Larotonda spaces: homogeneous spaces and conditional expectations

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Abstract

We define a *Larotonda space* as a quotient space $\mathcal{P} = \mathcal{U}_\mathcal{A}/\mathcal{U}_\mathcal{B}$ of the unitary groups of C*-algebras $1 \in \mathcal{B} \subset \mathcal{A}$ with a faithful unital conditional expectation

$$\Phi : \mathcal{A} \to \mathcal{B}.$$

In particular, $\mathcal{B}$ is complemented in $\mathcal{A}$, a fact which implies that $\mathcal{P}$ has C^∞ differentiable structure, with the topology induced by the norm of $\mathcal{A}$. The conditional expectation also allows one to define a reductive structure (in particular, a linear connection) and a $\mathcal{U}_\mathcal{A}$-invariant Finsler metric in $\mathcal{P}$.

Given a point $\rho \in \mathcal{P}$ and a tangent vector $X \in (T\mathcal{P})_\rho$, we consider the problem of whether the geodesic δ of the linear connection satisfying these inital data is (locally) minimal for the metric. We find a sufficient condition. Several examples are given, of locally minimal geodesics.

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1 Introduction

Let $\mathcal{A}$ be a unital C* algebra and denote by $\mathcal{U}_\mathcal{A}$ its unitary group. We are interested in homogeneous spaces of $\mathcal{U}_\mathcal{A}$. By this we mean quotients $\mathcal{P} = \mathcal{U}_\mathcal{A}/\mathcal{U}_\mathcal{B}$, where $1 \in \mathcal{B} \subset \mathcal{A}$ is a sub-C*-algebra. In order that the quotient $\mathcal{P}$ have a C^∞ manifold structure, we require that the subalgebra $\mathcal{B}$ be the range of a conditional expectation

$$\Phi : \mathcal{A} \to \mathcal{B}.$$

This conditional expectation allows us to introduce a reductive structure (in particular, a linear connection) in $\mathcal{P}$, and a natural $\mathcal{U}_\mathcal{A}$-invariant metric: to ensure the latter, we require additionally that the conditional expectation be faithful. The metric is induced by the pre-Hilbert C*-$\mathcal{B}$-module structure of $\mathcal{A}$, with the inner product given by Φ. We refer the reader to the book [6] for basic facts on the theory of Hilbert C^*-modules, though we will not venture beyond the elementary features of this theory. In fact we will not require the $\mathcal{B}$-valued inner product metric induced by Φ to be complete.

As we describe below, there are several examples of this type of space: the Grassmann manifold of $\mathcal{A}$, the projective space of $\mathcal{U}_{\mathcal{A}}$, the flag manifold, the space of unitary representations of a compact group.

A.R. Larotonda had the idea that one should pursue this natural course, where the conditional expectation provides the reductive structure (covariant derivative, curvature tensor, geodesics) and the metric. He co-authored our first attempts at this study [7], [1].

We are interested in the following problem: given $\rho \in \mathcal{P}$ and $X \in (T\mathcal{P})_\rho$, does the geodesic δ of the linear connection which satisfies

$$\delta(0) = \rho \quad \text{and} \quad \dot{\delta}(0) = X$$

have minimal length for $|t| \leq r$ (for some $r = r(\rho, X)$)?

Since $\mathcal{P}$ is a homogeneous reductive space [8], geodesics of the linear connection which start at $\rho_0 =$ (class of 1 in the quotient $\mathcal{P}$) are obtained as one parameter groups e^{tx} acting on ρ_0. Here $x \in \mathcal{A}$ satisfies $x^* = -x$, $\Phi(x) = 0$, and its image under the differential of the quotient map is X (x is uniquely determined by these conditions). Geodesics starting at other points of $\mathcal{P}$ are left translations of these.

There is no general theory in this context, implying the local minimality of geodesics (for instance, the metric is not smooth, nor complete). We introduce a sufficient condition on X in order that geodesics with this velocity vector are locally minimal: if there exists a Φ-invariant state φ of $\mathcal{A}$ such that

$$\varphi(x^4) = \varphi(x^2)^2 = \|\Phi(x^2)\|^2.$$

We call such pairings x, φ *minimal*. Examples are given, of spaces and vectors where this condition holds. Our main result states that if X satisfies this condition, then a geodesic δ of $\mathcal{P}$ with $\dot{\delta}(0) = X$ is minimal for

$$|t| \leq \frac{\pi}{2\|\Phi(x^2)\|^{1/2}}.$$

The contents of the paper are the following. In Section 2 we introduce the basic definitions and notations, and give some examples of Larotonda spaces. In Section 3 we introduce a map from $\mathcal{P}$ onto the Grassmann manifold of a GNS Hilbert space, which is 2-times a contraction at the differential level. In Section 4 we prove our main result, by embedding $\mathcal{P}$ in a suitable Hilbert space sphere. In Section 5 we give examples of spaces and tangent vectors where our result holds.

This paper is affectionately dedicated to our friend Angel Rafael Laotonda (1939-2005).

2 Preliminary facts

The left action of $\mathcal{U}_{\mathcal{A}}$ on $\mathcal{P}$ will be denoted by

$$L_u\rho, \text{ for } u \in \mathcal{U}_{\mathcal{A}} \text{ and } \rho \in \mathcal{P}$$

i.e. if $\rho = [w]$ (the class of $w \in \mathcal{U}_a$), then $L_u\rho = [uw]$. For any fixed $\rho \in \mathcal{P}$ the action induces a C^∞ map

$$\pi_\rho : \mathcal{U}_{\mathcal{A}} \to \mathcal{P} \ , \quad \pi_\rho(u) = L_u\rho.$$

We shall denote by δ_ρ the differential of π_ρ at $1 \in \mathcal{U}_{\mathcal{A}}$,

$$\delta_\rho = d(\pi_\rho)_1 : \mathcal{A}_{ah} \to T\mathcal{P}_\rho,$$

where $\mathcal{A}_{ah} = \{x \in \mathcal{A} : x^* = -x\}$ (the space of anti-hermitian elements of $\mathcal{A}$) is the Banach-Lie algebra of $\mathcal{U}_\mathcal{A}$.

The fact that $\mathcal{B}_{ah}$ is complemented in $\mathcal{A}_{ah}$ (by the the anti-hermitian part of the nullspace of Φ), implies that the quotient has a natural C^∞ structure (see for instance [3]).

Note that for any $\rho \in \mathcal{P}$, the isotropy group at ρ (i.e. the subgroup unitaries in $\mathcal{U}_\mathcal{A}$ which fix ρ), is the unitary group $\mathcal{U}_{\mathcal{B}_\rho}$ of a sub-C^*-algebra $\mathcal{B}_\rho \subset \mathcal{A}$. Indeed, if we pick $\rho = [u]$ ($u \in \mathcal{A}$), then $\mathcal{B}_\rho = u\mathcal{B}u^*$.

We shall introduce a reductive structure and a metric. Both structures will be defined in terms of a distribution

$$\mathcal{P} \ni \rho \mapsto \Phi_\rho \in \mathcal{B}(\mathcal{A}, \mathcal{A}), \tag{1}$$

of conditional expectations

$$\Phi_\rho : \mathcal{A} \to \mathcal{B}_\rho \subset \mathcal{A}.$$

Namely, for $\rho = [u]$, put

$$\Phi_{\rho_0} = u\Phi_0(u^* \cdot u)u^*.$$

This distribution is well defined. If $[u] = [w]$, then $u^*w, w^*u \in \mathcal{B}$, and therefore

$$w\Phi(w^*xw)w^* = uu^*w\Phi(w^*xw)w^*uu^* = u\Phi(u^*ww^*xww^*u)u^* = u\Phi(u^*xu)u^*.$$

Also, this distribution is smooth because it is equivariant under the action of $\mathcal{U}_\mathcal{A}$: if $\mathcal{U} \in \mathcal{U}_\mathcal{A}$ and $\rho' = L_u\rho$, then

$$\Phi_{\rho'}(x) = u\Phi_\rho(u^*xu)u^*.$$

Definition 2.1. (Reductive strucutre)
The reductive structure in $\mathcal{P}$ is given by the distribution

$$\mathcal{P} \ni \rho \mapsto \mathcal{K}_\rho := \{x \in \mathcal{A}_{ah} : \Phi_\rho(x) = 0\}.$$

Remark 2.2. Let us check that this definition fills the requirements of a reductive structure [5] (or [8] for the infinite dimensional setting)

1. $$\mathcal{K}_\rho \oplus (T\mathcal{U}_{\mathcal{B}_\rho})_1 = \mathcal{K}_\rho \oplus \mathcal{B}_{ah} = \mathcal{A}_{ah},$$

   because $\mathcal{K}_\rho$ and $\mathcal{B}_{ah}$ are the anti-herimitian parts of the nullspace and the range of the conditional expectation Φ_ρ (note that $\Phi_\rho(x^*) = \Phi_\rho(x)^*$).

2. The mapping $\rho \mapsto \mathcal{K}_\rho$ is smooth. This means that if $X_\rho \in (T\mathcal{P})_\rho$ is a smooth tangent field, and $x_\rho \in \mathcal{K}_\rho$ are defined by

   $$\delta_\rho(x_\rho) = X_\rho$$

   then the map $\rho \mapsto x_\rho$ is smooth. Indeed, fix $\rho_0 \in \mathcal{P}$. Since the maps $\pi_\rho : \mathcal{U}_\mathcal{A} \to \mathcal{P}$ are C^∞ submersions, on a neighbourhood of ρ_0 the tangent field X_ρ can be lifted to a smooth map $\rho \mapsto z_\rho \in \mathcal{A}_{ah}$, such that $\delta_\rho(z_\rho) = X_\rho$. On the other hand $Id - \Phi_\rho : \mathcal{A}_{ah} \to \mathcal{K}_\rho$ is a projection onto $\mathcal{K}_\rho$, thus

   $$\rho \mapsto x_\rho = (Id - \Phi_\rho)(z_\rho).$$

   Note that $\rho \mapsto \Phi_\rho \in \mathcal{B}(\mathcal{A}, \mathcal{A})$ is also smooth: locally, using a smooth local cross section σ for π_{ρ_0} near ρ_0,

   $$\Phi_\rho = \sigma(\rho)\Phi_{\rho_0}(\sigma(\rho)^* \cdot \sigma(\rho))\sigma(\rho)^*.$$

3. If $v \in \mathcal{U}_{\mathcal{B}_\rho}$, then

$$v\mathcal{K}_\rho v^* = \mathcal{K}_\rho.$$

Indeed, $v\Phi_\rho(x)v^* = \Phi_\rho(vxv^*)$.

A reductive structure, as in classical differnetial geometry, induces a linear connection in $\mathcal{P}$. Since we shall be concerned with geodesics. let us point out that given $\rho \in \mathcal{P}$ and $X \in (T\mathcal{P})_\rho$, the geodesic of $\mathcal{P}$ with $\delta(0) = \rho$ and $\dot{\delta}(0) = X$ is

$$\delta(t) = L_{e^{tx}}\rho,$$

where as above, $x \in \mathcal{K}_\rho$ with $\delta_\rho(x) = X$.

The Finsler metric is given as follows.

Definition 2.3. (Finsler metric)
If $\rho \in \mathcal{P}$ and $X \in (T\mathcal{P})_\rho$,

$$|X|_\rho = \|\Phi_\rho(x^2)\|^{1/2},$$

where x is the unique element in $\mathcal{K}_\rho$ such that $\delta_\rho(x) = X$. We emphazise that it is not a smooth distribution of norms: in general, the norm of a C^*-algebra is non smooth. Is is clearly a continuous distribution of norms at every tangent space of $\mathcal{P}$.

One important feature of this metric if that it is invariant under the left action of $\mathcal{U}_\mathcal{A}$ on $\mathcal{P}$.

Proposition 2.4. *If $\rho \in \mathcal{P}$, $X \in (T\mathcal{P})_\rho$ and $u \in \mathcal{U}_\mathcal{A}$, then*

$$|d(L_u)_\rho(X)|_{L_u\rho} = |X|_\rho.$$

Proof. If $w \in \mathcal{U}_\mathcal{A}$, denote by ℓ_w and R_w the right and left multiplication (by w) in $\mathcal{A}$. Let $\rho' = L_u\rho$. Then

$$\pi_{\rho'}(w) = L_w\rho' = L_{wu}\rho = \pi_\rho \circ R_u(w),$$

thus differentiating at $1 \in \mathcal{U}_\mathcal{A}$,

$$\delta_{\rho'} = d(\pi_\rho)_u \circ R_u.$$

On the other hand

$$\pi_\rho \circ \ell_u(w) = L_{uw}\rho = L_u(L_w\rho) = L_u \circ \pi_\rho(w),$$

and differentiating at 1

$$d(\pi_\rho)_u = d(L_u)_\rho \circ \delta_\rho \circ \ell_{u^*}.$$

Therefore

$$\delta_{\rho'} = d(L_u)_\rho \circ \delta_\rho \circ \ell_{u^*} \circ R_w.$$

Let $X \in (T\mathcal{P})_\rho$ and $x \in \mathcal{K}_\rho$ such that $\delta_\rho(x) = X$. Then

$$\Phi_{\rho'}(uxu^*) = u\Phi_\rho(x)u^* = 0 \ , \ \text{and} \quad (uxu)^* = -uxu^*,$$

(i.e. $uxu^* \in \mathcal{K}_{\rho'}$). Also

$$\delta_{\rho'}(uxu^*) = d(L_u)_\rho \circ \delta_\rho \circ \ell_{u^*} \circ R_u(uxu^*) = d(L_u)_\rho(\delta_\rho(x)) = d(L_u)_\rho(X).$$

Thus

$$|d(L_u)_\rho(X)|_{\rho'} = \|\Phi_{\rho'}(ux^2u^*)\|^{1/2} = \|u\Phi_\rho(x^2)u^*\|^{1/2} = \|\Phi_\rho(x^2)\|^{1/2} = |X|_\rho.$$

□

We shall call a space $\mathcal{P}$ (with the action, the expectation and the metric) a *Larotonda space*, honoring A.R. Larotonda who first saw the relevance of this class.

Let us describe examples of Larotonda spaces:

Examples 2.5.

1. Let $\epsilon_0 \in \mathcal{A}$ be a selfadjoint unitary (or symmetry), and let

$$\mathcal{P} = \{u\epsilon_0 u^* : u \in \mathcal{U}_\mathcal{A}\}.$$

If $\mathcal{A} = \mathcal{B}(\mathcal{H})$, the symmetry ϵ_0 is a reflection with respect to a closed subspace $\mathcal{S}_0 \subset \mathcal{H}$ (ϵ_0 equals the identity in $\mathcal{S}_0$ and minus the identity in $\mathcal{S}_0^\perp$) and $u\epsilon_0 u^*$ is the reflection with respect to $u(\mathcal{S}_0)$. Then the space $\mathcal{P}$ coincides with the set of reflections with respect to closed subspaces of $\mathcal{H}$ with the same dimension and co-dimension as $\mathcal{S}_0$, and thus $\mathcal{P}$ can be regarded as an operator parametrization of (a connected component of) the Grassmann manifold of $\mathcal{H}$, with the usual action of the unitary group. In the general case, the group $\mathcal{U}_\mathcal{A}$ may not be connected and the connected components of $\mathcal{P}$ cannot be characterized by dimensions of range and nullspace.

The subalgebra $\mathcal{B}$ is $\{b \in \mathcal{A} : b\epsilon_0 = \epsilon_0 b\}$, and consists of the elements in $\mathcal{A}$ which are diagonal with respect to the projection $\frac{1}{2}(\epsilon_0 + 1)$. The conditional expectation is $\Phi(a) = \frac{1}{2}\{a + \epsilon_0 a \epsilon_0\}$.

2. Let Ω be a compact topological space and $\mathcal{A} = C(\Omega, M_n(\mathbb{C}))$ the algebra of continuous functions from Ω to $\mathcal{B}(\mathbb{C}^n) = M_n(\mathbb{C})$, with norm

$$\|f\| = \sup_{t\in\Omega} \|f(t)\|.$$

Consider the subalgebra

$$\mathcal{B} = \{g \in \mathcal{A} : g(t) \in \mathbb{C}\cdot 1,\ t \in \Omega\}.$$

There is a natural conditional expectation

$$\Phi : \mathcal{A} \to \mathcal{B},\ \Phi(f) = Tr(f(\cdot)),$$

where Tr denotes the normalized trace. Apparently Φ is faithful. The unitary group $\mathcal{U}_\mathcal{A}$ is the group of functions with values in $U(n)$, and $\mathcal{U}_\mathcal{B}$ identifies with $\{g \in C(\Omega, \mathbb{C}) : |g(t)| = 1\}$. Fix $\rho_0 = 1$ the constant function equal to the identity.

The Lie algebra of $\mathcal{U}_\mathcal{A}$ consists of continuous functions with anti-hermitian values.

$$\mathcal{A}_{ah} = \{w \in C(\Omega, M_n(\mathbb{C})) : w(t)^* = -w(t)\},$$

and if $\delta_{\rho_0}(w) = W \in T(\mathcal{P}_0)_{\rho_0}$,

$$|W|_{\rho_0} = \sup_{t\in\Omega} \|Tr(w^2(t))\|^{1/2}.$$

3. Let $\mathcal{A}$ be a unital C*-algebra with a faithful state φ. Consider the subalgebra $\mathcal{B} = \mathbb{C}1$ and the conditional expectation

$$\Phi : \mathcal{A} \to \mathbb{C}, \quad \Phi(a) = \varphi(a)1.$$

In this case the homogeneous space $\mathcal{P}$ equals the projective unitary group $\mathcal{U}_{\mathcal{A}}/\mathbb{S}_1$. Denote by $[u]$ = class of u in this projective space. Put $\rho_0 = [1]$. A tangent vector X at $[1]$ is a class $[a]$ in $\mathcal{A}_{ah}/i\mathbb{R}$, and the unique element $x \in \mathcal{K}_{[1]}$ such that $\delta_{[1]}(x) = X$ is $x = a - \varphi(a)1$. Thus

$$|X|_{[1]} = \varphi(x^2) = \varphi(a - \varphi(1)1)^2)^{1/2}.$$

3 A 2-contraction onto the Grassmannian

We fix $\rho_0 = [1] \in \mathcal{P}$, and denote $\Phi = \Phi_{\rho_0}$ and $\mathcal{B} = \mathcal{B}_{\rho_0}$.

Let φ_0 be a state in $\mathcal{B}$. Consider its extension to $\mathcal{A}$ given by

$$\varphi = \varphi_0 \Phi.$$

Note that is φ is Φ-invariant: $\varphi(x) = \varphi(\Phi(x))$. We do not require that φ_0 be faithful. Let $\mathcal{H}_\varphi$ be the Hilbert space obtained from the pair $(\mathcal{A}, \varphi)$ by the GNS construction. Namely

$$\mathcal{A}_\varphi = \mathcal{A}/\mathcal{N}_\varphi,$$

where $\mathcal{N}_\varphi = \{z \in \mathcal{A} : \varphi(z^*z) = 0\}$, endowed with the definite inner product

$$< [x], [y] >= \varphi(x^*y).$$

$\mathcal{H}_\varphi$ is the completion of $\mathcal{A}_\varphi$. We shall define a map R_φ from the space $\mathcal{P}$ to the Grassmann manifold $Gr(\mathcal{H}_\varphi)$ of $\mathcal{H}_\varphi$. This manifold is the set of all closed subspaces of $\mathcal{H}_\varphi$. As remarked before, the closed subspaces of $\mathcal{H}_\varphi$ are in one to one correspondence with the symmetries of $\mathcal{H}_\varphi$. A symmetry (or reflection) is a selfadjoint unitary operator $S = S^* = S^{-1}$. A symmetry has two eigenspaces with eigenvalues ± 1, and the correspondence is given by

$$S \longleftrightarrow N(S - 1).$$

Equivalently,

$$\mathcal{S} \longleftrightarrow 2P_{\mathcal{S}} - 1,$$

where $P_{\mathcal{S}}$ is the orthogonal projection onto $\mathcal{S}$. We shall represent $Gr(\mathcal{H}\varphi)$ using symmetries. For an account of the geometry of the Grassmann manifold of an inifnite dimensional Hilbert space see [9], [4].

Recall that the conditional expectation decomposes $\mathcal{A}$

$$\mathcal{A} = \mathcal{B} \oplus N(\Phi).$$

This decomposition induces a $\varphi-$orthogonal decomposition of $\mathcal{A}_\varphi$. Indeed, the fact that φ is Φ-invariant implies that

$$\mathcal{A}_\varphi = \mathcal{A}/\mathcal{N}_\varphi = \mathcal{B}_\varphi \oplus N(\Phi)_\varphi$$

with $\mathcal{B}_\varphi \perp N(\Phi)_\varphi$, where

$$\mathcal{B}_\varphi = \mathcal{B}/(\mathcal{N}_\varphi \cap \mathcal{B}) \quad \text{and} \quad N(\Phi)_\varphi = N(\Phi)/(\mathcal{N}_\varphi \cap N(\Phi)).$$

Let R_0 be the symmetry which equals $+1$ at the closed subspace $\bar{\mathcal{B}}_\varphi$ of $\mathcal{H}_\varphi$, which is the completion of $\mathcal{B}_\varphi$. Note that the orthogonal projection onto this closed subspace is induced by Φ,

$$P_{\bar{\mathcal{B}}_\varphi}([x]) = [\Phi(x)],$$

So that $R_0([x]) = 2[\Phi(x)] - [x]$, for $x \in \mathcal{A}$.

The left action of $\mathcal{U}_\mathcal{A}$ on $\mathcal{A}$ induces unitary operators in $\mathcal{H}_\varphi$ (i.e. the images of elements in $\mathcal{U}_\mathcal{A}$ by the GNS representation): if $u \in \mathcal{U}_\mathcal{A}$

$$\lambda_\varphi(u)([x]) = [ux],$$

for $x \in \mathcal{A}$.

The mapping from $\mathcal{P}$ to $Gr(\mathcal{H}_\varphi)$ is given by

$$R_\varphi : \mathcal{P} \to Gr(\mathcal{H}_\varphi) \ , \quad R_\varphi(\rho) = \lambda_\varphi(u) R_0 \lambda_\varphi(u^*),$$

if $L_u \rho_0 = \rho$.

Proposition 3.1. *The map R_φ is well defined and smooth.*

Proof. Suppose that $L_u\rho_0 = L_w\rho_0$, then $w = uv$ for $v \in \mathcal{U}_\mathcal{B}$. Then for any $x \in \mathcal{A}$

$$w\Phi(w^*x) = uv\Phi(v^*u^*x) = u\Phi(u^*x),$$

i.e.

$$\lambda_\varphi(w)(P_{\bar{\mathcal{B}}_\varphi}(\lambda_\varphi(w^*)([x]))) = \lambda_\varphi(u)(P_{\bar{\mathcal{B}}_\varphi}(\lambda_\varphi(u^*)([x]))),$$

and therefore R_φ is well defined. Once it is well defined, it is clearly C^∞. Using a smooth local cross section σ for

$$\pi_{\rho_0} : \mathcal{U}_\mathcal{A} \to \mathcal{P},$$

the map R_φ can be locally described near ρ_0 as $R_\varphi(\rho) = \lambda_\varphi(\sigma(\rho))R_0\lambda_\varphi((\sigma(\rho)^*)$, which is apparently a C^∞ function of ρ. A standard argument using the action of $\mathcal{U}_\mathcal{A}$ on $\mathcal{P}$, shows that R_φ is C^∞ in all $\mathcal{P}$. □

The following Lemma will be useful to estimate the norm of the differential of R.

Lemma 3.2. *Let $T \in \mathcal{B}(\mathcal{H})$ be a selfadjoint operator which is co-diagonal with respect to a an orthogonal projection P, i.e.*

$$T(R(P)) \subset R(P)^\perp = N(P) \text{ and } T(N(P)) \subset N(P)^\perp = R(P).$$

Then $\|T\| = \|TP\| = \|PT\|$.

Proof. Writing operators in $\mathcal{H}$ as 2×2 matrices in terms of the decompositon $\mathcal{H} = R(P)\oplus N(P)$, using the fact that T is P-codiagonal,

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & T_{12} \\ T_{12}^* & 0 \end{pmatrix} \quad \text{and} \quad PT = \begin{pmatrix} 0 & T_{12} \\ 0 & 0 \end{pmatrix}.$$

Then

$$\|T\|^2 = \|T^2\| = \| \begin{pmatrix} T_{12}T_{12}^* & 0 \\ 0 & T_{12}^*T_{12} \end{pmatrix} \| = \max\{\|T_{12}T_{12}^*\|, \|T_{12}^*T_{12}\|\} = \|T_{12}\|^2 = \|PT\|^2.$$

Also $\|TP\| = \|(PT)^*\| = \|PT\|$. $\square$

The importance of the map R_φ for our problem lies in the following result.

Theorem 3.3. *The differential of R_φ is 2 times a contraction, i.e., for every $\rho \in \mathcal{P}$ and $X \in T\mathcal{P}_\rho$,*

$$\|d(R_\varphi)_\rho(X)\| \leq 2|X|_\rho.$$

The norm on the left hand side of this inequality is the usual operator norm in $\mathcal{B}(\mathcal{H}_\varphi)$

Proof. The usual norm of an operator is invariant under left and right multiplication by unitaries, the metric of $\mathcal{P}$ is left invariant. Therefore it suffices to prove the above inequality for the case $\rho = \rho_0$. Let $X \in T\mathcal{P}_{\rho_0}$ and $x \in \mathcal{K}_{\rho_0}$ such that $\delta_{\rho_0}(x) = X$. Recall that $x \in \mathcal{A}_{ah}$ and $\Phi(x) = 0$. Then

$$|X|_{\rho_0} = \|\Phi(x^2)\|^{1/2}.$$

The curve $\gamma(t) = L_{e^{tx}}\rho_0 = \pi_{\rho_0}(e^{tx})$ is a smooth curve in $\mathcal{P}$ with $\gamma(0) = \rho_0$ and $\dot\gamma(0)$ equal to

$$\dot\gamma(0) = \delta_{\rho_0}(x) = X.$$

Then

$$\begin{aligned} d(R_\varphi)_{\rho_0}(X) &= \frac{d}{dt} R_{\gamma(t)}|_{t=0} = \frac{d}{dt}\lambda_\varphi(e^{tx})R_0\lambda_\varphi(e^{-tx})|_{t=0} \\ &= \lambda_\varphi(x)R_0 - R_0\lambda_\varphi(x) = [\lambda_\varphi(x), R_0]. \end{aligned}$$

Note that $R_0 = 2P_{\bar{\mathcal{B}}_\varphi} - 1$, so that $[\lambda_\varphi(x), R_0] = 2[\lambda_\varphi(x), P_{\bar{\mathcal{B}}_\varphi}]$.

Pick $a \in \mathcal{A}$. At elements in $\mathcal{A}_\varphi \subset \mathcal{H}_\varphi$, this commutant is

$$[\lambda_\varphi(x), P_{\bar{\mathcal{B}}_\varphi}]([a]) = x\Phi(a) - \Phi(xa).$$

Let us prove that the selfadjoint operator $[\lambda_\varphi(x), P_{\bar{\mathcal{B}}_\varphi}] \in \mathcal{B}(\mathcal{H}_\varphi)$ is codiagonal with respect to $P_{\bar{\mathcal{B}}_\varphi}$. We must check that

$$[\lambda_\varphi(x), P_{\bar{\mathcal{B}}_\varphi}](R(P_{\bar{\mathcal{B}}_\varphi})) \subset N(P_{\bar{\mathcal{B}}_\varphi}) \text{ and } [\lambda_\varphi(x), P_{\bar{\mathcal{B}}_\varphi}](N(P_{\bar{\mathcal{B}}_\varphi})) \subset R(P_{\bar{\mathcal{B}}_\varphi}).$$

Since $\mathcal{B}_\varphi$ is dense in $R(P_{\bar{\mathcal{B}}_\varphi})$ and $N(\Phi)_\varphi$ is dense in $N(P_{\bar{\mathcal{B}}_\varphi})$, it suffices to show that

$$x\Phi(b) - \Phi(xb) \subset N(\Phi)$$

for $b \in \mathcal{B}$, and that

$$x\Phi(z) - \Phi(xz) \in \mathcal{B}$$

for all $z \in N(\phi)$. These assertions are straightforward to verify. Recall that $x \in N(\Phi)$. The first assertion: pick $b \in \mathcal{B}$, then

$$\Phi(x\Phi(b) - \Phi(xb)) = \Phi(x)b - \Phi(x)b = 0.$$

The second: pick $z \in N(\Phi)$, then

$$x\Phi(z) - \Phi(xz) = -\Phi(xz) \in \mathcal{B}.$$

Then, by the above Lemma, it suffices to estimate $\|[\lambda_\varphi(x), P_{\bar{\mathcal{B}}_\varphi}]P_{\bar{\mathcal{B}}_\varphi}\|$. Again, by a density argument, it sufices to consider vectors

$$[b] = P_{\bar{\mathcal{B}}_\varphi}([a]) = [\Phi(a)] \in \mathcal{B}_\varphi, \ a \in \mathcal{A}.$$

Then

$$\|[\lambda_\varphi(x), P_{\bar{\mathcal{B}}_\varphi}]([b])\|_\varphi^2 = \|x\Phi(b) - \Phi(xb)\|_\varphi^2 = \|xb\|_\varphi^2 = -\varphi(b^*x^2b).$$

Since φ is Φ-invariant,

$$-\varphi(b^*x^2b) = -\varphi(\Phi(b^*x^2b)) = -\Phi(b^*\Phi(x^2)b).$$

Aparently $0 \leq -\Phi(x^2) \leq \|\Phi(x^2)\|$ implies that $0 \leq -b^*\Phi(x^2)b \leq \|\Phi(x^2)\|b^*b$. Then

$$-\varphi(b^*x^2b) \leq \|\Phi(x^2)\|\varphi(b^*b) = \|\Phi(x^2)\|\|[b]\|_\varphi^2 \leq \|\Phi(x^2)\|\|[a]\|_\varphi^2,$$

Because $\|[b]\|_\varphi = \|P_{\bar{\mathcal{B}}_\varphi}([a])\|_\varphi \leq \|[a]\|_\varphi$. Therefore

$$\|[\lambda_\varphi(x), P_{\bar{\mathcal{B}}_\varphi}](P_{\bar{\mathcal{B}}_\varphi}[a])\|_\varphi^2 \leq \|\Phi(x^2)\|\|[a]\|_\varphi^2,$$

And thus

$$\|[\lambda_\varphi(x), P_{\bar{\mathcal{B}}_\varphi}]P_{\bar{\mathcal{B}}_\varphi}\| \leq \|\Phi(x^2)\|^{1/2},$$

which proves our statement. □

4 Minimality of geodesics

Here we address the problem of local minimality of geodesics of the linear connection in a Larotonda space $\mathcal{P}$. Geodesics in $\mathcal{P}$ are curves of the form [8]

$$\gamma(t) = L_{e^{tx}}\rho$$

for x a horizontal vector at ρ: $x^* = -x$, $\Phi_\rho(x) = 0$. Since the metric in $\mathcal{P}$ is invariant under the action of $\mathcal{U}_\mathcal{A}$, we may consider the case $\rho = \rho_0 = [1]$ and $\Phi_\rho = \Phi$.

To this effect, let φ_0 be a state in $\mathcal{B}$ which attains the norm of $\Phi(x^2)$. Namely,

$$\varphi_0(\Phi(x^2)) = -\|\Phi(x^2)\|. \tag{2}$$

Recall that $-\Phi(x^2) \geq 0$ in $\mathcal{B}$. It is an elementary fact (using basic C*-theory and the Hahn-Banach Theorem) that for any positive element in a C*-algebra there is a state attainning its norm. Let us fix from now on the state

$$\varphi = \varphi_0\Phi$$

in $\mathcal{A}$. Thus

$$-\varphi(x^2) = -\varphi_0(\Phi(x^2)) = \|\Phi(x^2)\|.$$

We follow the notations and definitions of the previous section. Let us point out the following elementary facts.

Remark 4.1. Let $S(t) \in Gr(\mathcal{H}_\varphi)$ be a smooth curve of symmetries. Then $S(t)[1]$ is a smooth curve in the unit sphere $\mathbb{S}(\mathcal{H}_\varphi)$, and

$$\mathcal{L}_{\mathbb{S}(\mathcal{H}_\varphi)}(S[1]) \leq \mathcal{L}_{Gr(\mathcal{H}_\varphi)}(S),$$

where $\mathcal{L}_X$ denotes the length of the curve in the corresponding space X measured in the usual way. Indeed, the map $S(t) \mapsto S(t)[1]$ is the restriction of the linear contraction

$$\mathcal{B}(\mathcal{H}_\varphi) \to \mathcal{H}_\varphi, \; T \mapsto T[1],$$

and both assertions follow.

For the given Φ-invariant state φ, we shall consider the map Δ_φ obtained as the composition of the map $R_\varphi : \mathcal{P} \to Gr(\mathcal{H}_\varphi)$ of the previous section with the evaluation map at the vector $[1] \in \mathcal{H}_\varphi$,

$$\Delta_\varphi : \mathcal{P} \to \mathbb{S}(\mathcal{H}_\varphi) \quad , \quad \Delta_\varphi(\rho) = R_\varphi(\rho)[1]. \tag{3}$$

This map is apparently C^∞, and its differential is a 2-contraction: if $\rho \in \mathcal{P}$ and $X \in T\mathcal{P}_\rho$,

$$\|d(\Delta_\varphi)_\rho(X)\|_{\mathcal{H}_\varphi} \leq 2|X|_\rho.$$

Indeed, this follows from Theorem (3.3) and the fact that the map $T \mapsto T[1]$ is contractive. In particular, if $\rho(t) \in \mathcal{P}$ is a smooth curve, then

$$\mathcal{L}_{\mathbb{S}(\mathcal{H}_\varphi)}(\Delta_\varphi(\rho)) \leq 2\mathcal{L}_\mathcal{P}(\rho). \tag{4}$$

Remark 4.2. Let us compute the length of a geodesic γ in $\mathcal{P}$: $\gamma(t) = L_{e^{tx}}\rho_0$. Recall that $\ell_u : \mathcal{U}_\mathcal{A} \to \mathcal{U}_\mathcal{A}$ denotes the left multiplication map (which is the restricition of a linear map). Note that $\pi_{\rho_0}\ell_u(w) = L_u L_w \rho_0 = L_u \pi_{\rho_0}(w)$, so that differentiating at $1 \in \mathcal{U}_\mathcal{A}$ we get

$$d(\pi_{\rho_0})_u \ell_u = d(L_u)_{\rho_0} d(\pi_{\rho_0})_{\rho_0} = d(L_u)_{\rho_0}\delta_{\rho_0},$$

so that

$$d(\pi_{\rho_0})_u = d(L_u)_{\rho_0}\delta_{\rho_0}\ell_{u^*} = d(L_u)_{\rho_0}(u^*\delta_{\rho_0}).$$

The derivative of γ is (by the above identity)

$$\dot\gamma(t) = d(\pi_{\rho_0})_{\gamma(t)}(e^{tx}x) = d(L_{e^{tx}})_{\rho_0}\delta_{\rho_0}(x).$$

By the invariance of the metric under the left action of $\mathcal{U}_\mathcal{A}$,

$$|\dot\gamma(t)|_{\gamma(t)} = |d(L_{e^{tx}})_{\rho_0}\delta_{\rho_0}(x)|_{\gamma(t)} = |\delta_{\rho_0}(x)|_{\rho_0} = \|\Phi(x^2)\|^{1/2}.$$

Therefore, if γ is parametrized in the interval $[t_0, t_1]$,

$$\mathcal{L}(\gamma) = \int_{t_0}^{t_1} |\dot\gamma(t)|_{\gamma(t)} dt = (t_1 - t_0)\|\Phi(x^2)\|^{1/2}.$$

In order to establish the minimality of certain geodesics, we shall use the map Δ_φ to compare lenghts of curves in the sphere of $\mathcal{H}_\varphi$, for an appropriate φ. To this end, let us consider the following condition.

Definition 4.3. *Suppose that $x^* = -x$, $\Phi(x) = 0$ and φ is Φ-invariant. We say that the pair (x, φ) is minimal if*

1. $\varphi(x^2) = -\|\Phi(x^2))\|$.
2. *There exists $\lambda \in \mathbb{R}$ such that*

$$\varphi((x^2 - \lambda)^2) = 0.$$

Proposition 4.4. *Let $x \in \mathcal{A}$ such that $x^* = -x$, $\Phi(x) = 0$ and φ is Φ-invariant with $\varphi(x^2) = -\|\Phi(x^2))\|$. Then the following conditions are equivalent*

1. *The pair (x, φ) is minimal.*
2. $[1] \in \mathcal{H}_\varphi$ *is an eigenvector of* $\lambda_\varphi(x^2)$.
3. $\varphi(x^2)^2 = \varphi(x^4)$.

Proof.
1. implies 2.:
$\varphi((x^2 - \lambda)^2) = 0$, written as an inner product is

$$< (\lambda_\varphi(x^2) - \lambda 1)[1], (\lambda_\varphi(x^2) - \lambda 1)[1] >= 0,$$

i.e. $\lambda_\varphi(x^2)[1] = \lambda[1]$.
2. implies 3.:
If $\lambda_\varphi(x^2)[1] = \alpha[1]$ for some $\alpha \in \mathbb{R}$, then

$$\alpha =< \lambda_\varphi(x^2)[1], [1] >= \varphi(x^2) = -\|\Phi(x^2)\|.$$

and

$$\varphi(x^4) =< \lambda_\varphi(x^4)[1], [1] >=< \lambda_\varphi(x^2)[1], \lambda_\varphi(x^2)[1] >= \alpha^2.$$

Then $\varphi(x^4) = \alpha^2 = \varphi(x^2)^2$.
3. implies 1.:
If $\varphi(x^4) = \varphi(x^2)^2$, taking $\lambda = -\|\Phi(x^2)\|$, a straightforward computation shows that

$$\varphi((x^2 - \lambda)^2) = 0.$$

□

Note that if the pair (x, φ)-minimal, then necessarilly $\lambda = -\|\Phi(x^2)\|$.

Remark 4.5. There is another, more restrictive condition, which is

$$\|x\| = \|x\|_\Phi.$$

Indeed, if φ verifies $\varphi(x^2) = -\|\Phi(x^2)\|$ (with φ Φ-invariant), then

$$\varphi(x^2)^2 = \|\Phi(x^2)\|^2 = \|x\|^4 = \|x^4\| \geq \varphi(x^4).$$

The other inequality occurs always.

Theorem 4.6. *Let $X \in (T\mathcal{P})_{\rho_0}$ and let $x \in \mathcal{K}_{\rho_0}$ the unique element such that $\delta_{\rho_0}(x) = X$. Suppose that there is a Φ-invariant state φ such that the pair (x, φ) is minimal. Then the geodesic*

$$\gamma(t) = L_{e^{tx}}\rho_0$$

has minimal length along its path for $|t| \leq \frac{\pi}{2\|\Phi(x^2)\|^{1/2}}$.

Proof. Consider the map Δ_φ,

$$\Delta_\varphi : \mathcal{P} \to \mathbb{S}(\mathcal{H}_\varphi)$$

defined above. Note that

$$\Delta_\varphi(\gamma(t)) = R_\varphi(L_{e^{tx}}\rho_0)[1] = \lambda_\varphi(e^{tx})R_0\lambda_\varphi(e^{-tx})[1] = [e^{tx}(2\Phi(e^{-tx}) - e^{-tx})] = [2e^{tx}\Phi(e^{-tx}) - 1].$$

We claim that condition (4.3) means that $\Delta_\varphi(\gamma)$ is a geodesic of $\mathbb{S}(\mathcal{H}_\varphi)$ (with its usual Hilbert-Riemann metric). Indeed, since the pair (x, φ) is minimal,

$$[x^2] = \lambda_\varphi(x^2)[1] = \lambda[1] = -\|\Phi(x^2)\|[1].$$

Then, if $n = 2k$ is even

$$[x^n] = (-\|\Phi(x^2)\|)^k[1],$$

and if $n = 2k + 1$ is odd

$$[x^n] = (-\|\Phi(x^2)\|)^k[x].$$

Therefore

$$[e^{tx}] = [1] - \frac{t^2}{2}\|\Phi(x^2)\|[1] + \frac{t^4}{4!}\|\Phi(x^2)\|^2[1] - \ldots + t[x] - \frac{t^3}{3!}\|\Phi(x^2)\|[x] + \frac{t^5}{5!}\|\Phi(x^2)\|^2[x] - \ldots$$

$$= \cos(t\|\Phi(x^2)\|^{1/2})[1] + \frac{1}{\|\Phi(x^2)\|^{1/2}}\sin(t\|\Phi(x^2)\|^{1/2})[x].$$

Recall that $\Delta_\varphi(\gamma(t)) = [2e^{tx}\Phi(e^{-tx})] - [1]$, which equals

$$2\lambda_\varphi(e^{tx}P_{\bar{\mathcal{B}}_\varphi}([e^{-tx}])) - [1] = 2\lambda_\varphi(P_{\bar{\mathcal{B}}_\varphi}(\cos(t\|\Phi(x^2)\|^{1/2})e^{tx}[1]) - [1],$$

because $P_{\bar{\mathcal{B}}_\varphi}(x) = [\Phi(x)] = 0$. Thus

$$\Delta_\varphi(\gamma(t)) = 2\cos(t\|\Phi(x^2)\|^{1/2})[e^{tx}] - [1]$$

$$= 2(\cos^2(t\|\Phi(x^2)\|^{1/2}) - 1)[1] + \frac{1}{\|\Phi(x^2)\|^{1/2}}2\cos(t\|\Phi(x^2)\|^{1/2})\sin(t\|\Phi(x^2)\|^{1/2})[x]$$

$$= \cos(2t\|\Phi(x^2)\|^{1/2})[1] + \sin(2t\|\Phi(x^2)\|^{1/2})\frac{[x]}{\|\Phi(x^2)\|^{1/2}},$$

which is a great circle (=minimal geodesic) of the sphere $\mathbb{S}(\mathcal{H}_\varphi)$ because

$$< [x], [1] >= \varphi(x) = 0,$$

(note that $\frac{[x]}{\|\Phi(x^2)\|^{1/2}}$ is a unit vector). This geodesic will remain minimal as long as

$$|2t|\|\Phi(x^2)\|^{1/2} \leq \pi.$$

Note also that the length of $\Delta_\varphi(\gamma)$ restricted to the interval $[t_1, t_2]$ is $2(t_2 - t_1)\|\Phi(x^2)\|^{1/2}$, which is 2 times the length of γ in this interval.

Let $\rho = \rho(t)$ be a smooth curve in $\mathcal{P}$ with endpoints $\gamma(t_1)$ and $\gamma(t_2)$, for $[t_1, t_2]$ with $|t_i| \leq \frac{\pi}{2\|\Phi(x^2)\|^{1/2}}$. Then by the inequality (4) (below Remark (4.1)),

$$\mathcal{L}_{\mathbb{S}(\mathcal{H}_\varphi)}(\Delta_\varphi(\rho)) \leq 2\mathcal{L}_{\mathcal{P}}(\rho).$$

Since $\Delta_\varphi(\gamma)$ is a minimal geodesic of $\mathbb{S}(\mathcal{H}_\varphi)$,

$$\mathcal{L}_{\mathbb{S}(\mathcal{H}_\varphi)}(\Delta_\varphi(\gamma)) \leq \mathcal{L}_{\mathbb{S}(\mathcal{H}_\varphi)}(\Delta_\varphi(\rho)),$$

and finally, as remarked above,

$$\mathcal{L}_{\mathbb{S}(\mathcal{H}_\varphi)}(\Delta_\varphi(\gamma)) = 2\mathcal{L}_{\mathcal{P}}(\gamma).$$

It follows that

$$\mathcal{L}_{\mathcal{P}}(\gamma) \leq \mathcal{L}_{\mathcal{P}}(\rho).$$

□

5 Examples

We consider examples where Theorem (4.6) applies. In the first example any geodesic is minimal (up to the critical value of t). The next examples show spaces where special geodesics are minimal.

Example 5.1. Let $\mathcal{P}$ be the connected component of a fixed symmetry ρ_0 in the Grassmann manifold of a C*-algebra $\mathcal{A}$ as in Example 2.5 1. The group $\mathcal{U}_{\mathcal{A}}$ acts on $\mathcal{P}$ by inner conjugation: $u \cdot \rho = u\rho u^*$. It is well known (see for instance [4]) that the action is transitive on $\mathcal{P}$. The isotropy group at ρ_0 is the set of unitaries which commute with ρ_0. Thus

$$\mathcal{B} = \{b \in \mathcal{A} : b\rho_0 = \rho_0 b\}.$$

Consider the conditional expectation

$$\Phi : \mathcal{A} \to \mathcal{B}, \quad \Phi(a) = \frac{1}{2}(a + \rho_0 a \rho_0).$$

Apparently Φ projects onto $\mathcal{B}$. Note that $\Phi(x) = 0$ if and only if x anticommutes with ρ_0. This implies that if $\Phi(x) = 0$, then x^2 commutes with ρ_0, i.e. $x^2 \in \mathcal{B}$. In particular, this implies that if φ is any Φ-invariant state in $\mathcal{A}$ such that

$$\varphi(x^2) = -\|\Phi(x^2)\| = -\|x^2\| = -\|x\|^2,$$

then the pair (x, φ) is minimal. Thus any geodesic in $\mathcal{P}$ is minimal up to the critical value of t.

Let us describe the Finsler metric of this Larotonda space. Let $X \in (T\mathcal{P})_{\rho_0}$. Then X is of the form

$$X = z\rho_0 - \rho_0 z,$$

for some $z^* = -z$ in $\mathcal{A}$. Note that tangent vectors at ρ_0 are selfadjoint elements which anticommute with ρ_0. In this example, δ_{ρ_0} is given by

$$\delta_{\rho_0}(a) = a\rho_0 - \rho_0 a.$$

Then a straightforward computation shows that the unique anti-hermitian x with $\Phi(x) = 0$ such that $\delta_{\rho_0}(x) = X$ is

$$x = \frac{1}{2}X\rho_0.$$

Then

$$|X|_{\rho_0} = \frac{1}{2}\|x\|,$$

i.e. $|\ |_\rho$ is essentially the usual norm of $\mathcal{A}$. Since the norm of $\mathcal{A}$ is invariant by multiplication by unitaries, it follows that the left invariant Finsler metric of $\mathcal{P}$ is $\frac{1}{2}$ times the usual norm of $\mathcal{A}$ at every tangent space.

This minimality result was proved in [9].

Example 5.2. Consider Example 2.5 2: $\mathcal{A} = C(\Omega, M_n(\mathbb{C}))$, with norm

$$\|f\| = \max_{t\in\Omega} \|f(t)\|,$$

the subalgebra $\mathcal{B}$ is

$$\mathcal{B} = \{g \in \mathcal{A} : g(t) \in \mathbb{C}\cdot 1\},$$

and the conditional expectation is

$$\Phi : \mathcal{A} \to \mathcal{B},\ \Phi(f) = Tr(f(t)).$$

Pick $\rho_0 = 1$. Let $W \in (T\mathcal{P})_{\rho_0}$ and let $w \in \mathcal{K}_{\rho_0}$ such that $\delta_{\rho_0}(W) = w$. Note that w is a matrix valued function with anti-hermitian values and $Tr(w(t)) = 0$ for all $t \in \Omega$. There exists $t_0 \in \Omega$ such that

$$\|Tr(w(t_0))^2\| = \sup_{t\in\Omega} \|Tr(w(t))^2\| = |W|^2_{\rho_0}.$$

Consider the state φ of $\mathcal{A}$ given by

$$\varphi(f) = Tr(f(t_0)),$$

which is the composition of evaluation at t_0 with Φ_{ρ_0}. Note that

$$\varphi(w^2) = Tr(w(t_0))^2) = -\sup_{t\in\Omega} \|Tr(w(t))^2\| = -\|\Phi(w^2)\|.$$

Suppose additionally that at this point t_0 one has that $w(t_0)^2 \in \mathbb{C}\cdot 1$, i.e. $w(t_0) = i\lambda\cdot 1$ for $\lambda \in \mathbb{R}$. Then the pair (w, φ) is minimal. Indeed, it is apparent that

$$\varphi(w^2)^2 = \lambda^4 = \varphi(w^4).$$

Therefore the curve

$$\gamma(t) = L_{e^{itW}}\rho_0 = e^{itw}$$

is minimal in $\mathcal{A}$ for $|t| \leq \frac{\pi}{2|W|_{\rho_0}}$.

Example 5.3. Consider Example 2.5 3 of the projective unitary space,

$$\mathcal{P} = \mathcal{U}_{\mathcal{A}}/\mathbb{S}_1.$$

Put $\rho_0 = [1]$. We fix a (faithful) conditional expectation

$$\Phi : \mathcal{A} \to \mathcal{B} = \mathbb{C}.1,$$

which is given by a faithful state φ: $\Phi(a) = \varphi(a).1$. The only Φ-invariant state is φ itself. Thus the requirement that

$$\varphi(x^2) = -\|\Phi(x^2)\| = -|\varphi(x^2)|$$

is fulfilled. Any $x \in \mathcal{A}$ such that $\varphi(x^2)^2 = \varphi(x^4)$ and $\varphi(x) = 0$ would produce a minimal geodesic in $\mathcal{P}$. In this case, Hypothesis (4.3) means that $x = ir\, S$, with $r \geq 0$ and S a symmetry. Indeed, if $\varphi(x^2)^2 = \varphi(x^4)$, then in the Cauchy-Schwarz inequality, we have equality:

$$-\varphi(x^2) = |< x^2, 1 >| \leq \|x^2\|_\varphi \|1\|_\varphi = \varphi(x^4)^{1/2} = |\varphi(x^2)| = -\varphi(x^2),$$

and thus x^2 is a multiple of 1: $x^2 = -\|x\|^2.1$ Thus $S = \frac{-i}{\|x\|}x$ is selfadjoint operator such that $S^2 = 1$. The spectral resolution of x is therefore

$$x = ire_+ - ire_-,$$

with e_+, e_- selfajoint mutually orthogonal projections such that $e_+ + e_- = 1$ The fact that $\varphi(x) = 0$ means that $\varphi(e_+) = \varphi(e_-)$

Consider $X = [x] \in \mathcal{A}/\mathbb{C}1$, which is a tangent vector to $\mathcal{P}$ at $[1]$. Apparently x is the unique horizontal element such that $\delta_{\rho_0}(x) = X$. Then the curve

$$\gamma(t) = [e^{tx}]$$

is a minimal geodesic in $\mathcal{P}$ (up to the critical value of t). In this case, $e^{tx} = e^{itr}e_+ + e^{-itr}e_-$, and thus

$$\gamma(t) = e^{itr}[e_+] + e^{-itr}[e_-] = \cos(tr)[1] + i\sin(tr)[e_+ - e_-].$$

Example 5.4. Consider $\mathcal{A} = M_3(\mathbb{C})$ and $\mathcal{B} = D_3(\mathbb{C})$ the subalgebra of diagonal matrices. Put $\Phi : \mathcal{A} \to \mathcal{B}$, $\Phi(a) =$ diagonal matrix formed with the diagonal entries of a. Consider x

$$x = \begin{pmatrix} 0 & \alpha & \beta \\ -\bar{\alpha} & 0 & 0 \\ -\bar{\beta} & 0 & 0 \end{pmatrix}.$$

Clearly $x^* = -x$ and $\Phi(x) = 0$ and x^2 is not diagonal. The quotient $\mathcal{U}_{\mathcal{A}}/\mathcal{U}_{\mathcal{B}}$ is the flag manifold of order 3. Consider the state $\varphi(a) =< ae_1, e_1 >$, where e_1 is the first vector in the canonical basis of $\mathbb{C}^3$. Then apparently φ is Φ-invariant and

$$-\varphi(x^2) = |\alpha|^2 + |\beta|^2 = \|\Phi(x^2)\|.$$

Note that $x^2 \notin \mathcal{B}$, nevertheless, elementary computations show that $\varphi(x^2)^2 = \varphi(x^4)$, i.e. the pair (x, φ) is minimal. Thus the curve $[e^{itx}]$ is minimal in $\mathcal{P}$ for $t \leq \frac{\pi}{2(|\alpha|^2+|\beta|^2)}$.

Example 5.5. Let G be a compact group and ρ_0 a strong operator continuous unitary representation of G on a Hilbert space $\mathcal{H}$,

$$\rho_0 : G \to \mathcal{U}(\mathcal{H}).$$

Let $\mathcal{A} = \mathcal{B}(\mathcal{H})$ and

$$\mathcal{B} = \rho_0(G)' = \{b \in \mathcal{B}(\mathcal{H}) : b\rho_0(g) = \rho_0(g)b \text{ for all } g \in G\}.$$

Consider the map

$$\Phi_0 : \mathcal{A} \to \mathcal{B}, \quad \Phi(a) = \int_G \rho_0(g) a \rho_0(g)^* d\mu(g),$$

where μ denotes the Haar measure of G, and the integral is considered in the strong operator sense. It is well known that Φ_0 is a (well defined) faithful conditional expectation.

For any $\tau \in \hat{G}$, let $\mathcal{B}_\tau = \{b \in \mathcal{A} : \rho(g)b\rho(g)^* = \tau(g)b \text{ for all } g \in G\}$. Note that $\mathcal{B} = \mathcal{B}_{\tau_0}$ where τ_0 is the trivial character. Also it is clear that if $b \in \mathcal{B}_\tau$ (for $\tau \neq \tau_0$), then

$$\Phi_0(b) = \int_G \rho_0(g) b \rho_0(g)^* d\mu(g) = (\int_G \tau(g) d\mu(g))b = 0,$$

because $b \in \mathcal{B} \cap \mathcal{B}_\tau = \{0\}$.

If $u \in \mathcal{U}(\mathcal{H})$, then $Ad(u)\rho_0$ (where $Ad(u)\rho_0(a) = u\rho_0(a)u^*$) is also a strong operator continuous representation of G. Thus we consider

$$\mathcal{P}_0 = \{Ad(u)\rho_0 : u \in \mathcal{U}(h)\}.$$

This space $\mathcal{P}_0$ was shown to be a differentiable manifold in [2]. Note that if $x \in \mathcal{B}_\tau$ ($\tau \neq \tau_0$), then $x^* \in \mathcal{B}_{\bar{\tau}}$. Therefore $\mathcal{B}_\tau$ contains no (non trivial) anti-hermitian elements, unless $\tau^2(g) = 1$ for all $g \in G$. In order to obtain examples of minimal pairs, we make the assumption that G is of **order** 2.

In this case also $\hat{G}$ is also of order two. Pick any $x \in \mathcal{B}_\tau$ ($\tau \neq \tau_0$), with $x^* = -x$ (for instance, pick any $a \in \mathcal{B}_\tau$ and put $x = a - a^*$). Note that $x^2 \in \mathcal{B}$:

$$\rho_0(g)x^2\rho_0(g)^* = \rho_0(g)x\rho_0(g)^*\rho_0(g)x\rho_0(g)^* = \tau(g)^2x^2 = x^2,$$

for any $g \in G$. Let φ_0 be a state of $\mathcal{B}$ such that $\phi_0(x^2) = -\|\Phi_0(x^2)\| = -\|x^2\|$. Then clearly the pair (x, φ) is minimal , and thus $\gamma(t) = Ad(e^{tx})\rho_0$ is a minimal geodesic of $\mathcal{P}_0$ up to $|t| \leq \frac{\pi}{2\|x\|}$.

Note also that $\rho_0(G) \subset \rho_0(G)'' = \mathcal{M}$, the von Neumann algebra generated by the unitaries $\rho_0(g)$, $g \in G$. One could also consider the action restricted to the unitary group $\mathcal{U}_\mathcal{M}$. Apparently, the integral $\Phi_0(m)$ takes values in $\mathcal{M}$, if $m \in \mathcal{M}$ (because it converges in the strong operator topology). Thus the example above could be stated for

$$\mathcal{P}_{0,\mathcal{M}} \simeq \mathcal{U}_\mathcal{M}/\mathcal{U}_{\mathcal{M}\cap\rho_0(G)'}.$$

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