# INDEFINITE LEAST-SQUARES PROBLEMS AND PSEUDO-REGULARITY

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Abstract. Given two Krein spaces $\mathcal{H}$ and $\mathcal{K}$, a (bounded) closed-range operator $C : \mathcal{H} \to \mathcal{K}$ and a vector $y \in \mathcal{K}$, the indefinite least-squares problem consists in finding those vectors $u \in \mathcal{H}$ such that

$$[\,Cu - y, Cu - y\,] = \min_{x \in \mathcal{H}} [\,Cx - y, Cx - y\,].$$

The indefinite least-squares problem has been thoroughly studied before with the assumption that the range of $C$ is a uniformly $J$-positive subspace of $\mathcal{K}$. Along this article the range of $C$ is only supposed to be a $J$-nonnegative pseudo-regular subspace of $\mathcal{K}$.

This work is devoted to present a description for the set of solutions of this abstract problem in terms of the family of $J$-normal projections onto the range of $C$.

## 1. Introduction

In signal processing applications it is frequently assumed that the mathematical model, describing the physical phenomena under study, satisfies the following equation:

$$z = Hx + \eta,$$

where $H \in \mathbb{R}^{m\times n}$ is known and $x \in \mathbb{R}^n$ is a parameter that needs to be determined. Sometimes, due to physical restrictions, it is not possible to measure $x$, and it is necessary to estimate this vector based on the measurement $z$. But $z$ is corrupted by noise $\eta$. According to the characteristics of the noise, different techniques may be used to estimate $x$. For instance, when no statistical information about the noise measurement is available, the $\mathcal{H}^\infty$-estimation technique has been proved to be an appropriate approach for several engineering problems. Given $\gamma > 0$, the $\mathcal{H}^\infty$-estimation technique in $\mathbb{R}^n$ consists in finding an estimation $\hat{x}$ of the vector $x$, such that:

$$\max_{x\in\mathbb{R}^n} \frac{\|x-\hat{x}\|^2}{\|z-Hx\|^2} \leq \gamma^2, \tag{1.1}$$

or equivalently,

$$\min_{x\in\mathbb{R}^n} \left( \|z-Hx\|^2 - \frac{1}{\gamma^2}\|x-\hat{x}\|^2 \right) \geq 0. \tag{1.2}$$

Note that (1.2) can be modeled as the minimization of an indefinite inner product on an affine manifold. In fact, $\mathbb{R}^{m+n}$ can be endowed with the indefinite inner product $[\,x, y\,] := x^T Jy$, $x, y \in \mathbb{R}^{m+n}$, where $J \in L(\mathbb{R}^{m+n})$ is the fundamental symmetry given by $J = \begin{pmatrix} I_m & 0 \\ 0 & -I_n \end{pmatrix}$. Then, considering $C := \begin{pmatrix} H \\ \gamma^{-1} I_n \end{pmatrix} \in L(\mathbb{R}^n, \mathbb{R}^{m+n})$ and $y := \begin{pmatrix} z \\ \gamma^{-1}\hat{x} \end{pmatrix} \in \mathbb{R}^{m+n}$, the $\mathcal{H}^\infty$-estimation problem is equivalent to finding

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a vector $y$ (which depends on $z$) such that the following indefinite least-squares problem (ILSP) admits a solution:

$$\min_{x\in\mathbb{R}^n} [\, y - Cx, y - Cx\,], \tag{1.3}$$

and to show that this minimum is nonnegative, see [8].

This work is devoted to study an abstract ILSP: Given arbitrary Krein spaces $\mathcal{H}$ and $\mathcal{K}$, a closed-range operator $C \in L(\mathcal{H},\mathcal{K})$ and a vector $y \in \mathcal{K}$, find the vectors $u \in \mathcal{H}$ such that

$$[\, y - Cu, y - Cu\,] = \min_{x\in\mathcal{H}} [\, y - Cx, y - Cx\,].$$

In finite-dimensional spaces, the ILSP has been exhaustively studied see e.g. [13, 14, 22, 8, 15, 20, 7]. In these papers, if $J$ is the fundamental symmetry of $\mathcal{K}$, it is assumed that $C^TJC$ is a positive-definite matrix, which is a sufficient condition for the existence of a unique solution for the ILSP. This is equivalent to assuming that $C$ is injective and the range of $C$ (hereafter denoted by $R(C)$) is a uniformly $J$-positive subspace of $\mathcal{K}$. Then, the regularity of $R(C)$ plays an essential role, since it guarantees the existence of a $J$-selfadjoint projection onto $R(C)$, which determines the unique solution of the ILS problem (1.3).

Even for the general setting it is known that the ILSP admits a solution if and only if $R(C)$ is $J$-nonnegative and $y \in R(C) + R(C)^{[\perp]}$, see e.g. [6, Thm. 8.4]. Then, the ILSP is well-posed only for the vectors $y$ in the (not necessarily closed) subspace $R(C)+R(C)^{[\perp]}$. Moreover, given $y \in R(C)+R(C)^{[\perp]}$, $u \in \mathcal{H}$ is a solution of the ILSP if and only if $y - Cu \in R(C)^{[\perp]}$ (see Lemma 3.1), i.e. if $u$ is a solution of the normal equation associated to $Cx = y$:

$$C^{\#}(Cx - y) = 0,$$

where $C^{\#}$ stands for the $J$-adjoint operator of $C$.

The assumption that $R(C)$ is a uniformly $J$-positive subspace of $\mathcal{K}$ implies that the ILSP is properly defined for every $y \in \mathcal{K}$, but this is a quite restrictive condition. Along this article (most of the time) it is assumed that $R(C)$ is a $J$-nonnegative pseudo-regular subspace of $\mathcal{K}$. Thus, the ILSP admits solutions for every vector in the (proper) closed subspace $R(C) + R(C)^{[\perp]}$. The pseudo-regularity of $R(C)$ is equivalent to the closedness of $R(C^{\#}C)$, see Lemma 3.4. Hence, under this assumption, the Moore-Penrose inverse $(C^{\#}C)^{\dagger}$ of $C^{\#}C$ is bounded and the solutions of the normal equation, and therefore of the ILSP, are exactly those

$$u \in u_y + N(C^{\#}C),$$

where $u_y = (C^{\#}C)^{\dagger}C^{\#}y$ is the unique solution in $N(C^{\#}C)^{\perp}$.

It is also worthy to mention that if $\kappa := \min\{\dim\mathcal{K}_+, \dim\mathcal{K}_-\} < \infty$ for a fundamental decomposition $\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-$ (i.e. $\mathcal{K}$ is a Pontryagin space) then every closed subspace turns out to be pseudo-regular. Therefore, in this case the assumption reduces to assume that $R(C)$ is just $J$-nonnegative.

Another advantage of considering an operator $C$ with pseudo-regular range is that there is a family of $J$-normal projections onto $R(C)$. These projections, which have been previously studied in [19], are the main technical tool used along this work in order to characterize the set of solutions of the ILSP.

The article is organized as follows: Section 2 introduces the notation and terminology used along. It also contains some preliminaries on Krein spaces, mainly on pseudo-regularity and $J$-normal projections.

The indefinite least-squares problem is described in Section 3. After a brief reminder of the state of the art of the problem, it is studied under the assumption that the range of $C$ is a $J$-nonnegative pseudo-regular subspace of $\mathcal{K}$. Also, some considerations are made in order to compare the ILSP associated to $Cx = y$ and the ILSP associated to another equation $C'x = y$, where $C'$ is a closed-range operator such that $R(C')$ is a uniformly $J$-positive subspace of $R(C)$.

Until this point the Krein space structure of $\mathcal{H}$, the domain of $C$, was unnecessary. However, Section 4 is devoted to consider a minimization problem among the indefinite least-squares solutions of $Cx = y$. A minimal least-squares solution (MILSS) of $Cx = y$ is a vector $w \in u_y + N(C^\# C)$ such that

$$[\,w, w\,] = \min_{u \in u_y + N(C^\# C)} [\,u, u\,].$$

If the ILSP associated to $Cx = y$ admits solutions, in order to guarantee the existence of a MILSS of $Cx = y$ it is necessary and sufficient that $N(C^\# C)$ is $J$-nonnegative and that the affine manifold $u_y + N(C^\# C)$ intersects $N(C^\# C)^{[\perp]}$, see Proposition 4.1. If it is also assumed that $N(C^\# C)$ and $R(C)$ are pseudo-regular subspaces of $\mathcal{H}$ and $\mathcal{K}$, respectively, then the set of MILSS can be computed in terms of the $J$-normal projections onto these subspaces and the Moore-Penrose inverse of $C$, see Theorem 4.3.

Finally, in Section 5 the operators used in Theorem 4.3 to describe the MILSS of $Cx = y$ are shown to be a family of generalized inverses of a fixed operator $C'$ with regular range.

## 2. Preliminaries

Along this work $\mathcal{H}$ denotes a complex (separable) Hilbert space. If $\mathcal{K}$ is another Hilbert space then $L(\mathcal{H}, \mathcal{K})$ is the algebra of bounded linear operators from $\mathcal{H}$ into $\mathcal{K}$ and $L(\mathcal{H}) = L(\mathcal{H}, \mathcal{H})$.

If $T \in L(\mathcal{H}, \mathcal{K})$ then $R(T)$ stands for its range and $N(T)$ for its nullspace.

Given two closed subspaces $\mathcal{S}$ and $\mathcal{T}$ of a Hilbert space $\mathcal{H}$ , $\mathcal{S} \dot{+} \mathcal{T}$ denotes the direct sum of them. Moreover, $\mathcal{S} \oplus \mathcal{T}$ stands for their (direct) orthogonal sum and $\mathcal{S} \ominus \mathcal{T} := \mathcal{S} \cap (\mathcal{S} \cap \mathcal{T})^\perp$.

If $\mathcal{H} = \mathcal{S} \dot{+} \mathcal{T}$, $P_{\mathcal{S}//\mathcal{T}}$ denotes the (unique, bounded) projection onto $\mathcal{S}$ along $\mathcal{T}$. In the particular case of $\mathcal{T} = \mathcal{S}^\perp$, the *orthogonal projection onto* $\mathcal{S}$ is denoted by $P_{\mathcal{S}}$.

In what follows we present the standard notation and some basic results on Krein spaces. For a complete exposition on the subject see [6, 2, 1].

Given a Krein space $(\mathcal{H}, [\ ,\ ])$ with a *fundamental decomposition* $\mathcal{H} = \mathcal{H}_+ \dot{+} \mathcal{H}_-$, the direct (orthogonal) sum of the Hilbert spaces $(\mathcal{H}_+, [\ ,\ ])$ and $(\mathcal{H}_-, -[\ ,\ ])$ is denoted by $(\mathcal{H}, \langle\ ,\ \rangle)$.

Observe that the inner products of $\mathcal{H}$ are related by means of a *fundamental symmetry*, i.e. a unitary selfadjoint operator $J \in L(\mathcal{H})$ which satisfies:

$$[\,x, y\,] = \langle\, Jx, y\,\rangle\,, \qquad x, y \in \mathcal{H}.$$

If $\mathcal{H}$ and $\mathcal{K}$ are Krein spaces, $L(\mathcal{H}, \mathcal{K})$ stands for the vector space of linear transformations which are bounded with respect to the associated Hilbert spaces $(\mathcal{H}, \langle\ ,\ \rangle_{\mathcal{H}})$ and $(\mathcal{K}, \langle\ ,\ \rangle_{\mathcal{K}})$. Given $T \in L(\mathcal{H}, \mathcal{K})$, the $J$-adjoint operator of $T$ is defined by $T^\# = J_{\mathcal{H}} T^* J_{\mathcal{K}}$, where $J_{\mathcal{H}}$ and $J_{\mathcal{K}}$ are the fundamental symmetries associated to $\mathcal{H}$ and $\mathcal{K}$, respectively. An operator $T \in L(\mathcal{H})$ is $J$-selfadjoint if $T = T^\#$.

A vector $x \in \mathcal{H}$ is *$J$-positive* if $[\,x, x\,] > 0$. A subspace $\mathcal{S}$ of $\mathcal{H}$ is *$J$-positive* if every $x \in \mathcal{S}$, $x \neq 0$, is a $J$-positive vector. $J$-nonnegative, $J$-neutral, $J$-negative and $J$-nonpositive vectors and subspaces are defined analogously.

Given a subspace $\mathcal{S}$ of a Krein space $\mathcal{H}$, the *$J$-orthogonal subspace* to $\mathcal{S}$ is defined by

$$\mathcal{S}^{[\perp]} = \{x \in \mathcal{H} : [\,x, s\,] = 0, \text{ for every } s \in \mathcal{S}\}.$$

The isotropic part of $\mathcal{S}$, $\mathcal{S}^\circ := \mathcal{S} \cap \mathcal{S}^{[\perp]}$ can be a non-trivial subspace. It holds that

$$\mathcal{H} = \overline{\mathcal{S} + \mathcal{S}^{[\perp]}} \oplus J(\mathcal{S}^\circ),$$

see [2, Prop. 1.7.6]. A subspace $\mathcal{S}$ of $\mathcal{H}$ is *$J$-non-degenerated* if $\mathcal{S} \cap \mathcal{S}^{[\perp]} = \{0\}$. Otherwise, it is a $J$-degenerated subspace of $\mathcal{H}$.

A (closed) subspace $\mathcal{S}$ of $\mathcal{H}$ is *regular* if $\mathcal{S} \dotplus \mathcal{S}^{[\perp]} = \mathcal{H}$. Equivalently, $\mathcal{S}$ is regular if and only if there exists a (unique) $J$-selfadjoint projection $E$ onto $\mathcal{S}$, see e.g. [2, Thm. 1.7.16].

On the other hand, a closed subspace $\mathcal{S}$ of $\mathcal{H}$ is called *pseudo-regular* if the algebraic sum $\mathcal{S} + \mathcal{S}^{[\perp]}$ is closed. Equivalently, $\mathcal{S}$ is pseudo-regular if there exists a regular subspace $\mathcal{M}$ such that $\mathcal{S} = \mathcal{S}^\circ [\dotplus] \mathcal{M}$, where $[\dotplus]$ stands for the $J$-orthogonal direct sum of the subspaces, see [9].

The importance of pseudo-regular subspaces lies in the fact that they enable to generalize some Pontryagin spaces arguments to general Krein spaces. They have also been used as a technical tool for the study of spectral functions (and distributions) for particular classes of operators in Krein spaces [10, 11, 17, 18, 21] and to extend the Beurling-Lax theorem for shifts in indefinite metric spaces [3, 4].

Also, $\mathcal{S}$ is pseudo-regular if and only if $\mathcal{S}$ is the range of a $J$-normal projection, i.e. if there exists a projection $Q \in L(\mathcal{H})$ with $R(Q) = \mathcal{S}$ such that $QQ^\# = Q^\# Q$, see [19, Thm. 4.3]. In particular, given a pseudo-regular subspace $\mathcal{S}$, $Q_0 = P_{\mathcal{S}//\mathcal{S}^{[\perp]} \ominus \mathcal{S}^\circ + J(\mathcal{S}^\circ)}$ is a $J$-normal projection onto $\mathcal{S}$. However, if $\mathcal{S}^\circ \neq \{0\}$ then there are infinitely many $J$-normal projections $Q$ satisfying $R(Q) = \mathcal{S}$. In what follows, $\mathcal{Q}_\mathcal{S}$ stands for the set of $J$-normal projections onto the pseudo-regular subspace $\mathcal{S}$, i.e.

$$\mathcal{Q}_\mathcal{S} = \{Q \in L(\mathcal{H}) : \ Q^2 = Q, \ QQ^\# = Q^\# Q \text{ and } R(P) = \mathcal{S} \ \}.$$

The next remark is a technical tool that is going to be frequently used along this work. It shows that, given a vector $y \in \mathcal{S} + \mathcal{S}^{[\perp]}$, the $J$-normal projections onto $\mathcal{S}$ provide the different decompositions of $y$ as a sum of a vector in $\mathcal{S}$ and a vector in $\mathcal{S}^{[\perp]}$, i.e. if $Q \in \mathcal{Q}_\mathcal{S}$ then

$$y = Qy + (I - Q)y, \quad \text{where} \quad Qy \in \mathcal{S} \quad \text{and} \quad (I - Q)y \in \mathcal{S}^{[\perp]}.$$

**Remark 2.1.** If $\mathcal{S}$ is a pseudo-regular subspace of $\mathcal{H}$ and $y \in \mathcal{S} + \mathcal{S}^{[\perp]}$, given any $Q \in \mathcal{Q}_\mathcal{S}$, then

$$Q^\#(I - Q)y = 0.$$

Indeed, if $P = Q(I - Q)^\#$ then $R(P) = \mathcal{S} \cap N(Q^\#) = \mathcal{S} \cap \mathcal{S}^{[\perp]} = \mathcal{S}^\circ$ and $N(P^\#) = R(P)^{[\perp]} = (\mathcal{S}^\circ)^{[\perp]} = \mathcal{S} + \mathcal{S}^{[\perp]}$. Therefore, if $y \in \mathcal{S} + \mathcal{S}^{[\perp]}$ then $Q^\#(I - Q)y = P^\# y = 0$. In particular, $(I - Q)y \in N(Q^\#) = R(Q)^{[\perp]} = \mathcal{S}^{[\perp]}$.

The following results belong to [19]. Their statements are included in order to make the paper self-contained.

**Proposition 2.2.** *A bounded projection $Q$ acting on $\mathcal{H}$ is $J$-normal if and only if there exist a $J$-selfadjoint projection $E \in L(\mathcal{H})$ and a projection $P \in L(\mathcal{H})$ satisfying $PP^\# = P^\# P = 0$ such that*

$$Q = E + P.$$

*The projections $E$ and $P$ are uniquely determined by $Q$. More precisely, $E = QQ^\#$ and $P = Q(I - Q^\#)$.*

Projections $P \in L(\mathcal{H})$ satisfying $PP^\# = P^\# P = 0$ were previously considered in [17, 11], in connection with neutral dual companions. If $\mathcal{S}$ is a fixed (closed) $J$-neutral subspace of $\mathcal{H}$, a *neutral dual companion of* $\mathcal{S}$ is another (closed) $J$-neutral subspace $\mathcal{T}$ of $\mathcal{H}$ such that $\mathcal{H} = \mathcal{S} \dotplus \mathcal{T}^{[\perp]}$ holds. If $\mathcal{T}$ is a neutral dual companion of

$\mathcal{S}$ then also $\mathcal{H} = \mathcal{T} \dotplus \mathcal{S}^{[\perp]}$ holds. So, the pair of subspaces $(\mathcal{S}, \mathcal{T})$ is called a *neutral dual pair*. Note that in this case $\mathcal{S} \dotplus \mathcal{T}$ is a regular subspace of $\mathcal{H}$.

A $J$-neutral subspace $\mathcal{N}$ of $\mathcal{H}$ is said to be a *hypermaximal $J$-neutral* subspace if it is simultaneously both maximal $J$-nonnegative and maximal $J$-nonpositive. Equivalently, $\mathcal{N}$ is a hypermaximal $J$-neutral subspace if and only if $\mathcal{N} = \mathcal{N}^{[\perp]}$, see [2, Prop. 1.4.19].

Given $C \in L(\mathcal{H}, \mathcal{K})$, its restriction $C|_{N(C)^\perp} : N(C)^\perp \to R(C)$ admits a linear inverse $(C|_{N(C)^\perp})^{-1} : R(C) \to N(C)^\perp$. Then, the Moore-Penrose inverse of $C$ is the linear operator $C^\dagger : R(C) + R(C)^\perp \to \mathcal{H}$ defined by

$$C^\dagger y = \begin{cases} (C|_{N(C)^\perp})^{-1} y & \text{if } y \in R(C); \\ 0 & \text{if } y \in R(C)^\perp. \end{cases}$$

Note that $C^\dagger$ is densely-defined on $\mathcal{K}$, and it is well-known that $C^\dagger \in L(\mathcal{K}, \mathcal{H})$ if and only if $R(C)$ is closed.

Hereafter, given two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, let $CR(\mathcal{H}, \mathcal{K})$ denotes the set of bounded closed-range operators from $\mathcal{H}$ into $\mathcal{K}$. The following are some properties of the Moore-Penrose inverse of a closed-range operator:

**Proposition 2.3.** *Given $C \in CR(\mathcal{H}, \mathcal{K})$,*

1. *$CC^\dagger = P_{R(C)}$ and $C^\dagger C = P_{N(C)^\perp}$, the orthogonal projections onto $R(C)$ and $N(C)^\perp$, respectively. In particular, $CC^\dagger C = C$ and $C^\dagger CC^\dagger = C^\dagger$.*
2. *$C^* \in CR(\mathcal{K}, \mathcal{H})$ and $(C^*)^\dagger = (C^\dagger)^*$.*
3. *If $U \in L(\mathcal{K}), V \in L(\mathcal{H})$ are unitary operators, then $(UCV)^\dagger = V^* C^\dagger U^*$.*

The Moore-Penrose inverse has been thoroughly studied along the years, see e.g. [5] for a complete exposition on this subject.

As a consequence of Proposition 2.3, if $\mathcal{H}$ and $\mathcal{K}$ are two Krein spaces and $C \in CR(\mathcal{H}, \mathcal{K})$ then $C^\# \in CR(\mathcal{K}, \mathcal{H})$ and $(C^\#)^\dagger = (C^\dagger)^\#$.

## 3. Indefinite least-squares problems

Along this work, the following indefinite least-squares problem is considered: Let $\mathcal{H}$ and $\mathcal{K}$ be two Krein spaces with fundamental symmetries $J_\mathcal{H}$ and $J_\mathcal{K}$, respectively. Given an operator $C \in CR(\mathcal{H}, \mathcal{K})$ and a vector $y \in \mathcal{K}$, find $u \in \mathcal{H}$ such that

$$[\, y - Cu, y - Cu \,]_\mathcal{K} = \min_{x \in \mathcal{K}} [\, y - Cx, y - Cx \,]_\mathcal{K}. \tag{3.1}$$

The next lemma shows necessary and sufficient conditions for the existence of indefinite least-squares solutions (ILSS) of the equation $Cx = y$. A proof can be found in [6, Theorem 8.4] or in [12, Lemma 3.1].

**Lemma 3.1.** *Let $C \in CR(\mathcal{H}, \mathcal{K})$ and $y \in \mathcal{K}$. Then, $u \in \mathcal{H}$ is an ILSS of the equation $Cx = y$ if and only if $R(C)$ is $J_\mathcal{K}$-nonnegative and $y - Cu \in R(C)^{[\perp]}$.*

Hence, in order to have a well-posed indefinite least-squares problem it is necessary that $y \in R(C) + R(C)^{[\perp]}$. Note that the set of admissible points $R(C) + R(C)^{[\perp]}$ is always dense in $(R(C)^\circ)^{[\perp]}$.

**Proposition 3.2.** *Let $C \in CR(\mathcal{H}, \mathcal{K})$. Then, $Cx = y$ admits an ILSS for every $y \in (R(C)^\circ)^{[\perp]}$ if and only if $R(C)$ is a $J_\mathcal{K}$-nonnegative pseudo-regular subspace of $\mathcal{K}$.*

**Proof.** Note that $Cx = y$ admits an ILSS for every $y \in (R(C)^\circ)^{[\perp]}$ if and only if $(R(C)^\circ)^{[\perp]} \subseteq R(C) + R(C)^{[\perp]}$ and $R(C)$ is $J_\mathcal{K}$-nonnegative. But

$$(R(C)^\circ)^{[\perp]} = \overline{R(C) + R(C)^{[\perp]}},$$

and the equivalence follows. □

In particular, $Cx = y$ admits an ILSS for every $y \in \mathcal{K}$ if and only if $R(C)$ is a uniformly $J$-positive subspace of $\mathcal{K}$, see also [12, Proposition 3.2].

Before describing the indefinite least-squares solutions of $Cx = y$, observe that the minimum value of $L(x) = [\, y - Cx, y - Cx\,]$, $x \in \mathcal{H}$, is attained at the projections (by means of normal projectors) of $y$ onto $R(C)$.

**Lemma 3.3.** *Given $C \in CR(\mathcal{H}, \mathcal{K})$ such that $R(C)$ is a $J$-nonnegative pseudo-regular subspace of $\mathcal{K}$ and $y \in R(C) + R(C)^{[\perp]}$,*

$$\min_{x \in \mathcal{K}} [\, y - Cx, y - Cx\,] = [\,(I - Q)y, (I - Q)y\,],$$

*where $Q \in L(\mathcal{K})$ is any $J$-normal projection onto $R(C)$.*

**Proof.** Since $R(C)$ is pseudo-regular, by [19, Thm. 4.3] there exists a $J$-normal projection $Q \in L(\mathcal{K})$ onto $R(C)$. Then, for any $x \in \mathcal{H}$,

$$\begin{aligned} [\, y - Cx, y - Cx\,] &= [\,(y - Qy) + (Qy - Cx), (y - Qy) + (Qy - Cx)\,] \\ &= [\,(I - Q)y, (I - Q)y\,] + 2\,\mathrm{Re}[\,(I - Q)y, Qy - Cx\,] + [\,Qy - Cx, Qy - Cx\,] \\ &\geq [\,(I - Q)y, (I - Q)y\,] + 2\,\mathrm{Re}[\,(I - Q)y, Qy - Cx\,], \end{aligned} \tag{3.2}$$

because $Qy - Cx \in R(C)$ which is a $J_{\mathcal{K}}$-nonnegative subspace. Furthermore, by Remark 2.1, $y \in R(C) + R(C)^{[\perp]}$ implies that $Q^{\#}(I - Q)y = 0$ and

$$[\,(I - Q)y, Qy - Cx\,] = [\,(I - Q)y, Q(y - Cx)\,] = [\,Q^{\#}(I - Q)y, y - Cx\,] = 0.$$

Therefore,

$$[\, y - Cx, y - Cx\,] \geq [\,(I - Q)y, (I - Q)y\,].$$

□

Also, note that the pseudo-regularity of $R(C)$ is equivalent to the boundedness of the Moore-Penrose inverse of $C^{\#}C$:

**Lemma 3.4.** *Given $C \in CR(\mathcal{H}, \mathcal{K})$, $R(C)$ is pseudo-regular if and only if $R(C^{\#}C)$ is closed.*

**Proof.** Since $R(C)$ is closed, note that $R(C^{\#}C)$ is closed if and only if $R(C) + N(C^{\#}) = R(C) + R(C)^{[\perp]}$ is closed, see [16, Corollary 2.5]. Thus, $R(C^{\#}C)$ is closed if and only if $R(C)$ is a pseudo-regular subspace of $\mathcal{K}$. □

Given $C \in CR(\mathcal{H}, \mathcal{K})$ and $y \in R(C) + R(C)^{[\perp]}$, observe that $C^{\#}y \in R(C^{\#}C)$. Then,

$$u_y := (C^{\#}C)^{\dagger} C^{\#} y, \tag{3.3}$$

is a solution of the *normal equation*:

$$C^{\#}(Cx - y) = 0. \tag{3.4}$$

In particular, $u_y$ is the unique solution of the normal equation in $N(C^{\#}C)^{\perp}$ and the set of solutions of (3.4) is the affine manifold

$$u_y + N(C^{\#}C).$$

The following is the main result of this section. It shows that the solutions of the ILSP associated to the equation $Cx = y$ are the solutions of the normal equation $C^{\#}(Cx - y) = 0$, but it also characterizes them in terms of the $J$-normal projections onto $R(C)$.

**Theorem 3.5.** *Given $C \in CR(\mathcal{H}, \mathcal{K})$, if $R(C)$ is a $J$-nonnegative pseudo-regular subspace of $\mathcal{K}$ and $y \in R(C) + R(C)^{[\perp]}$, the following conditions are equivalent:*

*(1) $u \in \mathcal{H}$ is an ILSS of $Cx = y$;*

*(2) $u \in \mathcal{H}$ is a solution of the normal equation $C^{\#}(Cx - y) = 0$;*

*(3) $Cu - Qy \in R(C)^\circ$ for any $J$-normal projection $Q$ onto $R(C)$.*

*If $y \notin R(C)$ the above conditions are also equivalent to:*

*4. there exists a $J$-normal projection $Q$ onto $R(C)$ such that $Cu = Qy$.*

*Moreover, the set of ILSS of $Cx = y$ coincides with the affine manifold*

$$u_y + N(C^\# C),$$

*where $u_y = (C^\# C)^\dagger C^\# y$.*

**Proof.** By Lemma 3.1, assuming the $J$-nonnegativity of $R(C)$, $u$ is an ILSS of $Cx = y$ if and only if $y - Cu \in R(C)^{[\perp]} = N(C^\#)$. Then, the equivalence *1.* $\leftrightarrow$ *2.* follows.

*2.* $\leftrightarrow$ *3.* : By Remark 2.1, $(I - Q)y \in R(C)^{[\perp]} = N(C^\#)$ for any $J$-normal projection $Q \in L(\mathcal{K})$ onto $R(C)$. Hence, $u \in \mathcal{H}$ is a solution of $C^\#(Cx - y) = 0$ if and only if $C^\#(Cu - Qy) = 0$, or equivalently, $Cu - Qy \in R(C)^\circ$.

*2.* $\leftrightarrow$ *4.* : Assume that $y \notin R(C)$ and $u$ is a solution of $C^\#(Cx - y) = 0$. Then, $y = Cu + z$ with $z \in R(C)^{[\perp]} \setminus R(C)$. So, there exists a regular subspace $\mathcal{T}$ of $R(C)^{[\perp]}$ such that $z \in \mathcal{T}$ and $R(C)^{[\perp]} = \mathcal{T}[\dot{+}]R(C)^\circ$. Also, consider a regular subspace $\mathcal{M}$ of $R(C)$ such that $R(C) = \mathcal{M}[\dot{+}]R(C)^\circ$. Then, note that $R(C)^\circ$ is a $J$-neutral subspace of the Krein space $\mathcal{K}' = (\mathcal{M} + \mathcal{T})^{[\perp]}$. So, it is well-known that there exists a neutral dual companion $\mathcal{N}$ of $R(C)^\circ$ in $\mathcal{K}'$, see [11]. Furthermore, $R(C)^\circ$ is a hypermaximal neutral subspace of $\mathcal{K}'$ [2, Prop. 1.4.19] because

$$\begin{aligned}
(R(C)^\circ)^{[\perp]_{\mathcal{K}'}} &= (R(C)^\circ)^{[\perp]} \cap \mathcal{K}' \\
&= (R(C) + R(C)^{[\perp]}) \cap (\mathcal{M} + \mathcal{T})^{[\perp]} = \\
&= (\mathcal{M} \dot{+} \mathcal{T} \dot{+} R(C)^\circ) \cap (\mathcal{M} + \mathcal{T})^{[\perp]} = R(C)^\circ.
\end{aligned}$$

Thus, $(\mathcal{M} + \mathcal{T})^{[\perp]} = \mathcal{K}' = \mathcal{N} \dot{+} (R(C)^\circ)^{[\perp]_{\mathcal{K}'}} = \mathcal{N} \dot{+} R(C)^\circ$ and the following decomposition of $\mathcal{K}$ holds:

$$\mathcal{K} = \mathcal{M}[\dot{+}](R(C)^\circ \dot{+} \mathcal{N})[\dot{+}]\mathcal{T}.$$

Given the projection $Q = P_{R(C)//\mathcal{T}+\mathcal{N}} \in L(\mathcal{K})$, it is easy to see that $Q^\# = P_{\mathcal{M}+\mathcal{N}//R(C)^{[\perp]}}$. Therefore, $Q$ is $J$-normal and it satisfies $Qy = Q(Cu + z) = Cu$.

Conversely, if $Cu = Qy$ for some $J$-normal projection $Q \in L(\mathcal{K})$ onto $R(C)$ then, by Remark 2.1, $y - Cu = (I - Q)y \in R(C)^{[\perp]} = N(C^\#)$. Therefore, $C^\#(Cu - y) = 0$.

Finally, recall that the set of solutions of the normal equation (which in this case coincides with the ILSS of $Cx = y$) is the affine manifold $u_y + N(C^\# C)$, where $u_y = (C^\# C)^\dagger C^\# y$. □

**Remark 3.6.** Given $C \in CR(\mathcal{H}, \mathcal{K})$ with pseudo-regular range $R(C)$, the equivalences *2.* $\leftrightarrow$ *3.* $\leftrightarrow$ *4.* in Theorem 3.5 holds independently of the (semi)definiteness of the range. Hence, Theorem 3.5 also characterizes the solutions of the normal equation $C^\#(Cx - y) = 0$ for $C \in CR(\mathcal{H}, \mathcal{K})$ with an arbitrary pseudo-regular range $R(C)$.

If $C \in CR(\mathcal{H}, \mathcal{K})$ and $R(C)$ is pseudo-regular, the set $\mathcal{Q}_{R(C)}$ of $J$-normal projections onto $R(C)$ is related to a family of inner inverses of $C$, where $X \in L(\mathcal{K}, \mathcal{H})$ is an inner inverse of $C$ if $CXC = C$. Let $\mathcal{I}$ denote the set of solutions $D \in L(\mathcal{K}, \mathcal{H})$ of the equations

$$CXC = C, \quad (CX)^\# CX = CX(CX)^\#. \tag{3.5}$$

Then, $D \in \mathcal{I}$ if and only if there exist $Q \in \mathcal{Q}_{R(C)}$ and $T \in L(\mathcal{K}, \mathcal{H})$ with $R(T) \subseteq N(C)$ such that

$$D = C^\dagger Q + T.$$

Indeed, if $D \in L(\mathcal{K}, \mathcal{H})$ is a solution of (3.5) then $Q := CD \in \mathcal{Q}_{R(C)}$ and $C^\dagger Q = C^\dagger CD = P_{N(C)^\perp} D$. So, $T := P_{N(C)} D \in L(\mathcal{K}, \mathcal{H})$ satisfies $R(T) \subseteq N(C)$ and $D = C^\dagger Q + T$.

Conversely, given $Q \in \mathcal{Q}_{R(C)}$ and $T \in L(\mathcal{K}, \mathcal{H})$ with $R(T) \subseteq N(C)$, consider $D := C^\dagger Q + T$. Then, $CD = CC^\dagger Q = P_{R(C)} Q = Q$ implies that $D$ is a solution of (3.5).

The following result describes the solutions of the ILSP associated to $Cx = y$ in terms of these generalized inverses.

**Proposition 3.7.** *Given $C \in CR(\mathcal{H}, \mathcal{K})$, if $R(C)$ is a $J$-nonnegative pseudo-regular subspace of $\mathcal{K}$ and $y \in R(C) + R(C)^{[\perp]}$, the following conditions are equivalent:*

*(1) $u \in \mathcal{H}$ is an ILSS of $Cx = y$;*

*(2) $Dy - u \in N(C^\# C)$ for any solution $D \in L(\mathcal{K}, \mathcal{H})$ of* (3.5).

*If $y \notin R(C)$ the above conditions are also equivalent to:*

*3. there exists a solution of* (3.5) *such that $Dy = u$.*

**Proof.** *1.* $\leftrightarrow$ *2.* : Given a solution $D \in L(\mathcal{K}, \mathcal{H})$ of (3.5), consider $Q \in \mathcal{Q}_{R(C)}$ and $T \in L(\mathcal{K}, \mathcal{H})$ with $R(T) \subseteq N(C)$ such that $D = C^\dagger Q + T$. For $u \in \mathcal{H}$, follows that $Dy - u \in N(C^\# C)$ if and only if $C^\#(Qy - Cu) = 0$, or equivalently, $Qy - Cu \in R(C)^\circ$. Thus the equivalence follows from Theorem 3.5.

*1.* $\leftrightarrow$ *3.* : Given $y \in (R(C) + R(C)^{[\perp]}) \setminus R(C)$, suppose that $u = Dy$ where $D \in L(\mathcal{K}, \mathcal{H})$ is a solution of (3.5). It is easy to see that $Q = CD$ is a $J$-normal projection with $R(Q) = R(C)$. Furthermore, $Cu = CDy = Qy$. By Theorem 3.5, this implies that $u$ is an ILSS of $Cx = y$.

Conversely, if $u \in \mathcal{H}$ is an ILSS of $Cx = y$, Theorem 3.5 states that $Cu = Qy$ for some $J$-normal projection $Q \in L(\mathcal{K})$. Then, $u = C^\dagger Qy + w$, where $w \in N(C)$. Consider $T \in L(\mathcal{K}, \mathcal{H})$ with $R(T) \subseteq N(C)$ such that $Ty = w$ and define $D = C^\dagger Q + T$. Thus, $D$ is a solution of (3.5) and $Dy = C^\dagger Qy + Ty = C^\dagger Qy + w = u$. $\square$

In the following it is shown that the ILSP associated to the equation $Cx = y$ can be rewritten as an ILSP associated to another equation $C'x = y$, where $C' \in CR(\mathcal{H}, \mathcal{K})$ and $R(C')$ is a uniformly $J$-positive subspace of $\mathcal{K}$. But this is only true if the vector $y \in \mathcal{K}$ is admissible for the ILSP associated to the equation $Cx = y$ (recall that the ILSP associated to the equation $C'x = y$ is always well-posed).

If $R(C)$ is a $J_\mathcal{K}$-nonnegative pseudo-regular subspace of $\mathcal{K}$ and $y \in R(C) + R(C)^{[\perp]}$, then

$$u \in \mathcal{H} \text{ is an ILSS of } Cx = y \quad \Leftrightarrow \quad u \in \mathcal{H} \text{ is an ILSS of } (EC)x = y,$$

where $E = QQ^\#$ and $Q$ is any $J$-normal projection onto $R(C)$.

First, observe that $R(EC) = E(R(C) + N(E)) = R(E)$ since $R(E) \subset R(C)$. Hence, $R(EC)$ is uniformly $J_\mathcal{K}$-positive and the indefinite least-squares problem associated to the equation $ECx = y$ is well-posed. Then, by Theorem 3.5, $u \in \mathcal{H}$ is an ILSS of $Cx = y$ if and only if $Cu - Qy \in R(C)^\circ$. But, $R(C)^\circ \subset N(E)$ implies that

$$ECu = E(Cu - Qy) + EQy = Ey,$$

and $E$ is the $J$-selfadjoint projection onto $R(EC)$. Then, $u$ is an ILSS of $ECx = y$, see e.g. [12, Prop. 3.2].

**Proposition 3.8.** *Let $C \in CR(\mathcal{H}, \mathcal{K})$ be such that $R(C)$ is a pseudo-regular subspace of $\mathcal{K}$. Then, $C' = CP_{N(C^\# C)^\perp} \in CR(\mathcal{H}, \mathcal{K})$ has regular range and, if $y \in R(C) + R(C)^{[\perp]}$,*

$$u \in \mathcal{H} \textit{ is an ILSS of } Cx = y \quad \Leftrightarrow \quad u \in \mathcal{H} \textit{ is an ILSS of } C'x = y.$$

**Proof.** Given $C \in CR(\mathcal{H}, \mathcal{K})$, consider the operator $E_0 := C(C^\# C)^\dagger C^\#$. By Lemma 3.4, $E_0 \in L(\mathcal{K})$ and it is easy to check that $E_0^2 = E_0$. As a consequence of Proposition 2.3 the projection $E_0$ is $J$-selfadjoint, and $R(E_0)$ is obviously contained in $R(C)$. Then, $R(E_0C) = E_0(R(C) + N(E_0)) = R(E_0)$ and note that

$$E_0 C = C(C^\# C)^\dagger C^\# C = C P_{N(C^\# C)^\perp} = C'.$$

Therefore, $R(C') = R(E_0C) = R(E_0)$ is regular.

Also, $R(E_0) \cap R(C)^\circ = \{0\}$ because $R(C)^\circ \subseteq R(C)^{[\perp]} = N(C^\#) \subseteq N(E_0)$. Since $C = C' + CP_{N(C^\# C)}$ and the range of $CP_{N(C^\# C)}$ coincides with $R(C)^\circ$, it follows that

$$R(C) = R(CP_{N(C^\# C)^\perp}) + R(C)^\circ = R(E_0C) + R(C)^\circ = R(E_0) \dot{+} R(C)^\circ.$$

Therefore, $E_0$ is a $J$-sefadjoint projection onto a regular complement of $R(C)^\circ$ in $R(C)$ and, by [19, Thm. 6.9] there exist (at least) a $J$-normal projection $Q \in L(\mathcal{K})$ such that $E_0 = QQ^\#$. Finally, if $y \in R(C) + R(C)^{[\perp]}$ the discussion above shows that the ILSS of $Cx = y$ and $C'x = y$ coincide. □

## 4. Minimizers among indefinite least-squares solutions

The following paragraphs are devoted to consider a minimization problem among the indefinite least-squares solutions of $Cx = y$, where $C \in CR(\mathcal{H}, \mathcal{K})$ and $y \in R(C) + R(C)^{[\perp]}$.

**Definition.** A vector $w \in \mathcal{H}$ is a minimal least-squares solution (hereafter MILSS) of $Cx = y$ if $w$ is an ILSS of $Cx = y$ and

$$[\,w, w\,]_{\mathcal{H}} \leq [\,u, u\,]_{\mathcal{H}}, \quad \text{for every ILSS } u \text{ of } Cx = y.$$

It follows from Theorem 3.5 that, if $R(C)$ is a pseudo-regular $J_{\mathcal{K}}$-nonnegative subspace of $\mathcal{K}$ and $y \in R(C) + R(C)^{[\perp]}$, the set of ILSS of $Cx = y$ coincides with

$$u_y + N(C^\# C),$$

where $u_y = (C^\# C)^\dagger C^\# y$. So, $w \in \mathcal{H}$ is a MILSS of $Cx = y$ if and only if

$$[\,w, w\,] = \min_{z \in N(C^\# C)} [\,u_y + z, u_y + z\,]. \tag{4.1}$$

Thus, if $P_{N(C^\# C)}$ is the orthogonal projection onto $N(C^\# C)$ and $w = u_y + z_w$ is the orthogonal decomposition of $w$ according to $\mathcal{H} = N(C^\# C)^\perp \oplus N(C^\# C)$, note that (4.1) can be rewritten as

$$\begin{aligned} [\,u_y + z_w, u_y + z_w\,] &= \min_{z \in N(C^\# C)} [\,u_y + z, u_y + z\,] \\ &= \min_{x \in \mathcal{H}} [\,u_y + P_{N(C^\# C)}x, u_y + P_{N(C^\# C)}x\,]. \end{aligned}$$

Hence, if $w = u_y + z_w \in u_y + N(C^\# C)$,

$$w \text{ is a MILSS of } Cx = y \quad \Leftrightarrow \quad z_w \text{ is an ILSS of } P_{N(C^\# C)}x = -u_y. \tag{4.2}$$

By Lemma 3.1, the existence of an ILSS of $P_{N(C^\# C)}x = -u_y$ is equivalent to

$$u_y \in N(C^\# C) + N(C^\# C)^{[\perp]},$$

and the $J_{\mathcal{H}}$-nonnegativity of $N(C^\# C)$. Therefore,

**Proposition 4.1.** *Let $C \in CR(\mathcal{H}, \mathcal{K})$ be such that $R(C)$ is a $J_{\mathcal{K}}$-nonnegative pseudo-regular subspace of $\mathcal{K}$ and consider $y \in R(C) + R(C)^{[\perp]}$. Then, there exists a MILSS $w \in \mathcal{H}$ of $Cx = y$ if and only if $N(C^\# C)$ is $J_{\mathcal{H}}$-nonnegative and $u_y \in N(C^\# C) + N(C^\# C)^{[\perp]}$. In this case, the set of MILSS of $Cx = y$ coincides with*

$$(u_y + N(C^\# C)) \cap N(C^\# C)^{[\perp]}.$$

**Proof.** The equivalence between the existence of a MILSS for $Cx = y$ and the conditions on $N(C^\# C)$ and $u_y$ follows from the discussion above. Also, note that $u_y \in N(C^\# C) + N(C^\# C)^{[\perp]}$ if and only if

$$(u_y + N(C^\# C)) \cap N(C^\# C)^{[\perp]} \neq \varnothing.$$

Now, assume that $w \in \mathcal{H}$ is a MILSS of $Cx = y$. Then, there exists $z_w \in N(C^\# C)$ such that $w = u_y + z_w$ and $z_w$ is an ILSS of $P_{N(C^\# C)}x = -u_y$. By Lemma 3.1, $-u_y - P_{N(C^\# C)}z_w \in N(C^\# C)^{[\perp]}$. So,

$$w = u_y + z_w = u_y + P_{N(C^\# C)}z_w \in (u_y + N(C^\# C)) \cap N(C^\# C)^{[\perp]}.$$

Conversely, suppose that $w \in (u_y + N(C^\# C)) \cap N(C^\# C)^{[\perp]}$. Then, $w$ is an ILSS of $Cx = y$ because $w \in u_y + N(C^\# C)$. Also, there exists $z_w \in N(C^\# C)$ such that $w = u_y + z_w$. Furthermore, since

$$-u_y - P_{N(C^\# C)}z_w = -u_y - z_w = -w \in N(C^\# C)^{[\perp]},$$

$z_w \in N(C^\# C)$ is an ILSS of $P_{N(C^\# C)}x = -u_y$. So, (4.2) implies that $w = u_y + z_w$ is a MILSS of $Cx = y$. □

In the rest of this section it is assumed that $N(C^\# C)$ is a $J_{\mathcal{H}}$-nonnegative pseudo-regular subspace of $\mathcal{H}$, aiming to describe the set of MILSS of $Cx = y$ in terms of $J$-normal projections.

Let $C \in CR(\mathcal{H}, \mathcal{K})$ be such that $R(C)$ is pseudo-regular and consider $y \in R(C) + R(C)^{[\perp]}$. Then, note that

$$u_y = (C^\# C)^\dagger C^\# y = 0 \quad \text{if and only if} \quad y \in R(C)^{[\perp]}.$$

In this case, $u \in \mathcal{H}$ is an ILSS of $Cx = y$ if and only if $u \in N(C^\# C)$. Moreover, by Proposition 4.1, $u \in \mathcal{H}$ is a MILSS of $Cx = y$ if and only if $u \in N(C^\# C)^\circ$.

**Lemma 4.2.** *Let $C \in CR(\mathcal{H}, \mathcal{K})$ be such that $R(C)$ is a $J_{\mathcal{K}}$-nonnegative pseudo-regular subspace of $\mathcal{K}$ and consider $y \in (R(C) + R(C)^{[\perp]}) \setminus R(C)^{[\perp]}$. Assume also that $N(C^\# C)$ is a $J_{\mathcal{H}}$-nonnegative pseudo-regular subspace of $\mathcal{H}$. Then, $w \in \mathcal{H}$ is a MILSS of $Cx = y$ if and only if there exists $P \in \mathcal{Q}_{N(C^\# C)}$ such that*

$$w = (I - P)u_y. \tag{4.3}$$

**Proof.** Given $C \in CR(\mathcal{H}, \mathcal{K})$ with $J_{\mathcal{K}}$-nonnegative pseudo-regular range $R(C)$, let $y \in (R(C) + R(C)^{[\perp]}) \setminus R(C)^{[\perp]}$. By the above remark, $u_y \neq 0$.

If $w \in \mathcal{H}$ is a MILSS of $Cx = y$, consider its orthogonal decomposition $w = u_y + z$, where $z \in N(C^\# C)$. Then, by (4.2), $z$ is an ILSS of the equation $P_{N(C^\# C)}x = -u_y$. Also $u_y \in N(C^\# C)^\perp$ and, by Theorem 3.5, there exists $P \in \mathcal{Q}_{N(C^\# C)}$ such that

$$z = P_{N(C^\# C)}z = P(-u_y) = -Pu_y.$$

Thus, $w = u_y + z = u_y - Pu_y = (I - P)u_y$ for some $P \in \mathcal{Q}_{N(C^\# C)}$.

Conversely, if $w = (I - P)u_y$ for some $P \in \mathcal{Q}_{N(C^\# C)}$ then, since $u_y \in N(C^\# C) + N(C^\# C)^{[\perp]}$,

$$w = (I - P)P^\# u_y + (I - P)(I - P)^\# u_y = (I - P)(I - P)^\# u_y,$$

because, by Proposition 4.1,

$$u_y \in R(P) + R(P)^{[\perp]} = R(P) + N(P^\#) = N((I - P)P^\#).$$

Then, $w \in N(C^\# C)^{[\perp]}$ and, by Proposition 4.1, $w$ is a MILSS of $Cx = y$. □

If $R(C)$ is a pseudo-regular subspace of $\mathcal{K}$, consider $E_0 = C(C^\#C)^\dagger C^\#$. If $y \in R(C) + R(C)^{[\perp]}$ then $Cu_y = E_0 y$ and

$$u_y = C^\dagger C u_y = C^\dagger E_0 y,$$

because $u_y \in N(C^\#C)^\perp \subseteq N(C)^\perp$. Moreover, if $y \in (R(C) + R(C)^{[\perp]}) \setminus R(C)^{[\perp]}$, applying this identity in (4.3) it follows that if $w \in \mathcal{H}$ is a MILSS of $Cx = y$ then there exists $P \in \mathcal{Q}_{N(C^\#C)}$ such that

$$w = (I - P)u_y = (I - P)C^\dagger E_0 y.$$

Furthermore, following the construction made in the proof of Theorem 3.5, it is easy to see that there exists $Q_0 \in \mathcal{Q}_{R(C)}$ such that $E_0 = Q_0^\# Q_0$. Hence, by Remark 2.1, $Q_0^\#(I - Q_0)y = 0$ and

$$w = (I - P)C^\dagger E_0 y = (I - P)C^\dagger Q_0^\# y.$$

**Theorem 4.3.** *Let $C \in CR(\mathcal{H}, \mathcal{K})$ such that $R(C)$ is a $J_{\mathcal{K}}$-nonnegative pseudo-regular subspace of $\mathcal{K}$ and consider $y \in (R(C) + R(C)^{[\perp]}) \setminus R(C)^{[\perp]}$. Assume also that $N(C^\#C)$ is a $J_{\mathcal{H}}$-nonnegative pseudo-regular subspace of $\mathcal{H}$. Then, $w \in \mathcal{H}$ is a MILSS of $Cx = y$ if and only if there exists $P \in \mathcal{Q}_{N(C^\#C)}$ and $Q \in \mathcal{Q}_{R(C)}$ such that*

$$w = (I - P)C^\dagger Q^\# y = (I - P)C^\dagger E y, \tag{4.4}$$

*where $E = QQ^\#$.*

**Proof.** Under these assumptions, there exists a MILSS of $Cx = y$. Furthermore, in the discussion above it was shown that, if $w \in \mathcal{H}$ is a MILSS of $Cx = y$ then there exists $P \in \mathcal{Q}_{N(C^\#C)}$ and $Q_0 \in \mathcal{Q}_{R(C)}$ such that

$$w = (I - P)u_y = (I - P)C^\dagger Q_0^\# y = (I - P)C^\dagger E_0 y.$$

Conversely, given $P \in \mathcal{Q}_{N(C^\#C)}$ and $Q \in \mathcal{Q}_{R(C)}$, consider the vector $w = (I - P)C^\dagger Q^\# y$. By Remark 2.1 it follows that $Q^\#(I - Q)y = 0$ and $x := C^\dagger Q^\# y = C^\dagger QQ^\# y$. Then, $Cx = P_{R(C)} QQ^\# y = Q^\# Q y$ and

$$Qy - Cx = Qy - QQ^\# y = Q(I - Q^\#)y \in R(C)^\circ.$$

So, by Theorem 3.5, $x \in u_y + N(C^\#C)$. Also, $w = (I - P)x = (I - P)u_y$ and, following the same arguments as in Lemma 4.2, $w \in N(C^\#C)^{[\perp]}$. Therefore, by Proposition 4.1, $w$ is a MILSS of $Cx = y$. □

In the description obtained for the MILSS of $Cx = y$ in the above theorem, the family of operators

$$\{(I - P)C^\dagger E : \ P \in \mathcal{Q}_{N(C^\#C)}\}$$

appears, where $E$ is the $J$-selfadjoint projection onto an arbitrary complement of $R(C)^\circ$ in $R(C)$. Along the next section, this family is related to some of the generalized inverses of $C' := EC$. Note that, under the assumptions of Theorem 4.3, $R(C') = R(E)$ is regular and $N(C') = N(C^\#C)$ is pseudo-regular.

## 5. Generalized inverses related to indefinite least-squares problems

The next result describes a family of generalized inverses of a closed-range operator with pseudo-regular range and nullspace.

**Proposition 5.1.** *Suppose that $C \in CR(\mathcal{H}, \mathcal{K})$ is such that $R(C)$ and $N(C)$ are pseudo-regular subspaces of $\mathcal{K}$ and $\mathcal{H}$, respectively. Then, $D \in L(\mathcal{K}, \mathcal{H})$ is a solution of*

$$\begin{cases} CXC = C, \\ XCX = X, \\ (CX)(CX)^{\#} = (CX)^{\#}(CX), \\ (XC)(XC)^{\#} = (XC)^{\#}(XC), \end{cases} \tag{5.1}$$

*if and only if there exist $Q \in \mathcal{Q}_{R(C)}$ and $P \in \mathcal{Q}_{N(C)}$ such that $D = (I-P)C^{\dagger}Q$.*

**Proof.** Given $Q \in \mathcal{Q}_{R(C)}$ and $P \in \mathcal{Q}_{N(C)}$, consider $D = (I-P)C^{\dagger}Q$. Since $CP = 0$,

$$CD = C(I-P)C^{\dagger}Q = CC^{\dagger}Q = P_{R(C)}Q = Q.$$

Also,

$$DC = (I-P)C^{\dagger}QC = (I-P)C^{\dagger}C = (I-P)P_{N(C)^{\perp}} = I - P,$$

because $R(P) = N(C)$. Therefore, $CD$ is a $J_{\mathcal{K}}$-normal projection and $DC$ is a $J_{\mathcal{H}}$-normal projection. Furthermore,

$$CDC = (CD)C = QC = C \quad \text{and} \quad DCD = (DC)D = (I-P)D = D.$$

Conversely, assume that $D \in L(\mathcal{K}, \mathcal{H})$ satisfies the equations in (5.1). Then, note that $Q := CD \in \mathcal{Q}_{R(C)}$, $P := I - DC \in \mathcal{Q}_{N(C)}$ and

$$(I-P)C^{\dagger}Q = (DC)C^{\dagger}(CD) = D(CC^{\dagger}C)D = DCD = D.$$

□

Let $E \in L(\mathcal{K})$ be a $J$-selfadjoint projection such that $R(E) \dotplus R(C)^{\circ} = R(C)$. Applying the above proposition to $C' = EC$ it is possible to reinterpret the operators of the form $(I-P)C^{\dagger}E$ (with $P \in \mathcal{Q}_{N(C^{\#}C)}$) as a particular family of generalized inverses of $C'$.

**Corollary 5.2.** *Suppose that $C \in CR(\mathcal{H}, \mathcal{K})$ is such that $R(C)$ and $N(C^{\#}C)$ are pseudo-regular subspaces of $\mathcal{K}$ and $\mathcal{H}$, respectively. Consider a $J$-selfadjoint projection $E \in L(\mathcal{K})$ such that $R(E) \dotplus R(C)^{\circ} = R(C)$. If $C' = EC$ then the operators in the set*

$$\{(I-P)C^{\dagger}E : \ P \in \mathcal{Q}_{N(C^{\#}C)}\},$$

*are the solutions in $L(\mathcal{K}, \mathcal{H})$ of*

$$\begin{cases} C'XC' &= C', \\ XC'X &= X, \\ C'X &= E, \\ (XC')(XC')^{\#} &= (XC')^{\#}(XC'). \end{cases} \tag{5.2}$$

**Proof.** Consider a $J$-selfadjoint projection $E \in L(\mathcal{K})$ such that $R(E) \dotplus R(C)^{\circ} = R(C)$. If $C' = EC$ note that

$$R(C') = R(E) \quad \text{and} \quad N(C') = N(C^{\#}C).$$

Then, apply Proposition 5.1 to $C'$. □

Thus, the statement of Theorem 4.3 can be rephrased as: $w \in \mathcal{H}$ is a MILSS of $Cx = y$ if and only if $u = Dy$ where $D \in L(\mathcal{K}, \mathcal{H})$ is a solution of Eq. (5.2).

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