HIGHER ORDER ELLIPTIC EQUATIONS IN HALF SPACE

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ABSTRACT. We consider $2m^{th}$ order elliptic equations with Holder coefficients in half space.We solve the Dirichlet problem with m conditions on the boundary of the upper half space. We first analyze the constant coefficient case, finding the Greens function and a representation formula. We then prove Schauder estimates.

1. INTRODUCTION

In this paper we solve the Dirichlet problem for elliptic operators of order $2m$ in the upper half space, with complex valued H*ö*lder coefficient. Our method is based on finding explicit formulas for the Green function for the constant coefficient equation and proving a priori estimates in the Holder spaces for solutions having homogeneos boundary data. This work differs in this sense from the well known paper of Agmon, Douglis and Nirenberg in which they solve first, the homogeneos problem with non homogeneus boundary data for an elliptic operator with only principal part. A Radon type transformation due to F. John is key to their work. See references [1] and [4]. Our aproach is based on transforming Fourier on the first variables the nonhomogeneos equation with homogeneos data. We find a representation formula which is key to the apriori estimates. We first treat the constant coefficient case and then, using the freezing coefficient method, we treat the variable coefficient equation. We remark that troughout the paper we will use the letter C to denote a constant (not always the same) that depends only on structure.In the case that a constant depends on any aditional quantity we will mention this explicitly. Look reference [5] for a treatment of the problem in full space and [8] for the case of smooth coefficients.

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2. Preliminaries

In this short section we introduce the Hölder spaces that will be relevant later on.

Set $\Omega = R^{n+1}_+$.

Define $|u|_0 = \sup\{|u(x)| : x \in \Omega\}$

$[u]_\delta = \sup\{\frac{|u(x)-u(y)|}{|x-y|^\delta} : x, y \in \Omega\}$

$|D^k u|_0 = \max\{|D^\alpha u|_0 : |\alpha| = k\}$ and $[D^k u]_\delta = \max\{[D^\alpha u]_\delta : |\alpha| = k\}$

And define,

$|u|_{m+\delta} = \sum_{k=0}^m |D^k u|_0 + [D^m u]_\delta$.

The space $C^{m+\delta}(\Omega) = \{u : |u|_{m+\delta} < \infty\}$ is a Banach space.

We will need to use the following interpolation result which we state without proof. See reference [3].

interpolation **Lemma 2.1.** *For every $\epsilon > 0$ and $0 \leq k \leq m$ there exists $C_{\epsilon,k}$ such that $[D^k u]_\delta \leq C_{\epsilon,k}|u|_0 + \epsilon|u|_{m+\delta}$*

For every $\epsilon > 0$ and $0 \leq k \leq m-1$ there exists $C_{\epsilon,k}$ such that $|D^k u|_0 \leq C_{\epsilon,k}|u|_0 + \epsilon|D^m u|_0$

And for every $\epsilon > 0$ and $0 \leq k \leq m-1$ there exists $C_{\epsilon,k}$ such that $[D^k u]_\delta \leq C_{\epsilon,k}|u|_0 + \epsilon|D^m u|_0$

for all $u \in C^{m+\delta}(\Omega)$

3. Constant coefficients

We consider operators of the form $Lu(x,t) = \sum_{|\alpha|+l \leq 2m} a_{\alpha,l} D^\alpha_x D^l_t u(x,t)$

where $a_{\alpha,l}$ are complex numbers, $x \in R^n$ and $t > 0$.

Let $L^\star u(x,t) = \sum_{|\alpha|+l \leq 2m} (-1)^{|\alpha|+l} a_{\alpha,l} D^\alpha_x D^l_t u(x,t)$

We will assume that the characteristic polynomial $p(i\xi, it) = \sum_{|\alpha|+l \leq 2m} a_{\alpha,l}(i\xi)^\alpha (it)^l$ satisfies the strong ellipticity condition

ellipticity
$$|p(i\xi, it)| \geq \lambda(1 + |\xi|^2 + t^2)^m \tag{3.1}$$

for all $(\xi, t) \in R^{n+1}$ Notice that $p(i\xi, z)$ as a polynomial in the z variable has no pure imaginary roots. It follows that we can write

$$p(i\xi, z) = p^+(i\xi, z)p^-(i\xi, z) \tag{3.2}$$

where $p^-(i\xi, z) = \prod_{j=1}^m (z - \lambda_j^-(\xi))$ and $p^+(i\xi, z) = \prod_{j=1}^m (z - \lambda_j^+(\xi))$ and

$\{\lambda_j^- : j = 1, ..., m\}$ are the roots of $p(i\xi, z)$ with negative real part and $\{\lambda_j^+ : j = 1, ..., m\}$ are the roots of $p(i\xi, z)$ with positive real part. This is true for all $\xi \in R^n$.

We will be able to impose m initial conditions on a solution u at $t = 0$

roots1 **Lemma 3.1.** *If* $p(i\xi, z) = 0$ *then* $|Re(z)| \geq \frac{\lambda}{C}(1 + |\xi|)$ *and* $|z| \leq C(1 + |\xi|)$, *for some* C *depending on* $\max\{|a_{\alpha,l}| : \alpha, l\}$

Proof. The proof is a straightforward application of (3.1) ellipticity □

We believe that the next property is also a consequence of the ellipticity condition (3.1) ellipticity and some general theorem on the roots of a complex polynomial but at this point we dont have a proof so we need to assume it.

Essential assumption about the roots $\lambda(\xi)$ of $p(i\xi, z)$:

We assume that there exists a constant C depending only on n, α, λ and $\max\{|a_{\alpha,l}| : \alpha, l\}$ such that if $\lambda(\xi)$ is a root of $p(i\xi, z)$, then for any multiindex α, we have

roots2 (3.3)
$$|D_\xi^\alpha \lambda(\xi)| \leq C(1 + |\xi|)^{1-\alpha}$$

for all $\xi \in R^n$.

We will now construct the Green function.

Let us denote by $\gamma^-(\xi)$ any simple closed curve such that $Re(\gamma^-(\xi)) \leq 0$ and $\gamma^-(\xi)$ encloses the m roots of $p^-(i\xi, z)$. And similarly, $Re(\gamma^+(\xi)) \geq 0$ and $\gamma^+(\xi)$ encloses the m roots of $p^+(i\xi, z)$. In the sequel, we will use different curves $\gamma^-(\xi)$, and $\gamma^+(\xi)$.

Define,

$h(y, s) = \int \int_{\gamma^-(\xi)} \frac{e^{sz}}{p(i\xi,z)} dz \;\; e^{iy\dot{\xi}} d\xi$ for $s > 0$ and

$h(y, s) = \int \int_{\gamma^+(\xi)} \frac{-e^{sz}}{p(i\xi,z)} dz \;\; e^{iy\cdot\xi} d\xi$ for $s < 0$

and define, for any positive integer l such that $2l + m \geq n + 2$ the function,

$h^\star(y, s) = \frac{1}{(|y|^2+s^2)^l} \int_{R^n} \int_{-\infty}^{+\infty} e^{i(y\cdot\xi + s\xi_{n+1})} (-\Delta)^l (\frac{1}{p(i\xi, i\xi_{n+1})}) d\xi_{n+1} d\xi$

hstar **Lemma 3.2.** $h = h^\star$

Proof. Formally, it follows by enlarging the contour of integration: Say, $s > 0$, then $\int_{\gamma^-(\xi)} \frac{e^{sz}}{p(i\xi,z)} dz = \int_{-\infty}^{+\infty} \frac{e^{is\xi_{n+1}}}{p(i\xi, i\xi_{n+1})} d\xi_{n+1}$ and an integration by parts show

$\int_{-\infty}^{+\infty} \int_{R^n} \frac{e^{i(y\cdot\xi + s\xi_{n+1})}}{p(i\xi, i\xi_{n+1})} d\xi d\xi_{n+1} = h^\star(y, s)$.

We proceed with a rigourous proof.

Let $\beta \in C_0^\infty(B_2(0))$ with $\beta = 1$ on $B_1(0)$ and $\phi \in C_0^\infty([-2, 2])$ and $\phi = 1$ in $[-1, 1]$.

Fix (y, s) with $y \in R^n$ and $s > 0$ and let $\epsilon > 0$. We show $|h(y, s) - h^\star(y, s)| \leq \epsilon$.

Write $h(y, s) - h^\star(y, s) = A + B + C + D$ where

$A = h(y, s) - \int \beta(\frac{\xi}{R}) \int_{\gamma^-(\xi)} \frac{e^{sz}}{p(i\xi,z)} dz \, e^{iy\cdot\xi} d\xi$

$B = \int \beta(\frac{\xi}{R}) \int_{\gamma^-(\xi)} \frac{e^{sz}}{p(i\xi,z)} dz\, e^{iy\cdot\xi} d\xi - \int \beta(\frac{\xi}{R}) \int_{-\infty}^{+\infty} \frac{e^{is\xi_{n+1}}}{p(i\xi,i\xi_{n+1})} d\xi_{n+1}\, e^{iy\cdot\xi} d\xi.$
$C = \int \beta(\frac{\xi}{R}) \int_{-\infty}^{+\infty} \frac{e^{is\xi_{n+1}}}{p(i\xi,i\xi_{n+1})} d\xi_{n+1}\, e^{iy\cdot\xi} d\xi - \int_{-\infty}^{+\infty} \int_{R^n} \beta(\frac{\xi}{R}) \phi(\frac{\xi_{n+1}}{\tilde{R}}) \frac{e^{i(y\cdot\xi+s\xi_{n+1})}}{p(i\xi,i\xi_{n+1})} d\xi d\xi_{n+1}$
$D = \int_{-\infty}^{+\infty} \int_{R^n} \beta(\frac{\xi}{R}) \phi(\frac{\xi_{n+1}}{\tilde{R}}) \frac{e^{i(y\cdot\xi+s\xi_{n+1})}}{p(i\xi,i\xi_{n+1})} d\xi d\xi_{n+1} - h^\star(y,s).$
We will choose R and $\tilde{R}$ large enough to make each term small.

To estimate A we can take the contour $\gamma^-(\xi)$ so that $Re(\gamma^-(\xi)) \le -C(1+|\xi|)$ and $|p(i\xi,z)| \ge C$ for all $z \in \gamma^-(\xi)$.

We have $|A| \le |\int (1-\beta(\frac{\xi}{R})) \int_{\gamma^-(\xi)} \frac{e^{sz}}{p(i\xi,z)} dz\, e^{iy\cdot\xi} d\xi| \le \int_{|\xi|\ge R} |\int_{\gamma^-(\xi)} \frac{e^{sz}}{p(i\xi,z)} dz| d\xi \le$

$$\int_{|\xi|\ge R} e^{-Cs(1+|\xi|)} \int_{\gamma^-(\xi)} \frac{1}{|p(i\xi,z)|} dz d\xi \le C \int_{|\xi|\ge R} lenght(\gamma^-(\xi)) e^{-Cs(1+|\xi|)} d\xi$$

$\le C \int_{|\xi|\ge R} (1+|\xi|) e^{-Cs(1+|\xi|)} d\xi.$

This last expression goes to 0 as R goes to ∞

To estimate B, take $\gamma^-(\xi) = I \cup C_M = \{i\xi_{n+1} : -M \le \xi_{n+1} \le M\} \cup \{Me^{i\theta} : \theta \in [\frac{\pi}{2}, \frac{3\pi}{2}]\}$.

Where $M = M(|\xi|) = K(1+|\xi|)^r$ and r and K are chosen large.

So, $B \le |\int \beta(\frac{\xi}{R})) \int_{C_M} \frac{e^{sz}}{p(i\xi,z)} dz\, e^{iy\cdot\xi} d\xi| + |\int \beta(\frac{\xi}{R}) \int_{|\xi_{n+1}|\ge M} \frac{e^{is\xi_{n+1}}}{p(i\xi,i\xi_{n+1})} d\xi_{n+1} e^{iy\cdot\xi} d\xi| = |B_1| + |B_2|$.

Notice that we can write $p(i\xi,z) = \sum_{j=0}^m a_j(\xi) z^j$ whith $|a_j(\xi)| \le C|\xi|^{m-j}$. Hence for $z \in C_M$ we have $|p(i\xi,z)| \ge C|z|^m$ and hence

$|B_1| \le C \int_{|\xi|\le 2R} \frac{lenght(C_M)}{M^m} d\xi \le C \int_{|\xi|\le 2R} \frac{1}{M^{m-1}} d\xi \le \frac{C}{K^{m-1}} \int \frac{1}{(1+|\xi|)^{r(m-1)}} d\xi$ which can be made arbitrarily small choosing r and K large.

And $|B_2| \le C \int_{|\xi|\le 2R} \int_{|\xi_{n+1}|\ge M} \frac{d\xi_{n+1}}{1+|\xi|^m+|\xi_{n+1}|^m} d\xi \le C \int_{|\xi|\le 2R} \frac{1}{M} d\xi$ which again can be made arbitrarily small choosing r and K large.

$|C| \le \int \beta(\frac{\xi}{R}) \int_{|\xi_{n+1}|\ge \tilde{R}} \frac{d\xi_{n+1}}{1+|\xi|^m+|\xi_{n+1}|^m} d\xi \le C\frac{R^n}{\tilde{R}}$, which we make small making $\tilde{R}$ large depending on R.

To estimate D, first notice that $\int_{-\infty}^{+\infty} \int_{R^n} \beta(\frac{\xi}{R}) \phi(\frac{\xi_{n+1}}{\tilde{R}}) \frac{e^{i(y\cdot\xi+s\xi_{n+1})}}{p(i\xi,i\xi_{n+1})} d\xi d\xi_{n+1} =$

$$\frac{1}{(|y|^2+s^2)^l} \int_{-\infty}^{+\infty} \int_{R^n} (-\Delta)^l \left(\frac{\beta(\frac{\xi}{R})\phi(\frac{\xi_{n+1}}{\tilde{R}})}{p(i\xi,i\xi_{n+1})}\right) e^{i(y\cdot\xi+s\xi_{n+1})} d\xi d\xi_{n+1} =$$

$\frac{1}{(|y|^2+s^2)^l} \int_{-\infty}^{+\infty} \int_{R^n} (-\Delta)^l (\frac{1}{p(i\xi,i\xi_{n+1})}) e^{i(y\cdot\xi+s\xi_{n+1})} d\xi d\xi_{n+1} + E$

where we have the estimate $|E| \le \frac{C}{(|y|^2+s^2)^l} \frac{1}{R}$ which holds for $\tilde{R} \ge R \ge 1$. Therefore,

$|D| \le \frac{1}{(|y|^2+s^2)^l} \int_{-\infty}^{+\infty} \int_{R^n} (1-\beta(\frac{\xi}{R})) \phi(\frac{\xi_{n+1}}{\tilde{R}}) (-\Delta)^l (\frac{1}{p(i\xi,i\xi_{n+1})}) e^{i(y\cdot\xi+s\xi_{n+1})} d\xi d\xi_{n+1} + \frac{C}{(|y|^2+s^2)^l} \frac{1}{R} \le \frac{1}{(|y|^2+s^2)^l} \int_{|\xi_{n+1}|\ge R} \int_{|\xi|\ge R} \frac{d\xi d\xi_{n+1}}{(1+|\xi|+|\xi_{n+1}|)^{m+2l}} + \frac{C}{(|y|^2+s^2)^l} \frac{1}{R} \le \frac{C}{(|y|^2+s^2)^l} \frac{1}{R}$

Choose R large enough to make $|A|$ and $|D|$ smaller than ϵ and then for fixed R, choose $\tilde{R} \ge R$ to make $|C| \le \epsilon$ $\square$

We state in a lemma some well known properties of the function h. See reference [5].

h **Lemma 3.3.**

$$h \in C^\infty(R^{n+1} \backslash \{0\})$$

$$|D_y^\alpha D_s^k h(y,s)| \leq \frac{C}{(|y|+|s|)^{n+1-2m+|\alpha|+k}}$$

for all $|y|^2 + s^2 \leq 1$ *and for any* k,$|\alpha|$.

$$|D_y^\alpha D_s^k h(y,s)| \leq \frac{C}{(|y|^2+s^2)^l}$$

for any l,$|\alpha| \geq 0$,$k \geq 0$ *and* $|y|^2 + s^2 \geq 1$.

$$Lh(y,s) = 0$$

for $(y,s) \neq (0,0)$.

Let us also define

$$h_R(y,s) = \int_{R^{n+1}} \frac{\varphi(\frac{\xi,\xi_{n+1}}{R}) e^{i(y\cdot\xi + s\xi_{n+1})}}{p(i\xi, i\xi_{n+1})} d\xi d\xi_{n+1}$$

Note, as in the estimation of term D in lemma hstar (3.2) that $h_R(y,s) \to h(y,s)$ as $R \to \infty$ for any $(y,s) \neq (0,0)$ and uniformly away from $(0,0)$.

Here $\varphi \in C_0^\infty(B_2(0,0))$ and $\varphi = 1$ on $B_1(0,0)$.

We next define a function k to account for the boundary values.

For $y \in R^n$ and $t > 0$, $\tau > 0$, define

$$k(y,t,\tau) = \int \int_{\gamma^+(\xi)} \frac{e^{-\tau z}}{p^+(i\xi,z)} \int_{\gamma^-(\xi)} \frac{e^{tw}}{p^-(i\xi,w)(w-z)} dw\, dz\, e^{iy\cdot\xi} d\xi$$

$$\textit{Let } \Phi(\xi,t,\tau) = \int_{\gamma^+(\xi)} \frac{e^{-\tau z}}{p^+(i\xi,z)} \int_{\gamma^-(\xi)} \frac{e^{tw}}{p^-(i\xi,w)(w-z)} dw\, dz.$$

To analize the function k we take into account that according to roots1 (3.1) we have that the m roots of $p^-(i\xi,z)$ satisfy $Re(\lambda^-(\xi)) \leq -\lambda(1+|\xi|)$ and $|\lambda^-(\xi)| \leq \Lambda(1+|\xi|)$ so we take $\gamma^-(\xi)$ a piecewise smooth contour parametrized by an angle θ which is the arc of the circle centered at 0 of radius $2\Lambda(1+|\xi|)$ joinig the points with real part equal to $\frac{-\lambda}{2}(1+|\xi|)$ in counterclockwise sense, followed by the vertical segment joining these two points. Denote by $w(\theta,\xi)$ the points on this contour.

Similarly, since the m roots of $p^+(i\xi, z)$ satisfy $Re(\lambda^+(\xi)) \geq \lambda(1+|\xi|)$ and $|\lambda^+(\xi)| \leq \Lambda(1+|\xi|)$ we take $\gamma^+(\xi)$ piecewise smooth contour parametrized by an angle ϕ which is the arc of the circle centered at 0 of radius $2\Lambda(1+|\xi|)$ joinig the points with real part equal to $\frac{\lambda}{2}(1+|\xi|)$ in counterclockwise sense, followed by the vertical segment joining these two points. Denote by $z(\phi, \xi)$ the points on this contour.

The following properties follow directly:

For any multiindex α and for all ξ, we have

$|D^\alpha_\xi w(\theta, \xi)| \leq C(1+|\xi|)^{1-|\alpha|}$, $|D^\alpha_\xi z(\phi, \xi)| \leq C(1+|\xi|)^{1-|\alpha|}$, $|D^\alpha_\xi D_\theta w(\theta, \xi)| \leq C(1+|\xi|)^{1-|\alpha|}$, $|D^\alpha_\xi D_\phi z(\phi, \xi)| \leq C(1+|\xi|)^{1-|\alpha|}$.

And for all ξ, we have

$\frac{\lambda}{2}(1+|\xi|) \leq |w(\theta, \xi) - \lambda^-(\xi)| \leq 2\Lambda(1+|\xi|)$ and $\frac{\lambda}{2}(1+|\xi|) \leq |z(\phi, \xi) - \lambda^+(\xi)| \leq 2\Lambda(1+|\xi|)$ and $\frac{\lambda}{2}(1+|\xi|) \leq |w(\theta, \xi) - z(\phi, \xi)| \leq 4\Lambda(1+|\xi|)$.

Using the above properties toghether with (3.3) we can prove the following estimtate.

Phi

$$|D^\alpha_\xi D^l_\tau \Phi(\xi, t, \tau)| \leq C \frac{e^{\frac{-(t+\tau)(1+|\xi|)}{2}}}{(1+|\xi|)^{2m-1+|\alpha|-l}} \tag{3.4}$$

for any α and any l; and for any $\xi \in R^n$, and any $\tau > 0$,$t > 0$.

To prove the estimate, we note that we can write $\Phi(\xi, t, \tau)$ as a sum of four terms of the form

$$\int_\theta \int_\phi \frac{e^{-\tau z(\xi,\phi)}}{p^+(i\xi, z(\xi, \phi))} \frac{e^{tw(\xi,\theta)}}{p^-(i\xi, w(\xi, \theta))} \frac{z_\phi(\xi, \phi) w_\theta(\xi, \theta)}{w(\xi, \theta) - z(\xi, \phi)} d\phi\, d\theta$$

for the apropiate limits of integrations in θ and ϕ which do not depend on ξ.

Write the integrand as $\frac{g_1 g_2}{g_3}$ where $g_1 = e^{-\tau z(\xi,\phi)} e^{tw(\xi,\theta)}$, $g_2 = z_\phi(\xi, \phi) w_\theta(\xi, \theta)$ and $g_3 = p^+(i\xi, z(\xi, \phi)) p^-(i\xi, w(\xi, \theta))(w(\xi, \theta) - z(\xi, \phi))$.

The following estimates follow by direct computation.

$|D^\gamma_\xi g_1| \leq C \frac{e^{\frac{-(t+\tau)(1+|\xi|)}{2}}}{(1+|\xi|)^{|\gamma|}}$.

$|D^\gamma_\xi g_2| \leq C(1+|\xi|)^{2-|\gamma|}$

Also note that we can write $g_3 = \prod_{j=1}^{2m+1} (\phi_j(\xi) - \psi_j(\xi))$ where $|\phi_j(\xi) - \psi_j(\xi)| \approx 1 + |\xi|$ and $|D^\alpha_\xi(\phi_j(\xi) - \psi_j(\xi))| \leq C(1+|\xi|)^{1-|\alpha|}$.

It follows that $|D^\gamma_\xi g_3| \leq C(1+|\xi|)^{2m+1-|\gamma|}$ and hence $|D^\gamma_\xi (g_3)^{-1}| \leq \frac{C}{(1+|\xi|)^{2m+|\gamma|+1}}$.

Putting the estimates toghether proves the claim.

Now using, the estimate (Phi 3.4), we can prove the following estimates for the function k.

k **Lemma 3.4.** *The function $k(y,t,\tau)$ is C^∞ in all its variables.*

$$|D_y^\alpha D_\tau^l k(y,t,\tau)| \le \frac{C}{(|y|+t+\tau)^{n+1-2m+|\alpha|+l}}$$

for all α and l.

$$\int_0^\infty \int_{R^{n-1}} |D_y^\alpha D_\tau^k k(y,t,\tau)||_{|x_k-y_k|=R} dy'\, d\tau \to 0$$

as $R\to\infty$ for any α and any k and $t>0$.

$$\int_{R^n} |D_y^\alpha D_\tau^k k(y,t,R)| dy \to 0$$

as $R\to\infty$, for any α and any k and $t>0$. And,

$$\int_0^\infty \int_{R^n} |k(y,t,\tau)| dy d\tau \le C$$

for any $t>0$. C is independent of t.

Proof. We have $k(y,t,\tau) = \int e^{iy\cdot\xi}\Phi(\xi,t,\tau)d\xi$ and $D_y^\alpha D_\tau^l k(y,t,\tau) = \int e^{iy\cdot\xi}(i\xi)^\alpha D_\tau^l \Phi(\xi,t,\tau)d\xi$ and the first assertion follows inmediately.

Let $\eta \in C_0^\infty(B_2(0))$, such that $\eta = 1$ in $B_1(0)$.

To prove the first estimate, write $D_y^\alpha D_\tau^l k(y,t,\tau) = \int e^{iy\cdot\xi}(i\xi)^\alpha D_\tau^l\Phi(\xi,t,\tau)d\xi =$
$\frac{1}{|y|^{2k}}\int e^{iy\cdot\xi}(-\Delta_\xi)^k((i\xi)^\alpha D_\tau^l\Phi(\xi,t,\tau))d\xi =$
$\frac{1}{|y|^{2k}}\int e^{iy\cdot\xi}(-\Delta_\xi)^k((i\xi)^\alpha D_\tau^l\Phi(\xi,t,\tau))\eta(|y|\xi)d\xi +$
$\frac{1}{|y|^{2k}}\int e^{iy\cdot\xi}(-\Delta_\xi)^k((i\xi)^\alpha D_\tau^l\Phi(\xi,t,\tau))(1-\eta(|y|\xi))d\xi =$
$A+B$.

And we have $A = \frac{1}{|y|^{2k}}\int(-\Delta_\xi)^k(e^{iy\cdot\xi}(i\xi)^\alpha)D_\tau^l\Phi(\xi,t,\tau))\eta(|y|\xi)d\xi$ and hence,
$|A| \le C\int_{|\xi|\le\frac{2}{|y|}} |\xi^\alpha D_\tau^l\Phi(\xi,t,\tau))|d\xi \le C\int_{|\xi|\le\frac{2}{|y|}} \frac{e^{-(t+\tau)|\xi|}}{(1+|\xi|)^{2m-1-|\alpha|-l}}d\xi.$

Note that $\int_{|\xi|\le\frac{2}{|y|}} \frac{e^{-(t+\tau)|\xi|}}{(1+|\xi|)^{2m-1-|\alpha|-l}}d\xi \le \int_{|\xi|\le\frac{2}{|y|}} \frac{1}{(1+|\xi|)^{2m-1-|\alpha|-l}}d\xi \le \frac{C}{|y|^{n+1-2m+|\alpha|+l}}$ and also
$\int_{|\xi|\le\frac{2}{|y|}} \frac{e^{-(t+\tau)|\xi|}}{(1+|\xi|)^{2m-1-|\alpha|-l}}d\xi \le \frac{C}{(t+\tau)^{n+1-2m+|\alpha|+l}}\int_{|\xi|\le\frac{2(t+\tau)}{|y|}} \frac{e^{-|\xi|}}{(t+\tau+|\xi|)^{2m-1-|\alpha|-l}}d\xi \le \frac{C}{(t+\tau)^{n+1-2m+|\alpha|+l}}\int_{R^n}\frac{e^{-|\xi|}}{|\xi|^{2m-1-|\alpha|-l}}d\xi \le \frac{C}{(t+\tau)^{n+1-2m+|\alpha|+l}}$

Therefore, $|A| \le \min\{\frac{1}{|y|^{n+1-2m+|\alpha|+l}}, \frac{1}{(t+\tau)^{n+1-2m+|\alpha|+l}}\} \le \frac{C}{(|y|+t+\tau)^{n+1-2m+|\alpha|+l}}$

To estimate the term B, we note that $|B| \le \frac{C}{|y|^{2k}}\int_{|\xi|\ge\frac{1}{|y|}} \frac{e^{-(t+\tau)|\xi|}}{(1+|\xi|)^{2m-1-|\alpha|-l+2k}}.$

And $\frac{C}{|y|^{2k}} \int_{|\xi| \geq \frac{1}{|y|}} \frac{e^{-(t+\tau)|\xi|}}{(1+|\xi|)^{2m-1-|\alpha|-l+2k}} \leq \frac{C}{|y|^{2k}} \int_{|\xi| \geq \frac{1}{|y|}} \frac{1}{(|\xi|^{2m-1-|\alpha|-l+2k}} \leq \frac{C}{|y|^{n+1-2m+|\alpha|+l}}$.

But also, for $|y| \leq t+\tau$, we have $\frac{C}{|y|^{2k}} \int_{|\xi| \geq \frac{1}{|y|}} \frac{e^{-(t+\tau)|\xi|}}{(1+|\xi|)^{2m-1-|\alpha|-l+2k}} \leq$ $\frac{C}{(t+\tau)^{n+1-2m+|\alpha|+l}} (\frac{t+\tau}{|y|})^{2k} \int_{|\xi| \geq \frac{t+\tau}{|y|}} \frac{e^{-|\xi|}}{(t+\tau+|\xi|)^{2m-1-|\alpha|-l+2k}} d\xi \leq \frac{C}{(t+\tau)^{n+1-2m+|\alpha|+l}} (\frac{t+\tau}{|y|})^{2k} \int_{|\xi| \geq \frac{t+\tau}{|y|}} \frac{C_r}{|\xi|^r} d\xi \leq \frac{C}{(t+\tau)^{n+1-2m+|\alpha|+l}}$, by choosing $r = 2k+n$.

Therefore, $|B| \leq \min\{\frac{1}{|y|^{n+1-2m+|\alpha|+l}}, \frac{1}{(t+\tau)^{n+1-2m+|\alpha|+l}}\} \leq \frac{C}{(|y|+t+\tau)^{n+1-2m+|\alpha|+l}}$

which finishes the proof of the first estimate.

To prove the other estimates, we note that for any k, we have

$D_y^\alpha D_\tau^l k(x-y,t,\tau) = \frac{1}{|x-y|^{2k}} \int e^{i(x-y)\cdot\xi} (-\Delta_\xi)^k ((i\xi)^\alpha D_\tau^l \Phi(\xi,t,\tau)) d\xi$. And, hence,

$|D_y^\alpha D_\tau^l k(x-y,t,\tau)| \leq \frac{1}{|x-y|^{2k}} \int \frac{e^{-(t+\tau)(1+|\xi|)}}{(1+|\xi|)^{2m-1-l-|\alpha|+2k}} d\xi$

Therefore,

$\int_{R^{n-1}} \int_0^\infty |D_y^\alpha D_\tau^l k(x-y,t,\tau)||_{|x_k-y_k|=R} d\tau dy' \leq$

$\int_{R^{n-1}} \frac{dy'}{(|x'-y'|^2+R^2)^{2k}} \int_0^\infty \int \frac{e^{-(t+\tau)(1+|\xi|)}}{(1+|\xi|)^{2m-1-l-|\alpha|+2k}} d\xi d\tau$

$\leq \int_{R^{n-1}} \frac{dy'}{(|x'-y'|^2+R^2)^{2k}} \int \frac{d\xi}{(1+|\xi|)^{2m-l-|\alpha|+2k}} \to 0$ as $R \to \infty$.

To prove $\int_{R^n} |D_y^\alpha D_\tau^k k(y,t,R)| dy \to 0$, use the first estimate $|D_y^\alpha D_\tau^l k(y,t,R)| \leq \frac{C}{(|y|+t+R)^{n+1-2m+|\alpha|+l}}$ for $|y| \leq 1$ and use $|D_y^\alpha D_\tau^l k(y,t,R)| \leq \frac{1}{|y|^{2k}} \int \frac{e^{-(t+R)(1+|\xi|)}}{(1+|\xi|)^{2m-1-l-|\alpha|+2k}} d\xi$ for $|y| \geq 1$.

And similarly for the last estimate.

□

We next state a lemma from complex analysis which follows by shifting contour of integration.

CA **Lemma 3.5.** *We let $\gamma(\xi)$ be a simple closed contour enclosing all roots of $p(i\xi, z)$ and $\gamma^+(\xi)$, $\gamma^-(\xi)$ as before.*

We have

$\frac{1}{2\pi i} \int_{\gamma(\xi)} \frac{z^k}{p(i\xi,z)} dz = 0$ *for* $0 \leq k \leq 2m-2$ *and* $= 1$ *for* $k = 2m-1$

$\frac{1}{2\pi i} \int_{\gamma^+(\xi)} \frac{z^j}{p^+(i\xi,z)(w-z)} dz = \frac{w^j}{p^+(i\xi,w)}$ *for all* $w \in \gamma^-(\xi)$ *and* $0 \leq j \leq 2m-1$

$\frac{1}{2\pi i} \int_{\gamma^-(\xi)} \frac{w^j}{p^-(i\xi,w)(w-z)} dw = \frac{-z^j}{p^-(i\xi,z)}$ *for all* $z \in \gamma^+(\xi)$ *and* $0 \leq j \leq 2m-1$

We define the Green function for the upper half space $\mathbb{R}_+^{n+1}$ to be

$$g(x-y,t,\tau) = h(x-y,t-\tau) - k(x-y,t,\tau) \quad (3.5)$$

We state in a theorem the properties of g.

PG **Theorem 3.6.** *$L_{x,t}(g(x-y,t,\tau)) = 0$ for $(x,t) \neq (y,\tau)$ and $L^\star_{y,\tau}(g(x-y,t,\tau)) = 0$ for $(y,\tau) \neq (x,t)$.*

In addition, g satisfies the following m boundary conditions at $t=0$, for any $\tau>0$
$g(x-y,0,\tau)=0$, $\frac{\partial g}{\partial t}(x-y,0,\tau)=0,\dots,\frac{\partial^{m-1}g}{\partial t^{m-1}}(x-y,0,\tau)=0$
and g satisfies the m boundary conditions at $\tau=0$, for any $t>0$,
$g(x-y,t,0)=0$, $\frac{\partial g}{\partial \tau}(x-y,t,0)=0,\dots,\frac{\partial^{m-1}g}{\partial \tau^{m-1}}(x-y,t,0)=0$

Proof. The proof is a straightforward computation using (3.5).

To prove $L_{x,t}(g(x-y,t,\tau))=0$, we prove $L_{x,t}(h(x-y,t-\tau))=0$ and $L_{x,t}(k(x-y,t,\tau))=0$.

We have, say for $t>\tau$, $D_x^\alpha D_t^l h(x-y,t-\tau)=\int\int_{\gamma^-(\xi)}\frac{z^l(i\xi)^\alpha}{p(i\xi,z)}e^{(t-\tau)z}dz\, e^{i(x-y)\cdot\xi}d\xi$
which implies that $L_{x,t}(h(x-y,t-\tau))=\int\int_{\gamma^-(\xi)}e^{(t-\tau)z}dz\, e^{i(x-y)\cdot\xi}d\xi=0$
Also, $D_x^\alpha D_t^l k(x-y,t,\tau)=\int\int_{\gamma^+(\xi)}\frac{e^{-\tau z}}{p^+(i\xi,z)}\int_{\gamma^-(\xi)}\frac{w^l(i\xi)^\alpha e^{wt}}{p^-(i\xi,w)(w-z)}dw\, dz\, e^{i(x-y)\cdot\xi}d\xi$.
Hence, $L_{x,t}(k(x-y,t,\tau))=\int\int_{\gamma^+(\xi)}\frac{e^{-\tau z}}{p^+(i\xi,z)}\int_{\gamma^-(\xi)}\frac{p(i\xi,w)e^{wt}}{p^-(i\xi,w)(w-z)}dw\, dz\, e^{i(x-y)\cdot\xi}d\xi=$
$\int\int_{\gamma^+(\xi)}\frac{e^{-\tau z}}{p^+(i\xi,z)}\int_{\gamma^-(\xi)}\frac{p^+(i\xi,w)e^{wt}}{(w-z)}dw\, dz\, e^{i(x-y)\cdot\xi}d\xi=0$,
since $\int_{\gamma^-(\xi)}\frac{p^+(i\xi,w)e^{wt}}{(w-z)}dw=0$ because for each $z\in\gamma^+(\xi)$, the function $f(w)=\frac{p^+(i\xi,w)e^{wt}}{w-z}$ is analytic inside $\gamma^-(\xi)$.

Next, we show $L^\star_{y,\tau}(g(x-y,t,\tau))=0$. Since $D_y^\alpha D_\tau^l(h(x-y,t-\tau))=(-1)^{|\alpha|+l}D_x^\alpha D_t^l(h(x-y,t-\tau))$, it follows that $L^\star_{y,\tau}(h(x-y,t-\tau))=0$.

And $D_y^\alpha D_\tau^l(k(x-y,t,\tau))=(-1)^{|\alpha|+l}\int\int_{\gamma^-(\xi)}\frac{e^{tw}}{p^-(i\xi,w)}\int_{\gamma^+(\xi)}\frac{z^l(i\xi)^\alpha e^{-\tau z}}{p^+(i\xi,z)(w-z)}dz\, dw\, e^{i(x-y)\cdot\xi}d\xi$.
And hence, $L^\star_{y,\tau}(k(x-y,t,\tau))=$
$\int\int_{\gamma^-(\xi)}\frac{e^{tw}}{p^-(i\xi,w)}\int_{\gamma^+(\xi)}\frac{p(^-(i\xi,z)e^{-\tau z}}{(w-z)}dz\, dw\, e^{i(x-y)\cdot\xi}d\xi=0$,
since $\int_{\gamma^+(\xi)}\frac{p(^-(i\xi,z)e^{-\tau z}}{(w-z)}dz=0$.

We now check the boundary conditions.

$g(x-y,0,\tau)=$

$$-\int\int_{\gamma^+(\xi)}\frac{e^{-\tau z}}{p(i\xi,z)}dz\, e^{i(x-y)\cdot\xi}\,d\xi-\int\int_{\gamma^+(\xi)}\frac{e^{-\tau z}}{p^+(i\xi,z)}\int_{\gamma^-(\xi)}\frac{dw}{p(i\xi,w)(w-z)}\,dz\, e^{i(x-y)\cdot\xi}d\xi=0$$

since $\int_{\gamma^-(\xi)}\frac{dw}{p^-(i\xi,w)(w-z)}=-\frac{1}{p^-(i\xi,z)}$.

And for $t<\tau$, we have $\frac{\partial g}{\partial t}(x-y,t,\tau)=$

$$-\int\int_{\gamma^+(\xi)}\frac{ze^{(t-\tau)z}}{p(i\xi,z)}dz\, e^{i(x-y)\cdot\xi}\,d\xi-\int\int_{\gamma^+(\xi)}\frac{e^{-\tau z}}{p^+(i\xi,z)}\int_{\gamma^-(\xi)}\frac{w\, e^{tw}\,dw}{p^-(i\xi,w)(w-z)}\,dz\, e^{i(x-y)\cdot\xi}d\xi$$

and hence, $\frac{\partial g}{\partial t}(x-y,0,\tau)=$

$$-\int\int_{\gamma^+(\xi)}\frac{ze^{(-\tau)z}}{p(i\xi,z)}dz\, e^{i(x-y)\cdot\xi}\,d\xi-\int\int_{\gamma^+(\xi)}\frac{e^{-\tau z}}{p^+(i\xi,z)}\int_{\gamma^-(\xi)}\frac{w\,dw}{p^-(i\xi,w)(w-z)}\,dz\, e^{i(x-y)\cdot\xi}d\xi=0$$

since
$\int_{\gamma^-(\xi)} \frac{w\,dw}{p^-(i\xi,w)(w-z)} = -\frac{z}{p^-(i\xi,z)}$.
Continue up to derivative of order $m-1$.

We now proceed to prove the m boundary conditions at $\tau = 0$.

$g(x-y,t,0) =$

$$\int\int_{\gamma^-(\xi)} \frac{e^{tz}}{p(i\xi,z)} dz\, e^{i(x-y)\cdot\xi}\, d\xi - \int\int_{\gamma^-(\xi)} \frac{e^{tw}}{p^-(i\xi,w)} \int_{\gamma^+(\xi)} \frac{dz}{p^+(i\xi,z)(w-z)}\, dw\, e^{i(x-y)\cdot\xi} d\xi = 0$$

since $\int_{\gamma^+(\xi)} \frac{dz}{p^+(i\xi,z)(w-z)} = \frac{1}{p^+(i\xi,w)}$

For $\tau < t$ we have $\frac{\partial g}{\partial \tau}(x-y,t,\tau) =$

$$-\int\int_{\gamma^-(\xi)} \frac{ze^{(t-\tau)z}}{p(i\xi,z)} dz\, e^{i(x-y)\cdot\xi}\, d\xi + \int\int_{\gamma^-(\xi)} \frac{e^{tw}}{p^-(i\xi,w)} \int_{\gamma^+(\xi)} \frac{z\,e^{-\tau z}\,dz}{p^+(i\xi,z)(w-z)}\, dw\, e^{i(x-y)\cdot\xi} d\xi$$

And ,hence $\frac{\partial g}{\partial \tau}(x-y,t,0) =$

$$-\int\int_{\gamma^-(\xi)} \frac{ze^{(t)z}}{p(i\xi,z)} dz\, e^{i(x-y)\cdot\xi}\, d\xi + \int\int_{\gamma^-(\xi)} \frac{e^{tw}}{p^-(i\xi,w)} \int_{\gamma^+(\xi)} \frac{z\,dz}{p^+(i\xi,z)(w-z)}\, dw\, e^{i(x-y)\cdot\xi} d\xi = 0$$

since $\int_{\gamma^+(\xi)} \frac{z\,dz}{p^+(i\xi,z)(w-z)} = \frac{w}{p(i\xi,w)}$
Continue up to derivative of order $m-1$. □

We use the function$g$ to prove a representation formula, which is the basis for the apriori estimates.

RF **Theorem 3.7.** *Let $u \in C^{2m+\delta}(R^{n+1}_+)$ satisfy the m boundary conditions at $t = 0$*

$$u(x,0) = 0, \frac{\partial u}{\partial t}(x,0) = 0, ..., \frac{\partial^{m-1} u}{\partial t^{m-1}}(x,0) = 0$$

then

$$u(x,t) = \int_0^\infty \int Lu(y,\tau) g(x-y,t,\tau) dy\, d\tau$$

for all $x \in R^n$, $t > 0$.

Proof. Fix $(x,t) \in R^{n+1}_+$ and $\epsilon > 0$.

Write the m conditions for u in (y,τ) variables,
$u(y,0) = 0, \frac{\partial u}{\partial \tau}(y,0) = 0, ..., \frac{\partial^{m-1} u}{\partial \tau^{m-1}}(y,0) = 0$
and let $v(y,\tau) = g(x-y,t,\tau)$. Recall $L^\star v(y,\tau) = 0$ and $v(y,0) = 0$, $\frac{\partial v}{\partial \tau} v(y,0) = 0, ..., \frac{\partial^{m-1} v}{\partial \tau^{m-1}} v(y,0) = 0$.

First we need to prove the next three claims.

*Claim*1

For $|\alpha| + l = 2m$ and $l \geq 1$ we have

$$\int_{R_+^{n+1}\backslash B_\epsilon(x,t)} D_y^\alpha D_\tau^l u(y,\tau)v(y,\tau)dy\,d\tau =$$

$$\int_{R_+^{n+1}\backslash B_\epsilon(x,t)} u(y,\tau)D_y^\alpha D_\tau^l v(y,\tau)dy\,d\tau - \int_{\partial B_\epsilon(x,t)} u(y,\tau)D_y^\alpha D_\tau^{l-1}v(y,\tau)\eta_\tau(y,\tau)\,dS(y,\tau)$$

$$+O(\epsilon)$$

where $\eta_\tau(y,\tau) = \frac{t-\tau}{\epsilon}$

*Claim*2

For $|\alpha| = 2m$ we have

$$\int_{R_+^{n+1}\backslash B_\epsilon(x,t)} D_y^\alpha u(y,\tau)v(y,\tau)dy\,d\tau =$$

$$\int_{R_+^{n+1}\backslash B_\epsilon(x,t)} u(y,\tau)D_y^\alpha v(y,\tau)dy\,d\tau - \int_{\partial B_\epsilon(x,t)} u(y,\tau)D_y^{\alpha'}v(y,\tau)\eta_{\alpha'}(y,\tau)\,dS(y,\tau)$$

$$+O(\epsilon)$$

where$\alpha = \alpha' + e_i$ and $\eta_{\alpha'}(y,\tau) = \frac{x_i-y_i}{\epsilon}$ and i is any integer $1 \leq i \leq n$

and

*Claim*3

For $|\alpha| + l < 2m$, we have

$$\int_{R_+^{n+1}\backslash B_\epsilon(x,t)} D_y^\alpha D_\tau^l u(y,\tau)v(y,\tau)dy\,d\tau =$$

$$(-1)^{||\alpha|+l} \int_{R_+^{n+1}\backslash B_\epsilon(x,t)} u(y,\tau)D_y^\alpha D_\tau^l v(y,\tau)dy\,d\tau + O(\epsilon)$$

The proof of the claims follows by succesive integration by parts.

Use the boundary conditions of u and v at $\tau = 0$ that complement each other so that no boundary terms appear at $\tau = 0$ and the decay of v and its derivatives at ∞.

The terms $O(\epsilon)$ come from the derivatives of v of orders less than $2m - 1$ integrated over $\partial B_\epsilon(x,t)$.

We illustrate with two cases to see how it goes.

$$\int_{R_+^{n+1}\setminus B_\epsilon(x,t)} D_\tau^{2m}u(y,\tau)v(y,\tau)dy\,d\tau =$$

$$\int_{R_+^{n+1}\setminus B_\epsilon(x,t)} D_\tau^{2m-1}u(y,\tau)D_\tau v(y,\tau)dy\,d\tau + \int_{\partial B_\epsilon(x,t)} D_\tau^{2m-1}u(y,\tau)v(y,\tau)\,\eta_\tau\,dS +$$

$$\int_{R^n} \lim_{R\to\infty}(D_\tau^{2m-1}u(y,R)\,v(y,R)) - D_\tau^{2m-1}u(y,0)\,v(y,0)dy$$

The second term is $O(\epsilon)$ and the last term is 0.

After m steps, using the m conditions of v at $\tau = 0$, we get

$$\int_{R_+^{n+1}\setminus B_\epsilon(x,t)} D_\tau^{2m}u(y,\tau)v(y,\tau)dy\,d\tau =$$

$$(-1)^m \int_{R_+^{n+1}\setminus B_\epsilon(x,t)} D_\tau^{m}u(y,\tau)D_\tau^{m}v(y,\tau)dy\,d\tau + O(\epsilon)$$

Continue integrating by parts, now using the m conditions of u at $\tau = 0$, to get after m steps,

$$\int_{R_+^{n+1}\setminus B_\epsilon(x,t)} D_\tau^{2m}u(y,\tau)v(y,\tau)dy\,d\tau =$$

$$\int_{R_+^{n+1}\setminus B_\epsilon(x,t)} u(y,\tau)D_\tau^{2m}v(y,\tau)dy\,d\tau - \int_{\partial B_\epsilon(x,t)} u(y,\tau)D_\tau^{2m-1}v(y,\tau)\,\eta_\tau\,dS + O(\epsilon)$$

Again as an illustration, we have

$$\int_{R_+^{n+1}\setminus B_\epsilon(x,t)} D_\tau^{2m-1}D_{y_j}u(y,\tau)v(y,\tau)dy\,d\tau =$$

$$(-1)^m \int_{R_+^{n+1}\setminus B_\epsilon(x,t)} D_\tau^{m-1}D_{y_j}u(y,\tau)D_\tau^{m}v(y,\tau)dy\,d\tau + O(\epsilon)$$

Up to now using the m conditions of v at $\tau = 0$,

and continue integrating by parts, now using $D_{y_j}D_\tau^l u = 0$ and $u = 0$ at $\tau = 0$ for $l = 0, ..., m-2$ to get

$$\int_{R_+^{n+1}\setminus B_\epsilon(x,t)} D_\tau^{2m-1}D_{y_j}u(y,\tau)v(y,\tau)dy\,d\tau =$$

$$\int_{R_+^{n+1}\setminus B_\epsilon(x,t)} u(y,\tau)D_\tau^{2m-1}D_{y_j}v(y,\tau)dy\,d\tau - \int_{\partial B_\epsilon(x,t)} u(y,\tau)D_\tau^{2m-1}v(y,\tau)\,\eta_{y_j}\,dS + O(\epsilon)$$

From the three claims we get

$$\int_{R_+^{n+1}\setminus B_\epsilon(x,t)} Lu(y,\tau)v(y,\tau)dy\,d\tau =$$

$$\int_{R^{n+1}_+\setminus B_\epsilon(x,t)} u(y,\tau)L^\star v(y,\tau)dy\,d\tau$$

$$-\sum_{|\alpha|+l=m,l\geq 1} a_{\alpha,l}\int_{\partial B_\epsilon(x,t)} u(y,\tau)D^\alpha_y D^{l-1}_\tau v(y,\tau)\,\eta_\tau\,dS+$$

$$-\sum_{|\alpha|=m} a_{\alpha,0}\int_{\partial B_\epsilon(x,t)} u(y,\tau)D^{\alpha'}_y v(y,\tau)\,\eta_{\alpha'}\,dS + O(\epsilon)$$

Since $L^\star v(y,\tau)=0$ for all $(y,\tau)\neq(x,t)$ the theorem will follow once we show that

$$\lim_{\epsilon\to 0}\{\sum_{|\alpha|+l=m,l\geq 1} a_{\alpha,l}\int_{\partial B_\epsilon(x,t)} u(y,\tau)D^\alpha_y D^{l-1}_\tau v(y,\tau)\,\eta_\tau\,dS+$$

$$+\sum_{|\alpha|=m} a_{\alpha,0}\int_{\partial B_\epsilon(x,t)} u(y,\tau)D^{\alpha'}_y v(y,\tau)\,\eta_{\alpha'}\,dS\}$$

$$= cu(x,t)$$

for some c depending on n.

Since $v(y,\tau)=h(x-y,t-\tau)-k(x-y,t,\tau)$ and k is regular for all (y,τ), we have to prove the limit above with h replacing v.

We now proceed to prove this limit. Notice that there is no longer the need to emphasize the last variable, so we just write $h=h(y)$ and $u=u(y)$,

and $y\in R^n$.

Let $|\alpha|=2m-1$ and write

$$\int_{\partial B_\epsilon(x)} u(y)D^\alpha h(x-y)\frac{(x_j-y_j)}{\epsilon}dS(y)=$$

$$\int_{\partial B_\epsilon(x)}(u(y)-u(x))D^\alpha h(x-y)\frac{(x_j-y_j)}{\epsilon}dS(y)+u(x)\int_{\partial B_\epsilon(x)} D^\alpha h(x-y)\frac{(x_j-y_j)}{\epsilon}dS(y)$$

The first term goes to 0 as $\epsilon\to 0$, since $|u(y)-u(x)|\leq C|x-y|$ and $|D^\alpha h(x-y)|\leq \frac{C}{|x-y|^{n-1}}$.

So, it is enough to show that

$$\lim_{\epsilon\to 0}\sum_{|\alpha|=2m} a_\alpha\int_{\partial B_\epsilon(x)} D^{\alpha-e_j}h(x-y)\frac{(x_j-y_j)}{\epsilon}dS(y)=c_n$$

Recall that $h_R\to h$ uniformly on compacts of $R^n\setminus\{0\}$

and $D^\alpha h_R\to D^\alpha h$ uniformly on compacts of $R^n\setminus\{0\}$ as $R\to\infty$

where $h_R(y)=\int\varphi(\frac{\xi}{R})\frac{e^{iy\cdot\xi}}{p(i\xi)}d\xi$.

Now,

$$\int_{\partial B_\epsilon(x)} D^{\alpha-e_j}h(x-y)\frac{(x_j-y_j)}{\epsilon}dS(y) = \int_{\partial B_1(0)} \epsilon^{n-1}D^{\alpha-e_j}h(\epsilon y)y_j dS(y)$$

Let $\beta = \alpha - e_j$, where $|\alpha| = 2m$, so $|\beta| = 2m-1$.

We have

$$\epsilon^{n-1}D^\beta h_R(\epsilon y) = \int \varphi(\frac{\xi}{\epsilon R})\frac{(i\xi)^\beta e^{iy\cdot\xi}}{\epsilon^m p(\frac{i\xi}{\epsilon})}d\xi$$

and

$$\int_{\partial B_1(0)} \epsilon^{n-1}D^\beta h_R(\epsilon y)y_j dS(y) = \int \varphi(\frac{\xi}{\epsilon R})\frac{(i\xi)^\beta}{\epsilon^{2m}p(\frac{i\xi}{\epsilon})}\int_{\partial B_1(0)} e^{iy\cdot\xi}y_j dS(y)d\xi =$$

$$\int \varphi(\frac{\xi}{\epsilon R})\frac{(i\xi)^\beta}{\epsilon^{2m}p(\frac{i\xi}{\epsilon})}\xi_j\int_{B_1(0)} e^{iy\cdot\xi}dyd\xi = \int_{B_1(0)}\int \varphi(\frac{\xi}{\epsilon R})\frac{(i\xi)^\alpha}{\epsilon^{2m}p(\frac{i\xi}{\epsilon})}e^{iy\cdot\xi}d\xi dy$$

Therefore, for fixed ϵ

$$\sum_{|\alpha|=2m} a_\alpha \int_{\partial B_\epsilon(x)} D^{\alpha-e_j}h(x-y)\frac{(x_j-y_j)}{\epsilon}dS(y) = \lim_{R\to\infty}\int_{B_1(0)}\int \varphi(\frac{\xi}{\epsilon R})\frac{p_{2m}(i\xi)}{\epsilon^{2m}p(\frac{i\xi}{\epsilon})}e^{iy\cdot\xi}d\xi dy$$

where p_{2m} is the principal part of p We show this last limit is independent of ϵ.

Note $\frac{p_{2m}(i\xi)}{\epsilon^{2m}} = p_{2m}(\frac{i\xi}{\epsilon})$ and write

$$\int_{B_1(0)}\int \varphi(\frac{\xi}{\epsilon R})\frac{p_{2m}(i\xi)}{\epsilon^{2m}p(\frac{i\xi}{\epsilon})}e^{iy\cdot\xi}d\xi dy =$$

$$\int_{B_1(0)}\int \varphi(\frac{\xi}{\epsilon R})\frac{p_{2m}(\frac{i\xi}{\epsilon})-p(\frac{i\xi}{\epsilon})}{p(\frac{i\xi}{\epsilon})}e^{iy\cdot\xi}d\xi dy + \int_{B_1(0)}\int \varphi(\frac{\xi}{\epsilon R})e^{iy\cdot\xi}d\xi dy = A + B$$

The first term goes to 0 and the second has a limit independent of ϵ.

To see this, let $p_{2m}(\frac{i\xi}{\epsilon}) - p(\frac{i\xi}{\epsilon}) = \tilde{p}(\frac{i\xi}{\epsilon})$, $\tilde{p}$ has degree $\leq 2m-1$.

Write $A = \int_{|y|\leq\frac{1}{\epsilon R}} ...dy + \int_{\frac{1}{\epsilon R}\leq|y|\leq 1} ...dy = A_1 + A_2$

Note that

$$A_1 = \int_{|y|\leq 1}\int \varphi(\xi)\frac{\tilde{p}(iR\xi)}{p(iR\xi)}e^{iy\cdot\xi}d\xi dy \to 0\, as\, R\to\infty$$

$$A_2 = \int_{1\leq|y|\leq\epsilon R}\int \varphi(\xi)\frac{\tilde{p}(iR\xi)}{p(iR\xi)}e^{iy\cdot\xi}d\xi dy =$$

$$\int_{1\leq|y|\leq\epsilon R}\frac{1}{|y|^{2l}}\int(-\Delta_\xi)^l(\varphi(\xi)\frac{\tilde{p}(iR\xi)}{p(iR\xi)})e^{iy\cdot\xi}d\xi dy \to 0\, as\, R\to\infty$$

where, in the last limit we note that setting $q(\xi) = \frac{\tilde{p}(i\xi)}{p(i\xi)}$, we have, for any α, $|D^\alpha q(\xi)| \leq \frac{C}{1+|\xi|^{|\alpha|+1}}$, and hence $|D^\alpha(q(R\xi))| \leq C\frac{R^{|\alpha|}}{1+(R|\xi|)^{|\alpha|+1}}$

The same argument shows that the limit of B is

$$\int_{|y|\leq 1}\int \varphi(\xi)e^{iy\cdot\xi}d\xi dy + \int_{1\leq|y|}\frac{1}{|y|^{2l}}\int(-\Delta)^l(\varphi(\xi))e^{iy\cdot\xi}d\xi dy$$

where l is any integer such that $2l \geq n+1$. This finishes de proof of the theorem.

□

In the next theorem we solve the Dirichlet problem for the upper half space R_+^{n+1} with m conditions at the boundary $t = 0$.

The next theorem is also used in combination with (3.7) to prove apriori estimates in the space $C^{2m+\delta}$

DP **Theorem 3.8.** *Let $f \in C^\delta(R_+^{n+1})$ and define $u(x,t) = \int_0^\infty \int f(y,\tau)g(x-y,t,\tau)dyd\tau$. Then,*

$$u \in C^{2m+\delta}(R_+^{n+1})\, and\, Lu(x,t) = f(x,t)$$

$$u(x,0) = 0,\ \frac{\partial u}{\partial t}(x,0) = 0,\ ....,\ \frac{\partial^{m-1}u}{\partial t^{m-1}}(x,0) = 0$$

Moreover,

$$|u|_{2m+\delta;R_+^{n+1}} \leq C|f|_{\delta;R_+^{n+1}}$$

Proof. The boundary conditions $u(x,0) = 0$, $\frac{\partial u}{\partial t}(x,0) = 0$, ..., $\frac{\partial^{m-1}u}{\partial t^{m-1}}(x,0) = 0$ follow inmediately from the analogous properties of g.

To prove $Lu = f$, write $u = v - w$ where $v(x,t) = \int_0^\infty \int f(y,\tau)h(x-y,t-\tau)dyd\tau$. And $w(x,t) = \int_0^\infty \int f(y,\tau)k(x-y,t,\tau)dyd\tau$.

Clearly, $Lw(x,t) = 0$.

We show $Lv(x,t) = f(x,t)$

Fix (x_0,t_0) and $\epsilon > 0$ such that $B_{3\epsilon}(x_0,t_0) \subseteq R_+^{n+1}$ and $\eta \in C_0^\infty(B_{3\epsilon}(x_0,t_0))$,with $\eta = 1$ on $B_{2\epsilon}(x_0,t_0)$

Write $v(x,t) = \int_0^\infty \int \eta(y,\tau)f(y,\tau)h(x-y,t-\tau)dyd\tau + \int_0^\infty \int(1-\eta(y,\tau))f(y,\tau)h(x-y,t-\tau)dyd\tau = v_1(y,\tau) + v_2(y,\tau)$

Clearly, $Lv_2 = 0$ on $B_\epsilon(x_0,t_0)$.

Note that $\bar{f} = \eta f \in C^\delta(R^{n+1})$ and it follows (see argument below) that for $1 \leq |\alpha| + l \leq 2m$,

$$D_x^\alpha D_t^l v_1(x,t) = \int_{R^{n+1}}(\bar{f}(y,\tau) - \bar{f}(x,t))D_x^\alpha D_t^l h(x-y,t-\tau)dy\, d\tau$$

Therefore,

$$Lv_1(x,t) = a_{0,0}v_1(x,t) + \int_{R^{n+1}} (\bar{f}(y,\tau) - \bar{f}(x,t)) \sum_{1\le|\alpha|+l} a_{\alpha,l} D_x^\alpha D_t^l h(x-y,t-\tau)dy\,d\tau =$$

$$\int_{R^{n+1}} (\bar{f}(y,\tau)-\bar{f}(x,t))L_{x,t}(h(x-y,t-\tau))dy\,d\tau + a_{0,0}\bar{f}(x,t)\int_{R^{n+1}} h(x-y,t-\tau)dy\,d\tau = \bar{f}(x,t)$$

We proceed to show $|u|_{2m+\delta;R_+^{n+1}} \le C|f|_{\delta;R_+^{n+1}}$.

First we prove a claim.

For $|\alpha| + l = 2m$ such that $|\alpha| \ge 1$ we have

$$D_x^\alpha D_t^l u(x,t) = \int_{R^{n+1}} (f(y,\tau) - f(x,t)) D_x^\alpha D_t^l (g(x-y,t,\tau)dy\,d\tau$$

Write $\alpha = \beta + e_j$. Since $|\beta| + l = 2m - 1$, it follows that

$$D_x^\alpha D_t^l u(x,t) = \int_{R^{n+1}} f(y,\tau) D_x^\beta D_t^l (g(x-y,t,\tau))dy\,d\tau$$

Let $w_\epsilon(x,t) = \int_{R^{n+1}} f(y,\tau)\eta_\epsilon(x-y,t-\tau)D_x^\beta D_t^l(g(x-y,t,\tau))dy\,d\tau$,

where now $\eta_\epsilon(x-y,t-\tau) = 0$ for $|x-y|^2 + (t-\tau)^2 \le \epsilon^2$ and $\eta_\epsilon(x-y,t-\tau) = 1$ for $|x-y|^2 + (t-\tau)^2 \ge 4\epsilon^2$

Then

$$w_\epsilon \to D_x^\beta D_t^l u$$

uniformly in R_+^{n+1} as $\epsilon \to 0$ and

$$\frac{\partial w_\epsilon(x,t)}{\partial x_j} = \int_{R^{n+1}} f(y,\tau)\frac{\partial}{\partial x_j}(\eta_\epsilon(x-y,t-\tau)D_x^\beta D_t^l(g(x-y,t,\tau)))dy\,d\tau =$$

$$\int_{R^{n+1}} (f(y,\tau) - f(x,t))\frac{\partial}{\partial x_j}(\eta_\epsilon(x-y,t-\tau)D_x^\beta D_t^l(g(x-y,t,\tau)))dy\,d\tau$$

$$-f(x,t)\int_{R^{n+1}} \frac{\partial}{\partial y_j}(\eta_\epsilon(x-y,t-\tau)D_x^\beta D_t^l(g(x-y,t,\tau)))dy\,d\tau$$

$$\to \int_{R^{n+1}} (f(y,\tau) - f(x,t))D_x^\alpha D_t^l(g(x-y,t,\tau)dy\,d\tau$$

uniformly in R_+^{n+1} as $\epsilon \to 0$ which proves the claim.

Now, let $|\alpha| + l = 2m$ and $|\alpha| \ge 1$ and write $\alpha = \beta + e_j$.

Let (x,t), $(\bar{x},\bar{t}) \in R_+^{n+1}$. Let $(\hat{x},\hat{t}) = \frac{1}{2}(x+\bar{x},t+\bar{t})$. Let $r = 2((x-\bar{x})^2 + (t-\bar{t})^2)^{1\backslash 2}$ and $B^+ = B_r(\hat{x},\hat{t}) \cap R_+^{n+1}$

We have,

$$D_x^\alpha D_t^l u(x,t) - D_x^\alpha D_t^l u(\bar{x},\bar{t}) =$$

$$\int_{B^+}(f(y,\tau)-f(x,t))D_{x,t}^{|\alpha|+l}g(x-y,t,\tau)dy\,d\tau-\int_{B^+}(f(y,\tau)-f(\bar{x},\bar{t}))D_{x,t}^{|\alpha|+l}g(\bar{x}-y,\bar{t},\tau)dy\,d\tau+$$

$$\int_{R_+^{n+1}\backslash B^+}(f(y,\tau)-f(\bar{x},\bar{t}))(D_{x,t}^{|\alpha|+l}g(x-y,t,\tau)-D_{x,t}^{|\alpha|+l}g(\bar{x}-y,\bar{t},\tau))dy\,d\tau+$$

$$(f(\bar{x},\bar{t})-f(x,t))\int_{R_+^{n+1}\backslash B^+}D_{x,t}^{|\alpha|+l}g(x-y,t,\tau)dy\,d\tau = A-B+C+(f(\bar{x},\bar{t})-f(x,t))D$$

We have $|A|\leq C[f]_\delta\int_{B^+}\frac{(|x-y|^2+(t-\tau)^2)^{\delta\backslash 2}}{(|x-y|^2+(t-\tau)^2)^{(n+1)\backslash 2}}dy\,d\tau\leq C[f]_\delta r^\delta$

$|B|\leq C[f]_\delta r^\delta$

$C\leq C[f]_\delta((|\bar{x}-x|^2+(\bar{t}-t)^2)^{1\backslash 2})\int_{R_+^{n+1}\backslash B^+}\frac{(|\bar{x}-y|^2+(\bar{t}-\tau)^2)^{\delta\backslash 2}}{(|\tilde{x}-y|^2+(\tilde{t}-\tau)^2)^{(n+2)\backslash 2}}dy\,d\tau\leq C[f]_\delta r^\delta$

To estimate D, we use that $|\alpha|\geq 1$ to write $D=\int_{R_+^{n+1}\backslash B^+}\frac{\partial}{\partial x_j}(D_{x,t}^{\beta+l}g(x-y,t,\tau))dy\,d\tau=$ $\int_{R_+^{n+1}\backslash B^+}\frac{\partial}{\partial x_j}(D_{x,t}^{\beta+l}h(x-y,t-\tau))dy\,d\tau+\int_{R_+^{n+1}\backslash B^+}\frac{\partial}{\partial x_j}(D_{x,t}^{\beta+l}k(x-y,t,\tau))dy\,d\tau=D_1+D_2$

$|D_1|=|\int_{R_+^{n+1}\backslash B^+}\frac{\partial}{\partial y_j}(D_{x,t}^{\beta+l}h(x-y,t-\tau))dy\,d\tau|=$

$|\int_{(\partial B^+)\backslash\{\tau=0\}}D_{x,t}^{\beta+l}h(x-y,t-\tau))dS(y,\tau)|\leq C.$

And similarly, $|D_2|\leq C$

This proves that $[D_x^\alpha D_t^l u]_{\delta;R_+^{n+1}}\leq C[f]_{\delta;R_+^{n+1}}$ for $|\alpha|+l=2m$ and $|\alpha|\geq 1$.

To estimate $D_t^{2m}u$, note

$D_t^{2m}u(x,t)=f(x,t)-\sum_{|\alpha|+l=2m;|\alpha|\geq 1}a_{\alpha,l}D_x^\alpha D_t^l u(x,t)-\sum_{|\alpha|+l\leq 2m-1}a_{\alpha,l}D_x^\alpha D_t^l u(x,t),$

which implies that

$[D_t^{2m}u]_{\delta;R_+^{n+1}}\leq C[f]_\delta+|u|_{2m-1+\delta}$

Because $g\in L^1(R_+^{n+1})$, we also have $|u|_{0;R_+^{n+1}}\leq C|f|_{0;R_+^{n+1}}$.

By interpolation, it follows that

$|u|_{2m+\delta;R_+^{n+1}}\leq C|f|_{\delta;R_+^{n+1}}$. This finishes the proof of the theorem.

Note that we have used in an essential way the estimates of lemmas (3.3) and (3.4).

□

In order to consider operators with variable coefficients and apply the freezing coefficient method we need the following observation, which we state as a theorem.

L0 **Theorem 3.9.** *Let* $u\in C^{2m+\delta}(R_+^{n+1})$ *and* $u(x,0)=0$, $\frac{\partial u}{\partial t}(x,0)=0$, ..., $\frac{\partial^{m-1}u}{\partial t^{m-1}}(x,0)=0$.

Let $(x_0,t_0)\in(R_+^{n+1})$ *be any point,* $R>0$, *and* $\eta\in C_0^\infty(B_{2R}(x_0,t_0))$, *with* $\eta=1$ *in* $B_R(x_0,t_0)$.

Then,

$|u|_{2m+\delta;B^+_R(x_0,t_0)} \le C|L(u\eta)|_{\delta;B^+_{2R}(x_0,t_0)}$

Proof. The proof follows from theorem (3.7) and (3.8).

Since $u\eta$ satisfies the same hypothesis as u, we have

$|u|_{2m+\delta;B^+_R(x_0,t_0)} \le |u\eta|_{2m+\delta;R^{n+1}_+} \le C|L(u\eta)|_{\delta;R^{n+1}_+} = C|L(u\eta)|_{\delta;B^+_{2R}(x_0,t_0)}$.

Finishing the proof. □

4. Variable Coefficients

In this section, we apply the constant coefficient estimates to prove Schauder estimates for variable Hölder coefficients. There will be a restriction on the size of the Hölder constant of the coefficient to get full estimates. We make a sligth change in notation: Take $x \in R^n_+$, write $x = (x', x_n)$ and consider $u \in C^{2m+\delta}(R^n_+)$,

such that $u(x', 0) = 0$, $\frac{\partial u(x',0)}{\partial x_n} = 0$, ..., $\frac{\partial^{m-1} u(x',0)}{\partial x_n^{m-1}} = 0$.

And we consider an operator $Lu(x) = \sum_{|\alpha|\le m} a_\alpha(x) D^\alpha u(x)$, where we assume $a_\alpha(x)$ are complex valued functions which are Hölder continuous.

Let $K_0 = \sum_{|\alpha|\le 2m} |a_\alpha|_{0;R^n_+}$ and $K_\delta = \sum_{\alpha|\le 2m} [a_\alpha]_{\delta;R^n_+}$.

Let $p(x, i\xi) = \sum_{|\alpha|\le m} a_\alpha(x)(i\xi)^\alpha$ and assume the ellipticity condition $|p(x, i\xi)| \ge \lambda(1 + |\xi|^{2m})$, for all $x \in R^n_+$ and for all $\xi \in R^n$.

As before, let $\{\lambda^-_1(x, \xi'), ..., \lambda^-_m(x, \xi')\}$ denote roots of the equation $p(x, i\xi', z) = 0$ with negative real part and $\{\lambda^+_1(x, \xi'), ..., \lambda^+_m(x, \xi')\}$ denote roots of the equation $p(x, i\xi', z) = 0$ with positive real part.

And, in order to apply the estimates for the constant coefficient case, we assume that if $\lambda(x_0, \xi')$ denotes any root of $p(x_0, i\xi', z) = 0$,then

$$|D^\alpha_\xi(\lambda(x_0, \xi')| \le C(1 + |\xi|)^{1-|\alpha|} \tag{4.6}$$

with a constant that depends only on n, α, λ and K_0

Finally, let for $x_0 \in R^n_+, L_0(x) = \sum_{|\alpha|\le 2m} a_\alpha(x_0) D^\alpha u(x)$.

Theorem 4.1. *There exists a constant γ depending on K_0 such that if $K_\delta \le \gamma$, then $|u|_{2m+\delta;R^n_+} \le C|Lu|_{\delta;R^n_+}$. The constant C is universal.*

Proof. Let x_0 and $\bar{x}$ be points in R^n_+ and $R \ge 1$ to be chosen. Fix $|\alpha| = 2m$. Let $\eta = 1$ in $B_R(x_0)$ and $\eta \in C^\infty_0(B_{2R}(x_0))$.

If $\bar{x} \notin B_R(x_0)$, then $\frac{|D^\alpha u(\bar{x}) - D^\alpha u(x_0|}{|\bar{x}-x_0|^\delta} \le \frac{2}{R}[u]_{2m;R^n_+}$

If $\bar{x} \in B_R(x_0)$, then $\frac{|D^\alpha u(\bar{x}) - D^\alpha u(x_0|}{|\bar{x}-x_0|^\delta} \leq [u\eta]_{2m+\delta;B_R^+(x_0)} \leq C|L_0(u\eta)|_{\delta;B_{2R}^+(x_0)}$, by theorem (L0 3.9) applied to the constant coefficient operator L_0.

We have $L_0(u\eta)(x) = \eta(x)L_0u(x) + \sum_{|\alpha|\leq 2m}\sum_{|\beta|+|\gamma|=|\alpha|;|\gamma|\geq 1} a_{\gamma;\beta}(x_0)D^\beta u(x)D^\gamma\eta(x)$

We have the estimates,

$|L_0(u\eta)|_{0;B_{2R}^+(x_0)} \leq |L_0(u)|_{0;B_{2R}^+(x_0)} + \frac{CK_0}{R}|u|_{2m;R_+^n}$

and

$[L_0(u\eta)]_{\delta;B_{2R}^+(x_0)} \leq [L_0(u)]_{\delta;B_{2R}^+(x_0)} + \frac{CK_0}{R^\delta}|u|_{2m;R_+^n}$

The estimates above follow easily from the following four facts:

$[uv]_\delta \leq |u|_0[v]_\delta + |v|_0[u]_\delta$

$|D^\gamma\eta|_0 \leq \frac{C}{R^{|\gamma|}}$ and $[D^\gamma\eta]_\delta \leq \frac{C}{R^{|\gamma|+\delta}}$

Interpolation to absorve $[D^\beta u]_\delta \leq C|u|_{2m}$, for $|\beta| \leq 2m-1$

And $R \geq 1$.

To continue,

write $L_0u(x) = Lu(x) + (L_0 - L)u(x)$, so $[L_0u]_{\delta;B_{2R}^+(x_0)} \leq [Lu]_{\delta;R_+^n} + [(L_0-L)u]_{\delta;B_{2R}^+(x_0)}$

And

$[(L_0-L)u]_{\delta;B_{2R}^+(x_0)} \leq \sum_{|\alpha|\leq 2m}[a_\alpha(.) - a_\alpha(x_0))D^\alpha u]_{\delta;B_{2R}^+(x_0)} \leq$
$\sum_{|\alpha|\leq 2m}[a_\alpha(.)-a_\alpha(x_0))]_{\delta;B_{2R}^+(x_0)}|D^\alpha u|_{0;B_{2R}^+(x_0)} + \sum_{|\alpha|\leq 2m}|a_\alpha(.)-a_\alpha(x_0))|_{0;B_{2R}^+(x_0)}[D^\alpha u]_{\delta;B_{2R}^+(x_0)} \leq$
$C(K_\delta|u|_{2m;R_+^n} + K_\delta R^\delta|u|_{2m+\delta;R_+^n})$

Similarly,

$|L_0u|_{0;B_{2R}^+(x_0)} \leq |Lu|_{0;R_+^n} + |(L_0-L)u|_{0;B_{2R}^+(x_0)}$

And

$|(L_0-L)u|_{0;B_{2R}^+(x_0)} \leq \sum_{|\alpha|\leq 2m}|a_\alpha(.) - a_\alpha(x_0)|_{0;B_{2R}^+(x_0)}|D^\alpha u|_{0;R_+^n} \leq CK_\delta R^\delta|u|_{2m;R_+^n}$.

Combining estimates, we obtain

$|L_0(u\eta)|_{\delta;B_{2R}^+(x_0)} \leq |Lu|_{\delta;R_+^n} + C(K_\delta + \frac{K_0}{R^\delta})|u|_{2m;R_+^n} + CK_\delta R^\delta|u|_{2m+\delta;R_+^n}$

Now, take sup over $\bar{x}$, $x_0 \in R_+^n$ and maximum over $|\alpha| = 2m$, to obtain

1 (4.7)
$$[u]_{2m+\delta;R_+^n} \leq C|Lu|_{\delta;R_+^n} + C(\frac{K_0}{R^\delta} + K_\delta(1+R^\delta))|u|_{2m+\delta;R_+^n}$$

with a constant C that depends only on n, λ,δ and K_0

Also, for $x_0 \in R_+^n$ and $R \geq 1$, we have $|u|_{0;B_R^+(x_0)} \leq |u\eta|_{0;R_+^n} \leq C|L_0(u\eta)|_{0;B_{2R}^+(x_0)} \leq$
$C|L_0(u)|_{0;B_{2R}^+(x_0)} + \frac{C}{R}|u|_{2m;R_+^n} \leq C|Lu|_{0;R_+^n} + C(K_\delta R^\delta + \frac{1}{R})|u|_{2m+\delta;R_+^n}$.

Taking sup over $x_0 \in R_+^n$, with $R \geq 1$ fixed, we get

2 (4.8)
$$|u|_{0;R_+^n} \leq C|Lu|_{0;R_+^n} + C(K_\delta R^\delta + \frac{1}{R})|u|_{2m+\delta;R_+^n}$$

.

By (4.7) and (4.8) and interpolation we have the estimate

3 (4.9) $$|u|_{2m+\delta;R^n_+} \leq C|Lu|_{\delta;R^n_+} + C(\frac{1}{R^\delta} + K_\delta(1+R^\delta))|u|_{2m+\delta;R^n_+}$$

with a constant C that depends only on n, λ, δ, and K_0

We now choose $R \geq 1$ large enough so that $C\frac{1}{R^\delta} \leq \frac{1}{4}$ and then γ small enough so that $C\gamma(1+R^\delta) \leq \frac{1}{4}$.

Then, if $K_\delta \leq \gamma$, from (4.9) we get

$|u|_{2m+\delta;R^n_+} \leq C|Lu|_{\delta;R^n_+}$

□

Notice that in the theorem above we can take γ of the form

g (4.10) $$\gamma = \frac{1}{4C(1+4C)}$$

for some constant C that depends only n, λ, δ, and K_0

In order to apply the continuity method to solve the Dirichlet problem we need estimates with no restriction on the coefficients.

We can overcome this difficulty by scaling each term as follows.

For $s > 0$ let $L_s u(x) = \sum_{|\alpha|\leq 2m} a_\alpha(x) s^{2m-|\alpha|} D^\alpha u(x)$.

And $p_s(x, i\xi) = \sum_{|\alpha|\leq 2m} a_\alpha(x) s^{2m-|\alpha|}(i\xi)^\alpha = s^{2m} p(x, \frac{i\xi}{s})$, hence $|p_s(x,i\xi)| \geq \lambda(s^{2m} + |\xi|^{2m})$.

It follows that $p_s(x, i\xi', z) = 0$ has m roots with negative real part for all $x \in R^n_+$, for all $\xi' \in R^{n-1}$ and for all $s > 0$.

S **Theorem 4.2.** *There exists s_0 depending on K_δ and K_0 so that if $s \geq s_0$, then* $|u|_{2m+\delta;R^n_+} \leq C|L_s u|_{\delta;R^n_+}$. *The constant C depends on s*

Proof. Let $v(x) = u(\frac{x}{\lambda})$ and $\tilde{a}_\alpha(x) = a_\alpha(\frac{x}{s})$.

Note $\tilde{L}v(x) = \sum_{|\alpha|\leq 2m} \tilde{a}_\alpha(x) D^\alpha v(x) = s^{-2m} L_s u(\frac{x}{s}) := \tilde{f}_s(x)$.

It follows that $\tilde{K}_0 = K_0$ and $\tilde{K}_\delta = s^{-\delta} K_\delta$. Therefore, the constant C in (4.1) is the same.

In order to apply (4.1) to v, we need, according to (4.10) that $\tilde{K}_\delta \leq \frac{1}{4C(1+4C)}$ which amounts to $\frac{K_\delta}{s^\delta} \leq \frac{1}{4C(1+4C)}$ and so we take $s_0 = (4CK_\delta(1+4C))^{\frac{1}{\delta}}$, where we emphasize that C depends only n, λ, δ, and K_0

An application of theorem (4.1) to v gives, for $s \geq s_0$

$|v|_{2m+\delta;R^n_+} \leq C|\tilde{L}v|_{\delta;R^n_+}$ with C universal.

Therefore,

$|u|_{2m+\delta;R^n_+} \leq C|L_s u|_{\delta;R^n_+}$.

A consequence of the apriori estimate of theorem (4.2) following a standard application of the continuity method is the following theorem on solvability of the Dirichlet problem.

Theorem 4.3. *Let s_0 be define as in theorem (4.2) and $s \geq s_0$. Then, given $f \in C^{\delta}(R^n_+)$, there exists (and is unique) solution $u \in C^{2m+\delta}(R^n_+)$ of*

$L_s u = f$ in R^n_+ such that

$u(x', 0) = 0$, $\frac{\partial u(x',0)}{\partial x_n} = 0$, ..., $\frac{\partial^{m-1} u(x',0)}{\partial x_n^{m-1}} = 0$.

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