PRODUCTS OF PROJECTIONS AND SELF-ADJOINT OPERATORS

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Abstract. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators from $\mathcal{H}$ to $\mathcal{H}$. Our goal in this article is to study the set $\mathcal{P}\cdot\mathcal{L}^h$ of operators in $\mathcal{L}(\mathcal{H})$ that can be factorized as the product of an orthogonal projection and a self-adjoint operator. We describe $\mathcal{P}\cdot\mathcal{L}^h$ and present optimal factorizations, in different senses, for an operator in this set.

1. Introduction

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators from $\mathcal{H}$ to $\mathcal{H}$. This article is devoted to the study of the set

$$\mathcal{P}\cdot\mathcal{L}^h = \{T \in \mathcal{L}(\mathcal{H}) : T = PA,\ P \in \mathcal{P},\ A \in \mathcal{L}^h\},$$

where $\mathcal{P}$ and $\mathcal{L}^h$ denote the sets of orthogonal projections and self-adjoint operators of $\mathcal{L}(\mathcal{H})$, respectively.

Previous works on factorizations of operators in terms of particular classes of operators are in [3], [5], [7], [9] and [21] among others. In particular, the sets $\mathcal{P}\cdot\mathcal{P}$ and $\mathcal{P}\cdot\mathcal{L}^+$ where $\mathcal{L}^+$ denotes the cone of the semidefinite positive operators of $\mathcal{L}(\mathcal{H})$, are studied in [9] and [5], respectively. In these articles different characterizations of the sets $\mathcal{P}\cdot\mathcal{P}$ and $\mathcal{P}\cdot\mathcal{L}^+$ are developed and also optimal factorizations are presented. Our goal in this article is to obtain similar results for the bigger set $\mathcal{P}\cdot\mathcal{L}^h$.

Now we summarize some of the results for $\mathcal{P}\cdot\mathcal{P}$ and $\mathcal{P}\cdot\mathcal{L}^+$ than can be found in [9] and [5]. For the set $\mathcal{P}\cdot\mathcal{P}$, Crimmins (see [21, Theorem 8]) showed that $T \in \mathcal{P}\cdot\mathcal{P}$ if and only if $T^2 = TT^*T$. Later, Corach and Maestripieri in [9] showed that if $T \in \mathcal{P}\cdot\mathcal{P}$ then it can always be factorized as

$$T = P_{\overline{\mathcal{R}(T)}} P_{\mathcal{N}(T)^\perp}, \tag{1}$$

where $P_{\overline{\mathcal{R}(T)}}$ and $P_{\mathcal{N}(T)^\perp}$ denote the orthogonal projections onto the closure of the range of T and onto the orthogonal complement of the nullspace of T, respectively. They also proved that the factorization (1) is optimal in the following two senses: if $T = P_{\mathcal{M}}P_{\mathcal{N}} \in \mathcal{P}\cdot\mathcal{P}$ then

a) $P_{\overline{\mathcal{R}(T)}} \leq P_{\mathcal{M}}$ and $P_{\mathcal{N}(T)^\perp} \leq P_{\mathcal{N}}$;

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b) $\|(P_{\overline{\mathcal{R}(T)}} - P_{\mathcal{N}(T)^{\perp}})x\| \leq \|(P_{\mathcal{M}} - P_{\mathcal{N}})x\|$ for all $x \in \mathcal{H}$.

On the other hand, for the set $\mathcal{P} \cdot \mathcal{L}^+$ in [5] it was proved that $T \in \mathcal{P} \cdot \mathcal{L}^+$ if and only if there exists $\lambda \geq 0$ such that $TT^* \leq \lambda T P_{\overline{\mathcal{R}(T)}}$. Furthermore, for $T \in \mathcal{P} \cdot \mathcal{L}^+$ it is always possible to find $A \in \mathcal{L}^+$ with $\mathcal{N}(A) = \mathcal{N}(T)$ such that $T = PA$, for some $P \in \mathcal{P}$. Between all elements of $\mathcal{L}^+$ with this property there exists one, denoted by $\hat{A}$, such that the factorization

$$T = P_{\overline{\mathcal{R}(T)}}\hat{A},$$

is optimal in the following senses:

a) $P_{\overline{\mathcal{R}(T)}} \leq P$ for all $P \in \mathcal{P}$ such that $T = PA$ for some $A \in \mathcal{L}^+$;
b) $\hat{A} \leq A$ and therefore $\|\hat{A}\| \leq \|A\|$ for all $A \in \mathcal{L}^+$ such that $T = PA$ for some $P \in \mathcal{P}$.

In this article we present general properties of operators in $\mathcal{P} \cdot \mathcal{L}^h$ and we compare the sets $\mathcal{P} \cdot \mathcal{L}^h$ and $\mathcal{P} \cdot \mathcal{L}^+$. In Section 2 we describe $\mathcal{P} \cdot \mathcal{L}^h$ and for a given $T \in \mathcal{P} \cdot \mathcal{L}^h$ we study the projection sets $\mathcal{P}_T = \{P \in \mathcal{P} : T = PA \text{ for some } A \in \mathcal{L}^h\}$ and $\mathcal{A}_T = \{A \in \mathcal{L}^h : T = PA \text{ for some } P \in \mathcal{P}\}$. Moreover, we see that given an operator $T \in \mathcal{P} \cdot \mathcal{L}^h$ it is not always possible to find $A \in \mathcal{L}^h$ with $\mathcal{N}(A) = \mathcal{N}(T)$ such that $T = PA$ for some $P \in \mathcal{P}$. We prove that this can be guaranteed under the extra hypothesis that $\overline{\mathcal{R}(T)} \dot{+} \mathcal{N}(T) = \mathcal{H}$. In such case, we find an element $A_{\mathcal{N}} \in \mathcal{L}^h$, with $\mathcal{N}(A_{\mathcal{N}}) = \mathcal{N}(T)$ such that the factorization

$$T = P_{\overline{\mathcal{R}(T)}}A_{\mathcal{N}},$$

is optimal in the following senses:

a) $P_{\overline{\mathcal{R}(T)}} \leq P$ for all $P \in \mathcal{P}$ such that $T = PA$ for some $A \in \mathcal{L}^h$;
b) $P_{\overline{\mathcal{R}(T)}} \leq^- P$ for all $P \in \mathcal{P}$ such that $T = PA$ for some $A \in \mathcal{L}^h$;
c) $A_{\mathcal{N}} \leq^- A$, for all $A \in \mathcal{L}^h$ such that $T = PA$, for some $P \in \mathcal{P}$;

Here, $\leq^-$ means the minus order defined for operators in $\mathcal{L}(\mathcal{H})$. Also, we distinguish another two factorizations for $T \in \mathcal{P} \cdot \mathcal{L}^h$ denoted by

$$T = P_{\overline{\mathcal{R}(T)}}A_0$$

and

$$T = P_{\overline{\mathcal{R}(T)}}A_T,$$

which are optimal in the following senses:

a) $\|A_0\| \leq \|A\|$ for all $A \in \mathcal{L}^h$ such that $T = PA$, for some $P \in \mathcal{P}$;
b) $\|(T^* - A_T)x\| \leq \|(T^* - A)x\|$ for all $x \in \mathcal{H}$ and for all $A \in \mathcal{L}^h$ such that $T = PA$, for some $P \in \mathcal{P}$;
c) $\|T - A_T\| \leq \|T - A\|$ for all $A \in \mathcal{L}^h$ such that $T = PA$, for some $P \in \mathcal{P}$.

See Theorems 2.2 and 3.2 for the definitions of A_T and A_0. The results about optimal factorizations can be found in Section 3.

2. The products of projections and self-adjoint operators

We begin this section by introducing some notation. For each $X \in \mathcal{L}(\mathcal{H})$, $\mathcal{R}(X)$ and $\mathcal{N}(X)$ are the range and nullspace of X, respectively. Besides, P_X stands for the orthogonal projection from $\mathcal{H}$ onto $\overline{\mathcal{R}(X)}$. The adjoint of X is X^* and the Moore-Penrose generalized inverse of X is $X^\dagger$. We recall that $X^\dagger \in \mathcal{L}(\mathcal{H})$ if and only if X has closed range. On the other hand, if $\mathcal{V}, \mathcal{W}$ are closed subspaces of $\mathcal{H}$ such that $\mathcal{H} = \mathcal{V} \dot{+} \mathcal{W}$ (direct sum), the symbol $Q_{\mathcal{V}//\mathcal{W}}$ identifies the oblique projection onto $\mathcal{V}$ along $\mathcal{W}$, that is, the operator $Q \in \mathcal{L}(\mathcal{H})$ with range $\mathcal{V}$ and nullspace $\mathcal{W}$ such that $Q^2 = Q$. Given $T \in \mathcal{L}(\mathcal{H})$, $T = V_T|T|$ denotes the polar decomposition of T where V_T is a partial isometry with $\mathcal{N}(V_T) = \mathcal{N}(T)$ and $|T| = (T^*T)^{1/2}$. Finally, as we have announced in the Introduction, we shall denote by

$$\mathcal{P} \cdot \mathcal{L}^h := \{PA : P \in \mathcal{P}, A \in \mathcal{L}^h\},$$

where $\mathcal{P} := \{P \in \mathcal{L}(\mathcal{H}) \ : \ P^2 = P = P^*\}$ and $\mathcal{L}^h := \{A \in \mathcal{L}(\mathcal{H}) \ : \ A = A^*\}$.

The next result will be frequently used along the article.

Proposition 2.1. *If $T = PA \in \mathcal{P} \cdot \mathcal{L}^h$ then $T = P_T A$.*

Proof. If $T = PA$ for $P \in \mathcal{P}$ and $A \in \mathcal{L}^h$ then $\mathcal{R}(P_T) = \overline{\mathcal{R}(T)} \subseteq \mathcal{R}(P)$ and, therefore, $P_T A = P_T P A = P_T T = T$. □

The following result characterizes the set $\mathcal{P} \cdot \mathcal{L}^h$. The equivalence of conditions $a)$ and $c)$ in the above theorem is [21, Theorem 9].

Theorem 2.2. *Let $T \in \mathcal{L}(\mathcal{H})$ be given. The following statements are equivalent:*

a) $T \in \mathcal{P} \cdot \mathcal{L}^h$.
b) $TP_T \in \mathcal{L}^h$.
c) $T^*T^2 \in \mathcal{L}^h$.
d) $T^n \in \mathcal{P} \cdot \mathcal{L}^h$ *for all* $n \in \mathbb{N}$.
e) $|T|V_T \in \mathcal{L}^h$.
f) $A_T = T + T^* - P_T T^* \in \mathcal{L}^h$.

Proof. $a) \leftrightarrow b)$ Assume that $T \in \mathcal{P} \cdot \mathcal{L}^h$. Then $T = P_T A$ for some $A \in \mathcal{L}^h$. So that $TP_T \in \mathcal{L}^h$. Conversely, if $TP_T \in \mathcal{L}^h$ then $A = T + T^* - TP_T \in \mathcal{L}^h$ and $T = P_T A \in \mathcal{P} \cdot \mathcal{L}^h$.

$b) \leftrightarrow c)$ Observe that $TP_T \in \mathcal{L}^h$ then for all $x, y \in \mathcal{H}$, $\langle T^*T^2x, y\rangle = \langle T^2x, Ty\rangle = \langle TP_T Tx, Ty\rangle = \langle P_T T^* Tx, Ty\rangle = \langle Tx, T^2y\rangle = \langle x, T^*T^2y\rangle$, which is to say that $T^*T^2 \in \mathcal{L}^h$. Now, if $T^*T^2 = (T^2)^*T$ then by left multiplication by $(T^*)^\dagger$ and then taking adjoint we get that $(T^2)^* = T^*TP_T$. Then, again by left multiplication by $(T^*)^\dagger$ we obtain that $TP_T \in \mathcal{L}^h$.

$a) \leftrightarrow d)$ Assume that $a)$ holds, so that $T = PA$ for some $(P, A) \in \mathcal{P} \times \mathcal{L}^h$. Pick any $k \in \mathbb{N}$. Then $T^{2k} = (PA)^{2k} = P(AP)^k(PA)^k = P(T^*)^kT^k$. On the other hand, $T^{2k+1} = TT^{2k} = PAP(T^*)^kT^k = P(T^*)^{k+1}T^k$. Note that $(T^*)^{k+1}T^k$ is self-adjoint since $(T^*)^{k+1}T^k = T^{*k}APT^k = T^{*k}AT^k$. Whence d) follows and the proof is complete.

$a) \leftrightarrow e)$ Let $T = V_T|T|$ be the polar decomposition of T. If $T \in \mathcal{P} \cdot \mathcal{L}^h$ then $T = P_T A$ for some $A \in \mathcal{L}^h$. So that $V_T|T| = T = P_T A = V_T V_T^* A$ and therefore $V_T(V_T^*A - |T|) = 0$. Then $\mathcal{R}(V_T^*A - |T|) \subseteq N(T) \cap \overline{\mathcal{R}(T^*)} = \{0\}$.

Thus $V_T^*A = |T|$ and $|T|V_T$ is self-adjoint. Conversely, if $|T|V_T$ is self-adjoint then there exists $A \in \mathcal{L}^h$ such that $|T| = V_T^*A$ (see [18, Theorem 2.2] and [10, Theorem 3.5]). Then $T = V_T|T| = V_T V_T^* A = P_T A$ and the assertion follows.

a) $\leftrightarrow$ *f*) If $T = P_T A$, for some $A \in \mathcal{L}^h$ then $TP_T \in \mathcal{L}^h$ and so $A_T = T + T^* - TP_T \in \mathcal{L}^h$. Conversely, if $A_T \in \mathcal{L}^h$ then $P_T A_T = T \in \mathcal{P} \cdot \mathcal{L}^h$. □

Remark 2.3. *From now on, we denote by* $A_T := T + T^* - TP_T$.

Corollary 2.4. *a)* $\mathcal{P} \cdot \mathcal{L}^h$ *is closed.*

b) If $T \in \mathcal{P} \cdot \mathcal{L}^h$ *then* $T = P_T A_T$.

c) If $T \in \mathcal{P} \cdot \mathcal{L}^h$ *then* $T^{2k} \in \mathcal{P} \cdot \mathcal{L}^+$ *for all* $k \in \mathbb{N}$.

d) If $T \in \mathcal{P} \cdot \mathcal{L}^h$ *and* $|T|V_T \in \mathcal{L}^+$ *then* $T^{2k+1} \in \mathcal{P} \cdot \mathcal{L}^+$ *for all* $k \in \mathbb{N}$.

Proof. *a*) It follows from item *c*) of Theorem 2.2.

b) It follows from the proof of *a*) $\rightarrow$ *d*) of Theorem 2.2.

c) From *a*) $\rightarrow$ *b*) of Theorem 2.2 we know that $T^{2k} = P_T(T^*)^k T^*$. Then $T \in \mathcal{P} \cdot \mathcal{L}^+$.

d) Since $|T|V_T \in \mathcal{L}^+$ then $(T^*)^2 T = T^*|T|^2 = |T|V_T^*|T||T| \in \mathcal{L}^+$. Now, as $T \in \mathcal{P} \cdot \mathcal{L}^h$ then $T^{2k+1} \in \mathcal{P} \cdot \mathcal{L}^h$ for all $k \in \mathbb{N}$. From the proof of Theorem 2.2 we get that $T^{2k+1} = P(T^*)^{k+1}T^k$. Observe that $(T^*)^{k+1}T^k = (T^*)^{k-1}(T^*)^2 T T^{k-1} \in \mathcal{L}^+$. So that, $T^{2k+1} \in \mathcal{P} \cdot \mathcal{L}^+$. □

Remark 2.5. *Observe that given* $T \in \mathcal{P} \cdot \mathcal{L}^h$, T^{2k+1} *is not necessarily in* $\mathcal{P} \cdot \mathcal{L}^+$ *for all* $k \in \mathbb{N}$. *For example, consider* $T = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} \in \mathcal{P} \cdot \mathcal{L}^h$. *Note that* $T^{2k+1} = T$ *for all* $k \in \mathbb{N}$. *However, since* $TP_T = \begin{pmatrix} -1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} \notin \mathcal{L}^+$ *then, by* [5, Theorem 3.2], $T^{2k+1} \notin \mathcal{P} \cdot \mathcal{L}^+$.

Remark 2.6. *The following example shows that:*

a) If $T \in \mathcal{P} \cdot \mathcal{L}^h$ *then* $\overline{\mathcal{R}(T)} \cap \mathcal{N}(T) \neq \{0\}$, *in general;*

b) $\mathcal{P} \cdot \mathcal{L}^+ \subsetneq \mathcal{P} \cdot \mathcal{L}^h$;

c) $\mathcal{P} \cdot \mathcal{L}^h$ *is not closed by adjunction.*

In fact, consider $T = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{L}(\mathbb{C}^3)$. Since $T^*T^2 \in \mathcal{L}^h$ then $T \in \mathcal{P} \cdot \mathcal{L}^h$. Then $\mathcal{R}(T) \cap \mathcal{N}(T) \neq \{0\}$. On the other hand, note that $\mathcal{P} \cdot \mathcal{L}^+ \subsetneq \mathcal{P} \cdot \mathcal{L}^h$ because if $T \in \mathcal{P} \cdot \mathcal{L}^+$ then $\overline{\mathcal{R}(T)} \cap \mathcal{N}(T) = \{0\}$ (see [5, Lemma 3.1]). Finally, to see that $\mathcal{P} \cdot \mathcal{L}^h$ is not closed by adjunction, it is sufficient to check that $T(T^*)^2 \notin \mathcal{L}^h$.

From now on, given $T \in \mathcal{P} \cdot \mathcal{L}^h$, we set

$$\mathcal{P}_T := \{P \in \mathcal{P} \ : \ PA = T \text{ for some } A \in \mathcal{L}^h\}$$

and

$$\mathcal{A}_T := \{A \in \mathcal{L}^h \ : \ PA = T \text{ for some } P \in \mathcal{P}\}.$$

In the next two results, we study the projection sets $\mathcal{P}_T$ and $\mathcal{A}_T$.

Proposition 2.7. *Let* $T \in \mathcal{P} \cdot \mathcal{L}^h$ *and* $P \in \mathcal{P}$. *The following assertions are equivalent:*

a) $P \in \mathcal{P}_T$.
b) $\mathcal{R}(T) \subseteq \mathcal{R}(P)$ *and* $TP = PT^*$.
c) $PA_T = T$.

Proof. *a)* $\to$ *b)* Suppose that there exists $A \in \mathcal{L}^h$ such that $PA = T$. Then $\mathcal{R}(T) \subseteq \mathcal{R}(P)$ and $TP = PAP = PT^*$.

b) $\to$ *c)* If $\mathcal{R}(T) \subseteq \mathcal{R}(P)$ and $TP = PT^*$ then $PA_T = PT + PT^* - PP_T T^* = PT = T$.

c) $\to$ *a)* Since $T \in \mathcal{P} \cdot \mathcal{L}^h$ then $A_T \in \mathcal{L}^h$ and so $P \in \mathcal{P}_T$. □

Proposition 2.8. *Let* $T \in \mathcal{P} \cdot \mathcal{L}^h$ *and* $A \in \mathcal{L}^h$. *The following assertions are equivalent:*

a) $A \in \mathcal{A}_T$.
b) $P_T A = T$.
c) $A = A_T + X$ *for some* $X \in \mathcal{L}^h$ *with* $\mathcal{R}(X) \subseteq \mathcal{R}(T)^\perp$.

Proof. *a)* $\to$ *b)* If $PA = T$ for some $P \in \mathcal{P}$, then $\mathcal{R}(T) \subseteq \mathcal{R}(P)$ and, whence, $T = P_T A$.

b) $\to$ *c)* Note that $A = P_T A P_T + P_T A(I - P_T) + (I - P_T)AP_T + (I - P_T)A(I - P_T) = TP_T + T(I - P_T) + (I - P_T)T^* + (I - P_T)A(I - P_T) = T + T^* - P_T T^* + (I - P_T)A(I - P_T) = A_T + (I - P_T)A(I - P_T)$ and the assertion follows.

c) $\to$ *a)* Since $P_T A = P_T A_T = T$ then $A \in \mathcal{A}_T$. □

Proposition 2.9. *The set* $\mathcal{A}_T$ *is a closed (in norm)* $\mathbb{R}$*-affine manifold.*

Proof. Item c) of Proposition 2.8 shows that $\mathcal{A}_T$ is a $\mathbb{R}$-affine manifold. Now let us see that $\mathcal{A}_T$ is closed. If $\{A_n\} \subseteq \mathcal{A}_T$ and $A_n \xrightarrow[n\to\infty]{\|\cdot\|} A$ then $A \in \mathcal{L}^h$ and $A_n x \underset{n\longrightarrow\infty}{\rightarrow} Ax$ for all $x \in \mathcal{H}$. Then $Tx = P_T A_n x \underset{n\to\infty}{\longrightarrow} P_T A x$. So that $P_T A = T$ and therefore $A \in \mathcal{A}_T$. □

In [5, Proposition 4.1] it was proved that if $T \in \mathcal{P} \cdot \mathcal{L}^+$ then it always exists $A \in \mathcal{L}^+$ such that $T = P_T A$ and $\mathcal{N}(A) = \mathcal{N}(T)$. Furthermore, it was shown that this special factor in $\mathcal{L}^+$ turns to have optimal properties among all $A \in \mathcal{L}^+$ such that $T = PA$ for some $P \in \mathcal{P}$. Motivated by this, given $T \in \mathcal{P} \cdot \mathcal{L}^h$ we are interested in finding $A \in \mathcal{A}_T$ such that $\mathcal{N}(A) = \mathcal{N}(T)$. Unfortunately, it is not always possible in $\mathcal{P} \cdot \mathcal{L}^h$. For instance, consider $T = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{L}(\mathbb{C}^3)$. By Remark 2.6 it holds that $T \in \mathcal{P} \cdot \mathcal{L}^h \setminus \mathcal{P} \cdot \mathcal{L}^+$.

It is easy to check that $A_T = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 0 \end{pmatrix}$. Then, by Proposition 2.8, every $A \in \mathcal{A}_T$ is $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & x \end{pmatrix}$, with $x \in \mathbb{R}$. Since $\det(A) \neq 0$ for all $x \in \mathbb{R}$ then A is invertible. Therefore, $\mathcal{N}(A) \neq \mathcal{N}(T)$ for all $A \in \mathcal{A}_T$.

The next result will be useful in order to study when it is possible to find an $A \in \mathcal{A}_T$ with $\mathcal{N}(A) = \mathcal{N}(T)$.

Proposition 2.10. *Let* $T \in \mathcal{P} \cdot \mathcal{L}^h$ *and* $A \in \mathcal{A}_T$. *The following statements hold:*

a) $\overline{\mathcal{R}(T)} \cap \mathcal{N}(A) = \{0\}$ *(and therefore* $\mathcal{R}(A) + \mathcal{R}(T)^{\perp}$ *is dense in* $\mathcal{H}$*);*

b) T *has closed range if and only if* $\mathcal{H} = \mathcal{R}(A) + \mathcal{R}(T)^{\perp}$*;*

c) $\mathcal{R}(T)^{\perp} \cap \mathcal{R}(A) = \{0\}$ *if and only if* $\mathcal{N}(A) = \mathcal{N}(T)$*.*

Proof. *a*) Take $x \in \overline{\mathcal{R}(T)} \cap \mathcal{N}(A)$. Then $x = P_T x$ and $0 = Ax = AP_T x = T^* x$. So that $x \in \overline{\mathcal{R}(T)} \cap \mathcal{N}(T^*) = \{0\}$.

b) First, let us see that $\mathcal{R}(T) \subseteq \mathcal{R}(A) + \mathcal{R}(T)^{\perp}$. In fact, if $y \in \mathcal{R}(T)$ then $y = Tx = P_T Ax$ for some $x \in \mathcal{H}$. Then $P_T Tx = P_T Ax$ and so $Tx - Ax \in \mathcal{R}(T)^{\perp}$. Therefore the inclusion is obtained. Now, if T has closed range then $\mathcal{H} = \mathcal{R}(T) + \mathcal{R}(T)^{\perp} \subseteq \mathcal{R}(A) + \mathcal{R}(T)^{\perp}$. Conversely, suppose that $\mathcal{H} = \mathcal{R}(A) + \mathcal{R}(T)^{\perp}$ and $T = P_T A$. Hence, $\overline{\mathcal{R}(T)} = P_T(\mathcal{H}) = P_T(\mathcal{R}(A) + \mathcal{R}(T)^{\perp}) = \mathcal{R}(P_T A) = \mathcal{R}(T)$, i.e., T has closed range.

c) Let $T = P_T A$. Suppose that $\mathcal{N}(A) = \mathcal{N}(T)$. If $y \in \mathcal{R}(A) \cap \mathcal{R}(T)^{\perp}$ then $y = Ax$ for some $x \in \mathcal{H}$ and $P_T y = 0$. So that, $0 = P_T Ax = Tx$. Hence $x \in \mathcal{N}(T) = \mathcal{N}(A)$ and, therefore $y = 0$. Conversely, since $T = P_T A$ it is clear that $\mathcal{N}(A) \subseteq \mathcal{N}(T)$. Let $x \in \mathcal{N}(T)$. Then $0 = P_T Ax$ and so $Ax \in \mathcal{R}(A) \cap \mathcal{R}(T)^{\perp} = \{0\}$. Then $x \in \mathcal{N}(A)$ and then the assertion follows. □

Theorem 2.11. *Let* $T \in \mathcal{P} \cdot \mathcal{L}^h$*. If there exists* $A \in \mathcal{A}_T$ *with* $\mathcal{N}(A) = \mathcal{N}(T)$ *then* $\overline{\mathcal{R}(T)} \dot{+} \mathcal{N}(T)$ *is dense in* $\mathcal{H}$*. Conversely, if* $\overline{\mathcal{R}(T)} \dot{+} \mathcal{N}(T) = \mathcal{H}$ *then there exists* $A \in \mathcal{A}_T$ *with* $\mathcal{N}(A) = \mathcal{N}(T)$*.*

Proof. Let $T \in \mathcal{P} \cdot \mathcal{L}^h$. Assume that $T = P_T A$ with $A \in \mathcal{L}^h$ and $\mathcal{N}(A) = \mathcal{N}(T)$. Then, by items a) and c) of Proposition 2.10, it holds that $\overline{R(T)} \cap \mathcal{N}(T) = \{0\}$ and $\mathcal{N}(A) + \overline{\mathcal{R}(T)}$ is dense in $\mathcal{H}$. Therefore $\overline{\mathcal{R}(T)} \dot{+} \mathcal{N}(T)$ is dense in $\mathcal{H}$.

On the other hand, if $\overline{\mathcal{R}(T)} \dot{+} \mathcal{N}(T) = \mathcal{H}$, take $Q = Q_{\overline{\mathcal{R}(T)}//\mathcal{N}(T)}$ and define $A := A_T Q$. Note that $A = T^* Q = Q^* P_T T^* Q \in \mathcal{L}^h$ because of Theorem 2.2. Furthermore $\mathcal{N}(T) \subseteq \mathcal{N}(A)$ and if $Ax = A_T Qx = 0$ then $Qx \in \mathcal{N}(A_T) \cap \overline{R(T)} = \{0\}$ (see Proposition 2.10). Then $x \in \mathcal{N}(Q) = \mathcal{N}(T)$ and so $\mathcal{N}(A) = \mathcal{N}(T)$. In addition $P_T A = P_T A_T Q = TQ = T$. The proof is complete. □

Corollary 2.12. *Let* $T \in \mathcal{P} \cdot \mathcal{L}^h$ *with closed range. The next conditions are equivalent:*

a) There exists $A \in \mathcal{A}_T$ *with* $\mathcal{N}(A) = \mathcal{N}(T)$*.*

b) $\mathcal{R}(T) \dot{+} \mathcal{N}(T) = \mathcal{H}$*.*

Proof. Assume that there exists $A \in \mathcal{A}_T$ with $\mathcal{N}(A) = \mathcal{N}(T)$. Then, by Theorem 2.11, $\overline{\mathcal{R}(T)} \dot{+} \mathcal{N}(T)$ is dense in $\mathcal{H}$. We claim that $\overline{\mathcal{R}(T)} \dot{+} \mathcal{N}(T)$ is closed. In fact, by Proposition 2.10, as T has closed range then $\mathcal{H} = \mathcal{R}(A) \dot{+} \mathcal{R}(T)^{\perp}$ and so $\mathcal{R}(T) \dot{+} \mathcal{N}(T) = \mathcal{H}$ as desired. The converse follows by Theorem 2.11. □

In the next proposition, given $T \in \mathcal{P} \cdot \mathcal{L}^h$ we present equivalent conditions to those of Theorem 2.11 in order to guarantee the existence of an $A \in \mathcal{A}_T$ with $\mathcal{N}(A) = \mathcal{N}(T)$. For that, given a pair $\mathcal{V}, \mathcal{W} \subseteq \mathcal{H}$ of closed subspaces we shall denote by $c_0(\mathcal{V}, \mathcal{W})$ to the cosine of the Dixmier angle between $\mathcal{V}$ and $\mathcal{W}$, i.e.,

$$c_0(\mathcal{V}, \mathcal{W}) := \sup\{|\langle v, w \rangle| \ : \ v \in \mathcal{V}, w \in \mathcal{W}, \|v\| = \|w\| = 1\}.$$

It holds that $c_0(\mathcal{V},\mathcal{W})<1$ if and only if $\mathcal{V}+\mathcal{W}$ is closed and $\mathcal{V}\cap\mathcal{W}=\{0\}$ (see [12, Theorem 1]).

Proposition 2.13. *Let $T\in\mathcal{P}\cdot\mathcal{L}^h$. The following conditions are equivalent:*

a) $c_0\left(\mathcal{R}(T)^\perp,\overline{A_T(\overline{\mathcal{R}(T)})}\right)<1$;

b) $c_0\left(\mathcal{N}(P),\overline{A(\mathcal{R}(P))}\right)<1$ *for all* $P\in\mathcal{P}, A\in\mathcal{L}^h$ *such that* $T=PA$;

c) $\mathcal{H}=\overline{\mathcal{R}(T)}\dot{+}\mathcal{N}(T)$.

Proof. $a)\to b)$ By Proposition 2.8 every $A\in\mathcal{A}_T$ can be written as $A=A_T+X$ for some $X\in\mathcal{L}^h$ with $\mathcal{R}(X)\subseteq\mathcal{R}(T)^\perp$. Now, if $P\in\mathcal{P}_T$ then $\mathcal{N}(P)\subseteq\mathcal{R}(T)^\perp$. Furthermore $\overline{A(\mathcal{R}(P))}=\overline{\mathcal{R}(T^*)}$ and $\overline{A_T\overline{\mathcal{R}(T)}}=\overline{A_T\mathcal{R}(P_T)}=\overline{\mathcal{R}(T^*)}$. Therefore the assertion follows because $c_0\left(\mathcal{N}(P),\overline{A(\mathcal{R}(P))}\right)\leq c_0\left(\mathcal{R}(T)^\perp,\overline{A_T\overline{\mathcal{R}(T)}}\right)$.

$b)\to c)$ Since $c_0\left(\mathcal{N}(P),\overline{A(\mathcal{R}(P))}\right)<1$ for all $P\in\mathcal{P}$ and $A\in\mathcal{L}^h$ such that $T=PA$ then, in particular, $c_0\left(\mathcal{R}(T)^\perp,\overline{A_T(\overline{\mathcal{R}(T)})}\right)<1$. Now observe that $\overline{A_T(\overline{\mathcal{R}(T)})}=\overline{\mathcal{R}(T^*)}$. Then we get that $\mathcal{R}(T)^\perp\dot{+}\overline{\mathcal{R}(T^*)}$ is closed. In consequence, $\overline{\mathcal{R}(T)}+\mathcal{N}(T)=\mathcal{H}$. In addition, if $x\in\overline{\mathcal{R}(T)}\cap\mathcal{N}(T)$ then $x=P_Tx$ and $0=TP_Tx=P_TT^*x$. So that $T^*x\in\mathcal{R}(T^*)\cap\mathcal{R}(T)^\perp\subseteq\overline{\mathcal{R}(T^*)}\cap\mathcal{R}(T)^\perp=\{0\}$. Therefore $x\in\mathcal{N}(T^*)\cap\overline{\mathcal{R}(T)}=\{0\}$ as desired.

$c)\to a)$ Since $\mathcal{N}(T)^\perp=\overline{A_T(\overline{\mathcal{R}(T)})}$ then the assertion follows by [12, Theorem 12 and Theorem 16]. □

For the next result we denote by $\mathcal{I}_0$ the set of split partial isometries of $\mathcal{L}(\mathcal{H})$, i.e, the set of partial isometries V such that $\mathcal{R}(V)\dot{+}\mathcal{N}(V)=\mathcal{H}$. This class of operators was studied in [1].

Proposition 2.14. *Let $T\in\mathcal{L}(\mathcal{H})$. The following assertions are equivalent:*

a) $T\in\mathcal{P}\cdot\mathcal{L}^h$ *and* $\overline{\mathcal{R}(T)}\dot{+}\mathcal{N}(T)=\mathcal{H}$;

b) $|T|V_T\in\mathcal{L}^h$ *and* $V_T\in\mathcal{I}_0$.

Proof. The proof follows from Theorem 2.2 and the facts that $\mathcal{R}(V_T)=\overline{\mathcal{R}(T)}$ and $\mathcal{N}(V_T)=\mathcal{N}(T)$. □

Remark 2.15. *Given a closed subspace $\mathcal{S}\subseteq\mathcal{H}$ and $A\in\mathcal{L}^h$, it is said that the pair $(A,\mathcal{S})$ is compatible if there exists $Q\in\mathcal{L}(\mathcal{H})$ such that $Q^2=Q$, $\mathcal{R}(Q)=\mathcal{S}$ and $AQ=Q^*A$. This notion was introduced and studied in* [19]. *It was proved that the pair $(A,\mathcal{S})$ is compatible if and only $c_0(\mathcal{S}^\perp,\overline{A(\mathcal{S})})<1$ (*[19, Theorem 4.7]*). Therefore, observe that given $T\in\mathcal{P}\cdot\mathcal{L}^h$, the conditions of Proposition 2.13 are equivalent to the compatibility of the pair $(A_T,\overline{\mathcal{R}(T)})$ and also to the compatibility of the pair $(A,\mathcal{R}(P))$ for all $A\in\mathcal{L}^h$ and $P\in\mathcal{P}$ such that $T=PA$.*

Definition 1. *Let $T\in\mathcal{P}\cdot\mathcal{L}^h$ be such that $\overline{\mathcal{R}(T)}\dot{+}\mathcal{N}(T)=\mathcal{H}$. If $Q=Q_{\overline{\mathcal{R}(T)}//\mathcal{N}(T)}$ we define*

$$A_{\mathcal{N}}=A_TQ.$$

Observe that, by the proof of Theorem 2.11, $A_{\mathcal{N}} \in \mathcal{A}_T$ and $\mathcal{N}(A) = \mathcal{N}(T)$.

Proposition 2.16. *The operator $A_{\mathcal{N}}$ satisfies the following properties:*

a) $A_{\mathcal{N}}$ is the unique operator in $\mathcal{A}_T$ with nullspace equal to $\mathcal{N}(T)$.
b) $\mathcal{R}(A_{\mathcal{N}})$ is closed if and only if $\mathcal{R}(T)$ is closed.

Proof. *a*). Suppose that there exists $A \in \mathcal{A}_T$ such that $\mathcal{N}(A) = \mathcal{N}(A_{\mathcal{N}}) = \mathcal{N}(T)$. Then $\mathcal{R}(A - A_{\mathcal{N}}) \subseteq \mathcal{N}(T^*)$ since $T^*(A - A_{\mathcal{N}}) = AP_T(A - A_{\mathcal{N}}) = A(T - T) = 0$. On the other hand, as $\mathcal{N}(A) = \mathcal{N}(A_{\mathcal{N}}) = \mathcal{N}(T)$ then $\mathcal{R}(A - A_{\mathcal{N}}) \subseteq \mathcal{N}(T)^{\perp}$. Hence, $\mathcal{R}(A - A_{\mathcal{N}}) \subseteq \mathcal{N}(T^*) \cap \mathcal{N}(T)^{\perp} = \{0\}$ because $\mathcal{H} = \overline{\mathcal{R}(T)} \dot{+} \mathcal{N}(T)$, so $A = A_{\mathcal{N}}$.

b) Suppose that $\mathcal{R}(A_{\mathcal{N}})$ is closed. Then, $\mathcal{R}(A_{\mathcal{N}}) = \overline{\mathcal{R}(T^*)}$ and so, $\mathcal{R}(A_{\mathcal{N}}) \dot{+} \mathcal{N}(T^*) = \mathcal{H}$ because $\overline{\mathcal{R}(T)} \dot{+} \mathcal{N}(T) = \mathcal{H}$. Therefore, by Proposition 2.10 , $\mathcal{R}(T)$ is closed.

Conversely, if $\mathcal{R}(T)$ is closed then, by Proposition 2.10, $\mathcal{R}(A_{\mathcal{N}}) \dot{+} \mathcal{R}(T)^{\perp} = \mathcal{H}$. Hence, applying [15, Theorem 2.3], we obtain that $\mathcal{R}(A_{\mathcal{N}})$ is closed. □

Remark 2.17. *Notice that if $T \in \mathcal{P} \cdot \mathcal{L}^+$ with $\overline{\mathcal{R}(T)} \dot{+} \mathcal{N}(T) = \mathcal{H}$ then $A_{\mathcal{N}}$ coincides with the optimal operator in $\mathcal{L}^+$ given in* [5, Remark 4.2]. *In fact, by* [5, Proposition 4.1]*, there exists a unique $A \in \mathcal{L}^+$ with $\mathcal{N}(A) = \mathcal{N}(T)$ such that $T = P_T A$. Therefore, it is sufficient to show that $A_{\mathcal{N}} \in \mathcal{L}^+$. Now, $A_{\mathcal{N}} = A_T Q = T^*Q = Q^*T^*Q = Q^*P_T T^*Q \in \mathcal{L}^+$ because by* [5, Theorem 3.2]*, $P_T T^* \in \mathcal{L}^+$.*

Proposition 2.18. *Let $T \in \mathcal{P} \cdot \mathcal{L}^h$ with $\overline{\mathcal{R}(T)} \dot{+} \mathcal{N}(\mathcal{T}) = \mathcal{H}$. Then the following assertions hold:*

a) For every $A \in \mathcal{A}_T$ there exists $X \in \mathcal{L}^h$ with $\mathcal{R}(X) \subseteq \mathcal{R}(T)^{\perp}$ such that $A = A_{\mathcal{N}} + X$. Furthermore $\mathcal{R}(A) = \mathcal{R}(A_{\mathcal{N}}) \dot{+} \mathcal{R}(X)$.
b) There exists $A \in \mathcal{A}_T$ with dense range.
c) There exists $A \in \mathcal{A}_{\mathcal{T}}$ invertible if and only if $\mathcal{R}(T)$ is closed.

Proof. *a*) It is easy to check that every $A \in \mathcal{A}_T$ can be written as $A = A_{\mathcal{N}} + X$, for some $X \in \mathcal{L}^h$ with $\mathcal{R}(X) \subseteq \mathcal{R}(T)^{\perp}$. Now, since $\overline{\mathcal{R}(A_{\mathcal{N}})} \dot{+} \overline{\mathcal{R}(X)} = \overline{\mathcal{R}(T^*)} \dot{+} \overline{\mathcal{R}(X)}$ is closed then, by [6, Theorem 3.10], we get that $\mathcal{R}(A) = \mathcal{R}(A_{\mathcal{N}}) \dot{+} \mathcal{R}(X)$.

b) Define $A = A_{\mathcal{N}} + (I - P_T)$. By the above item $A \in \mathcal{A}_T$ and, since $\mathcal{R}(A) = \mathcal{R}(A_{\mathcal{N}}) + \mathcal{N}(T^*)$ and $\overline{\mathcal{R}(A_{\mathcal{N}})} = \overline{\mathcal{R}(T^*)}$ it holds that A has dense range.

c) If there exists $A \in \mathcal{A}_T$ invertible then $\mathcal{R}(T) = \mathcal{R}(P_T A) = \mathcal{R}(P_T) = \overline{\mathcal{R}(T)}$. So that T has closed range. Conversely, if $\mathcal{R}(T)$ is closed then $A = A_N + (I - P_T) \in \mathcal{A}_T$ and $\mathcal{R}(A) = \mathcal{H}$. Therefore, A is invertible. □

Proposition 2.19. *Let $T \in \mathcal{P} \cdot \mathcal{L}^h$ with closed range such that $\mathcal{R}(T) \dot{+} \mathcal{N}(T) = \mathcal{H}$. Then the following assertions hold:*

a) $Q_{\mathcal{R}(T)//\mathcal{N}(T)} = (A_{\mathcal{N}} P_T)^{\dagger} A_{\mathcal{N}} = (T^*)^{\dagger} A_{\mathcal{N}}$;
b) $\{A \in \mathcal{A}_T : \mathcal{R}(A)$ is closed$\} = \{A_{\mathcal{N}} + X : X \in \mathcal{L}^h, \mathcal{R}(X)$ is closed and $\mathcal{R}(X) \subseteq \mathcal{N}(T^*)\}$;
c) $T^{\dagger} \in \mathcal{P} \cdot \mathcal{L}^h$.

Proof. *a*) This proof is similar to the proof of [5, Proposition 4.3].

b) It is clear that every $A \in \mathcal{A}_T$ can be written as $A = A_{\mathcal{N}} + X$, for some $X \in \mathcal{L}^h$ with $\mathcal{R}(X) \subseteq \mathcal{R}(T)^{\perp}$. Since $\mathcal{H} = \mathcal{R}(T)\dot{+}\mathcal{N}(T)$ then $\mathcal{H} = \mathcal{R}(T^*)\dot{+}\mathcal{N}(T^*)$. So, $c_0(\mathcal{R}(A_{\mathcal{N}}), \overline{\mathcal{R}(X)}) \leq c_0(R(T^*), \mathcal{N}(T^*)) < 1$. Thus $\mathcal{R}(A_{\mathcal{N}})\dot{+}\overline{\mathcal{R}(X)}$ is closed. Then by [6, Theorem 3.10] it holds that $\mathcal{R}(A) = \mathcal{R}(A_{\mathcal{N}})\dot{+}\mathcal{R}(X)$. Therefore it is clear that if $\mathcal{R}(X)$ is closed then $\mathcal{R}(A)$ is closed. Conversely, if $\mathcal{R}(A)$ is closed then by [15, Theorem 2.3] it holds that $\mathcal{R}(X)$ is closed.

c) By Proposition 2.18 there exists $A \in \mathcal{L}^h$ invertible such that $P_T A = T$. Define $C := P_{R(AP)} A^{-1} \in \mathcal{P} \cdot \mathcal{L}^h$. Therefore it holds that C has closed range, $TC = P_T$ and $R(C) \subseteq N(T)^{\perp}$. Thus, by [4, Theorem 3.1], $C = T^{\dagger}$ and so $T^{\dagger} \in \mathcal{P} \cdot \mathcal{L}^h$.

□

3. Optimal decompositions

This section is devoted to the study of optimal factors in $\mathcal{P}_T$ and $\mathcal{A}_T$ for $T \in \mathcal{P} \cdot \mathcal{L}^h$. We shall consider three different criteria of optimality: minimization with respect to usual order between self-adjoint operators, minimization with respect to the minus order in $\mathcal{L}(\mathcal{H})$ and minimization of the distance to T. By usual order between selfadjoint operators we mean that given $A, B \in \mathcal{L}^h$, $A \leq B$ if $B - A \in \mathcal{L}^+$. For the minus order we shall use the symbol $\leq^-$, given $A, B \in \mathcal{L}(\mathcal{H})$, it is said that $A \leq^- B$ if and only if there exist two idempotents Q_1 and Q_2 in $\mathcal{L}(\mathcal{H})$ such that $A = Q_1 B$ and $A^* = Q_2 B^*$. The minus order was introduced by Hartwig [17] and independently by Nambooripad [20] on semigroups. Later this order was extended to operators in $\mathcal{L}(\mathcal{H})$ by Antezana, Corach and Stojanoff [2] and by Šmerl [22].

Let us start studying the optimality in $\mathcal{P}_T$:

Proposition 3.1. *If* $T \in \mathcal{P} \cdot \mathcal{L}^h$ *then:*

- *a)* $P_T = \min\{P : P \in \mathcal{P}_T\}$, *where the minimum is taken with respect usual order between self-adjoint operators.*
- *b)* $P_T = \min\{P : P \in \mathcal{P}_T\}$, *where the minimum is taken with respect to the minus order.*

Proof. Let $P \in \mathcal{P}_T$. Then $\overline{\mathcal{R}(T)} \subseteq \mathcal{R}(P)$. So that, is clear that $P_T \leq P$. Furthermore, $P_T = P_T P$. Then $P_T \leq^- P$. □

In [5, Proposition 4.7] it was proven that if $T \in \mathcal{P} \cdot \mathcal{L}^+$ then there exists $\hat{A} \in \mathcal{L}^+$ with $\mathcal{N}(\hat{A}) = \mathcal{N}(T)$ and $T = P_T \hat{A}$ such that $\hat{A}$ realizes the minimum among all the positive operators A such that $T = P_T A$ in two ways: with respect to the operator norm and with respect to the usual order defined on the set of self-adjoint operators. Hence, one may wonder if a similar result can be obtained for $T \in \mathcal{P} \cdot \mathcal{L}^h$. But, as the next example shows, it is not possible, in general. For example, consider $T = PA = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} \in \mathcal{P} \cdot \mathcal{L}^h$.

It is easy to check that $A_{\mathcal{N}} = \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{L}^h$. Now, by Proposition 2.16 we know that $A_{\mathcal{N}}$ is the unique operator in $\mathcal{A}_T$ with nullspace $\mathcal{N}(T)$. But,

$\|A_{\mathcal{N}}\| = 2 \geq \sqrt{2} = \|T\|$. However, as we will see in the next result, the set $\mathcal{A}_T$ has a minimum with respect to the operator norm. We include its proof for the sake of completeness. However, the arguments are very similar to those in [11, Section 1] where the problem of finding the entry D in the block operator matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ so as to satisfy the norm bound $\left\| \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right\| \leq \mu$, for given Hilbert space operators A, B, C and prescribed μ, is fully studied.

Theorem 3.2. *Given $T \in \mathcal{P} \cdot \mathcal{L}^h$ it holds that*

$$\min_{A \in \mathcal{A}_T} \|A\| = \|T\|.$$

Moreover, the minimum is achieved in the operator A_0 defined in (3).

Proof. Write

$$T_1 := T|_{\overline{\mathcal{R}(T)}} \quad \text{and} \quad T_2 := T|_{\mathcal{N}(T^*)}.$$

For all $h \in \mathcal{H}$

$$\|T\|^2 \|P_T h\|^2 = \|T^*\|^2 \|P_T h\|^2 \geq \|T^* P_T h\|^2 = \|T_1 P_T h\|^2 + \|T_2^* P_T h\|^2$$

whence

$$\langle T_2 T_2^* P_T h, P_T h \rangle \leq \langle (\|T\|^2 - T_1^2) P_T h, P_T h \rangle. \tag{2}$$

Put $\alpha := \|T\|$,

$$D_\alpha := (\alpha^2|_{\overline{\mathcal{R}(T)}} - T_1^2)^{\frac{1}{2}} \quad \text{and} \quad \mathcal{D}_\alpha := \overline{\mathcal{R}(D_\alpha)}.$$

Then (2) yields a contraction $C_\alpha : \mathcal{D}_\alpha \to \mathcal{N}(T^*)$ such that $T_2^* = C_\alpha D_\alpha$. In particular, $A_T = \begin{pmatrix} T_1 & T_2 \\ T_2^* & 0 \end{pmatrix}$ can be written as

$$A_T = \begin{pmatrix} T_1 & D_\alpha C_\alpha^* \\ C_\alpha D_\alpha & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & C_\alpha \end{pmatrix} \begin{pmatrix} T_1 & D_\alpha \\ D_\alpha & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & C_\alpha^* \end{pmatrix}.$$

Take $X_0 := -C_\alpha T_1 C_\alpha^* \in \mathcal{L}^h(\mathcal{N}(T^*))$ and $A_0 := A_T + X_0 \in \mathcal{A}_T$, so that

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & C_\alpha \end{pmatrix} \begin{pmatrix} T_1 & D_\alpha \\ D_\alpha & -T_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & C_\alpha^* \end{pmatrix}. \tag{3}$$

It is well known that the block operator matrix $\begin{pmatrix} T_1 & D_\alpha \\ D_\alpha & -T_1 \end{pmatrix}$ is α times a unitary operator on $\overline{\mathcal{R}(T)} \oplus \mathcal{D}_\alpha$. Thus, for all $h \in \overline{\mathcal{R}(T)}$ and $x \in \mathcal{D}_\alpha$,

$$\left\| \begin{pmatrix} T_1 & D_\alpha \\ D_\alpha & -T_1 \end{pmatrix} \begin{pmatrix} h \\ u \end{pmatrix} \right\| = \alpha \left\| \begin{pmatrix} h \\ u \end{pmatrix} \right\|.$$

Therefore, $\|A_0\| \leq \alpha = \|T\|$. Indeed, as $\|T\| \leq \|A\|$, for all $A \in \mathcal{A}_T$, it turns out that $\|A_0\| = \|T\| = \min_{A \in \mathcal{A}_T} \|A\|$. □

Note that the operator A_T does not realize the minimum in Theorem 3.2. In fact, consider $T = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \in \mathcal{P} \cdot \mathcal{L}^h$. Here, $A_T = \begin{pmatrix} 3/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}$ and $\|A_T\| = \frac{1+\sqrt{6}}{2} > \sqrt{2} = \|T\|$. However A_T is optimal in the next sense:

Theorem 3.3. *Let $T \in \mathcal{P} \cdot \mathcal{L}^h$. Then*

$$\min_{A \in \mathcal{A}_T} \|(T^* - A)x\| = \|(T^* - A_T)x\| \text{ for all } x \in \mathcal{H}. \tag{4}$$

Moreover A_T is the unique operator in $\mathcal{A}_T$ which realizes the minimum in (4). In particular, it holds that

$$\min_{A \in \mathcal{A}_T} \|T - A\| = \|T - A_T\|, \tag{5}$$

Proof. Let $x \in \mathcal{H}$ and $A \in \mathcal{A}_T$. Then $\|(T^* - A)x\|^2 = \|T^* - A_T - X)x\|^2 = \|(T^* - T - (I - P_T)T^* - X)x\|^2 = \|(P_T T^* - T - X)x\|^2 = \|(TP_T - T - X)x\|^2 = \|T(P_T - I)x\|^2 + \|Xx\|^2 \geq \|T(P_T - I)x\|^2 = \|(T^* - A_T)x\|^2$. In addition, if there exists another $A_1 = A_T + X_1 \in \mathcal{A}_T$ such that $\|(T^* - A_1)x\| \leq \|(T^* - A)x\|$ for all $x \in \mathcal{H}$ then, in particular, $\|(T^* - A_1)x\| \leq \|(T^* - A_T)x\|$ for all $x \in \mathcal{H}$. Hence $\|X_1 x\| = 0$ for all $x \in \mathcal{H}$. So that $X_1 = 0$ and therefore $A_1 = A_T$. Finally, from the above we get that $\|T - A_T\| = \|T^* - A_T\| \leq \|T^* - A\| = \|T - A\|$. □

Finally, given $T \in \mathcal{P} \cdot \mathcal{L}^h$ with $\overline{\mathcal{R}(T)} \dot{+} \mathcal{N}(T) = \mathcal{H}$ we shall prove that $A_{\mathcal{N}}$ is optimal in $\mathcal{A}_T$ with respect to the minus order in $\mathcal{L}(\mathcal{H})$. For this we use the following result due to Dijić, Fongi and Maestripieri [13, Proposition 3.2]).

Proposition 3.4. *Let $A, B \in \mathcal{L}(\mathcal{H})$. The following assertions are equivalent:*

a) $A \leq^- B$;
b) $\mathcal{N}(A) + \mathcal{N}(B - A) = \mathcal{N}(A^*) + \mathcal{N}(B^* - A^*) = \mathcal{H}$.

Theorem 3.5. *If $T \in \mathcal{P} \cdot \mathcal{L}^h$ and $\overline{\mathcal{R}(T)} \dot{+} \mathcal{N}(T) = \mathcal{H}$ then*

$$A_{\mathcal{N}} = \min\{A : A \in \mathcal{A}_T\},$$

where the minimum is taken with respect to the minus order. Moreover, $A_{\mathcal{N}}$ is the unique element in $\mathcal{A}_T$ that realizes the minimum.

Proof. By Proposition 2.18 every $A \in \mathcal{A}_T$ can be written as $A = A_{\mathcal{N}} + X$, for some $X = X^*$ and $\mathcal{R}(X) \subseteq \mathcal{R}(T)^\perp$. Furthermore $\mathcal{R}(A) = \mathcal{R}(A_{\mathcal{N}}) + \mathcal{R}(X)$. Now, $\mathcal{H} = \overline{\mathcal{R}(T)} \dot{+} \mathcal{N}(T) \subseteq \mathcal{N}(A - A_{\mathcal{N}}) + \mathcal{N}(A_{\mathcal{N}})$. Then, by Proposition 3.4, we get that $A_{\mathcal{N}} \leq^- A$. Now, suppose that there exists $\tilde{A} \in \mathcal{A}_T$ such that $\tilde{A} \leq^- A$ for all $A \in \mathcal{A}_T$. In particular it holds that $\tilde{A} \leq A_{\mathcal{N}}$. Then there exists an idempotent $Q \in \mathcal{L}(\mathcal{H})$ such that $\tilde{A} = QA_{\mathcal{N}}$. Then $\mathcal{N}(T) = \mathcal{N}(A_{\mathcal{N}}) \subseteq \mathcal{N}(\tilde{A}) \subseteq \mathcal{N}(T)$. Thus $\mathcal{N}(\tilde{A}) = \mathcal{N}(T)$ and therefore, by Proposition 2.16, $\tilde{A} = A_{\mathcal{N}}$. □

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