ON THE CLASSIFICATION OF FREE ARAKI-WOODS FACTORS

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Abstract. Using Hjorth's theory of turbulence, we prove that the free Araki-Woods factors are not classifiable by countable structures.

1. Introduction

(A) A central program of von Neumann algebra theory is to develop effective tools for analyzing the structure of von Neumann algebras in general, and von Neumann factors in particular. The ultimate goal of this pursuit would be to classify von Neumann algebras completely up to isomorphism, by assigning invariants that can be concretely computed in examples. The program, which naturally focusses on the case of separably acting von Neumann algebras and factors, has been vastly successful, though it is far from complete, and the subject is still developing rapidly.

In the past 10 years, tools from descriptive set theory have successfully been applied to give quantitative information about the *complexity* of classifying von Neumann algebras. Interestingly, these descriptive set theoretic results give *delimitative*, or negative, information about the classification of von Neumann factors. Indeed, descriptive set theoretic techniques have been used to show that certain classes of invariants are insufficient as complete invariants for the general problem of classifying separably acting factors. Results of this kind have been obtained in [18, 19, 17] addressing the problem for many classes of factors, including II_1, II_∞, and type III_λ factors, as well as the classical *Araki-Woods* factors [20]. Similar results have also been obtained in the analogous C*-algebra setting, where eventually the efforts

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of [7, 8, 5, 6] culminated with a complete determination of the complexity of classifying nuclear, separable, simple C*-algebras in [16].

The present paper obtains new results about the complexity of classifying the *free* Araki-Woods factors. These factors form a distinguished class of full von Neumann algebras that were introduced by Shlyakhtenko in [22] and have been very much studied since then (see [23, 24, 25, 14, 11, 12, 1, 13] and references therein).

Starting from an orthogonal representation $(U_t)_{t\in\mathbb{R}}$ of $\mathbb{R}$ on a real Hilbert space $\mathcal{H}_\mathbb{R}$ of dimension at least 2, Shlyakhtenko constructed a free Araki-Woods von Neumann algebra $\Gamma(\mathcal{H}_\mathbb{R}, U_t)''$. When U_t is the trivial representation, the free Araki-Woods von Neumann algebras are factors of type II_1, isomorphic to $L(\mathbb{F}_n)$, the group von Neumann algebra of the free group on n generators, a fact that witnesses its origins in Voiculescu's free probability theory. Otherwise they are full factors of type III and they may be viewed as free analogs of the classical Araki-Woods factors. In [22] and [23] Shlyakhtenko showed that there is a unique free Araki-Woods factor of type III_λ with $0 < \lambda < 1$ up to isomorphism, and that free Araki-Woods factors are typically type III_1 factors. For a more thorough survey of free Araki-Woods factors see the exposé of Vaes in the Séminaire Bourbaki [26].

Our main theorem is the following:

Theorem 1.1. *The class of free Araki-Woods factors of type* III_1 *is not classifiable by countable structures.*

This implies in particular that their classification is not smooth in the classical sense, but the above statement is far stronger: See e.g. the discussion in the introductions of [8] and [19] for details.

(B) The proof of Theorem 1.1 uses Hjorth's turbulence theory, which was developed in [10]. Turbulence is a topological criterion that implies a very strong form of *generic ergodicity* for equivalence relations. A turbulent equivalence relation does not admit classification by *countable structures*, where by countable structures we mean e.g. countable groups, graphs, fields, orderings, etc. This is because the isomorphism relation on countable structures is induced by a natural action of S_∞ (the group of *all* permutations of $\mathbb{N}$), and turbulence implies generic ergodicity with respect to S_∞ orbit equivalence relations. (See section 4 for the precise definition of these notions.)

In section 3 below, we construct a large family of symmetric measures on $\mathbb{R}$, such that the isomorphism type of the free Araki-Woods factor that is associated to each such measure is extremely unstable across the family of measures. Eventually we prove in Theorem 4.6 that this instability tantamount to turbulence, and Theorem 1.1 follows.

To position us for a turbulence argument, we crucially use the τ-invariant. This invariant was introduced by Connes in [2] to study full von Neumann algebras of type III_1, and Shlyakthenko later used it in his paper [24] to develop the structural theory of free Araki-Woods factors, and in [25] to give a 1-parameter family of non-isomorphic free Araki-Woods factors.

We note that all the free Araki-Woods factors obtained in this article are associated to orthogonal representations of multiplicity function equal to 1. The descriptive set-theoretic complexity of classifying free Araki-Woods factors *not* of multiplicity 1 remains open. We mention that the τ-invariant is insensitive to multiplicity, and until recently, not much was known on the dependency of the isomorphism class with respect to the multiplicity of the representation. Indeed before the appearance of the article [14] in 2016, only one example was known of two non isomorphic free Araki-Woods factors that share the same τ-invariant, but could be distinguished by the multiplicity of their common representation [21], (see also [26, section 3.4] for an overview of this unpublished result of Shlyakhtenko).

(C) Beyond this introduction, the paper is organized into five sections as follows: In section 2, we recall Shlyakthenko's construction of a (multiplicity 1) free Araki-Woods factor from a symmetric Borel measure on $\mathbb{R}$. In section 3, we describe a large family of Borel probability measures with prescribed properties on a (specific) Cantor subset of $\mathbb{R}$. In section 4 we show that certain topologies related to the τ-invariant that these measures induce in $\mathbb{R}$ are not classifiable by countable structures using a turbulent argument. In section 5 we use this to show that the family of factors arising from the measures considered in section 3 is not classifiable by countable structures. For that we show that the construction is Borel computable, in the sense that the map which associates the corresponding free Araki-Woods factor to each symmetric, regular, non atomic and finite measure on $\mathbb{R}$ whose support is contained in $[-1, 1]$, is a Borel map when the set of separably acting factors is given the Effros Borel structure. Finally, in section 6, we give a brief discussion of some open problems.

(D) As a concluding remark, we briefly comment on how our result compares with the astounding recent article of Houdayer, Vaes and Shlyakthenko [14] where they show complete classification of certain family of free Araki-Woods factors of type III_1, with no restriction in the multiplicity within the family. A feature of the measures we construct in section 3 is that they are finite, supported on the interval $[-1, 1]$, and non atomic. While it might seem plausible to remove the first two conditions, the fact that the measures are non atomic seems inevitable with current techniques. Moreover being able to deal only with non atomic measures is also crucial in section 5 when we model our free Araki-Woods factors on a fixed Hilbert space (see remark 5.4 and Question 6.3). On the other hand the measures that are taken into account in the main theorem of [14] albeit they are finite, they are also required to have an atom at a point different from 0.

2. Free Araki-Woods factors

In this section we recall in some detail the construction of the free Araki-Woods factors when the representation has multiplicity 1. We will thus focus only on the construction that starts with a symmetric measure on the real line. Everything in this section can be found in the foundational articles of Shlyakhtenko [22] and [24].

Let ν be a symmetric measure on $\mathbb{R}$. Let $\mathcal{H}_\nu = \mathcal{L}^2(\mathbb{R},\nu)$ and $\mathcal{H}_{\nu,\mathbb{R}} = \{f \in \mathcal{H}_\nu : f(x) = \overline{f(-x)}\}$. Let $J : \mathcal{H}_\nu \to \mathcal{H}_\nu; f(x) \to \overline{f(-x)}$. J is an antilinear involution and $Jf = f \ \forall f \in \mathcal{H}_{\nu,\mathbb{R}}$. $\mathcal{H}_{\nu,\mathbb{R}}$ with the inner product inherited from $\mathcal{H}_\nu$ is a real Hilbert space.

Let $U_t(f)(x) = e^{itx}f(x)$, the multiplication operator by e^{itx} on $\mathcal{H}_\nu$. For every $t \in \mathbb{R}$, U_t is a unitary operator on $\mathcal{B}(\mathcal{H}_\nu)$ and $t \to U_t$ defines a strongly continuous one parameter group of automorphisms on $\mathcal{H}_\nu$. Moreover U_t leaves $\mathcal{H}_{\nu,\mathbb{R}}$ invariant, so it also defines an action of $\mathbb{R}$ on $\mathcal{H}_{\nu,\mathbb{R}}$. By Stone's Theorem, there exists a unique self adjoint, strictly positive operator A acting on $\mathcal{H}_\nu$ such that $U_t = A^{it}$, (in this case it is just multiplication by e^x). Note that A is bounded if and only if $supp(\nu)$ is compact. Observe that

$$JU_tJ(f)(x) = JU_t\overline{f(-x)} = J(e^{itx}\overline{f(-x)}) = \overline{e^{it(-x)}\overline{f(-(-x))}} = U_t(f)(x).$$

This shows that $JU_tJ = U_t$, but then it follows that $JAJ = A^{-1}$.

Define a new inner product on $\mathcal{H}_\nu$ by the formula

$$\langle f, g\rangle_U = \langle \frac{2}{1+A^{-1}} f, g\rangle.$$

Observe that even when A is unbounded, $\frac{2}{1+A^{-1}}$ is always bounded. Moreover since $\frac{2}{1+A^{-1}}$ is strictly positive, $\langle , \rangle_U$ is non degenerate. Denote with $\tilde{\mathcal{H}}_\nu$ the Hilbert space completion of $\mathcal{H}_\nu$ with respect to $\langle , \rangle_U$.

It is straightforward to check that if $f \in \mathcal{H}_{\nu,\mathbb{R}}$ then $\langle f, f\rangle_U = \langle f, f\rangle$, that is $\mathcal{H}_{\nu,\mathbb{R}}$ embeds isometrically in $\tilde{\mathcal{H}}_\nu$.

The next proposition records some of the key properties of this isometric embedding that will be useful for what follows (the proofs can be found in [22, pages 331-332]):

Proposition 2.1.

(1) The restriction of the real part of the inner product on $\tilde{\mathcal{H}}_\nu$ to $\mathcal{H}_{\nu,\mathbb{R}}$ is the inner product on $\mathcal{H}_{\nu,\mathbb{R}}$.

(2) The restriction of the imaginary part of the inner product on $\tilde{\mathcal{H}}_\nu$ to $\mathcal{H}_{\nu,\mathbb{R}}$ is given by $\langle i\frac{1-A^{-1}}{1+A^{-1}}\cdot, \cdot\rangle_{\mathcal{H}_{\nu,\mathbb{R}}}$

(3) $\mathcal{H}_{\nu,\mathbb{R}} \cap i\mathcal{H}_{\nu,\mathbb{R}} = \{0\}$.
(4) $\mathcal{H}_{\nu,\mathbb{R}} + i\mathcal{H}_{\nu,\mathbb{R}}$ *is dense in* $\tilde{\mathcal{H}}_\nu$.

The Full Fock space of $\tilde{\mathcal{H}}_\nu$ is the Hilbert defined as follows:

$$\mathcal{F}(\tilde{\mathcal{H}}_\nu) = \mathbb{C}\Omega \oplus \oplus_{n=1}^{\infty} \tilde{\mathcal{H}}_\nu^{\otimes n},$$

where the unit vector Ω is called the vacuum vector. Every $\xi \in \tilde{\mathcal{H}}_\nu$ defines a left creation operator on $\mathcal{F}(\tilde{\mathcal{H}}_\nu)$ by the formulas $l(\xi)\Omega = \xi$ and $l(\xi)(\xi_1 \otimes \cdots \otimes \xi_n) = \xi \otimes \xi_1 \otimes \cdots \otimes \xi_n$. If we denote with $s(\xi) = \frac{l(\xi)+l(\xi)^*}{2}$, the real part of $l(\xi)$, the free Araki-Woods factor associated to the measure ν is defined as the von Neumann algebra

$$\Gamma(\nu) = \Gamma(\mathcal{H}_{\nu,\mathbb{R}}, U_t)'' = \{s(f) : f \in \mathcal{H}_{\nu,\mathbb{R}}\}'' \subset \mathcal{B}(\mathcal{F}(\tilde{\mathcal{H}}_\nu)).$$

If two measures ν and μ are equivalent, then the representations are unitarily equivalent and thus the free Araki-Woods factors $\Gamma(\nu)$ and $\Gamma(\mu)$ are isomorphic. Moreover, if $\tau(\nu)$ denotes the weakest topology on $\mathbb{R}$ that makes the map $t \to U_t \in \mathcal{U}(\mathcal{L}^2(\mathbb{R}, \nu))$ continuous with respect to the strong operator topology on $\mathcal{U}(\mathcal{L}^2(\mathbb{R}, \nu))$, then $\tau(\nu)$ is an invariant of the von Neumann Algebra $\Gamma(\nu)$. Indeed, in sections 8.5 and 8.6 of [24] Shlyaktenko showed that it coincides with the τ-invariant for full type III_1 factors that Connes introduced in [2, Definition 5.1] (see also [26, Théorème 2.7] for a more streamlined proof of this crucial fact). In [25] Shlyaktenko used this invariant to construct a one parameter family of non isomorphic free Araki-Woods factors. Roughly, the strategy of [25] is to construct a one parameter family of symmetric Bernoulli convolution measures on $\mathbb{R}$ with the property that their τ-invariants are pairwise different. Due to the stiffness of Bernoulli convolution measures, the construction presented there does not seem to be adapted to show that there are E_0-many free Araki-Woods factors. In this article we will still use the topologies $\tau(\nu)$ as a tool to prove that the free Araki-Woods factors are not classifiable by countable structures. The construction of the measures we will use to achieve this is necessarily more involved than the one employed in [25]. The next section is devoted to construct such measures.

3. A construction of measures on a Cantor set

Consider the Polish space $\mathbb{R}^{\mathbb{N}}$, and for each $n \in \mathbb{N}$ consider the function $\gamma_n : \mathbb{R} \to (\frac{1}{n}, 1 - \frac{1}{n})$ given by the formula

$$\gamma_n(x) = (1 - \frac{2}{n})(\frac{\arctan x}{\pi} + \frac{1}{2}) + \frac{1}{n}.$$

That is γ_n is a contraction of $\mathbb{R}$ to the interval $(\frac{1}{n}, 1 - \frac{1}{n})$.

For each $a \in \mathbb{R}^{\mathbb{N}}$ define a measure on $3^{\mathbb{N}} := \{0,1,2\}^{\mathbb{N}}$ by

$$\mu_a = \prod_{n=1}^{\infty} \frac{1}{2}(1-\gamma_n(a(n)))(\delta_0+\delta_2)+\gamma_n(a(n))\delta_1,$$

where δ_i is the Dirac point mass at $i \in \{0,1,2\}$.

Observe that since $\frac{1}{n} < \gamma_n(a(n)) < 1-\frac{1}{n}$, cylinder sets can have arbitrarily small positive measure, thus μ_a is non atomic.

Remark 3.1. The key reason to introduce the functions γ_n is to restrict $\gamma_n(a(n))$ to be in the interval $(\frac{1}{n}, 1-\frac{1}{n})$. This allows us to have control on how fast the measure puts weights on the atoms δ_i , $i = 1,2,3$ when n goes to ∞. The next lemma shows an instance of the need of this control.

Recall that if $\omega : \mathbb{N} \to \mathbb{R}_{>0}$, the weighted Hilbert space for the weight ω is the real Hilbert space $\ell^2(\mathbb{N},\omega) = \{f : \mathbb{N} \to \mathbb{R},\ \sum_{n\in\mathbb{N}} f(n)^2\omega(n) < \infty\}$, where the inner product is given by $\langle f,g\rangle = \sum_{n\in\mathbb{N}} f(n)g(n)\omega(n)$.

Lemma 3.2. *If $f \in \ell^2(\mathbb{N},\omega)$ where ω is the weight $\omega(n) = n$, and $a \in \mathbb{R}^{\mathbb{N}}$ then the measures μ_a and μ_{a+f} are equivalent.*

Proof. By Kakutani's Criterium [15, Corollary 1], we have to show that the sum:

$$\sum_{n=1}^{\infty} \left[\sqrt{\gamma_n(a(n)+f(n))}-\sqrt{\gamma_n(a(n))}\right]^2+\left[\sqrt{1-\gamma_n(a(n)+f(n))}-\sqrt{1-\gamma_n(a(n))}\right]^2$$

is finite.

Let $\theta_n(x) = \sqrt{\gamma_n(x)}$. Then $\theta_n'(x) = \frac{2}{\sqrt{\gamma_n(x)}}\gamma_n'(x)$. Since $|\gamma_n'(x)| < 1$, then $|\theta_n'(x)| \leq \frac{2}{\sqrt{\gamma_n(x)}}$. By the intermediate value theorem:

$$[\theta_n(a(n)+f(n))-\theta_n(a(n))]^2 = ([\sqrt{\gamma_n(a(n)+f(n))}-\sqrt{\gamma_n(a(n))}\Big]^2 = [\theta_n'(c(n))\,f(n)]^2,$$

with $c(n) \in \mathbb{R}$ is in between $a(n)$ and $a(n)+f(n)$.

But then

$$[\sqrt{\gamma_n(a(n)+f(n))}-\sqrt{\gamma_n(a(n))}\Big]^2 \leq \frac{4}{\gamma_n(c(n))}f(n)^2$$

.

Now by definition, $\gamma_n(x) > \frac{1}{n}$ for all $x \in \mathbb{R}$, so $\frac{1}{\gamma_n(c(n))} < n$ and since $f \in \ell^2(\mathbb{N},\omega)$ then the series

$$\sum_{n=1}^{\infty} \frac{4}{\gamma_n(c(n))}f(n)^2$$

is convergent, thus the series

$$\sum_{n=1}^{\infty}\left[\sqrt{\gamma_n(a(n)+f(n))}-\sqrt{\gamma_n(a(n))}\right]^2$$

is convergent. The same procedure also shows that

$$\sum_{n=1}^{\infty}\left[\sqrt{1-\gamma_n(a(n)+f(n))}-\sqrt{1-\gamma_n(a(n))}\right]^2$$

is finite.

□

Let us define a ternary Cantor set on $\mathbb{R}$ inductively as follows. Consider

$$\tilde{C}_n = [-1,-1+\frac{2}{5^n}]\cup[-\frac{1}{5^n},\frac{1}{5^n}]\cup[1-\frac{2}{5^n},1]$$

Let $C_1 := \tilde{C}_1$. To define C_2, identify each interval of the three intervals of C_1 with the interval $[-1,1]$ and place a copy of $\tilde{C}_2$ inside each of them. Put in formulas: $C_2 = (\frac{1}{5}\tilde{C}_2 - \frac{4}{5}) \cup (\frac{1}{5}\tilde{C}_2) \cup (\frac{1}{5}\tilde{C}_2 + \frac{4}{5})$. C_n is constructed by identifying each of the 3^{n-1} intervals of C_{n-1} with the interval $[-1,1]$ and placing a copy of $\tilde{C}_n$ inside each of them. Observe that C_n consists of 3^n intervals of (euclidean) length $2.5^{-\frac{n(n+1)}{2}}$. Let $C_\infty := \cap_{n=1}^{\infty} C_n$.

C_∞ is a ternary Cantor set and we canonically identify it with $3^{\mathbb{N}}$. Via this identification, a *central interval of* C_n is an interval of C_n that corresponds to a cylinder in $3^{\mathbf{n}} := \{0,1,2\}^n$ of the form

$$\mathcal{I}_{(\alpha_1,\ldots,\alpha_{n-1},1)} = \{x\in 3^{\mathbf{n}} : \forall\, 1\le j<n:\ x(j)=\alpha_j\in\{0,1,2\}\,, x(n)=1\}.$$

Similarly, an *outer interval of* C_n is an interval of C_n that corresponds to a cylinder in $3^{\mathbf{n}}$ of the form:

$$\mathcal{I}_{(\alpha_1,\ldots,\alpha_n)} = \{x\in 3^{\mathbf{n}} : \forall\, 1\le j<n:\ x(j)=\alpha_j\in\{0,1,2\}\,, x(n)=\alpha_n\in\{0,2\}\}.$$

Under this identification of $3^{\mathbb{N}}$ with C_∞, for each $a\in\mathbb{R}^{\mathbb{N}}$ the measure μ_a becomes a probability measure on $C_\infty \subset [-1,1] \subset \mathbb{R}$. Moreover, as a measure on $[-1,1]$, μ_a is symmetric, nonatomic, regular and $\mu_a(\bigcup_{\mathcal{I}\in\{\text{central interval of } C_n\}}\mathcal{I}) = \gamma_n(a(n))$.

For each $a\in\mathbb{R}^{\mathbb{N}}$ we consider the free Araki-Woods factor $\Gamma(\mu_a)$. The goal is to distinguish between these factors by means of the τ-invariant described in the previous section. Observe that by Lemma 2.2 in [25], a sequence $\{t_n\}_{n\in\mathbb{N}}\subset\mathbb{R}$ converges to zero in the topology $\tau(\mu_a)$ if and only if $\hat{\mu}_a(t_n)\to 1$, where $\hat{\mu}_a(t) = \int e^{2\pi i s t} d\mu_a(s)$ is the Fourier transform of μ_a. Since the measure μ_a is symmetric and supported on $[-1,1]$, the integral is equal to $\int_{-1}^{1} cos(2\pi s t) d\mu_a(s)$.

Consider the function $\psi_0(s) := \cos(\frac{\pi}{2}s)$ and let

$$K_n = \inf\{\psi_0(s) : s \in [-\frac{1}{5^n}, \frac{1}{5^n}]\}$$

and

$$k_n = \sup\{\psi_0(s) : s \in [1-\frac{2}{5^n}, 1]\}.$$

Then $K_n \to 1$ and $k_n \to 0$.

Consider the functions $\psi_n(s) = \cos(\frac{\pi}{2}5^{\frac{(n-1)(n-2)}{2}}s)$. Observe that when restricted to each interval of C_{n-1}, ψ_n looks like ψ_0 on the interval $[-1,1]$. But then it follows that for each central interval $\mathcal{I}$ of C_n we have that $\psi_n(s) \geq K_n$ and for each outer interval $\mathcal{I}$ of C_n we have that $\psi_n(s) \leq k_n$.

Lemma 3.3. *If $a \in \mathbb{R}^{\mathbb{N}}$ and $n_i \to \infty$ such that $a(n_i) \to +\infty$ (and so $\gamma_{n_i}(a(n_i)) \to 1$) then*

$$\int_{-1}^{1} \psi_{n_i}(s)d\mu_a(s) \to 1 \text{ as } i \to \infty.$$

Proof. Since the measure μ_a is supported on $C_\infty \subset C_{n_i}$ then:

$$\int_{-1}^{1} \psi_{n_i}(s)d\mu_a(s) = \sum_{\mathcal{I}\in\{\text{interval of } C_{n_i}\}} \int_{\mathcal{I}} \psi_{n_i}(s)d\mu_a(s)$$

$$\geq \sum_{\mathcal{I}\in\{\text{central interval of } C_{n_i}\}} \int_{\mathcal{I}} \psi_{n_i}(s)d\mu_a(s)$$

$$\geq \sum_{\mathcal{I}\in\{\text{central interval of } C_{n_i}\}} \int_{\mathcal{I}} K_{n_i}d\mu_a(s)$$

$$= K_{n_i}\,\mu_a(\bigcup_{\mathcal{I}\in\{\text{ central interval of } C_{n_i}\}} \mathcal{I}) = K_{n_i}\gamma_{n_i}(a(n_i)) \underset{n_i\to\infty}{\longrightarrow} 1.$$

□

Lemma 3.4. *If $a \in \mathbb{R}^{\mathbb{N}}$ and $n_i \to \infty$ such that $a(n_i) \to 0$ (and so $\gamma_{n_i}(a(n_i)) \to \frac{1}{2}$) then*

$$\int_{-1}^{1} \psi_{n_i}(s)d\mu_a(s) \not\to 1 \text{ as } i \to \infty.$$

Proof.

$$\int_{-1}^{1} \psi_{n_i}(s)d\mu_a(s) = \sum_{\mathcal{I}\in\{\text{interval of } C_{n_i}\}} \int_{\mathcal{I}} \psi_{n_i}(s)d\mu_a(s)$$

$$= \sum_{\mathcal{I}\in\{\text{central interval of } C_{n_i}\}} \int_{\mathcal{I}} \psi_{n_i}(s)d\mu_a(s) + \sum_{\mathcal{I}\in\{\text{outer interval of } C_{n_i}\}} \int_{\mathcal{I}} \psi_{n_i}(s)d\mu_a(s)$$

$$\leq \sum_{\mathcal{I}\in\{\text{central interval of } C_{n_i}\}} \int_{\mathcal{I}} 1 d\mu_a(s) + \sum_{\mathcal{I}\in\{\text{outer interval of } C_{n_i}\}} \int_{\mathcal{I}} k_{n_i} d\mu_a(s)$$

$$= \mu_a(\bigcup_{\mathcal{I}\in\{\text{central interval } C_{n_i}\}} \mathcal{I}) + k_{n_i}\, \mu_a(\bigcup_{\mathcal{I}\in\{\text{outer interval of } C_{n_i}\}} \mathcal{I})$$

$$= \gamma_{n_i}(a(n_i)) + k_{n_i}(1-\gamma_{n_i}(a(n_i))) \underset{i\to\infty}{\longrightarrow} \frac{1}{2}$$

□

4. Non-classification of measures on the Cantor set

We recall the definition of generic S_∞-ergodicity from Hjorth's turbulence theory, [10, Definition 3.6]. (We emphasize again that S_∞ here denotes the Polish group of *all* permutations of $\mathbb{N}$.)

Definition 4.1. Let E and F be equivalence relations on a Polish spaces X and Y, respectively.

0) We will say that $f : X \to Y$ is (E, F)-equivariant if

$$xEx' \implies f(x)Ff(x').$$

1) We say that E is *generically F-ergodic* if every Baire measurable (E, F)-equivariant function $f : X \to Y$ must map a comeagre set in X into a single F-class.

2) E is generically *generically S_∞-ergodic* if E is generically F-ergodic whenever F is an orbit equivalence relation induced by a continuous (equivalently, Borel) action of S_∞ on a Polish space Y.

Hjorth turbulence theorem, [10, Theorem 3.18], provides a dynamical criterion for S_∞ ergodicity. Notice that in the previous definition, if the classes of E are meagre, then F-ergodicity implies that a Baire measurable (E, F)-equivariant map cannot distinguish the classes of E.

To put the above into our context, we make the following definition:

Definition 4.2. Given $a, b \in \mathbb{R}^{\mathbb{N}}$ we say that

(1) $a \sim b$ if the von Neumann algebras $\Gamma(\mu_a)$ and $\Gamma(\mu_b)$, are isomorphic.
(2) $a \approx b$ if the measure μ_a is equivalent to the measure μ_b.

Notice $\approx$ is a sub-equivalence relation of $\sim$, written in symbols $\approx\subseteq\sim$. Below we establish that $\sim$ has meagre classes, and that $\sim$ is generical S_∞-ergodic. The latter follows from Lemma 2.1 from [20], which indeed uses Hjorth's theory of turbulence in its proof.

Lemma 4.3. *For any $t \in [0,1]$ and any infinite set $I \subset \mathbb{N}$ the set*

$$A(I) = \{a \in \mathbb{R}^{\mathbb{N}} : (\exists (n_i)_{i\in\mathbb{N}} \subset I) \lim_{n_i \to \infty} \gamma_{n_i}(a(n_i)) = t\}$$

is comeagre.

Proof. We have that

$$A(I) = \bigcap_{N\in\mathbb{N}} \bigcap_{\varepsilon>0} \bigcup_{n>N, n\in I} \{a \in \mathbb{R}^{\mathbb{N}} : |\gamma_n(a(n)) - t| < \varepsilon\}$$

and since the sets $\bigcup_{n>N,n\in I}\{a \in \mathbb{R}^{\mathbb{N}} : |\gamma(a(n)) - t| < \varepsilon\}$ are open and dense, $A(I)$ is a dense G_δ set of $\mathbb{R}^{\mathbb{N}}$. □

Lemma 4.4. *The equivalence relation $\sim$ on $\mathbb{R}^{\mathbb{N}}$ has dense classes.*

Proof. By construction, for each $a \in \mathbb{R}^{\mathbb{N}}$, $\gamma_n(a(n)) \neq 0, 1$. That means that if $a, b \in \mathbb{R}^{\mathbb{N}}$ differ only in finitely many coordinates, then by Kakutani's criterium (or by a straightforward application of Lemma 3.2) the product measures μ_a and μ_b on $3^{\mathbb{N}}$, are equivalent. Since $\approx \subseteq \sim$, we have that $a \sim b$.

Thus $[a]_\sim \supset \{b \in \mathbb{R}^{\mathbb{N}} : b(n) = a(n)$ except for finitely many n's$\}$ and this set is dense in $\mathbb{R}^{\mathbb{N}}$. □

Lemma 4.5. *The equivalence relation $\sim$ on $\mathbb{R}^{\mathbb{N}}$ has meagre classes.*

Proof. Suppose that $\sim$ has a non-meagre class. Since classes are dense, it follows that it has a comeagre class, call it $[a]_\sim$. Therefore, if we take in Lemma 4.3 $t_0 = 1$, and $I = \mathbb{N}$, we can find $b \in [a]_\sim$ and some increasing sequence $n_i \in \mathbb{N}$ such that

$$\gamma_{n_i}(b(n_i)) \underset{n\to\infty}{\longrightarrow} 1.$$

Applying once again Lemma 4.3 to the set $I = \{n_i : n_i \in \mathbb{N}\}$ and $t = \frac{1}{2}$, there exists $c \in \mathbb{R}^{\mathbb{N}}$ such that $c \in [a]_\sim$ and a subsequence n_{i_k} such that $\gamma_{n_{i_k}}(c(n_{i_k})) \to \frac{1}{2}$ as $k \to \infty$. But then, if we let $t_n = \frac{1}{4}5^{\frac{(n-1)(n-2)}{2}}$ by Lemma 3.3:

$$\hat{\mu}_b(t_{n_{i_k}}) = \int \cos(2\pi \frac{1}{4} 5^{\frac{(n_{i_k}-1)(n_{i_k}-2)}{2}} s) d\mu_b(s) = \int \psi_{n_{i_k}}(s) d\mu_b \to 1$$

while Lemma 3.4 asserts that

$$\hat{\mu}_c(t_{n_{i_k}}) \not\to 1.$$

It follows that the topologies on $\mathbb{R}$ induced by the measures μ_b and μ_c do not coincide. This implies that the corresponding von Neumann algebras have different τ-invariants, thus $b \not\sim c$, a contradiction. □

Theorem 4.6. *The equivalence relation $\sim$ on $\mathbb{R}^{\mathbb{N}}$ is generically S_∞-ergodic and it is not classifiable by countable structures.*

Proof. Consider the weighted Hilbert space $\ell^2_{\mathbb{R}}(\mathbb{N},\omega)$, where ω is the weight $\omega(n) = n$. $\ell^2_{\mathbb{R}}(\mathbb{N},\omega)$ acts on $\mathbb{R}^{\mathbb{N}}$ by addition. Since $\ell^2_{\mathbb{R}}(\mathbb{N},\omega)$ is a Banach space that is a dense subspace of $\mathbb{R}^{\mathbb{N}}$, Lemma 2.1 in [20] asserts that this action by addition is turbulent, and has dense and meagre classes. Thus by Hjorth turbulence theorem [10, Theorem 3.18], the equivalence relation $E^{\mathbb{R}^{\mathbb{N}}}_{\ell^2_{\mathbb{R}}(\mathbb{N},\omega)}$ is generically S_∞-ergodic.

Now if $a \in \mathbb{R}^{\mathbb{N}}$ and $f \in \ell^2_{\mathbb{R}}(\mathbb{N},\omega)$, Lemma 3.2 tells that the measures μ_{a+f} and μ_a are equivalent. Thus $E^{\mathbb{R}^{\mathbb{N}}}_{\ell^2_{\mathbb{R}}(\mathbb{N},\omega)} \subseteq \approx \subseteq \sim$. This implies that $\sim$ is generically S_∞-ergodic.

Since $\sim$ has dense, meagre classes, by Hjorth's Theorem it follows that $\sim$ is not classifiable by countable structures. □

5. A common Hilbert Space for free Araki-Woods factors

Observe that from its construction, the free Araki-Woods factor $\Gamma(\nu)$ acts on the Hilbert space $\mathcal{F}(\tilde{\mathcal{H}}_\nu)$. However to apply the tools of Borel reducibility (defined below) from descriptive set theory, it is necessary to have a common Hilbert space on which all the $\Gamma(\nu)$ act.

Recall from [18] and [19] that if $\mathcal{H}$ is a separable complex Hilbert space, then $\mathrm{vN}(\mathcal{H})$ denotes the standard Borel space of von Neumann algebras acting on $\mathcal{H}$, equipped with the Effros Borel structure originally introduced in [3] and [4].

The goal of this section is to construct a Borel assignment $\hat{\theta}$ from the Borel space $\mathcal{X}$ of symmetric, non atomic, regular probability measures supported on the interval $[-1,1]$ to $\mathrm{vN}(\ell^2(\mathbb{N}))$, such that for each $\nu \in \mathcal{X}$, $\hat{\theta}(\nu)$ is isomorphic to the free Araki-Woods factor $\Gamma(\nu)$.

Applying $\hat{\theta}$ to the measures constructed in section 3 then achieves a *Borel reduction* (see definition below) of the equivalence relation $\sim$ (as defined in section 4) to isomorphism in the the Effros Borel space of von Neumann algebras, and so by Theorem 4.6 the isomorphism relation in the Effros Borel space is not classifiable by countable structures.

Before going further, we recall the definition of Borel reduction and Borel reducibility.

Definition 5.1. Given equivalence relations E and F on standard Borel spaces X and Y, we say that $f : X \to Y$ is a *Borel reduction* of E to F if f is a Borel function such that for all $x, x' \in X$,

$$xEx' \iff f(x)Ff(x').$$

If a Borel reduction of E to F exists, then we write $E \leq_B F$, and say that E is *Borel reducible* to F.

By hypothesis every measure $\nu \in \mathcal{X}$ is regular and supported on the interval $[-1,1]$. A classical application of the Urysohn's lemma proves that the continuous functions $\mathcal{C}([-1,1],\mathbb{C}) := \{f : [-1,1] \to \mathbb{C} : f \text{ is continuous}\}$ are dense in the Hilbert space $\mathcal{L}^2([-1,1],\nu) = \mathcal{L}^2(\mathbb{R},\nu) = \mathcal{H}_\nu$. But more is true: since the measure ν is non atomic (in particular it has no mass at $x = \pm 1$), a simple adaptation of the proof also shows that $\mathcal{A} := \{f \in \mathcal{C}([-1,1],\mathbb{C}) : f(1) = f(-1)\}$ is dense in $\mathcal{H}_\nu$ and that $\mathcal{B} := \{f \in \mathcal{A} : f(x) = \overline{f(-x)}\}$ is dense in $\mathcal{H}_{\nu,\mathbb{R}}$. On the other hand, the Stone Weierstrass Theorem asserts that the trigonometric polynomials, namely the $\mathbb{C}$-span of the set $\mathcal{Y} := \{e^{\pi i n x} : n \in \mathbb{Z}\}$ is dense in $\mathcal{A}$ when $\mathcal{A}$ is endowed with the uniform norm. Also note that $\mathcal{Y} \subset \mathcal{B}$. Moreover the formula

$$f(x) = \frac{f(x) + \overline{f(-x)}}{2} + i\,\frac{-if(x) + i\overline{f(-x)}}{2}$$

shows that $\mathcal{A} = \mathcal{B} + i\mathcal{B}$. It follows that the $\mathbb{R}$-span of $\mathcal{Y}$ is dense in $\mathcal{B}$ when $\mathcal{B}$ is endowed with the uniform norm. We then conclude that the $\mathbb{R}$-span of $\mathcal{Y}$ is dense in $\mathcal{H}_{\nu,\mathbb{R}}$. Recall that by the item (4) in Proposition 2.1 $\mathcal{H}_{\nu,\mathbb{R}} + i\mathcal{H}_{\nu,\mathbb{R}}$ is dense in $\tilde{\mathcal{H}}_\nu$. Then the $\mathbb{C}$-span of $\mathcal{Y}$ is dense in $\tilde{\mathcal{H}}_\nu$.

Since $\tilde{\mathcal{H}}_\nu \ni \xi \to s(\xi) \in \mathcal{B}(\mathcal{F}(\tilde{\mathcal{H}}_\nu))$ is an isometry and $\mathcal{H}_{\nu,\mathbb{R}}$ embeds isometrically in $\tilde{\mathcal{H}}_\nu$, then

$$\Gamma(\nu) = \{s(f) : f \in \mathcal{H}_{\nu,\mathbb{R}}\}'' = \{s(f) : f \in \mathcal{Y}\}''$$

At this stage we have shown that all the free Araki-Woods factors $\Gamma(\nu)$ arising from measures in $\nu \in \mathcal{X}$ are generated by operators defined from the common countable set $\mathcal{Y}$.

Let us now fix an order on the set $\mathcal{Y}$, for instance $\mathcal{Y} = \{f_l\}_{l \in \mathbb{N}}$ with $f_l(x) = e^{\pi i (-1)^l \lfloor l/2 \rfloor x}$. Since $\mathbb{C}$-span of $\mathcal{Y}$ is dense in $\tilde{\mathcal{H}}_\nu$, the Gram-Schmidt process applied to it with respect to the inner product in $\tilde{\mathcal{H}}_\nu$ constructs an orthonormal basis $\{v_k^\nu\}_{k \in \mathbb{N}}$ of the complex Hilbert space $\tilde{\mathcal{H}}_\nu$.

To fix the notation, recall that v_k^ν is defined inductively as follows: Define $w_1^\nu = f_1$ and $v_1^\nu = \frac{w_1^\nu}{\|w_1^\nu\|_{\tilde{\mathcal{H}}_\nu}}$. The k step of the Gram-Schmidt process is

$$w_k^\nu = f_k - \sum_{l=1}^{k-1} \langle f_k, v_l^\nu \rangle_{\tilde{\mathcal{H}}_\nu} v_l^\nu,$$

and

$$v_k^\nu = \frac{w_k^\nu}{\|w_k^\nu\|_{\tilde{\mathcal{H}}_\nu}}.$$

Remark 5.2. (1) We will show the crucial fact that $\|w_k^\nu\|_{\tilde{\mathcal{H}}_\nu} \neq 0$. This is never an issue in most applications of Gram-Schmidt, because if $\|w_k^\nu\|_{\tilde{\mathcal{H}}_\nu} = 0$ one removes v_k^ν from the final set of vectors. However, for each k fixed we will be interested in analyzing the dependency of the unit vectors v_k^ν with respect to the parameter ν, so we can not just "delete" a vector.

(2) For every $k \in \mathbb{N}$, and for every $\nu \in \mathcal{X}$ fixed, w_k^ν is a continuous function on $[-1,1]$, and $w_k^\nu \in \mathcal{H}_{\nu,\mathbb{R}} + i\mathcal{H}_{\nu,\mathbb{R}}$. More is true, w_k^ν is a trigonometric polynomial of degree equal to $\deg f_k = \lfloor k/2 \rfloor$. The reason is that each w_k^ν is a complex linear combination of continuous functions in $\mathcal{Y} \subset \mathcal{H}_{\nu,\mathbb{R}}$.

Theorem 5.3. *For every $l \in \mathbb{N}$ the assignment:*

$$\alpha_l : \mathcal{X} \to (\mathcal{C}[-1,1], \mathbb{C})$$
$$\alpha_l(\nu) = v_l^\nu$$

is continuous, when $\mathcal{X}$ is endowed with the weak topology and $(\mathcal{C}[-1,1],\mathbb{C})$ is endowed with the uniform norm.

Remark 5.4. The proof of this seemingly simple theorem is more technical than what it first looks. In fact the statement is false if one applies the Gram-Schmidt process to an arbitrary, countable dense set of continuous functions. It is also false if the measures have atoms. To prove the theorem we will need some preliminary lemmas.

Lemma 5.5. *Assume that for every measure $\nu \in \mathcal{X}$, f^ν is a trigonometric polynomial. Suppose that $\|f^{\nu_j} - f^\nu\|_{(\mathcal{C}[-1,1],\mathbb{C})} \underset{\nu_j \to \nu}{\longrightarrow} 0$.*

Then for every $f \in \mathcal{Y}$,

$$\langle f, f^{\nu_j}\rangle_{\tilde{\mathcal{H}}_{\nu_j}} \underset{\nu_j \to \nu}{\longrightarrow} \langle f, f^\nu\rangle_{\tilde{\mathcal{H}}_\nu}$$

Proof. Since $f^{\nu_j} \in \mathcal{H}_{\nu,\mathbb{R}} + i\mathcal{H}_{\nu,\mathbb{R}}$, write $f^{\nu_j} = g^{\nu_j} + ih^{\nu_j}$ with $g^{\nu_j}, h^{\nu_j} \in \mathcal{H}_{\nu,\mathbb{R}}$. Then

$$\langle f, f^{\nu_j}\rangle_{\tilde{\mathcal{H}}_{\nu_j}} = \langle f, g^{\nu_j} + ih^{\nu_j}\rangle_{\tilde{\mathcal{H}}_{\nu_j}} = \langle f, g^{\nu_j}\rangle_{\tilde{\mathcal{H}}_{\nu_j}} + i\langle f, h^{\nu_j}\rangle_{\tilde{\mathcal{H}}_{\nu_j}}$$
$$= Re\langle f, g^{\nu_j}\rangle_{\tilde{\mathcal{H}}_{\nu_j}} - Im\langle f, h^{\nu_j}\rangle_{\tilde{\mathcal{H}}_{\nu_j}} + i\Big[Im\langle f, g^{\nu_j}\rangle_{\tilde{\mathcal{H}}_{\nu_j}} + Re\langle f, h^{\nu_j}\rangle_{\tilde{\mathcal{H}}_{\nu_j}}\Big]$$

Since the functions inside each of the four inner products are in $\mathcal{H}_{\nu,\mathbb{R}}$, items (1) and (2) in Proposition 2.1 give the following identifications:

(1) $Re\langle f, g^{\nu_j}\rangle_{\tilde{\mathcal{H}}_{\nu_j}} = \langle f, g^{\nu_j}\rangle_{\mathcal{H}_{\nu_j,\mathbb{R}}}$.
(2) $Im\langle f, h^{\nu_j}\rangle_{\tilde{\mathcal{H}}_{\nu_j}} = \langle i\frac{1-e^{-x}}{1+e^{-x}} f, h^{\nu_j}\rangle_{\mathcal{H}_{\nu_j,\mathbb{R}}}$.
(3) $Im\langle f, g^{\nu_j}\rangle_{\tilde{\mathcal{H}}_{\nu_j}} = \langle i\frac{1-e^{-x}}{1+e^{-x}} f, g^{\nu_j}\rangle_{\mathcal{H}_{\nu_j,\mathbb{R}}}$
(4) $Re\langle f, h^{\nu_j}\rangle_{\tilde{\mathcal{H}}_{\nu_j}} = \langle f, h^{\nu_j}\rangle_{\mathcal{H}_{\nu_j,\mathbb{R}}}$.

The advantage now is that these four inner products are given by integration against the original measure $\nu_j \in \mathcal{X}$, and all the functions involved are in $(\mathcal{C}[-1,1],\mathbb{C})$. Moreover since $\|f^{\nu_j} - f^\nu\|_{(\mathcal{C}[-1,1],\mathbb{C})} \underset{\nu_j \to \nu}{\to} 0$, then

$\|g^{\nu_j} - g^\nu\|_{(\mathcal{C}[-1,1],\mathbb{C})} \underset{\nu_j\to\nu}{\to} 0$ and $\|h^{\nu_j} - h^\nu\|_{(\mathcal{C}[-1,1],\mathbb{C})} \underset{\nu_j\to\nu}{\to} 0$. Thus the proof of the Lemma reduces to prove the following statement:

Claim 1: If $f, \phi^\nu, \phi^{\nu_j} \in (\mathcal{C}[-1,1],\mathbb{C})$, and $\|\phi^{\nu_j} - \phi^\nu\|_{(\mathcal{C}[-1,1],\mathbb{C})} \underset{\nu_j\to\nu}{\longrightarrow} 0$ then

$$\langle f, \phi^{\nu_j}\rangle_{\mathcal{H}_{\nu_j}} \underset{\nu_j\to\nu}{\longrightarrow} \langle f, \phi^\nu\rangle_{\mathcal{H}_\nu}$$

Proof of Claim.

$$\begin{aligned}
\left|\langle f, \phi^{\nu_j}\rangle_{\mathcal{H}_{\nu_j}} - \langle f, \phi^\nu\rangle_{\mathcal{H}_\nu}\right| &= \left|\int \bar{f}\cdot\phi^{\nu_j}d\nu_j - \int \bar{f}\cdot\phi^\nu d\nu\right| \\
&= \left|\int \bar{f}\cdot\phi^{\nu_j}d\nu_j - \int \bar{f}\cdot\phi^\nu d\nu_j + \int \bar{f}\cdot\phi^\nu d\nu_j - \int \bar{f}\cdot\phi^\nu d\nu\right| \\
&\leq \left|\int \bar{f}\cdot(\phi^{\nu_j} - \phi^\nu)d\nu_j\right| + \left|\nu_j(\bar{f}\cdot\phi^\nu) - \nu(\bar{f}\cdot\phi^\nu)\right| \\
&\leq \|\phi^{\nu_j} - \phi^\nu\|_{(\mathcal{C}[-1,1],\mathbb{C})}\int|\bar{f}|d\nu_j + \left|\nu_j(\bar{f}\cdot\phi^\nu) - \nu(\bar{f}\cdot\phi^\nu)\right|
\end{aligned}$$

(1) Since all the measures in $\mathcal{X}$ are probability measures and f is continuous on the compact set $[-1,1]$ then $\int|\bar{f}|d\nu_j \leq \|f\|_{(\mathcal{C}[-1,1],\mathbb{C})}$.
(2) Since $\bar{f}\cdot\phi^\nu$ is continuous on $[-1,1]$ then by the definition of $\nu_j \to \nu$ it follows that $\nu_j(f\cdot\phi^\nu) \to \nu(f\cdot\phi^\nu)$.

So both summands in the last inequality converge to zero when $\nu_j \to \nu$. □

□

Lemma 5.6. *For each $k \in \mathbb{N}$ and for each $\nu \in \mathcal{X}$ $\|w_k^\nu\|_{\tilde{\mathcal{H}}_\nu} \neq 0$.*

Proof. A distinguished feature of the Gram-Schimdt process applied to an ordered set $\mathcal{S}$ is that the vectors obtained in step k, are in the complex span of the first k vectors of $\mathcal{S}$. In the case at hand it entails that each w_k^ν is in the $\mathbb{C}$-span of $\{f_l\}_{l=1}^k \subset \mathcal{Y}$, thus w_k^ν is a trigonometric polynomial. Moreover, it can be shown by an easy induction that its leading term is $f_k = e^{\pi i(-1)^l\lfloor k/2\rfloor t}$, in particular $\deg w_k^\nu = \deg f_k = \lfloor k/2\rfloor$ and then w_k^ν has at most $2\lfloor k/2\rfloor$ roots. Since every measure $\nu \in \mathcal{X}$ is non atomic, it follows that the ν-measure of the set $\{t \in [-1,1] : w_k^\nu(t) = 0\}$ is zero. But then $\|w_k^\nu\|^2_{\mathcal{H}_\nu} = \int_{-1}^1 |w_k^\nu|^2 d\nu > 0$.

Since $w_k^\nu \in \mathcal{H}_{\nu,\mathbb{R}} + i\mathcal{H}_{\nu,\mathbb{R}}$, write $w_k^\nu = g_k^\nu + ih_k^\nu$, with $g_k^\nu, h_k^\nu \in \mathcal{H}_{\nu,\mathbb{R}}$. Observe that the Gram-Schmidt process applied to the ordered set $\mathcal{Y}$ also implies that g_k^ν and h_k^ν are trigonometric polynomials and that one of them has degree $\lfloor k/2\rfloor$. But then the same argument used to show that $\|w_k^\nu\|^2_{\mathcal{H}_\nu} \neq 0$, proves that $\|g_k^\nu\|^2_{\mathcal{H}_{\nu,\mathbb{R}}} = \|g_k^\nu\|^2_{\mathcal{H}_\nu} \neq 0$ or $\|h_k^\nu\|^2_{\mathcal{H}_{\nu,\mathbb{R}}} = \|h_k^\nu\|^2_{\mathcal{H}_\nu} \neq 0$.

Suppose now that $\|w_k^\nu\|_{\tilde{\mathcal{H}}_\nu} = 0$. This means that $g_k^\nu + ih_k^\nu = 0$ when viewed as a vector in the Hilbert space $\tilde{\mathcal{H}}_\nu$. Proposition 2.1 asserts that

$\mathcal{H}_{\nu,\mathbb{R}} \cap i\mathcal{H}_{\nu,\mathbb{R}} = \{0\}$. It follows that $g_k^\nu = h_k^\nu = 0$ when viewed as vectors in the Hilbert space $\mathcal{H}_{\nu,\mathbb{R}} \subset \mathcal{H}_\nu$. This is a contradiction.

□

Lemma 5.7. *Assume that* $\|w_k^{\nu_j} - w_k^\nu\|_{(\mathcal{C}[-1,1],\mathbb{C})} \underset{\nu_j\to\nu}{\longrightarrow} 0$. *Then* $\|w_k^{\nu_j}\|_{\tilde{\mathcal{H}}_{\nu_j}} \underset{\nu_j\to\nu}{\longrightarrow} \|w_k^\nu\|_{\tilde{\mathcal{H}}_\nu}$.

Proof. The proof is similar to the proof of Lemma 5.5: start by writing the inner product in $\tilde{\mathcal{H}}_{\nu_j}$ in terms of four inner products in $\mathcal{H}_{\nu_j}$, and then apply Claim 1 and the triangle inequality. The details are left to the reader. □

Now we are in shape to prove Theorem 5.3.

Proof of Theorem 5.3. We proceed by induction on l.

<u>Case $l = 1$:</u>

Since for every measure $\nu \in \mathcal{X}$, w_1^ν is equal to $f_1 = 1$, it is clear that $\|w_1^{\nu_j} - w_1^\nu\|_{(\mathcal{C}[-1,1],\mathbb{C})} \underset{\nu_j\to\nu}{\to} 0$. By Lemma 5.7, $\|w_1^{\nu_j}\|_{\tilde{\mathcal{H}}_{\nu_j}} \underset{\nu_j\to\nu}{\longrightarrow} \|w_1^\nu\|_{\tilde{\mathcal{H}}_\nu}$, and by Lemma 5.6, $\|w_1^\nu\|_{\tilde{\mathcal{H}}_\nu} \neq 0$, then $1/\|w_1^{\nu_j}\|_{\tilde{\mathcal{H}}_{\nu_j}} \underset{\nu_j\to\nu}{\longrightarrow} 1/\|w_1^\nu\|_{\tilde{\mathcal{H}}_\nu}$. But then $\|v_1^{\nu_j} - v_1^\nu\|_{(\mathcal{C}[-1,1],\mathbb{C})} \underset{\nu_j\to\nu}{\to} 0$.

<u>Case $l = k$:</u>

Suppose the statement is true for all $l < k$. We first show that if $\nu_j \to \nu$ in $\mathcal{X}$ then $\|w_k^{\nu_j} - w_k^\nu\|_{(\mathcal{C}[-1,1],\mathbb{C})} \to 0$. Since

$$\|w_k^{\nu_j} - w_k^\nu\|_{(\mathcal{C}[-1,1],\mathbb{C})} \le \sum_{l=1}^{k-1} \left\| \langle f_k, v_l^{\nu_j}\rangle_{\tilde{\mathcal{H}}_{\nu_j}} v_l^{\nu_j} - \langle f_k, v_l^\nu\rangle_{\tilde{\mathcal{H}}_\nu} v_l^\nu \right\|_{(\mathcal{C}[-1,1],\mathbb{C})}$$

It is enough to prove that for each $l < k$,

$$\left\| \langle f_k, v_l^{\nu_j}\rangle_{\tilde{\mathcal{H}}_{\nu_j}} v_l^{\nu_j} - \langle f_k, v_l^\nu\rangle_{\tilde{\mathcal{H}}_\nu} v_l^\nu \right\|_{(\mathcal{C}[-1,1],\mathbb{C})} \underset{\nu_j\to\nu}{\to} 0.$$

the triangular inequality shows that this norm is less or equal than

$$|\langle f_k, v_l^{\nu_j}\rangle_{\tilde{\mathcal{H}}_{\nu_j}} - \langle f_k, v_l^\nu\rangle_{\tilde{\mathcal{H}}_\nu}|\, \|v_l^{\nu_j}\|_{(\mathcal{C}[-1,1],\mathbb{C})} + |\langle f_k, v_l^\nu\rangle_{\tilde{\mathcal{H}}_\nu}|\, \|v_l^{\nu_j} - v_l^\nu\|_{(\mathcal{C}[-1,1],\mathbb{C})}.$$

By the inductive hypothesis $\|v_l^{\nu_j} - v_l^\nu\|_{(\mathcal{C}[-1,1],\mathbb{C})} \underset{\nu_j\to\nu}{\to} 0$, so the second summand converges to zero and $\|v_l^{\nu_j}\|_{(\mathcal{C}[-1,1],\mathbb{C})} < C$, where the constant C is independent of ν_j. Then by Lemma 5.5 the first summand also converges to zero.

The same proof as in the case $l = 1$ combined with Lemma 5.6 now shows that

$$\left\| \frac{w_k^\nu}{\|w_k^\nu\|_{\tilde{\mathcal{H}}_\nu}} - \frac{w_k^{\nu_j}}{\|w_k^{\nu_j}\|_{\tilde{\mathcal{H}}_{\nu^j}}} \right\|_{(\mathcal{C}[-1,1],\mathbb{C})} = \|v_k^{\nu_j} - v_k^\nu\|_{(\mathcal{C}[-1,1],\mathbb{C})} \underset{\nu_j\to\nu}{\to} 0.$$

□

For each orthonormal basis $\{v_l^\nu\}_{l\in\mathbb{N}}$ of $\tilde{\mathcal{H}}_\nu$ constructed by the procedure described above, construct an orthonormal basis of the full Fock space $\mathcal{F}(\tilde{\mathcal{H}}_\nu)$ by doing all possible finite tensor products of the v_l^ν (in a prescribed order, independent of ν). Denote this basis of $\mathcal{F}(\tilde{\mathcal{H}}_\nu)$ by $\{e_n^\nu\}_{l\in\mathbb{N}}$.

The final ingredient needed in order to give the desired Borel function $\hat{\theta}$ announced at the beginning of this section, is [20, Lemma 3.14], which we repeat for the reader's convenience.

Lemma 5.8. *Suppose X is a standard Borel space and $(H_x : x \in X)$ is a family of infinite dimensional separable Hilbert spaces, and that $(e_n^x)_{n\in\mathbb{N}}$ is an orthonormal basis of H_x for each $x \in X$. Suppose further that Y is a standard Borel space space and $(T_y^x : x \in X, y \in Y)$ is a family of operators such that $T_y^x \in \mathcal{B}(H_x)$ for all $y \in Y$, $x \in X$ and that the functions*

$$X \times Y \to \mathbb{C} : (x, y) \mapsto \langle T_y^x e_n^x, e_m^x\rangle$$

are Borel for all n, m. Then there is a Borel function $\theta : X \times Y \to \mathcal{B}(\ell^2(\mathbb{N}))$ and a family $(\varphi_x : x \in X)$ such that

(1) $\varphi_x \in \mathcal{B}(H_x, \ell^2(\mathbb{N}))$ satisfies $\varphi_x(e_n^x) = e_n$, where $(e_n)_{n\in\mathbb{N}}$ is the standard basis for $\ell^2(\mathbb{N})$.
(2) For all $x \in X$, $y \in Y$ and $\xi \in H_x$ we have $\theta(x, y)(\varphi_x(\xi)) = \varphi_x(T_y^x(\xi))$

Moreover, if M_x is the von Neumann algebra generated by the family $(T_y^x : y \in Y)$, and there are Borel functions

$$\psi_n : X \to Y$$

such that $(T_{\psi_n(x)}^x : n \in \mathbb{N})$ generates M_x for each $x \in X$, then there is a Borel function $\hat{\theta} : X \to \mathrm{vN}(\ell^2(\mathbb{N}))$ such that $\hat{\theta}(x) \simeq M_x$ for all $x \in X$.

We are now in position to state and prove the required technical result of this section.

Theorem 5.9. *Let $\mathcal{X}$ be the set of regular non atomic symmetric measures supported in the interval $[-1, 1]$. There is a Borel function $\hat{\theta} : \mathcal{X} \to \mathrm{vN}(\ell^2(\mathbb{N}))$ such that for all $\nu \in \mathcal{X}$, $\hat{\theta}(\nu)$ is isomorphic to $\Gamma(\nu)$.*

Proof. Recall first that $\mathcal{X}$ is a Standard Borel space when viewed as a Borel subset of the finite measures in the compact space $[-1, 1]$ endowed with the weak topology.

We make the following identifications in Lemma 5.8:

(1) $X := \mathcal{X}$.
(2) $(H_x : x \in X)$ is $(\mathcal{F}(\tilde{\mathcal{H}}_\nu) : \nu \in \mathcal{X})$ where $\mathcal{F}(\tilde{\mathcal{H}}_\nu)$ is the Full Fock Space associated to the measure ν constructed in section 2.
(3) $Y := \mathcal{Y} = \{e^{2\pi i(-1)^l\lfloor l/2\rfloor t}\}_{l\in\mathbb{N}}$.

(4) For each $\nu \in \mathcal{X}$, $\{e_n^\nu\}_{l\in\mathbb{N}}$ is the orthogonal basis of the Full Fock Space $\mathcal{F}(\tilde{\mathcal{H}}_\nu)$ constructed from $\mathcal{Y}$ by the procedure described above.

(5) For each $f \in \mathcal{Y}$, for each $\nu \in \mathcal{X}$, $T_f^\nu := s(f) \in \mathcal{B}(\mathcal{F}(\tilde{\mathcal{H}}_\nu))$.

By Theorem 5.3, for each $k \in \mathbb{N}$ the assignment

$$\alpha_k : \mathcal{X} \to (C[-1,1], \mathbb{C})$$

$$\alpha_k(\nu) = v_k^\nu$$

is continuous. Moreover for each n fixed e_n^ν is of the form $v_{k_1}^\nu \otimes v_{k_2}^\nu \otimes \cdots \otimes v_{k_l}^\nu$, where $(k_1, k_2, \ldots, k_l)$ only depends on n, and $\mathcal{Y}$ is a countable set. We then have that for each $n, m \in \mathbb{N}$ fixed, the function

$$\mathcal{X} \times \mathcal{Y} \to \mathbb{C}$$

$$(\nu, f) \to \langle s(f)e_n^\nu, e_m^\nu \rangle_{\mathcal{F}(\tilde{\mathcal{H}}_\nu)}$$

is continuous. Since for each $\nu \in \mathcal{X}$ fixed, we have shown that $\Gamma(\nu) = \{s(f) : f \in \mathcal{Y}\}''$, the theorem now follows from Lemma 5.8. □

Theorem 5.10. *The isomorphism relation of free Araki-Woods factors, $\Gamma(\nu)$, with $\nu \in \mathcal{X}$ and where $\mathcal{X}$ is the set of regular non atomic symmetric probability measures supported in the interval $[-1,1]$, is not classifiable by countable structures.*

Proof. Let $\simeq^{\mathrm{vN}(\mathcal{H})}$ denote the isomorphism relation in $\mathrm{vN}(\mathcal{H})$. By Theorem 5.9, there exists $\hat{\theta} : \mathcal{X} \to \mathrm{vN}(\ell^2(\mathbb{N}))$ Borel such that $\hat{\theta}(\nu) \simeq \Gamma(\nu)$. The assignment $\Psi : \mathbb{R}^{\mathbb{N}} \to \mathcal{X}$, $\Psi(a) = \mu_a$ constructed in section 3 is clearly Borel. By Theorem 4.6 the equivalence relation $\sim$ on $\mathbb{R}^{\mathbb{N}}$ given by $a \sim b$ if and only if $\Gamma(\mu_a)$ is isomorphic with $\Gamma(\mu_b)$ is not classifiable by countable structures. These things combined show that $\sim \leq_B \simeq^{\mathrm{vN}(\ell^2(\mathbb{N}))}$ and thus that the isomorphism relation of free Araki-Woods factors is not classifiable by countable structures. □

6. Questions and discussion

We finish the paper by discussing a few open problems related to the theme of this paper. Below, "descriptive set-theoretic complexity", or just "complexity", always refers to the notion of complexity that arises from the Borel reducibility pre-ordering, $\leq_B$. (Borel reducibility as a notion of relative complexity is discussed at length in the context of functional analysis in the introduction of [8].)

The results of this paper only give information about the descriptive set-theoretic complexity of classifying free Araki-Woods factors of orthogonal representations of multiplicity 1. The following is open.

Question 6.1. What is the descriptive set theoretic complexity of classifying free Araki-Woods factors not of multiplicity 1?

Consider the family of all separably acting free Araki-Woods factors. By the nature of their construction, they are born acting on different Hilbert spaces. Theorem 5.9 only allows to model as algebras of operators on a fixed Hilbert space the free Araki-Woods factors constructed from measures in $\mathcal{X}$, the set of regular non atomic symmetric measures supported in the interval $[-1, 1]$. These considerations lead to the next two questions.

Question 6.2. Does the family of free Araki-Woods factors admit a (natural) standard Borel space structure? In particular, is the set of factors in $\mathrm{vN}(\ell^2(\mathbb{N}))$ that are isomorphic to free Araki-Woods factors a Borel set?

Question 6.3. Is it possible to remove the restrictions on the measure, and on the multiplicity function to model in a Borel way all free Araki-Woods factors as elements in $\mathrm{vN}(\ell^2(\mathbb{N}))$?

A positive answer to Questions 6.2 and 6.3 combined with [19, Theorem 16] would show that isomorphism of free Araki-Woods factors is implemented by the action of the unitary group of $\ell^2(\mathbb{N})$.

Question 6.4. Is isomorphism of free Araki-Woods factors Borel reducible to an orbit equivalence relation induced by a continuous (or, which is the same, Borel) action of a Polish group on a Polish space?

Observe that [19, Theorem 16] obtains an upper bound on the complexity of the isomorphism relation on the set of factors in $\mathrm{vN}(\ell^2(\mathbb{N}))$, owing to that the isomorphism relation naturally reduces to an orbit equivalence relation induced by an action of the unitary group (of ℓ^2). This is to some extend analogous to the upper bound for the complexity of classifying separable C^*-algebras obtained in [5]. However, unlike the situation for C^*-algebras, where the upper bound was eventually shown by Sabok in [16] to also be the lower bound (hence giving a complete determination of the complexity), no such theorem has been achieved yet for separably acting von Neumann algebras and von Neumann factors. That is, the following question remains wide open:

Question 6.5. Is the classification problem for separably acting von Neumann factors the maximal (in terms of complexity) orbit equivalence relation induced by an action of the unitary group?

We don't even know the answer to this in the special case of II_1 factors. The second author conjectures that the answer is "yes", even for separably acting II_1 factors.

There are most likely two major obstacles standing in the way of answering the previous question: One is to develop sufficiently sophisticated techniques in von Neumann algebras to construct and distinguish a sufficiently vast class of non-isomorphic separably acting factors.

The other obstacle is more descriptive set theoretic. We do not currently know very much about the complexity of orbit equivalence relations induced by the unitary group. The following open question exposes our ignorance well.

Question 6.6. Is there an orbit equivalence relation, induced by the unitary group acting continuously on a Polish space, which realizes the maximal possible complexity that the an orbit equivalence relation of a Polish group can have?

Note that there *is* an orbit equivalence relation, induced by a continuous action of a Polish group, of maximal complexity, and this was proved in [9]. (The isometry group of the Urysohn metric space is an example of a Polish group which admits such an action.)

Perhaps it is natural to think that the answer to Question 6.6 should be 'yes'. However, the world might be far more interesting if the answer is no. For one thing, it would also mean that the complexity of classifying separably acting factors would be strictly lower than the complexity of classifying nuclear, simple, separable C*-algebras. It might also mean that the unitary group has its own turbulence theory that somehow generalizes Hjorth's S_∞-centric theory.

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