employ the decomposition of a Hilbert space in the presence of two projections (see Dixmier [24], Halmos [32]). Consider

$$\mathcal{H}_{11} = R(P) \cap R(Q)\ ,\quad \mathcal{H}_{00} = N(P) \cap N(Q)\ ,\ \mathcal{H}_{10} = R(P) \cap N(Q)\ ,\quad \mathcal{H}_{01} = N(P) \cap R(Q)$$

and $\mathcal{H}_0$ the orthogonal complement of the sum of the above. This last subspace is usually called the *generic part* of the pair P, Q. Note also that

$$N(P-Q) = \mathcal{H}_{11} \oplus \mathcal{H}_{00}\ ,\quad N(P-Q-1) = \mathcal{H}_{10}\ \text{ and } N(P-Q+1) = \mathcal{H}_{01},$$

so that the generic part depends in fact of the difference $P - Q$.

Halmos [32] proved that there is an isometric isomorphism between $\mathcal{H}_0$ and a product Hilbert space $\mathcal{L} \times \mathcal{L}$ such that in the above decomposition (putting $\mathcal{L} \times \mathcal{L}$ in place of $\mathcal{H}_0$), the projections are

$$P = 1 \oplus 0 \oplus 1 \oplus 0 \oplus \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$Q = 1 \oplus 0 \oplus 0 \oplus 1 \oplus \begin{pmatrix} C^2 & CS \\ CS & S^2 \end{pmatrix},$$

where $C = cos(Z)$ and $S = sin(Z)$ for some operator $0 \leq Z \leq \pi/2$ in $\mathcal{L}$ with trivial nullspace.

Let us denote by $P_0 = P|_{\mathcal{H}_0}, Q_0 = Q|_{\mathcal{H}_0}$, and $X_0 = X_{P,Q}|_{\mathcal{H}_0}$. Then

$$X_{P,Q} = \begin{pmatrix} 0 & iZ \\ -iZ & 0 \end{pmatrix}.$$

Recall the definition of the *reduced minimum modulus* $\gamma(A)$ of an operator A:

$$\gamma(A) = \inf\{\|Af\| : \|f\| = 1, f \in N(A)^{\perp}\} = \inf\{\sigma(A) \setminus \{0\}\}.$$

Proposition 5.8. *Let φ, ψ be continuous unimodular functions in $\mathbb{T}$ with $w(\varphi) = w(\psi)$. Then*

$$Z = M_\varphi \cos^{-1}\left(|T_{\varphi\bar{\psi}}|\right) M_{\bar{\varphi}}$$

and, in particular,

$$\|X_{\varphi,\psi}\| = \cos^{-1}(\gamma(T_{\varphi\bar{\psi}})).$$

Proof. On the non generic part of P_φ and P_ψ, the operator $X = X_{\varphi,\psi}$ is trivial. Then, in order to compute its norm, we restrict to the generic part. In this subspace it can be described by Halmos' model,

$$X_0 = \begin{pmatrix} 0 & iZ \\ -iZ & 0 \end{pmatrix}.$$

Then

$$Q_0 P_0 Q_0 = \begin{pmatrix} C^2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Now

$$C^2 = P_\varphi P_\psi P_\varphi = (M_\varphi P_+ M_{\bar{\varphi}})(M_\psi P_+ M_{\bar{\psi}})(M_\varphi P_+ M_{\bar{\varphi}}) = M_\varphi T^*_{\varphi\bar{\psi}} T_{\varphi\bar{\psi}} M_{\bar{\varphi}} = M_\varphi |T_{\varphi\bar{\psi}}|^2 M_{\bar{\varphi}}.$$

Therefore $0 \leq C = \cos(Z) = M_\varphi |T_{\varphi\bar{\psi}}| M_{\bar{\varphi}}$, and thus, $Z = M_\varphi \cos^{-1}\left(|T_{\varphi\bar{\psi}}|\right) M_{\bar{\varphi}}$. From this formula, it follows that

$$\|X_{\varphi,\psi}\| = \|\cos^{-1}(|T_{\varphi\bar{\psi}}|)\| = \cos^{-1}(\lambda_0),$$

where

$$\lambda_0 = \inf\, \sigma(|T_{\varphi\bar{\psi}}|) = \inf\, \sigma(|T_{\varphi\bar{\psi}}|) \setminus \{0\} = \gamma(T_{\varphi\bar{\psi}}).$$

The second equality can be deduced from the assumption that $T_{\varphi\bar{\psi}}$ is injective, which implies that 0 cannot be an isolated point of $\sigma(|T_{\varphi\bar{\psi}}|)$. □

If φ, ψ are continuous unimodular functions with $w(\varphi) = w(\psi)$, by Arens-Royden's theorem there exists a unique continuous real function θ in $\mathbb{T}$, with $-\pi \leq \theta \leq \pi$ such that

$$e^{i\theta} = \bar{\varphi}\psi.$$

Let us call θ the *argument* of $\bar{\varphi}\psi$.

Theorem 5.9. *Let φ, ψ be continuous unimodular functions in $\mathbb{T}$ such that $w(\varphi) = w(\psi)$. Then*

$$\|[M_\theta, P_+]\| \geq \cos^{-1}(\gamma(T_{\bar{\varphi}\psi})),$$

where θ is the argument of $\bar{\varphi}\psi$.

Proof. We have that $e^{i\theta} = \bar{\varphi}\psi$. Consider the curve

$$\alpha(t) = M_{e^{it\theta}} P_\varphi M_{e^{-it\theta}}.$$

Apparently, $\alpha(t)$ is a smooth curve in Gr with $\alpha(0) = P_\varphi$ and $\alpha(1) = M_{\bar{\varphi}\psi} P_\varphi M_{\varphi\bar{\psi}} = P_\psi$. Then, $\alpha(t)$ is longer than the (unique) minimal geodesic which joins φH^2 and ψH^2, whose length is $\|X_{\varphi,\psi}\|$. Note that

$$\dot{\alpha}(t) = iM_{e^{it\theta}} M_\theta P_\varphi - iP_\varphi M_\theta M_{e^{-it\theta}} = iM_{e^{it\theta}} M_\varphi (M_\theta P_+ - P_+ M_\theta) M_{\bar{\varphi}} M_{e^{-it\theta}}\,.$$

Thus, we have that $\|\dot{\alpha}(t)\| = \|M_\theta P_+ - P_+ M_\theta\|$, and using Proposition 5.8, we obtain

$$\cos^{-1}(\gamma(T_{\bar{\varphi}\psi})) = \|X_{\varphi,\psi}\| \leq L(\alpha) = \int_0^1 \|\dot{\alpha}(t)\| dt = \|M_\theta P_+ - P_+ M_\theta\|.$$

□

Remark 5.10. In the above theorem, note that the operator M_θ is selfadjoint. Therefore the commutator $[M_\theta, P_+] = M_\theta P_+ - P_+ M_\theta$ is anti-hermitian. Also elementary computations show that

$$P_+[M_\theta, P_+]P_+ = P_-[M_\theta, P_+]P_- = 0,$$

i.e. $[M_\theta, P_+]$ is co-diagonal with respect to P_+. Thus, its norm equals the norm of the $1,2$ entry in the 2×2 matrix M_θ, which is the Hankel operator H_θ (with symbol θ):

$$\|[M_\theta, P_+]\| = \|P_- M_\theta P_+\| = \|H_\theta\|.$$

Then, by Nehari's theorem (see for instance [50]),

$$\|[M_\theta, P_+]\| = \inf\{\|\theta - f\|_\infty : f \in H^\infty\}.$$

Hence,

$$\|X_{\varphi,\psi}\| \leq \inf\{\|\theta - f\|_\infty : f \in H^\infty\}.$$

5.3 The third example

The idempotent E can be written as a 2×2 matrix in terms of the decomposition $\mathcal{H} = R(E) \oplus R(E)^\perp$

$$E = \begin{pmatrix} 1 & B \\ 0 & 0 \end{pmatrix},$$

where $B : R(E)^\perp \to R(E)$. Consider the operator $S = E + E^* - 1$. Note that S is selfadjoint and invertible. Indeed, its square is

$$S^2 = \begin{pmatrix} 1 + BB^* & 0 \\ 0 & 1 + B^*B \end{pmatrix}.$$

Also it is clear that $SE = E^*S$, $SE^* = ES$. Then $ES^2 = S^2E$ and $E^*S^2 = S^2E^*$. It follows that if Σ is the isometric part in the polar decomposition of S,

$$S = \Sigma|E| = |E|\Sigma$$

then $\Sigma = S|E|^{-1}$ is a selfadjoint unitary operator (i.e., a symmetry). Moreover, since E, E^* commute with S^2, they commute also with $|S| = (S^2)^{1/2}$, and Σ satisfies

$$\Sigma E = S|S|^{-1}E = SE|S|^{-1} = E^*S|S|^{-1} = E^*\Sigma.$$

Clearly also $\Sigma E^* = E\Sigma$. Recall from Corollary 3.3 ([15]),

$$P_{R(E)} = ES^{-1} \;\text{ and }\; P_{R(E^*)} = E^*S^{-1}.$$

Therefore $\Sigma P_{R(E)}\Sigma = P_{R(E^*)}$.

Remark 5.11. Any other unitary operator conjugating $P_{R(E)}$ and $P_{R(E^*)}$ will be of the form ΣW, with W a unitary operator commuting with $P_{R(E)}$. Let $Z^* = Z$ with $\|Z\| \leq \pi$ such that $e^{iZ} = \Sigma W$. Then

$$P(t) = e^{itZ}P_{R(E)}e^{-itZ}, \quad t \in [0, 1].$$

is a curve joining $P_{R(E)}$ and $P_{R(E^*)}$ with constant speed (and length) equal to

$$\|\dot{P}(t)\| = \|e^{itZ}[Z, P_{R(E)}]e^{-itZ}\| = \|[Z, P_{R(E)}]\|.$$

For instance one could choose $W = 1$. Note that since Σ is a symmetry, its spectral decomposition is very simple, namely

$$\Sigma = \frac{1}{2}(1 + \Sigma) - \frac{1}{2}(1 - \Sigma),$$

where $\frac{1}{2}(1 \pm \Sigma)$ are the eigenprojections corresponding to the eigenvalues ± 1. Then

$$\log(\Sigma) = i\frac{\pi}{2}(1 - \Sigma) \;\text{ and }\; [\log(\Sigma), P_{R(E)}] = i\frac{\pi}{2}[\Sigma, P_{R(E)}].$$

Then

$$\ell(P(t)) = \frac{\pi}{2}\|\Sigma, P_{R(E)}]\| = \frac{\pi}{2}\|\Sigma P_{R(E)} - P_{R(E)}\Sigma\| = \frac{\pi}{2}\|P_{R(E)} - \Sigma P_{R(E)}\Sigma\| = \frac{\pi}{2}\|P_{R(E)} - P_{R(E^*)}\|.$$

This curve $P(t)$ above is not a geodesic, though it is related to the geodesic which joins $P_{R(E)}$ and $P_{R(E^*)}$.

In [20] Chandler Davis considered for two projections P_1, P_2 such that P_1+P_2-1 has trivial kernel, the symmetry V obtained as above, by means of the polar decomposition $P_1+P_2-1 = V|P_1+P_2-1|$. If we put $P_1 = P_{R(E)}$ and $P_2 = P_{R(E^*)}$, then

$$P_1+P_2-1 = P_{R(E)} + P_{N(E)^\perp} - 1 = P_{R(E)} - P_{N(E)}.$$

It follows from Corollary 3.3 that this latter operator is precisely S^{-1}. Therefore, in this case Davis' symmetry V coincides with Σ: they are, respectively, the unitary parts in the polar decompositions of S^{-1} and S (the unitary part of a selfadjoint operator A with trivial nullspace is in fact the sign function $sign(A)$; clearly $sign(S) = sign(S^{-1})$) .

On the other hand, in [3] it was shown that if there exists a unique geodesic joining P and Q, then it is given by

$$V(2P-1) = e^{iX_{P,Q}}.$$

Thus, in our situation, $X = X_{P_{R(E)}, P_{R(E^*)}}$ is given by

$$\Sigma(2P_{R(E)}-1) = e^{iX}, \ \text{ i.e., } X = \log(\Sigma(2P_{R(E)}-1)).$$

In order to compute the norm of this operator, we shall need the matrix form of e^{iX}:

$$e^{iX} = S(S^2)^{-1/2}(2P_{R(E)}-1) =$$

$$= \begin{pmatrix} 1 & B \\ B^* & -1 \end{pmatrix} \begin{pmatrix} (1+BB^*)^{-1/2} & 0 \\ 0 & (1+B^*B)^{-1/2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} =$$

$$= \begin{pmatrix} (1+BB^*)^{-1/2} & -B(1+B^*B)^{-1/2} \\ B^*(1+BB^*)^{-1/2} & (1+B^*B)^{-1/2} \end{pmatrix}.$$

If $\xi, \psi \in \mathcal{H}$, let $\psi \otimes \xi$ denote the rank one operator given by $\eta \mapsto <\eta, \xi> \psi$. We shall also need the following result:

Lemma 5.12. *Suppose that $B : R(E)^\perp \to R(E)$ has a singular value decomposition*

$$B = \sum_{n\geq 1} r_n \psi_n \otimes \xi_n,$$

where $\{\psi_n\}$ and $\{\xi_n\}$ are orthonormal systems in $R(E)$ and $R(E)^\perp$, respectively. Then X is diagonalizable, with eigenvalues $\pm\arctan(r_n)$.

Proof. Note that $B^* = \sum_{n\geq 1} r_n \xi_n \otimes \psi_n$, $BB^* = \sum_{n\geq 1} r_n^2 \psi_n \otimes \psi_n$ and $B^*B = \sum_{n\geq 1} r_n^2 \xi_n \otimes \xi_n$. Fix $n_0 \geq 1$. Note that $\xi_{n_0} \in R(E)^\perp$ and $\psi_{n_0} \in R(E)$ are orthogonal, and span an invariant subspace for e^{iX}:

$$e^{iX}\xi_{n_0} = \begin{pmatrix} (1+BB^*)^{-1/2} & -B(1+B^*B)^{-1/2} \\ B^*(1+BB^*)^{-1/2} & (1+B^*B)^{-1/2} \end{pmatrix} \begin{pmatrix} 0 \\ \xi_{n_0} \end{pmatrix} = \begin{pmatrix} -B(1+B^*B)^{-1/2}\xi_{n_0} \\ (1+B^*B)^{-1/2}\xi_{n_0} \end{pmatrix} =$$

$$= \begin{pmatrix} -r_{n_0}(1+r_{n_0}^2)^{-1/2}\psi_{n_0} \\ (1+r_{n_0}^2)^{-1/2}\xi_{n_0} \end{pmatrix}.$$

Similarly,

$$e^{iX}\psi_{n_0} = \begin{pmatrix} (1+r_{n_0}^2)^{-1/2}\psi_{n_0} \\ r_{n_0}(1+r_{n_0}^2)^{-1/2}\xi_{n_0} \end{pmatrix}.$$

Moreover, the matrix of e^{iX} restricted to the subspace spanned by ψ_{n_0}, ξ_{n_0} (written in this orthonormal basis) is

$$\begin{pmatrix} (1+r_{n_0}^2)^{-1/2} & -r_{n_0}(1+r_{n_0}^2)^{-1/2} \\ r_{n_0}(1+r_{n_0}^2)^{-1/2} & (1+r_{n_0}^2)^{-1/2} \end{pmatrix},$$

whose eigenvalues are $(1+r_{n_0}^2)^{-1/2} \pm ir_{n_0}(1+r_{n_0}^2)^{-1/2}$. The full operator e^{iX} is therefore diagonalized as an orthogonal sum of 2×2 blocks, each block having these eigenvalues. Elementary computations show that the eigenvalues of X are $\pm\arctan(r_n)$. □

Proposition 5.13. *In the general case, for arbitrary $B : R(E)^\perp \to R(E)$, the norm of X is*

$$\|X\| = \arctan(\|B\|).$$

Proof. One can approximate B with B_k having singular values decompositions: for instance, put $B = V_0|B|$ the polar decomposition of B, and approximate $|B|$ with diagonalizable operators. Then, if we denote by X_k the exponent induced as above by the operator B_k, it is clear that $X_k \to X$ (=the exponent corresponding to B). Then $\|X_k\| \to \|X\|$. On the other hand, denoting by $\{r_{n,k} : n \geq 1\}$ the singular values of B_k, from the above Lemma, it is clear that

$$\|X_k\| = \sup_{n\geq 1}\arctan(r_{n,k}) = \arctan(\sup_{n\geq 1} r_{n,k}) = \arctan(\|B_k\|) \to \arctan(\|B\|).$$

□

Corollary 5.14. *Let $Z^* = Z$ such that $e^{iZ} = \Sigma W$, where W is a unitary operator that commutes with $P_{R(E)}$. Then*

$$\|[Z, P_{R(E)}]\| \geq \arctan(\|B\|).$$

Remark 5.15. Note that this implies that for any idempotent E,

$$\|P_{R(E)} - P_{R(E^*)}\| < 1.$$

Remark 5.16. Recall from Section 4, that the geodesic distance is related with the norm by the equation

$$d(P_{R(E)}, P_{N(E)^\perp}) = \sin^{-1}(\|P_{R(E)} - P_{R(E^*)}\|).$$

Combining these facts, after elementary computations, one gets that

$$\|P_{R(E)} - P_{R(E^*)}\| = \frac{\|B\|}{(1+\|B\|^2)^{1/2}}.$$

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(Esteban Andruchow) Instituto de Ciencias, Universidad Nacional de Gral. Sarmiento, J.M. Gutierrez 1150, (1613) Los Polvorines, Argentina and Instituto Argentino de Matemática, 'Alberto P. Calderón', CONICET, Saavedra 15 3er. piso, (1083) Buenos Aires, Argentina.

(Gustavo Corach) Instituto Argentino de Matemática, 'Alberto P. Calderón', CONICET, Saavedra 15 3er. piso, (1083) Buenos Aires, Argentina, and Depto. de Matemática, Facultad de Ingeniería, Universidad de Buenos Aires, Argentina.