Uncertainty principle and geometry of the infinite Grassmann manifold

Esteban Andruchow and Gustavo Corach

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Abstract

We study the pairs of projections

$$P_I f = \chi_I f, \quad Q_J f = \left(\chi_J \hat{f}\right)\check{}, \quad f \in L^2(\mathbb{R}^n),$$

where $I, J \subset \mathbb{R}^n$ are sets of finite positive Lebesgue measure, χ_I, χ_J denote the corresponding characteristic functions and $\hat{}, \check{}$ denote the Fourier-Plancherel transformation $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ and its inverse. These pairs of projections have been widely studied by several authors in connection with the mathematical formulation of Heisenberg's uncertainty principle. Our study is done from a differential geometric point of view. We apply known results on the Finsler geometry of the Grassmann manifold $\mathcal{P}(\mathcal{H})$ of a Hilbert space $\mathcal{H}$ to establish that there exists a unique minimal geodesic of $\mathcal{P}(L^2(\mathbb{R}^n))$, which is a curve of the form

$$\delta(t) = e^{itX_{I,J}} P_I e^{-itX_{I,J}}$$

which joins P_I and Q_J and has length $\pi/2$. Here $X_{I,J}$ is a selfadjoint operator determined by the sets I,J. As a consequence we obtain that if H is the logarithm of the Fourier-Plancherel map, then

$$\|[H, P_I]\| \geq \pi/2.$$

The spectrum of $X_{I,J}$ is denumerable and symmetric with respect to the origin, and it has a smallest positive eigenvalue $\gamma(X_{I,J})$ which satisfies

$$\cos(\gamma(X_{I,J})) = \|P_I Q_J\|.$$

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1 Introduction

Probably, the simplest statement of what is known as the uncertainty principle of Heisenberg is that a non zero function and its Fourier transform cannot be sharply localized. It had been anounced in 1927 by Heisenberg but it may have been mentioned before, in 1925, by Wiener in a lecture at Göttingen. Nowadays, uncertainty relations, inequalities and principles are central in many parts of complex, harmonic and functional analysis, PDEs, signal processing and mathematical physics, among other areas. The books by Havin and Jöricke [11] and Gröchenig [9], among many others, exemplify the interest of the subject.

As mentioned in the excellent survey by Folland and Sitaram [8] (pp. 228), it was Fuchs who first underlined the role of the projections P_I and Q_J, defined below, in the understanding, formulation and proof of the uncertainty principle. To fix the notations, suppose that I, J are two measurable subsets of $\mathbb{R}^n$ with finite positive measure and consider the Fourier-Plancherel transform $U_{\mathcal{F}} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$. If χ_A is the characteristic function of $A \subset \mathbb{R}^n$, define

$$P_I f = \chi_I f \ , \quad Q_J f = U_{\mathcal{F}}^* P_J U_{\mathcal{F}} f \tag{1}$$

for $f \in L^2(\mathbb{R}^n)$. Since $U_{\mathcal{F}}$ is a unitary operator in $L^2(\mathbb{R}^n)$, it follows that P_I, Q_J are selfadjoint projections acting in $L^2(\mathbb{R}^n)$.

The study of the operators $P_I Q_J$, $P_I Q_J P_I$, $Q_J P_I Q_J$ (sometimes called "concentration operators") in the uncertainty principles started with W.J-H. Fuchs, and continued with Landau, Pollak and Slepian, Lenard [12], Berthier and Jauch, Amrein and Berthier, Donoho and Stark [5], to mention a few authors (see the references in the article by Folland and Sitaram [8]). Precisely Donoho and Stark [5] emphasized the role that the condition $\|P_I Q_J\| < 1$ plays in what Folland and Sitaram call the uncertainty meta-principle. Donoho and Stark proved that if there exists $f \in L^2(\mathbb{R}^n)$ such that

- $\int_{\mathbb{R}^n} |f(t)|^2 dt = 1$;
- $\int_{\mathbb{R}^n \setminus I} |f(t)|^2 dt < \epsilon_I^2$, and
- $\int_{\mathbb{R}^n \setminus J} |\hat{f}(w)|^2 dw < \epsilon_J^2$,

then $|I||J| \geq (1 - \epsilon_I - \epsilon_J)^2$, where $| \quad |$ denotes the Lebesgue measure in $\mathbb{R}^n$. Their proof is elementary and essentially consists in noticing that $(|I||J|)^{1/2}$ is the Hilbert-Schmidt norm of $P_I Q_J$. This approach avoids a difficult task, which is to compute $\|P_I Q_J\|$, which is strictly smaller than $(|I||J|)^{1/2}$. In this paper we characterize $\|P_I Q_J\|$ by using the differential geometry of the set

$$\mathcal{P}(L^2(\mathbb{R}^n)) = \{P \in \mathcal{B}(L^2(\mathbb{R}^n)) : P^2 = P^* = P\}$$

of all (bounded linear) projections acting in $L^2(\mathbb{R}^n)$. This geometry has been studied in several papers, for instance [13], [3], [1].

In Section 2 we briefly describe the main results on the geometry of $\mathcal{P}(\mathcal{H})$ to be used later. We also describe what is known as the Halmos decomposition of a Hilbert space $\mathcal{H}$ in the presence of two selfadjoint projections P, Q. In Section 3 we recall results on the pair P_I, Q_J obtained by Lenard [12] which are relevant for our computations. As a first result, we prove that P_I and Q_J are joined by a unique geodesic in $\mathcal{P}(L^2(\mathbb{R}^n))$, which is a shortest smooth curve $\delta : [0, 1] \to \mathcal{P}(L^2(\mathbb{R}^n))$ such that $\delta(0) = P$, $\delta(1) = Q$. This curve δ is given by

$$\delta(t) = e^{itX_{I,J}} P_I e^{-itX_{I,J}},$$

where $X_{I,J}$ is a selfadjoint operator, with norm $\pi/2$, and such that

$$P X_{I,J} P = (1 - P) X_{I,J} (1 - P) = 0$$

for both $P = P_I, Q_J$. The reduced minimum modulus $\gamma_{X_{I,J}}$ of $X_{I,J}$, i.e.

$$\gamma_{X_{I,J}} = \inf\{\|X_{I,J} f\|_2 : f \in R(X_{I,J}), \|f\|_2 = 1\}$$

coincides with $\arccos\|P_I Q_J\|$. As a by-product, we prove that

$$\|P_I, \log U_{\mathcal{F}}]\| \geq \pi/2,$$

and analogously for Q_J. In Section 4 we study spatial properties of P_I and Q_J. For instance, we prove that $R(P_I) + R(Q_J)$ is a direct sum, and a proper closed subspace of $L^2(\mathbb{R}^n)$. Also, using results of the second named author and Maestripieri [2], we show that for any pair of projections P, Q such that $PQ = P_I Q_J$ one has that

$$\|P_I f - Q_J f\|_2 \leq \|Pf - Qf\|_2,$$

for any $f \in L^2(\mathbb{R}^n)$.

2 Basic properties

2.1 Halmos decomposition

Let $\mathcal{H}$ be a Hilbert space, $\mathcal{B}(\mathcal{H})$ the algebra of bounded linear operators in $\mathcal{H}$ and $\mathcal{P}(\mathcal{H})$ the set of selfadjoint (orthogonal) projections. If $T \in \mathcal{B}(\mathcal{H})$, we denote by $R(T)$ the range of T, and by $N(T)$ its nullspace.

A tool that will be useful in the study of the pairs P_I, Q_J is *Halmos decomposition* [10], which is the following orthogonal decomposition of $\mathcal{H}$: given a pair of projections P and Q, consider

$$\mathcal{H}_{11} = R(P) \cap R(Q) \ , \quad \mathcal{H}_{00} = N(P) \cap N(Q) \ , \ \mathcal{H}_{10} = R(P) \cap N(Q) \ , \quad \mathcal{H}_{01} = N(P) \cap R(Q)$$

and $\mathcal{H}_0$ the orthogonal complement of the sum of the above. This last subspace is usually called the *generic part* of the pair P, Q. Note also that

$$N(P-Q) = \mathcal{H}_{11} \oplus \mathcal{H}_{00} \ , \quad N(P-Q-1) = \mathcal{H}_{10} \ \text{ and } N(P-Q+1) = \mathcal{H}_{01},$$

so that the generic part depends in fact of the difference $P - Q$.

Halmos proved that there is an isometric isomorphism between $\mathcal{H}_0$ and a product Hilbert space $\mathcal{L} \times \mathcal{L}$ such that in the above decomposition (putting $\mathcal{L} \times \mathcal{L}$ in place of $\mathcal{H}_0$), the projections are

$$P = 1 \oplus 0 \oplus 1 \oplus 0 \oplus \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$Q = 1 \oplus 0 \oplus 0 \oplus 1 \oplus \begin{pmatrix} C^2 & CS \\ CS & S^2 \end{pmatrix},$$

where $C = \cos(X)$ and $S = sin(X)$ for some operator $0 < X \leq \pi/2$ in $\mathcal{L}$. Moreover, X, C and S have trivial nullspace.

Apparently, the pair (P, Q) satisfies that PQ is compact if and only if $\mathcal{H}_{11}$ is finite dimensional and $C = \cos(X)$ is compact. In this case, the spectral resolution of X can be easily described. Since $0 < \cos(X)$ is compact, it follows that

$$X = \sum_n \gamma_n P_n,$$

where $0 < \gamma_n < \pi/2$ is an increasing (finite or infinite) sequence, P_n are the projections onto the eigenspaces correspondig to γ_n. For all n, $\dim R(P_n) < \infty$, and

$$\oplus_{n \geq 1} R(P_n) = \mathcal{L}.$$

2.2 Finsler geometry of the Grassmann manifold of $\mathcal{H}$

Let us recall some basic facts on the differential geometry of the set $\mathcal{P}(\mathcal{H})$ (see for instance [3], [13], [1]).

1. The space $\mathcal{P}(\mathcal{H})$ is a homogeneous space under the action of the unitary group $\mathcal{U}(\mathcal{H})$ by inner conjugation: if $U \in \mathcal{U}(\mathcal{H})$ and $P \in \mathcal{P}(\mathcal{H})$, the action is given by

$$U \cdot P = UPU^*.$$

   This action is locally transitive: it is well known that two projections P_1, P_2 such that $\|P_1 - P_2\| < 1$, are conjugate. Therefore, since the unitary group $\mathcal{U}(\mathcal{H})$ is connected, the orbits of the action coincide with the connected components of $\mathcal{P}(\mathcal{H})$, which are parametrized by rank and nullity. P_I and Q_J, having infinite rank and nullity, belong to the same component. These components are C^∞-submanifolds of $\mathcal{B}(\mathcal{H})$.

2. There is a natural linear connection in $\mathcal{P}(\mathcal{H})$. If $\dim \mathcal{H} < \infty$, it is the Levi-Civita connection of the Riemannian metric which consists of considering the Frobenius inner product at every tangent space. It is based on the diagonal / co-diagonal decomposition of $\mathcal{B}(\mathcal{H})$. To be more specific, given $P_0 \in \mathcal{P}(\mathcal{H})$, the tangent space of $\mathcal{P}(\mathcal{H})$ at P_0 consists of all self-adjoint co-diagonal matrices (in terms of P_0). The linear connection in $\mathcal{P}(\mathcal{H})$ is induced by a reductive structure, where the horizontal elements at P_0 (in the Lie algebra of $\mathcal{U}(\mathcal{H})$: the space of antihermitian elements of $\mathcal{B}(\mathcal{H})$) are the co-diagonal antihermitian operators. The geodesics of $\mathcal{P}(\mathcal{H})$ which start at P_0 are curves of the form

$$\delta(t) = e^{itX} P_0 e^{-itX}, \tag{2}$$

   with $X^* = X$ co-diagonal with respect to P_0. Observe that X is co-diagonal with respect to every $P_t = \delta(t)$.

3. There exists a unique geodesic joining two projections P and Q if and only if

$$R(P) \cap N(Q) = N(P) \cap R(Q) = \{0\},$$

   (see [1]).

4. If $\mathcal{H}$ is infinite dimensional, the Frobenius metric is not available. However, if one endows each tangent space of $\mathcal{P}(\mathcal{H})$ with the usual norm of $\mathcal{B}(\mathcal{H})$, one obtains a continuous (non regular) Finsler metric,

$$d(P_0, P_1) = \inf\{\ell(\gamma) : \gamma \text{ a continuous piecewise smooth curve in } \mathcal{P}(\mathcal{H}) \text{ joining } P_0 \text{ and } P_1\}$$

   where $\ell(\gamma)$ denotes the length of γ (parametrized in the interval $[a, b]$):

$$\ell(\gamma) = \int_a^b \|\dot{\gamma}(t)\| dt.$$

   In [13] it was shown that the geodesics (2) remain minimal among their endpoints for all t such that

$$|t| \leq \frac{\pi}{2\|X\|}.$$

That is, for any pair $t_1 < t_2$ such that $|t_1|, |t_2| \leq \frac{\pi}{2\|X\|}$, and any piecewise smooth curve γ in $\mathcal{P}(\mathcal{H})$, joining $\delta(t_1)$ and $\delta(t_2)$, satisfies that $\ell(\delta|_{[t_1,t_2]}) \leq \ell(\gamma)$.

It holds that $d(P_0, P_1) < \pi/2$ if and only if $\|P_0 - P_1\| < 1$. In other words, $\|P_0 - P_1\| = 1$ if and only if $d(P_0, P_1) = \pi/2$.

3 Geometry of the pairs P_I, Q_J

Lenard proved in [12] (Proposition 7) that the projections $P_I, Q_J \in \mathcal{P}(L^2(\mathbb{R}^n))$ defined in example (1), satisfy

$$R(P_I) \cap N(Q_J) = R(Q_J) \cap N(P_I) = \{0\}. \tag{3}$$

Moreover, $\|P_I - Q_J\| = 1$.

Therefore one obtains the following:

Theorem 3.1. *Let I, J be measurable subsets of $\mathbb{R}^n$ of finite measure, and P_I, Q_J the above projections. Then there exists a unique selfadjoint operator $X_{I,J}$ satisfying:*

1. $\|X_{I,J}\| = \pi/2$.
2. *$X_{I,J}$ is P_I and Q_J co-diagonal. In other words, $X_{I,J}$ maps functions in $L^2(\mathbb{R}^n)$ with support in I to functions with support in $\mathbb{R}^n - I$, and functions whose Fourier transform has support in J to functions such that the Fourier transform has support in $\mathbb{R}^n - J$.*
3. $e^{iX_{I,J}} P_I e^{-iX_{I,J}} = Q_J$.
4. *If $P(t)$, $t \in [0,1]$ is a smooth curve in $\mathcal{P}(L^2(\mathbb{R}^n))$ with $P(0) = P_I$ and $P(1) = Q_J$, then*

$$\ell(P) = \int_0^1 \|\dot{P}(t)\| dt \geq \pi/2.$$

Proof. By the condition (3) above ([12], Proposition 7), it follows from [1] that there exists a unique minimal geodesic of $\mathcal{P}(L^2(\mathbb{R}^n))$, of the form

$$\delta_{I,J}(t) = e^{itX_{I,J}} P_I e^{itX_{I,J}}$$

with $X_{I,J}^* = X_{I,J}$ co-diagonal with respect to P_I (and Q_J) such that

$$\delta_{I,J}(1) = Q_J.$$

Condition 4. above is the minimality property of $\delta_{I,J}$. Finally, the fact that $\|P_I - Q_J\| = 1$ means that $\|X_{I,J}\| = \pi/2$. □

Remark 3.2. It is known [8] (Theorem 8.4) that $\lambda_1 = \|P_I Q_J P_I\| = \|P_I Q_J\|^2 < 1$, and moreover $\sqrt{\lambda_1}$ equals the cosine of the angle between the subspaces $R(P_I)$ and $R(Q_J)$.

One can also relate this number λ_1 with the operator $X_{I,J}$. Using Halmos decomposition (recall that it consists only of $\mathcal{H}_{00}$ and the generic part $\mathcal{H}_0$ in this case),

$$P_I Q_J P_I = 0 \oplus \begin{pmatrix} C^2 & 0 \\ 0 & 0 \end{pmatrix}$$

and thus $\lambda_1 = \|\cos(X)\|^2$. We shall see below that the spectrum of X is a strictly increasing sequence of positive eigenvalues $\gamma_n \to \pi/2$, with finite multiplicity. Moreover, since $P_I Q_J$ is a Hilbert-Schmidt operator in $L^2(\mathbb{R}^n)$, it follows that C is a Hilbert-Schmidt operator of $\mathcal{L}$. Thus

$$\{\cos(\gamma_n)\} \in \ell^2.$$

For a given $P \in \mathcal{P}(\mathcal{H})$, let $\mathcal{A}_P$ be

$$\mathcal{A}_P = \{X \in \mathcal{B}(\mathcal{H}) : [X, P] \text{ is compact}\}.$$

Apparently $\mathcal{A}_P$ is a C*-algebra.

Theorem 3.3. *Let I, J be measurable subsets of $\mathbb{R}^n$ of finite Lebesgue measure.*

1. *The selfadjoint operator $X_{I,J}$ has closed infinite dimensional range. Its spectrum is of the form*
$$\sigma(X_{I,J}) = \{0\} \cup \{\pm\gamma_n : 1 \le n < \infty\} \cup \{\pi/2\},$$
with $\gamma_n > 0$ an increasing sequence converging to $\pi/2$, γ_n are eigenvalues of finite multiplicity (γ_n and $-\gamma_n$ have the same multiplicity), and neither 0 nor $\pi/2$ are eigenvalues. In particular, $X_{I,J}$ is non compact.

2. *Let I_0 be another measurable set with finite measure such that $|I \cap I_0| = 0$, and let $P_0 = P_{I_0}$. Then, the commutant $[X_{I,J}, P_0]$ is compact.*

Proof. Easy matrix computations ([1]) show that, in the decomposition $\mathcal{H} = \mathcal{H}_{00} \oplus (\mathcal{L} \times \mathcal{L})$, $X_{I,J}$ is of the form

$$X_{I,J} = 0 \oplus \begin{pmatrix} 0 & -iX \\ iX & 0 \end{pmatrix}.$$

Note that the spectrum of this operator is symmetric with respect to the origin. Indeed, if V equals the symmetry

$$V = 1 \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

then apparently $V X_{I,J} V = -X_{I,J}$. Also note that

$$X_{I,J}^2 = 0 \oplus \begin{pmatrix} X^2 & 0 \\ 0 & X^2 \end{pmatrix}.$$

Therefore the spectrum of $X_{I,J}$ is

$$\sigma(X_{I,J}) = \{0\} \cup \{\gamma_n : n \ge 1\} \cup \{-\gamma_n : n \ge 1\},$$

with 0 of infinite multiplicity, and the multiplicity of γ_n equal to the multiplicity of $-\gamma_n$, and finite. The set $\{\gamma_n : n \ge 1\}$ is infinite, and is therefore an increasing sequence converging to $\pi/2$. This holds because otherwise, the operator C would have finite rank, and therefore $P_I Q_J P_I$ would be of finite rank, which is not the case (see [12]). Thus $X_{I,J}$ has closed range of infinite dimension. This finishes the proof of 1.

Let us prove 2. Note that $P_I P_0 = 0$ and $Q_J P_0 = Q_J P_{I_0}$ is compact, and therefore $P_I, Q_J \in \mathcal{A}_{P_0}$. $X_{I,J}$ is P_I co-diagonal. This means that the symmetry $2P_I - 1$ anti-commutes with $X_{I,J}$:

$$X_{I,J}(2P_I - 1) = -(2P_I - 1)X_{I,J}.$$

Then $e^{iX_{I,J}}P_I = P_I e^{-iX_{I,J}}$. Therefore

$$2Q_J - 1 = e^{iX_{I,J}}(2P_I - 1)e^{-iX_{I,J}} = e^{i2X_{I,J}}(2P_I - 1).$$

Then $e^{i2X_{I,J}} = (2Q_J - 1)(2P_I - 1) \in \mathcal{A}_{P_0}$. By the spectral picture of $X_{I,J}$ it is clear that $X_{I,J}$ can be obtained as an holomorphic function of $e^{i2X_{I,J}}$. Since $\mathcal{A}_{P_0}$ is a C*-algebra, this implies that $X_{I,J} \in \mathcal{A}_{P_0}$. □

Let us relate the operator $X_{I,J}$ with the mathematical version of the uncertainty principle, according to [5] and [8].

Let $A \in \mathcal{B}(\mathcal{H})$ be an operator with closed range, the *reduced minimum modulus* γ_A of A is the positive number

$$\gamma_A = \min\{\|A\xi\| : \xi \in N(A)^\perp, \|\xi\| = 1\} = \min\{|\lambda| : \lambda \in \sigma(A), \lambda \neq 0\}.$$

By the above Remark, we have:

Corollary 3.4. *With the current notations,*

$$\|Q_J P_I\| = \cos(\gamma_{X_{I.J}}).$$

Proof. Indeed, in the above description of the spectrum of $X_{I,J}$, the reduced minimum modulus $\gamma_{X_{I.J}}$ of $X_{I,J}$ coincides with γ_1. □

Let $X^0_{I,J}$ be the restriction of $X_{I,J}$ to the generic part of P_I and Q_J, i.e., its restriction to $N(X_{I,J})^\perp$. In Halmos decomposition

$$X^0_{I,J} = \begin{pmatrix} 0 & -iX \\ iX & 0 \end{pmatrix}.$$

Recall the formula by Donoho and Stark [5]

$$\|P_I Q_J\|_{HS} = |I|^{1/2}|J|^{1/2}.$$

From the preceeding facts, it also follows:

Corollary 3.5. *With the current notations*

$$|I|^{1/2}|J|^{1/2} = \|\cos(X)\|_{HS} = \frac{1}{\sqrt{2}}\|\cos(X^0_{I,J})\|_{HS} = \{\sum_{n=1}^{\infty}\frac{1}{2}\cos(\gamma_n)^2\}^{1/2}.$$

Proof.

$$|I||J| = \|P_I Q_J\|^2_{HS} = Tr(P_I Q_J P_I) = Tr(C^2) = \frac{1}{2}Tr\begin{pmatrix} C^2 & 0 \\ 0 & C^2 \end{pmatrix} = \frac{1}{2}Tr(\cos(X^0_{I,J})^2).$$

□

This co-diagonal exponent $X_{I,J}$ (with respect both to P_I and Q_J) has interesting features when $I = J$ and $|I| < \infty$. In this case denote by $X_I = X_{I,I}$; then, we have two unitary operators intertwining P_I and Q_I: the Fourier transform $U_{\mathcal{F}}$ and the exponential e^{iX_I}. That is

$$U_{\mathcal{F}}^* P_I U_{\mathcal{F}} = Q_I = e^{iX_I} P_I e^{-iX_I}.$$

Let $H = H^*$ be the natural logarithm of the Fourier transform, $e^{iH} = U_{\mathcal{F}}$. Namely, writing E_1, E_{-1}, E_i and E_{-i} the eigenprojections of $U_{\mathcal{F}}$,

$$H = -\pi E_{-1} + \frac{\pi}{2} E_i - \frac{\pi}{2} E_{-i}.$$

Note that $\|H\| = \pi$. Thus, one obtains a smooth path joining P_I and Q_I:

$$\varphi(t) = e^{-itH} P_I e^{itH}.$$

and, apparently, $\varphi(1) = Q_I$.

Since the Fourier transform intertwines P_I and Q_J, the norm of its commutant with either of these projections can be regarded as a measure of non commutativity between P_I and Q_J.

Let us state the following abstract result, which we shall apply then to our case at hand.

Lemma 3.6. *Let P, Q be projections with $\|P - Q\| = 1$ and such that there exists a minimal geodesic (unique or not) joining them. If U is a unitary operator such that $UPU^* = Q$, then*

$$\|[\log U, P]\| = \|[\log U, Q]\| \geq \pi/2,$$

for any selfadjoint operator $\log U$ satisfying $e^{i \log U} = U$.

Proof. There is a geodesic $\delta(t) = e^{itX} P e^{-itX}$ with minimal length in $\mathcal{P}(\mathcal{H})$ among the curves joining P and Q. Its length is $\pi/2$. There is another curve joining P and Q, namely

$$\varphi(t) = e^{it \log U} P e^{-it \log U}.$$

Then

$$\pi/2 \leq \ell(\varphi) = \int_0^1 \|\dot{\varphi}(t)\| dt = \int_0^1 \|e^{it \log U} [\log U, P] e^{-it \log U}\| dt = \|[\log U, P]\|.$$

Note that

$$U[\log U, P]U^* = [\log U, UPU^*] = [\log U, Q]$$

because U and $\log U$ commute. □

This result applies directly to our case:

Corollary 3.7. *For any Lebesgue measurable set $I \subset \mathbb{R}^n$ with $|I| < \infty$, one has*

$$\|[H, P_I]\| = \|[H, Q_I]\| \geq \pi/2.$$

Remark 3.8.

1. We may write H in terms of $U_{\mathcal{F}}$ using the well known formulas

$$E_{-1} = \frac{1}{4}(1 - U_{\mathcal{F}} + U_{\mathcal{F}}^2 - U_{\mathcal{F}}^3),\ E_i = \frac{1}{4}(1 - iU_{\mathcal{F}} - U_{\mathcal{F}}^2 + iU_{\mathcal{F}}^3),\ E_{-i} = \frac{1}{4}(1 + iU_{\mathcal{F}} - U_{\mathcal{F}}^2 - iU_{\mathcal{F}}^3),$$

   and thus

$$H = \frac{\pi}{4}\{-1 + (1+i)U_{\mathcal{F}} - U_{\mathcal{F}}^2 + (1+i)U_{\mathcal{F}}^3\}.$$

   Then

$$[H, P_I] = \frac{\pi}{4}\{(1+i)[U_{\mathcal{F}}, P_I] - [U_{\mathcal{F}}^2, P_I] + (1+i)[U_{\mathcal{F}}^3, P_I]\}.$$

   Inequality in Corollary 3.7 can be written

$$\|(1+i)[U_{\mathcal{F}}, P_I] - [U_{\mathcal{F}}^2, P_I] + (1+i)[U_{\mathcal{F}}^3, P_I]\| \geq 2.$$

2. In the special case when the set I is (essentially) symmetric with respect to the origin, P_I commutes with $U_{\mathcal{F}}^2$. Then

$$[U_{\mathcal{F}}^2, P_I] = 0 \quad \text{and} \quad [U_{\mathcal{F}}^3, P_I] = [U_{\mathcal{F}}, P_I]U_{\mathcal{F}}^2 = U_{\mathcal{F}}^2[U_{\mathcal{F}}, P_I].$$

   Therefore one has

$$[H, P_I] = \frac{(1+i)\pi}{4}[U_{\mathcal{F}}, P_I](1 + U_{\mathcal{F}}^2).$$

   Note that the operator $U_{\mathcal{F}}^2 f(x) = f(-x)$ is a symmetry, and that $\frac{1}{2}(1 + U_{\mathcal{F}}^2)$ is the orthogonal projection E_e onto the the subspace of essentially even functions ($f(x) = f(-x)$ *a.e.*). Then

$$[H, P_I] = \frac{(1+i)\pi}{2}[U_{\mathcal{F}}, P_I]E_e = \frac{(1+i)\pi}{2}E_e[U_{\mathcal{F}}, P_I].$$

We may combine the above facts to obtain the following inequalities involving the restrictions of P_I and Q_I to the subspace of essentially even functions of $L^2(\mathbb{R}^n)$.

Corollary 3.9. *Suppose that I is essentially symmetric, with finite measure.*

1.

$$\|E_e[U_{\mathcal{F}}, P_I]\| = \|E_e[U_{\mathcal{F}}, P_I]E_e\| \geq \frac{1}{\sqrt{2}}.$$

2.

$$\|E_e P_I - E_e Q_I\| \geq \frac{1}{\sqrt{2}},$$

where $E_e P_I = P_I E_e$ and $E_e Q_I = Q_I E_e$ are orthogonal projections.

Proof. Recall that E_e and $U_{\mathcal{F}}$ commute. Then

$$E_e[U_{\mathcal{F}}, P_I]E_e = E_e(U_{\mathcal{F}}P_I - P_I U_{\mathcal{F}})E_e = U_{\mathcal{F}}E_e(P_I - U_{\mathcal{F}}^* P_I U_{\mathcal{F}})E_e$$
$$= U_{\mathcal{F}}E_e(P_I - Q_I)E_e.$$

where E_e, as well as $U_{\mathcal{F}}$, and thus also $Q_I = U_{\mathcal{F}}^* P_I U_{\mathcal{F}}$ commute with E_e. □

The ranges of these two orthogonal projections $E_e P_I$ and $E_e Q_I$ consist of the elements of $L^2(\mathbb{R}^n)$ which are essentially even and vanish (essentially) outside I, and the analogous subspace for the Fourier transform.

4 Spatial properties of P_I and Q_J

Let us return to the general setting (I not necessarily equal to J, both of finite measure). The range $R(P_I)$ is the set of functions with support in I, and the range $R(Q_J)$ is the set of functions whose Fourier transform is supported in J (this latter set is usually called the Paley-Wiener space of J). Accordingly, $N(P_I)$ and $N(Q_J)$ have similar spatial interpretations. The ranges and nullspaces of P_I and Q_J have several interesting properties, with respect to their sums, which we examine below. First we need the following lemma:

Lemma 4.1. *Let P, Q be orthogonal projections such that $\|P - Q\| = 1$. Then one and only one of the following conditions hold:*

1. *$N(P) + R(Q) = \mathcal{H}$, with non direct sum (and this is equivalent to $R(P) + N(Q)$ being a direct sum and a closed proper subspace of $\mathcal{H}$).*
2. *$R(P) + N(Q) = \mathcal{H}$, with non direct sum (and this is equivalent to $N(P) + R(Q)$ being a direct sum and a closed proper subspace of $\mathcal{H}$).*
3. *$R(P) + N(Q)$ is non closed (and this is equivalent to $N(P) + R(Q)$ being non closed).*

Proof. By the Krein-Krasnoselskii-Milman formula

$$\|P - Q\| = \max\{\|P(1 - Q)\|, \|Q(1 - P)\|\},$$

we have that one and only one of the following hold:

1. $\|P(1 - Q)\| < 1$ and $\|Q(1 - P)\| = 1$,
2. $\|P(1 - Q)\| = 1$ and $\|Q(1 - P)\| < 1$, or
3. $\|P(1 - Q)\| = 1$ and $\|Q(1 - P)\| = 1$.

This alternative corresponds precisely with the three conditions in the Lemma. It is well known that for two orthogonal projections E and F, $\|EF\| < 1$ holds if and only if $R(E) \cap R(F) = \{0\}$ and $R(E) + R(F)$ closed ; the sum $\mathcal{M} + \mathcal{N}$ of two closed subspaces is closed if and only if the sum $\mathcal{M}^\perp + \mathcal{N}^\perp$ is closed (about these facts, see for instance [6], in particular Theorems 9.7 and 9.35). Therefore, $\|EF\| < 1$ is also equivalent to $N(E) + N(F) = \mathcal{H}$.

If we apply these facts to $E = P$ and $F = 1 - Q$, we obtain that the first alternative is equivalent to $R(P) \cap N(Q) = \{0\}$ and $R(P) + N(Q)$ closed, or to $N(P) + R(Q) = \mathcal{H}$.

Analogously, the second alternative is equivalent to $R(Q) \cap N(P) = \{0\}$ and $R(Q) + N(P)$ closed, or to $N(Q) + R(P) = \mathcal{H}$.

Note that in the first case, $R(P) + N(Q)$ is proper, otherwise its orthogonal complement would be $N(P) \cap R(Q) = \{0\}$, which together with the fact that $N(P) + R(Q) = \mathcal{H}$ (closed!), would lead us to the second alternative.

Analogously in the second alternative, $N(P) + R(Q)$ is proper.

If neither of these two happen, it is clear that neither $R(P) + N(Q)$ nor (equivalently) the sum of the orthogonals $N(P) + R(Q)$ is closed. □

We have the following:

Theorem 4.2. *Let $I, J \subset \mathbb{R}^n$ with finite Lebesgue measure. Then*

1. *$R(P_I)+R(Q_J)$ is a closed proper subset of $L^2(\mathbb{R}^n)$, with infinite codimension. The sum is direct ($R(P_I)\cap R(Q_J)=\{0\}$).*

2. *$N(P_I)+N(Q_J)=L^2(\mathbb{R}^n)$, and the sum is not direct ($N(P_I)\cap N(Q_J)$ is infinite dimensional).*

3. *$R(P_I)+N(Q_J)$ and $N(P_I)+R(Q_J)$ are proper dense subspaces of $L^2(\mathbb{R}^n)$, and $R(P_I)\cap N(Q_J)=N(P_I)\cap R(Q_J)=\{0\}$.*

Proof. By the cited result [6] (Theorem 9.35), two projections P, Q, satisfy that $R(P)+R(Q)$ is closed and $R(P)\cap R(Q)=\{0\}$ if and only if $\|PQ\|<1$. It is also known (see above, [8], Theorem 8.4 or [12], Proposition 9) that $\|P_IQ_J\|<1$. The intersection of these spaces is, in our case (using the notation of the Halmos decomposition)

$$R(P_I)\cap R(Q_J)=\mathcal{H}_{11}=\{0\}.$$

As remarked above, Lenard proved that $\mathcal{H}_{11}=\mathcal{H}_{10}=\mathcal{H}_{01}=\{0\}$, and $\mathcal{H}_{00}$ is infinite dimensional. The orthogonal complement of this sum is

$$(R(P_I)+R(Q_J))^\perp=N(P_I)\cap N(Q_J)=\mathcal{H}_{00}.$$

Thus the first assertion follows.

In our case $\|P_I-Q_J\|=1$ ([12] p. 421) thus we may apply the above Lemma.

The first condition cannot happen:

$$(N(P_I)+R(Q_J))^\perp=R(P_I)\cap N(Q_J)=\mathcal{H}_{10}=\{0\}.$$

By a similar argument, neither the second condition can happen. Thus $R(P_I)+N(Q_J)$ is non closed, and its orthogonal complement is trivial. Thus the second and third assertions follow. □

Remark 4.3. It is known (see for instance [7]), that if P, Q are projections with PQ compact and $R(P)\cap R(Q)=\{0\}$, then

$$\|PQ\|<1.$$

In [2], the second named author and A. Maestripieri studied the set of operators $T\in\mathcal{B}(\mathcal{H})$ which are of the form $T=PQ$. Among other properties, they proved that T may have many factorizations, but there is a minimal factorization (called *canonical factorization* of T), namely

$$T=P_{\overline{R(T)}}P_{N(T)^\perp},$$

which satisfies that if $T=PQ$, then $R(T)\subset R(P)$ and $N(T)^\perp\subset R(Q)$ (or equivalently $N(Q)\subset N(T)$). Following this notation,

Proposition 4.4. *The factorization P_IQ_J is canonical.*

Proof. Put $T=P_IQ_J$. Using Halmos decomposition in this particular case ($\mathcal{H}=\mathcal{H}_{00}\oplus(\mathcal{L}\times\mathcal{L})$), apparently

$$P_IQ_JP_I=0\oplus\begin{pmatrix} C^2 & 0\\ 0 & 0\end{pmatrix},$$

and thus $R(P_IQ_JP_I) = 0 \oplus (R(C^2) \times 0)$. Recall that C has trivial nullspace, then C^2 has trivial nullspace, and thus has dense range in $\mathcal{L}$. It follows that

$$\overline{R(T)} = \overline{R(P_IQ_J)} = \overline{R(P_IQ_JP_I)} = 0 \oplus (\mathcal{L} \times 0),$$

which is precisely the range of P_I: $\overline{R(T)} = R(P_I)$. Note the following elementary fact:

$$N(PQ) = N(Q) \oplus (R(Q) \cap N(P)).$$

For the factorization $T = P_IQ_J$ it is known ([12] Proposition 7) that $R(Q_J) \cap N(P_I) = 0$. Thus

$$N(T) = N(P_IQ_J) = N(Q_J)$$

and the proof follows. □

In [2] it was proved that if $T = PQ = P_0Q_0$, and the latter is the canonical factorization, then

$$\|P_0f - Q_0f\|_2 \leq \|Pf - Qf\|_2$$

for any $f \in L^(\mathbb{R}^n)$. In particular $\|P_0 - Q_0\| \leq \|P - Q\|$. In our case we get the following result

Corollary 4.5. *Let P, Q projections in $L^2(\mathbb{R}^n)$ such that $PQ = P_IQ_J$. Then for any $f \in L^2(\mathbb{R}^n)$ one has*

$$\|P_If - Q_Jf\|_2 \leq \|Pf - Qf\|_2.$$

In particular, $\|P_I - Q_J\| \leq \|P - Q\|$.

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(Esteban Andruchow) Instituto de Ciencias, Universidad Nacional de Gral. Sarmiento, J.M. Gutierrez 1150, (1613) Los Polvorines, Argentina and Instituto Argentino de Matemática, 'Alberto P. Calderón', CONICET, Saavedra 15 3er. piso, (1083) Buenos Aires, Argentina.

(Gustavo Corach) Instituto Argentino de Matemática, 'Alberto P. Calderón', CONICET, Saavedra 15 3er. piso, (1083) Buenos Aires, Argentina, and Depto. de Matemática, Facultad de Ingeniería, Universidad de Buenos Aires, Argentina.