LOCAL SOLVABILITY OF ELLIPTIC EQUATIONS OF EVEN ORDER WITH HOLDER COEFFICIENTS

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Abstract. We consider $2m^{th}$ order elliptic equations with Holder coefficients.We show local solvability of the Dirichlet problem with m conditions on the boundary of the upper half space.First we consider local solvability in free space and then we treat the boundary case.Our method is based on applying the operator to an approximate solution and iteration in the Holder spaces. A priori estimates for the approximate solution is the essential part of the paper.

Higher order elliptic equation,A priori estimates, Holder coefficients, Boundary estimates.

35B45;35C15;35G15

1. Introduction

This paper is about local solvability of elliptic equations of order 2m and Holder coefficients.

First we show local solvability in free space. Given a function $f \in C^{\delta}(\mathbf{R}^n)$ we define an approximate solution u which in the case that the coefficients are constant coincides with an actual solution. We apply the operator to the approximate solution and we estimate the Holder norm of the difference $Lu - f$; working locally, we show this Holder norm is smaller than the Holder norm of f. An iteration allows us to conclude local solvability. The essential hypothesis in this case is that the coefficients of the equation belong to a Holder space C^{β} for some $\beta > \delta$.

Next we treat the problem in half space. Here we show local solvability of the Dirichlet problem with m conditions at the boundary. In this case, given a function $f \in C^\delta(\mathbf{R}^n)$ we define an approximate solution u which in the case that the coefficients are constant coincides with an actual solution and which in addition satisfies the m boundary conditions. Again, we apply the operator to the approximate solution and we estimate the Holder norm of the difference $Lu - f$; working locally, we show this Holder norm is smaller than the Holder norm of f. For this case, in order to estimate the derivatives of the approximate solution we need to assume certain decay of the derivatives of the roots of the characteristic polynomial. We believe this property should follow from ellipticity only.

There are a number of results concerning this problem. We cite the works of F. John [5] and Agmon, Douglis and Nirenberg [1]. Our work differs in the method of proof. Essential to their work is a representation formula for the constant coefficient equation. This representation formula comes in turn as a result of a Radon type transformation due to F. John.

Our method is based on the Fourier transform. Our approximate solutions are defined by transforming and anti transforming Fourier as if the equation had constant coefficients.

A big part of the paper is about a priori estimates. The a priori estimates in free space are simpler and are done first. Next we do a priori estimates up to the boundary. The existence theorems are simple consequence of these estimates. We do not use the continuity method.

One advantage of this direct method is that a solution is shown to exist locally with no restrictions on the lower order terms.

We have proved similar results in [7] where we treat the global problem via Schauder estimates and the continuity method with restrictions on the lower order terms.

See [10] and [2] for related results with smooth coefficients.

In all of the estimates we will use the letter C to denote a constant that depends only on structure. This means that C can depend on the ellipticity constant λ, on the dimension n and on the fixed constants δ and K_0 introduced below. Any other dependence will be noted explicitly unless it is obvious.

2. Preliminaries

Let $\epsilon > 0$. All our estimates will take place in $B_\epsilon(0)$, from now on B_ϵ.

Throughout the paper η_ϵ will be a function such that $\eta_\epsilon \in C_0^\infty(B_\epsilon)$ and such that $\eta_\epsilon = 1$ in $B_{\frac{\epsilon}{2}}$.

And, throughout the paper φ will be a function such that $\varphi \in C_0^\infty(B_2)$ and such that $\varphi = 1$ in B_1.

We will frequently consider the truncation $\varphi(\frac{\xi}{R})$ with $R \to \infty$

We fix two real numbers δ and β such that $0 < \delta < \beta \leq 1$

We consider operators of the form $Lu(x) = \sum_{|\alpha| \leq 2m} a_\alpha(x) D^\alpha u(x)$, where $x \in \mathbf{R}^n$ and $a(x) \in \mathbf{C}$.

About the coefficients a_α we will assume that $|a_\alpha(x) - a_\alpha(y)| \leq K_{\epsilon,\beta}|x - y|^\beta$ for all $x, y \in B_\epsilon(0)$ and for all multi index α.

We notice that $K_{\epsilon,\delta} \leq \epsilon^{\beta-\delta} K_{\epsilon,\beta}$. Hence in our estimates, we will use $|a_\alpha(x) - a_\alpha(y)| \leq K_{\epsilon,\delta}|x - y|^\delta$ for all $x, y \in B_\epsilon(0)$ and at the end use $K_{\epsilon,\delta} \leq \epsilon^{\beta-\delta} K_{\epsilon,\beta}$.

This last constant we can make as small as we need by choosing ϵ small enough.

Also we set $\sum_{|\alpha| \leq 2m} |a_\alpha|_{0,B_\epsilon} = K_0$

Let us write the characteristic polynomial of the equation by $p(x, i\xi) = \sum_{|\alpha| \leq 2m} a_\alpha(x)(i\xi)^\alpha$.

We will assume the ellipticity condition $|p(x, i\xi)| \geq \lambda(1 + |\xi|)^{2m}$ for all $x \in \mathbf{R}^n$ and for all $\xi \in \mathbf{R}^n$.

A crucial consequence of the ellipticity assumption is the following estimate

$$|D_\xi^\alpha \frac{1}{p(x, i\xi)}| \leq \frac{C}{(1 + |\xi|)^{2m+|\alpha|}} \tag{2.1}$$

More generally, we note also that setting $a(x, y, \xi) = \dfrac{p(x, i\xi) - p(y, i\xi)}{p(y, i\xi)}$ we have

$$|D_\xi^\alpha a(x, y, \xi)| \leq C K_{\epsilon,\delta} \frac{|x - y|^\delta}{(1 + |\xi|)^{|\alpha|}} \tag{2.2}$$

which hold for all $x, y \in B_\epsilon$ and for all $\xi \in \mathbf{R}^n$.

We introduce here the concepts we need in order to treat the problem in half space.

We emphasize the last coordinate by writing

$p(x, i\xi', i\xi_n) = \sum_{k=0}^{2m} (i\xi_n)^{2m-k} \sum_{|\alpha'| \leq k} a_{\alpha'}(x)(i\xi')^{\alpha'} =$

$= a(x)(i\xi_n)^{2m} + \sum_{k=1}^{2m}(i\xi_n)^{2m-k}\sum_{|\alpha'|\leq k} a_{\alpha'}(x)(i\xi')^{\alpha'}$ and where by ellipticity, we have $|a(x)| \geq \lambda$.

Notice that by ellipticity, the polynomial $p(x, i\xi', z)$ as a polynomial in the complex variable z has no pure imaginary roots.

Write $p(x, i\xi', z) = 0, z = \lambda(x, i\xi')$ denote the roots.

We remark that exactly m of the $2m$ roots which we denote by $\lambda^-(x, i\xi')$ satisfy $Re(\lambda^-(x, i\xi')) < 0$ and exactly m of the $2m$ roots which we denote by $\lambda^+(x, i\xi')$ satisfy $Re(\lambda^+(x, i\xi')) > 0$.

Therefore we can factor $p(x, i\xi', z) = a(x)p^+(x, i\xi', z)p^-(x, i\xi', z)$ where $p^{+-}(x, i\xi', z) = \prod_{k=1}^{m}(z - \lambda_k^{+-}(x, i\xi'))$.

By ellipticity and the K_0 bound it follows that each root satisfies $|Re(\lambda(x, i\xi')| \geq C(1 + |\xi'|)$ and $|\lambda(x, i\xi')| \leq C(1 + |\xi'|)$.

In addition we make the following assumptions which are key to the estimates in half space.

We assume that

$$|D_{\xi'}^{\alpha}\lambda(y, \xi')| \leq C(1 + |\xi'|)^{1-|\alpha|} \tag{2.3}$$

for all $y \in B_\epsilon$ and $\xi' \in \mathbf{R}^{n-1}$.

And we assume that

$$|D_{\xi'}^{\alpha}\lambda(y, \xi') - D_{\xi'}^{\alpha}\lambda(x, \xi')| \leq CK_{\epsilon,\delta}|x - y|^{\delta}(1 + |\xi'|)^{1-|\alpha|} \tag{2.4}$$

for all $x, y \in B_\epsilon$ and $\xi' \in \mathbf{R}^{n-1}$.

3. Local existence in free space

Given $g \in C^\delta(B_\epsilon)$, let $|g| = \max\{|g(x)| : x \in \bar{B}_\epsilon\}$ and $[g] = \max\{\frac{|g(x)-g(\bar{x})|}{|x-\bar{x}|^\delta} : x, \bar{x} \in \bar{B}_\epsilon\}$

And $|g|_{\delta;B_\epsilon} = |g| + [g]$.

Let $\Omega = B_\epsilon$ or $\Omega = B_\epsilon^+$

$|D^k u|_0 = \max\{|D^\alpha u|_0 : |\alpha| = k\}$ and $[D^k u]_\delta = \max\{[D^\alpha u]_\delta : |\alpha| = k\}$

And define,

$|u|_{2m+\delta} = \sum_{k=0}^{2m}|D^k u|_0 + [D^{2m}u]_\delta$.

The space $C^{2m+\delta}(\Omega) = \{u : |u|_{m+\delta} < \infty\}$ is a Banach space.

Given $g \in C^\delta(B_\epsilon)$, we define the approximate solution $N(\eta_\epsilon g)$ by the following formula. For $x \in B_\epsilon$ let

$$N(\eta_\epsilon g)(x) = lim_{R\to\infty} \int g(y)\eta_\epsilon(y) \int \varphi(\frac{\xi}{R}) \frac{e^{i(x-y).\xi}}{p(y,i\xi)} d\xi \, dy \tag{3.5}$$

First we show $N(\eta_\epsilon g)$ has derivatives up to order $2m$ and in fact

$D^\alpha N(\eta_\epsilon g)(x) = lim_{R\to\infty} \int g(y)\eta_\epsilon(y) \int \varphi(\frac{\xi}{R}) \frac{(i\xi)^\alpha e^{i(x-y).\xi}}{p(y,i\xi)} d\xi \, dy,$

for any multi index α, $|\alpha| \leq 2m$.

We proceed to prove this. Let $|\alpha| = 2m$. We show the above limit exists and is uniform in x.

Write

$$\int g(y)\eta_\epsilon(y) \int \varphi(\frac{\xi}{R}) \frac{(i\xi)^\alpha e^{i(x-y).\xi}}{p(y,i\xi)} d\xi \, dy = I_R + g(x)II_R + g(x)III_R$$

, where

$$I_R = \int (g(y) - g(x))\eta_\epsilon(y) \int \varphi(\frac{\xi}{R}) \frac{(i\xi)^\alpha e^{i(x-y).\xi}}{p(y,i\xi)} d\xi \, dy.$$

$$II_R = \int \eta_\epsilon(y) \int \varphi(\frac{\xi}{R}) \frac{(i\xi)^\alpha e^{i(x-y).\xi}}{p(x,i\xi)} d\xi \, dy.$$

And,

$$III_R = \int \eta_\epsilon(y) \int \varphi(\frac{\xi}{R})(i\xi)^\alpha e^{i(x-y).\xi} (\frac{1}{p(y,i\xi)} - \frac{1}{p(x,i\xi)}) d\xi \, dy.$$

Let k be an integer $k \geq n+1$ and let

$$I = \int \frac{(g(y) - g(x))\eta_\epsilon(y)}{|x-y|^{2k}} \int e^{i(x-y).\xi} (-\Delta_\xi)^k (\frac{(i\xi)^\alpha}{p(y,i\xi)}) d\xi \, dy.$$

$$II = \int (1-\Delta)(\eta_\epsilon(y)) \frac{1}{|x-y|^{2k}} \int e^{i(x-y).\xi} (-\Delta_\xi)^k (\frac{(i\xi)^\alpha}{(1+|\xi|^2)p(x,i\xi)}) d\xi \, dy$$

$$III = \int \frac{\eta_\epsilon(y)}{|x-y|^{2k}} \int e^{i(x-y).\xi} (-\Delta_\xi)^k ((i\xi)^\alpha (\frac{1}{p(y,i\xi)} - \frac{1}{p(x,i\xi)})) d\xi \, dy.$$

Let us show that $III_R \to III$ as $R \to \infty$. The other are similar.

First, we write

$$III = \int \frac{\eta_\epsilon(y)}{|x-y|^{2k}} \int e^{i(x-y).\xi} (-\Delta_\xi)^k ((i\xi)^\alpha (\frac{1}{p(y,i\xi)} - \frac{1}{p(x,i\xi)})) \varphi(|x-y|\xi) d\xi \, dy +$$

$$\int \frac{\eta_\epsilon(y)}{|x-y|^{2k}} \int e^{i(x-y).\xi}(-\Delta_\xi)^k((i\xi)^\alpha(\frac{1}{p(y,i\xi)} - \frac{1}{p(x,i\xi)}))(1-\varphi(|x-y|\xi))d\xi\, dy = A + B$$

and note that

$$A = \int \frac{\eta_\epsilon(y)}{|x-y|^{2k}} \int (-\Delta_\xi)^k(e^{i(x-y).\xi}\varphi(|x-y|\xi))(i\xi)^\alpha(\frac{1}{p(y,i\xi)} - \frac{1}{p(x,i\xi)})d\xi\, dy.$$

We have

$$III_R = \int \frac{\eta_\epsilon(y)}{|x-y|^{2k}} \int e^{i(x-y).\xi}(-\Delta_\xi)^k((i\xi)^\alpha(\frac{1}{p(y,i\xi)} - \frac{1}{p(x,i\xi)})\varphi(\frac{\xi}{R}))d\xi\, dy.$$

and hence we can write,

$$III_R = A_R + B_R + C_R$$

where

$$A_R = \int \frac{\eta_\epsilon(y)}{|x-y|^{2k}} \int (-\Delta_\xi)^k(e^{i(x-y).\xi}\varphi(|x-y|\xi))(i\xi)^\alpha(\frac{1}{p(y,i\xi)} - \frac{1}{p(x,i\xi)})\varphi(\frac{\xi}{R})d\xi\, dy.$$

$$B_R = \int \frac{\eta_\epsilon(y)}{|x-y|^{2k}} \int e^{i(x-y).\xi}(-\Delta_\xi)^k((i\xi)^\alpha(\frac{1}{p(y,i\xi)} - \frac{1}{p(x,i\xi)}))\varphi(\frac{\xi}{R})(1-\varphi(|x-y|\xi))d\xi\, dy.$$

And

$$C_R = \sum_{|\gamma|+|\beta|=2k, |\beta|\geq 1} \int \frac{\eta_\epsilon(y)}{|x-y|^{2k}} \int e^{i(x-y).\xi} D^\gamma((i\xi)^\alpha(\frac{1}{p(y,i\xi)} - \frac{1}{p(x,i\xi)}))D^\beta(\varphi(\frac{\xi}{R}))(1-\varphi(|x-y|\xi))d\xi\, dy.$$

Therefore, $|III_R - III| \leq |A - A_R| + |B - B_R| + |C_R|$.

We estimate

$$|A - A_R| \leq C \int \eta_\epsilon(y)|x-y|^\delta \int_{|\xi|\leq \frac{2}{|x-y|}} 1 - \varphi(\frac{\xi}{R})d\xi\, dy \leq$$

$$C \int \eta_\epsilon(y)|x-y|^\delta \int_{R\leq|\xi|\leq \frac{2}{|x-y|}} 1 d\xi\, dy \leq CR^{-\delta}$$

$$|B - B_R| \leq C \int \eta_\epsilon(y)\frac{|x-y|^\delta}{|x-y|^{2k}} \int_{|\xi|\geq \frac{1}{|x-y|}} \frac{1-\varphi(\frac{\xi}{R})}{(1+|\xi|)^{2k}} d\xi\, dy \leq$$

$$\int_{|x-y|\leq \frac{1}{R}} \frac{|x-y|^\delta}{|x-y|^{2k}} \int_{\frac{1}{|x-y|}\leq|\xi|} \frac{1}{(1+|\xi|)^{2k}} d\xi\, dy + \int_{|x-y|\geq \frac{1}{R}} \frac{|x-y|^\delta}{|x-y|^{2k}} \int_{R\leq|\xi|} \frac{1}{(1+|\xi|)^{2k}} d\xi\, dy \leq CR^{-\delta}$$

and

$$|C_R| \leq C \int_{|x-y|\geq 2R} \frac{|x-y|^\delta}{|x-y|^{2k}} \frac{1}{R^{|\beta|}} \int_{R\leq|\xi|\leq 2R} \frac{1}{(1+|\xi|)^{|\gamma|}} d\xi\, dy \leq CR^{-\delta}$$

To show $II_R \to II$, first note that

$$II_R = \int (1-\Delta)\eta_\epsilon(y) \int e^{i(x-y).\xi} (\frac{(i\xi)^\alpha}{(1+|\xi|^2)p(x,i\xi)})\varphi(\frac{\xi}{R})d\xi\, dy =$$
$$\int (1-\Delta)(\eta_\epsilon(y))\frac{1}{|x-y|^{2k}} \int e^{i(x-y).\xi}(-\Delta_\xi)^k(\frac{(i\xi)^\alpha}{(1+|\xi|^2)p(x,i\xi)}\varphi(\frac{\xi}{R}))d\xi\, dy.$$

and then we proceed as before.

3.1. **Auxiliary integrals for free space.** Next, we prove two auxiliary integrals.

First, let us a emphasize the calculations above since they will be used several times in the sequel.

Consider, for $|\alpha| \leq 2m$, the integral

$$J = \int e^{i(x-y).\xi} \frac{\xi^\alpha}{p(x,i\xi)}\varphi(\frac{\xi}{R})d\xi$$

We write

$$J = \frac{1}{|x-y|^{2k}} \int e^{i(x-y).\xi}(-\Delta)^k(\frac{\xi^\alpha}{p(x,i\xi)}\varphi(\frac{\xi}{R}))d\xi = J_1 + J_2 + J_3$$

Where

$$J_1 = \frac{1}{|x-y|^{2k}} \int (-\Delta)^k(e^{i(x-y).\xi}\varphi(|x-y|\xi))\frac{\xi^\alpha}{p(x,i\xi)}\varphi(\frac{\xi}{R})d\xi$$
$$J_2 = \frac{1}{|x-y|^{2k}} \int e^{i(x-y).\xi}(-\Delta)^k(\frac{\xi^\alpha}{p(x,i\xi)})\varphi(\frac{\xi}{R})(1-\varphi(|x-y|\xi))d\xi.$$

And,

$$J_3 = \frac{1}{|x-y|^{2k}} \sum_{|\gamma|+|\beta|=2k, |\beta|\geq 1} \int e^{i(x-y).\xi} D^\gamma(\xi^\alpha \frac{1}{p(x,i\xi)}))D^\beta(\varphi(\frac{\xi}{R}))(1-\varphi(|x-y|\xi))d\xi.$$

We estimate each integral as follows,

$$|J_1| \leq \int_{|\xi|\leq \frac{2}{|x-y|}} \frac{1}{(1+|\xi|)^{2m-|\alpha|}}d\xi \leq \frac{C}{|x-y|^{n-2m+|\alpha|}}.$$
$$|J_2| \leq \frac{1}{|x-y|^{2k}} \int_{|\xi|\geq \frac{1}{|x-y|}} \frac{1}{(1+|\xi|)^{2m+2k-|\alpha|}}d\xi \leq \frac{C}{|x-y|^{n-2m+|\alpha|}}.$$

And

$$|J_3| \leq \frac{1}{|x-y|^{2k}}\frac{1}{R^{2k}}X_{|x-y|\geq \frac{1}{2R}}\frac{1}{R^{2m-|\alpha|}}.$$

Next, we prove two important bounds that we need in order to continue.

Let

$$I = \int \eta_\epsilon(y) \int e^{i(x-y).\xi} \frac{\xi^\alpha}{p(x,i\xi)} \varphi(\frac{\xi}{R}) d\xi\, dy$$

We claim $|I| \le C$.

To prove this write,

$$I = \int (1-\Delta)\eta_\epsilon(y) \int e^{i(x-y).\xi} \frac{\xi^\alpha}{(1+|\xi|^2)p(x,i\xi)} \varphi(\frac{\xi}{R}) d\xi\, dy$$

and note

$$|I| \le C\frac{1}{\epsilon} \int_{B_\epsilon} \frac{1}{|x-y|^{n-2}} dy \le C.$$

For the other auxiliary integral, let $x, \bar{x} \in B_\epsilon$, let $\hat{x} = \frac{x+\bar{x}}{2}$ and $\rho = 2|x-\bar{x}|$, so $\rho \le 4\epsilon$.

Let

$$I = \int_{B_\rho(\hat{x})} \eta_\epsilon(y) \int e^{i(x-y).\xi} \frac{\xi^\alpha}{p(x,i\xi)} \varphi(\frac{\xi}{R}) d\xi\, dy.$$

We claim $|I| \le C$.

To prove this write,

$$I = \int_{B_\rho(\hat{x})} \eta_\epsilon(y) - \eta_\epsilon(x) \int e^{i(x-y).\xi} \frac{\xi^\alpha}{p(x,i\xi)} \varphi(\frac{\xi}{R}) d\xi\, dy + \eta_\epsilon(x) \int_{B_\rho(\hat{x})} \int e^{i(x-y).\xi} \frac{\xi^\alpha}{p(x,i\xi)} \varphi(\frac{\xi}{R}) d\xi\, dy =$$

$$A + \eta_\epsilon(x) B.$$

We have

$$|A| \le C \int_{B_\rho(\hat{x})} \frac{|\eta_\epsilon(y) - \eta_\epsilon(x)|}{|x-y|^n} dy \le \frac{C}{\epsilon} \int_{B_\rho(\hat{x})} \frac{1}{|x-y|^{n-1}} dy \le C\frac{\rho}{\epsilon} \le C.$$

And

$$B = \int_{B_\rho(\hat{x})} (1-\Delta) \int e^{i(x-y).\xi} \frac{\xi^\alpha}{p(x,i\xi)(1+|\xi|^2)} \varphi(\frac{\xi}{R}) d\xi\, dy =$$

$$\int_{B_\rho(\hat{x})} \int e^{i(x-y).\xi} \frac{\xi^\alpha}{p(x,i\xi)(1+|\xi|^2)} \varphi(\frac{\xi}{R}) d\xi\, dy - \int_{\partial B_\rho(\hat{x})} \left\langle \nabla_y \int e^{i(x-y).\xi} \frac{\xi^\alpha}{p(x,i\xi)(1+|\xi|^2)} \varphi(\frac{\xi}{R}) d\xi, \frac{y-\hat{x}}{|y-\hat{x}|} \right\rangle dS_y$$

$= B_1 + B_2$.

And we can estimate

$$|B_1| \le C \int_{B_\rho(\hat{x})} \frac{1}{|x-y|^{n-2}} dy \le C\rho^2$$

and

$$|B_2| \le \int_{\partial B_\rho(\hat{x})} \frac{1}{|x-y|^{n-1}} dS_y \le C.$$

3.2. **Main results for free space.**

Lemma 3.1. *The function defined by $u = N(g\eta_\epsilon)$ is in $C^{2m+\delta}(B_\epsilon)$*

Proof. We have shown that u has derivatives up to order $2m$ and for $|\alpha| \le 2m$ we have $D^\alpha u(x) = \lim_{R\to\infty} \int g(y)\eta_\epsilon(y) \int \varphi(\frac{\xi}{R}) \frac{(i\xi)^\alpha e^{i(x-y).\xi}}{p(y,i\xi)} d\xi\, dy$. Let $|\alpha| = 2m$, and let

$$I_R = \int g(y)\eta_\epsilon(y) \int \varphi(\frac{\xi}{R}) \frac{(i\xi)^\alpha e^{i(x-y).\xi}}{p(y,i\xi)} d\xi\, dy.$$

Let $x, \bar{x} \in B_\epsilon$.

Write $I_R(x) - I_R(\bar{x}) = A + B$, where

$$A = \int g(y)\eta_\epsilon(y) \int (e^{i(x-y).\xi} - e^{i(\bar{x}-y).\xi})(i\xi)^\alpha \frac{1}{p(\bar{x},i\xi)} \varphi(\frac{\xi}{R}) d\xi\, dy$$

$$B = \int g(y)\eta_\epsilon(y) \int (e^{i(x-y).\xi} - e^{i(\bar{x}-y).\xi})(i\xi)^\alpha (\frac{1}{p(y,i\xi)} - \frac{1}{p(\bar{x},i\xi)}) \varphi(\frac{\xi}{R}) d\xi\, dy.$$

Write $A = B_1 + B_2$ where

$$B_1 = \int (g(y) - g(x))\eta_\epsilon(y) \int (e^{i(x-y).\xi} - e^{i(\bar{x}-y).\xi}) \frac{(i\xi)^\alpha}{p(\bar{x},i\xi)} \varphi(\frac{\xi}{R}) d\xi\, dy.$$

and

$$B_2 = g(x) \int \eta_\epsilon(y) \int (e^{i(x-y).\xi} - e^{i(\bar{x}-y).\xi}) \frac{(i\xi)^\alpha}{p(\bar{x},i\xi)} \varphi(\frac{\xi}{R}) d\xi\, dy.$$

$B_1 = C_1 - C_2 + C_3$ where

$$C_1 = \int_{B_\rho(\hat{x})} (g(y) - g(x))\eta_\epsilon(y) \int e^{i(x-y).\xi} \frac{(i\xi)^\alpha}{p(\bar{x},i\xi)} \varphi(\frac{\xi}{R}) d\xi\, dy.$$

$$C_2 = \int_{B_\rho(\hat{x})} (g(y) - g(x))\eta_\epsilon(y) \int e^{i(\bar{x}-y).\xi} \frac{(i\xi)^\alpha}{p(\bar{x},i\xi)} \varphi(\frac{\xi}{R}) d\xi\, dy.$$

and

$$C_3 = \int_{R^n \setminus B_\rho(\hat{x})} (g(y) - g(x))\eta_\epsilon(y) \int (e^{i(x-y).\xi} - e^{i(\bar{x}-y).\xi}) \frac{(i\xi)^\alpha}{p(\bar{x},i\xi)} \varphi(\frac{\xi}{R}) d\xi\, dy.$$

. We have $|C_1| \le C \int_{B_\rho(\hat{x})} |y-x|^\delta \frac{1}{|y-x|^n} dy \le C\rho^\delta \le C|x-\bar{x}|^\delta$

and $C_2 = D_1 + D_2$ where

$$D_1 = \int_{B_\rho(\hat{x})} (g(y) - g(\bar{x}))\eta_\epsilon(y) \int e^{i(\bar{x}-y).\xi} \frac{(i\xi)^\alpha}{p(\bar{x}, i\xi)} \varphi(\frac{\xi}{R}) d\xi\, dy.$$

and

$$D_2 = (g(x) - g(\bar{x})) \int_{B_\rho(\hat{x})} \eta_\epsilon(y) \int e^{i(\bar{x}-y).\xi} \frac{(i\xi)^\alpha}{p(\bar{x}, i\xi)} \varphi(\frac{\xi}{R}) d\xi\, dy.$$

Estimate $|D_1| \le C\rho^\delta \le C|x - \bar{x}|^\delta$ just like C_1 and

$$|D_2| \le C|x - \bar{x}|^\delta \left| \int_{B_\rho(\hat{x})} \eta_\epsilon(y) \int e^{i(\bar{x}-y).\xi} \frac{(i\xi)^\alpha}{p(\bar{x}, i\xi)} \varphi(\frac{\xi}{R}) d\xi\, dy \right| \le C|x - \bar{x}|^\delta$$

by auxiliary integral. Next,

$$C_3 = \int_{R^n \setminus B_\rho(\hat{x})} (g(y) - g(x))\eta_\epsilon(y) \int i \int_0^1 (e^{i(x_s-y).\xi} ds (x - \bar{x}).\xi \frac{(i\xi)^\alpha}{p(\bar{x}, i\xi)} \varphi(\frac{\xi}{R}) d\xi\, dy.$$

and we have

$$|C_3| \le C|x-\bar{x}| \int_0^1 \int_{R^n \setminus B_\rho(\hat{x})} |y-x|^\delta \frac{1}{|x_s - y|^{n+1}} dy\, ds \le C|x-\bar{x}| \int_{R^n \setminus B_\rho(\hat{x})} |y-x|^\delta \frac{1}{|x - y|^{n+1}} dy$$
$$\le C|x - \bar{x}|\rho^{\delta-1} \le C|x - \bar{x}|^\delta.$$

Finishing the estimation of B_1.

$$B_2 = \int (1 - \Delta)\eta_\epsilon(y) \int (e^{i(x-y).\xi} - e^{i(\bar{x}-y).\xi}) \frac{(i\xi)^\alpha}{(1 + |\xi|^2) p(\bar{x}, i\xi)} \varphi(\frac{\xi}{R}) d\xi\, dy.$$

and

$$|B_2| \le C|x - \bar{x}| \frac{1}{\epsilon^2} \int_{B_\epsilon} \int_0^1 \frac{1}{|x_s - y|^{n-1}} ds\, dy \le C \frac{|x - \bar{x}|}{\epsilon} \le \frac{C}{\epsilon^\delta} |x - \bar{x}|^\delta.$$

This finishes the estimation of the term A.

Write $B = C_1 + C_2 + C_3$ where

$$C_1 = \int_{B_\rho(\hat{x})} g(y)\eta_\epsilon(y) \int e^{i(x-y).\xi} (i\xi)^\alpha (\frac{1}{p(y, i\xi)} - \frac{1}{p(\bar{x}, i\xi)}) \varphi(\frac{\xi}{R}) d\xi\, dy.$$

$$C_2 = \int_{B_\rho(\hat{x})} g(y)\eta_\epsilon(y) \int e^{i(\bar{x}-y).\xi} (i\xi)^\alpha (\frac{1}{p(y, i\xi)} - \frac{1}{p(\bar{x}, i\xi)}) \varphi(\frac{\xi}{R}) d\xi\, dy.$$

$$C_3 = \int_{R^n \setminus B_\rho(\hat{x})} g(y)\eta_\epsilon(y) \int (e^{i(x-y).\xi} - e^{i(\bar{x}-y).\xi}) (i\xi)^\alpha (\frac{1}{p(y, i\xi)} - \frac{1}{p(\bar{x}, i\xi)}) \varphi(\frac{\xi}{R}) d\xi\, dy.$$

$C_1 = D_1 + D_2$ where

$$D_1 = \int_{B_\rho(\hat{x})} (g(y) - g(x))\eta_\epsilon(y) \int e^{i(x-y).\xi}(i\xi)^\alpha(\frac{1}{p(y,i\xi)} - \frac{1}{p(\bar{x},i\xi)})\varphi(\frac{\xi}{R})d\xi\, dy.$$

$$D_2 = g(x)\int_{B_\rho(\hat{x})} \eta_\epsilon(y) \int e^{i(x-y).\xi}(i\xi)^\alpha(\frac{1}{p(y,i\xi)} - \frac{1}{p(\bar{x},i\xi)})\varphi(\frac{\xi}{R})d\xi\, dy.$$

and we estimate

$$|D_1| \le C\int_{B_\rho(\hat{x})} |y-x|^\delta \frac{1}{|x-y|^n} dy \le C\rho^\delta \le C|x-\bar{x}|^\delta.$$

$D_2 = F_1 + F_2$ where

$$F_1 = \int_{B_\rho(\hat{x})} \eta_\epsilon(y) \int e^{i(x-y).\xi}(i\xi)^\alpha(\frac{1}{p(y,i\xi)} - \frac{1}{p(x,i\xi)})\varphi(\frac{\xi}{R})d\xi\, dy.$$

$$F_2 = \int_{B_\rho(\hat{x})} \eta_\epsilon(y) \int e^{i(x-y).\xi}(i\xi)^\alpha(\frac{1}{p(x,i\xi)} - \frac{1}{p(\bar{x},i\xi)})\varphi(\frac{\xi}{R})d\xi\, dy$$

and we estimtate

$$|F_1| \le C\int_{B_\rho(\hat{x})} |y-x|^\delta \frac{1}{|x-y|^n} dy \le C\rho^\delta \le C|x-\bar{x}|^\delta.$$

and

$$|F_2| \le \sum_{|\beta|\le 2m} |a_\beta(x) - a_\beta(\bar{x})| \left| \int_{B_\rho(\hat{x})} \eta_\epsilon(y) \int e^{i(x-y).\xi} \frac{(i\xi)^\alpha (i\xi)^\beta}{p(x,i\xi)p(\bar{x},i\xi)} \varphi(\frac{\xi}{R})d\xi\, dy \right| \le C|x-\bar{x}|^\delta$$

by auxiliary integral.

This finishes the estimation of C_1,

$$|C_2| \le C\int_{B_\rho(\hat{x})} \frac{1}{|\bar{x}-y|^{n-\delta}} dy \le C|x-\bar{x}|^\delta$$

and

$$|C_3| \le C|x-\bar{x}| \int_{R^n \setminus B_\rho(\hat{x})} \frac{1}{|y-\hat{x}|^{n+1-\delta}} dy \le C|x-\bar{x}|^\delta.$$

This finishes the estimation of the term B. The lemma is proved.

$\square$

We notice that

$$L(N(g\eta_\epsilon)(x) = lim_{R\to\infty}\int g(y)\eta_\epsilon(y)\int e^{i(x-y).\xi}\frac{p(x,i\xi)}{p(y,i\xi)}\varphi(\frac{\xi}{R})d\xi\, dy.$$

And also,

$$g\eta_\epsilon(x) = lim_{R\to\infty}\int g(y)\eta_\epsilon(y)\int e^{i(x-y).\xi}\varphi(\frac{\xi}{R})d\xi\, dy.$$

Hence, we define

$$T(g\eta_\epsilon)(x) := L(N(g\eta_\epsilon)(x) - g\eta_\epsilon(x) =$$

$$lim_{R\to\infty}\int g(y)\eta_\epsilon(y)\int e^{i(x-y).\xi}\frac{p(x,i\xi)-p(y,i\xi)}{p(y,i\xi)}\varphi(\frac{\xi}{R})d\xi\, dy.$$

We can now state the main theorem of this section.

Theorem 3.2. *We have the estimate* $|T(g\eta_\epsilon)|_{\delta;B_\epsilon} \leq CK_{\delta,\epsilon}|g|_{\delta;B_\epsilon}$

Proof. For fixed $R \geq 1$, let

$$I_R(x) = \int g(y)\eta_\epsilon(y)\int e^{i(x-y).\xi}\frac{p(x,i\xi)-p(y,i\xi)}{p(y,i\xi)}\varphi(\frac{\xi}{R})d\xi\, dy.$$

We write

$$I_R(x) - I_R(\bar{x}) = \int g(y)\eta_\epsilon(y)\int (e^{i(x-y).\xi} - e^{i(\bar{x}-y).\xi})\frac{p(x,i\xi)-p(y,i\xi)}{p(y,i\xi)}\varphi(\frac{\xi}{R})d\xi\, dy +$$

$$\int g(y)\eta_\epsilon(y)\int e^{i(\bar{x}-y).\xi}\frac{p(x,i\xi)-p(\bar{x},i\xi)}{p(y,i\xi)}\varphi(\frac{\xi}{R})d\xi\, dy = A+B$$

Write $A = A_1 + g(x)A_2$ where

$$A_1 = \int (g(y)-g(\bar{x}))\eta_\epsilon(y)\int (e^{i(x-y).\xi} - e^{i(\bar{x}-y).\xi})\frac{p(x,i\xi)-p(y,i\xi)}{p(y,i\xi)}\varphi(\frac{\xi}{R})d\xi\, dy.$$

$$A_2 = \int \eta_\epsilon(y)\int e^{i(\bar{x}-y).\xi}\frac{p(x,i\xi)-p(\bar{x},i\xi)}{p(y,i\xi)}\varphi(\frac{\xi}{R})d\xi\, dy.$$

Next, write $A_1 = B_1 - B_2 + B_3$ where

$$B_1 = \int_{B_\rho(\bar{x})} (g(y)-g(\bar{x}))\eta_\epsilon(y)\int e^{i(x-y).\xi}\frac{p(x,i\xi)-p(y,i\xi)}{p(y,i\xi)}\varphi(\frac{\xi}{R})d\xi\, dy$$

$$B_2 = \int_{B_\rho(\bar{x})} (g(y)-g(\bar{x}))\eta_\epsilon(y)\int e^{i(\bar{x}-y).\xi}\frac{p(x,i\xi)-p(y,i\xi)}{p(y,i\xi)}\varphi(\frac{\xi}{R})d\xi\, dy$$

and

$$B_3 = \int_{R^n\setminus B_\rho(\hat{x})} (g(y)-g(\bar{x}))\eta_\epsilon(y)\int (e^{i(x-y).\xi}-e^{i(\bar{x}-y).\xi})\frac{p(x,i\xi)-p(y,i\xi)}{p(y,i\xi)}\varphi(\frac{\xi}{R})d\xi\, dy.$$

We estimate

$$|B_1| \le K_{\delta;\epsilon}[g]\int_{B_\rho(\hat{x})} |y-\bar{x}|^\delta |y-x|^\delta \frac{1}{|y-x|^n}dy \le K_{\delta;\epsilon}[g]|x-\bar{x}|^\delta$$

$$|B_2| \le K_{\delta;\epsilon}[g]\int_{B_\rho(\hat{x})} |y-\bar{x}|^\delta |y-x|^\delta \frac{1}{|y-\bar{x}|^n}dy \le K_{\delta;\epsilon}[g]|x-\bar{x}|^\delta$$

and

$$|B_3| \le K_{\delta;\epsilon}[g]|x-\bar{x}|\int_0^1\int_{R^n\setminus B_\rho(\hat{x})} |y-\bar{x}|^\delta|y-x|^\delta\frac{1}{|y-x_s|^{n+1}}dy\, ds$$

$$\le K_{\delta;\epsilon}[g]|x-\bar{x}|\int_{R^n\setminus B_\rho(\hat{x})}\frac{1}{|y-\hat{x}|^{n+1-\delta}}dy \le K_{\delta;\epsilon}[g]|x-\bar{x}|^\delta.$$

Finishing the estimation of A_1

$A_2 = C_1 - C_2 + C_3$ where

$$C_1 = \int_{B_\rho(\hat{x})}\eta_\epsilon(y)\int e^{i(x-y).\xi}\frac{p(x,i\xi)-p(y,i\xi)}{p(y,i\xi)}\varphi(\frac{\xi}{R})d\xi\, dy.$$

$$C_2 = \int_{B_\rho(\hat{x})}\eta_\epsilon(y)\int e^{i(\bar{x}-y).\xi}\frac{p(x,i\xi)-p(y,i\xi)}{p(y,i\xi)}\varphi(\frac{\xi}{R})d\xi\, dy.$$

$$C_3 = \int_{R^n\setminus B_\rho(\hat{x})}\eta_\epsilon(y)\int (e^{i(x-y).\xi}-e^{i(\bar{x}-y).\xi})\frac{p(x,i\xi)-p(x,i\xi)}{p(y,i\xi)}\varphi(\frac{\xi}{R})d\xi\, dy.$$

and estimate

$$|C_1| \le K_{\delta;\epsilon}\int_{B_\rho(\hat{x})}|x-y|^\delta\frac{1}{|x-y|^n}dy \le K_{\delta;\epsilon}|x-\bar{x}|^\delta.$$

$C_2 = E_1 + E_2$ where

$$E_1 = \int_{B_\rho(\hat{x})}\eta_\epsilon(y)\int e^{i(\bar{x}-y).\xi}\frac{p(x,i\xi)-p(\bar{x},i\xi)}{p(\bar{x},i\xi)}\varphi(\frac{\xi}{R})d\xi\, dy.$$

and

$$E_2 = \int_{B_\rho(\hat{x})}\eta_\epsilon(y)\int e^{i(\bar{x}-y).\xi}\frac{p(x,i\xi)(p(\bar{x},i\xi)-p(y,i\xi))}{p(y,i\xi)p(\bar{x},i\xi)}\varphi(\frac{\xi}{R})d\xi\, dy.$$

$$|E_1| \le \sum_{|\alpha|\le 2m}|a_\alpha(x)-a_\alpha(\bar{x})|\left|\int_{B_\rho(\hat{x})}\eta_\epsilon(y)\int e^{i(\bar{x}-y).\xi}\frac{(i\xi)^\alpha}{p(\bar{x},i\xi)}\varphi(\frac{\xi}{R})d\xi\, dy\right| \le K_{\delta;\epsilon}|x-\bar{x}|^\delta.$$

by auxiliary integral. And

$$|E_2| \le K_{\delta;\epsilon} \int_{B_\rho(\hat{x})} |\bar{x}-y|^\delta \frac{1}{|\bar{x}-y|^n} dy \le K_{\delta;\epsilon} |x-\bar{x}|^\delta.$$

We also have

$$|C_3| \le K_{\delta;\epsilon} |x-\bar{x}| \int_{R^n \setminus B_\rho(\hat{x})} \int_0^1 \frac{1}{|y-x_s|^{n+1}} |y-x|^\delta ds\, dy \le K_{\delta;\epsilon} |x-\bar{x}|^\delta.$$

Finishing the estimation of A_2 and hence of A.

To estimate B write

$B = F_1 + g(\bar{x})F_2$ where

$$F_1 = \int (g(y) - g(\bar{x}))\eta_\epsilon(y) \int e^{i(\bar{x}-y).\xi} \frac{p(x,i\xi) - p(\bar{x},i\xi)}{p(y,i\xi)} \varphi(\frac{\xi}{R}) d\xi\, dy.$$

and

$$F_2 = \int \eta_\epsilon(y) \int e^{i(\bar{x}-y).\xi} \frac{p(x,i\xi) - p(\bar{x},i\xi)}{p(y,i\xi)} \varphi(\frac{\xi}{R}) d\xi\, dy.$$

and we have

$$|F_1| \le K_{\delta;\epsilon} [g] |x-\bar{x}|^\delta \int_{B_\epsilon} |y-\bar{x}|^\delta \frac{1}{|y-\bar{x}|^n} dy \le K_{\delta;\epsilon} [g] |x-\bar{x}|^\delta$$

Write $F_2 = G_1 + G_2$ where

$$G_1 = \int \eta_\epsilon(y) \int e^{i(\bar{x}-y).\xi} \frac{p(x,i\xi) - p(\bar{x},i\xi)}{p(\bar{x},i\xi)} \varphi(\frac{\xi}{R}) d\xi\, dy$$

and

$$G_2 = \int \eta_\epsilon(y) \int e^{i(\bar{x}-y).\xi} (p(x,i\xi) - p(\bar{x},i\xi)) \frac{(p(\bar{x},i\xi) - p(y,i\xi))}{p(\bar{x},i\xi)p(y,i\xi)} \varphi(\frac{\xi}{R}) d\xi\, dy.$$

We have

$$|G_1| \le \sum_{|\alpha|\le 2m} |a_\alpha(x) - a_\alpha(\bar{x})| \left| \int \eta_\epsilon(y) \int e^{i(\bar{x}-y).\xi} \frac{(i\xi)^\alpha}{p(\bar{x},i\xi)} \varphi(\frac{\xi}{R}) d\xi\, dy \right| \le K_{\delta;\epsilon} |x-\bar{x}|^\delta$$

by auxiliary integral. And

$$|G_2| \le K_{\delta;\epsilon} |x-\bar{x}|^\delta \int_{B_\epsilon} |y-\bar{x}|^\delta \frac{1}{|y-\bar{x}|^n} dy \le K_{\delta;\epsilon} |x-\bar{x}|^\delta.$$

This finishes the estimation of the term B.

So we have shown that $[T(g\eta_\epsilon)]_{\delta;B_\epsilon} \le CK_{\delta,\epsilon} |g|_{\delta;B_\epsilon}$.

The prove that $|T(g\eta_\epsilon)|_{0;B_\epsilon} \le CK_{\delta;\epsilon}|g|_{0;B_\epsilon}$ follows easily and we omit it.

□

We are now ready to prove local existence of solutions of $Lu = f$ when $f \in C^\delta(B_\epsilon)$

Theorem 3.3. *There exists $\epsilon > 0$ depending on structure so that given $f \in C^\delta(B_\epsilon)$, there exists $u \in C^{2m+\delta}(B_\epsilon)$, such that $Lu = f$ in $B_{\frac{\epsilon}{2}}$*

Proof. For $0 < \epsilon$ to be chosen, we define a sequence of functions g_k in $C^\delta(B_\epsilon)$ as follows:

Let $g_0 = f$ and $g_{k+1} = f - T(g_k\eta_\epsilon)$.

By (3.1), we have g_k in $C^\delta(B_\epsilon)$ and by (3.2) we have

$|g_{k+1} - g_k|_{\delta;B_\epsilon} = |T(\eta_\epsilon(g_k - g_{k-1}))|_{\delta;B_\epsilon} \le CK_{\delta;\epsilon}|g_k - g_{k-1}|_{\delta;B_\epsilon} \le \epsilon^{\beta-\delta}K_{\epsilon,\beta}|g_k - g_{k-1}|_{\delta;B_\epsilon}$.

Choosing ϵ small so that $\epsilon^{\beta-\delta}K_{\epsilon,\beta} < 1$, we can conclude that there exists $g \in C^\delta(B_\epsilon)$ such that

$g = f - T(g\eta_\epsilon)$, which gives $g = f - L(N(g\eta_\epsilon)) + g\eta_\epsilon$. In particular, since $\eta_\epsilon = 1$ in $B_{\frac{\epsilon}{2}}$, we have

$L(N(g\eta_\epsilon)) = f$ in $B_{\frac{\epsilon}{2}}$. Let $u = N(g\eta_\epsilon)$. So $u \in C^{2m+\delta}(B_\epsilon)$ and $Lu = f$ in $B_{\frac{\epsilon}{2}}$

□

4. Local existence in half space. Boundary conditions

In this section we prove local existence of solutions in half space.

We will show that there exists $\epsilon > 0$ such that given $f \in C^\delta(B_\epsilon^+)$ there exists $u \in C^{2m+\delta}(B_\epsilon^+)$ such that $Lu = f$ in B_ϵ^+ and such that u satisfies the m boundary conditions,

$u(x',0) = 0, \frac{\partial u(x',0)}{\partial x_n} = 0, ..., \frac{\partial^{m-1}u(x',0)}{\partial x_n^{m-1}} = 0.$

Given $g \in C^\delta(B_\epsilon)$, we define the approximate solution $N(\eta_\epsilon g)$ by the following formula. For $x \in B_\epsilon^+$ let

$$N(\eta_\epsilon g)(x) = lim_{R\to\infty} \int_{R^n_+} g(y)\eta_\epsilon(y) \int \varphi(\frac{\xi}{R})\frac{e^{i(x-y).\xi}}{p(y,i\xi)}d\xi\, dy \tag{4.6}$$

In order to achieve the boundary conditions, we define $x \in B_\epsilon^+$

$$H(\eta_\epsilon g)(x) = \int_{R^n_+} g(y)\eta_\epsilon(y) \int e^{i(x'-y').\xi'}\Phi(y,x,\xi')d\xi'\, dy \tag{4.7}$$

where $\Phi(y,x,\xi')$ is given by

$$\Phi(y,x,\xi') = \int_{\gamma^+(\xi')}\int_{\gamma^-(\xi')} \frac{e^{-y_n z}e^{x_n w}}{a(y)p^+(y,i\xi',z)p^-(y,i\xi',w)(w-z)}dw\,dz \tag{4.8}$$

where $\gamma^+(\xi')$ denotes any piecewise smooth contour on the right half plane enclosing the roots of $p^+(y,i\xi',z)$. And $\gamma^-(\xi')$ denotes any piecewise smooth contour on the left half plane enclosing the roots of $p^-(y,i\xi',z)$.

4.1. **Auxiliary integrals for half space for operator N.** We proceed to prove the auxiliary integrals we need in half space related to the operator N.

Let us begin by estimating the integral

$$A_R(x,\xi') = \int \frac{e^{ix_n\xi_n}\xi_n^{2m-1}}{p(x,i\xi)}\varphi(\frac{\xi_n}{R})d\xi_n.$$

We claim that $|A_R(x,\xi')| \le C$, for $x \in \mathbf{R}^n_+$, and $\xi' \in \mathbf{R}^{n-1}$.

And $lim_{R\to\infty}A_R(x,\xi')$ exists uniformly for $x \in \mathbf{R}^n_+$, and $\xi' \in \mathbf{R}^{n-1}$.

We prove the claim.

Write

$$p(x,i\xi) = a(x)(i\xi_n)^{2m} + \sum_{k=1}^{2m}(i\xi_n)^{2m-k}\sum_{|\alpha|\le k}a_\alpha(x)(i\xi')^\alpha.$$

Here α is an $n-1$ multi index.

A change of variables $s = x_n\xi_n$, $ds = x_n d\xi_n$, gives

$$A_R = \int \frac{e^{is}s^{2m-1}}{x_n^{2m}p(x,\xi',\frac{s}{x_n})}\varphi(\frac{s}{x_nR})\,ds$$

Note,

$$x_n^{2m}p(x,i\xi',\frac{is}{x_n}) = a(x)(is)^{2m} + \sum_{k=1}^{2m}x_n^k(is)^{2m-k}\sum_{|\alpha|\le k}a_\alpha(x)(i\xi')^\alpha = a(x)(is)^{2m} + q$$

And,

$$|x_n^{2m}p(x,i\xi',\frac{is}{x_n})| \ge C(x_n^{2m} + x_n^{2m}|\xi'|^{2m} + s^{2m}).$$

We write $|a(x)(is)^{2m} + q|^2 = |a(x)|^2s^{4m} + Q$, where $Q = Q(x,s,\xi')$ and note that

$$|a(x)|^2s^{4m} + Q \ge C(x_n^{4m} + x_n^{4m}|\xi'|^{4m} + s^{4m}).$$

And
$$|Q| \leq Cs^{2m}\sum_{k=1}^{2m} x_n^k s^{2m-k}(1+|\xi'|)^k + C\sum_{k=1}^{2m} x_n^{2k}s^{4m-2k}(1+|\xi'|)^{2k}.$$
Hence,
$$A_R = \bar{a}(x)\int \frac{e^{is}s^{4m-1}}{|a(x)|^2 s^{4m}+Q}\varphi(\frac{s}{x_nR})ds + \int \frac{e^{is}s^{2m-1}\bar{q}}{|a(x)|^2 s^{4m}+Q}\varphi(\frac{s}{x_nR})ds = \bar{a}(x)A + B.$$
Write
$$A = \int \frac{\cos(s)s^{4m-1}}{|a(x)|^2 s^{4m}+Q}\varphi(\frac{s}{x_nR})ds + i\int \frac{\sin(s)s^{4m-1}}{|a(x)|^2 s^{4m}+Q}\varphi(\frac{s}{x_nR})ds = A_1 + A_2.$$
Notice that
$$A_1 = \int_{-2x_nR}^{2x_nR} \frac{\cos(s)s^{4m-1}}{|a(x)|^2 s^{4m}+Q}\varphi(\frac{s}{x_nR})ds =$$
$$\int_0^{2x_nR}\cos(s)s^{4m-1}\varphi(\frac{s}{x_nR})(\frac{1}{|a(x)|^2s^{4m}+Q(s)} - \frac{1}{|a(x)|^2s^{4m}+Q(-s)})ds =$$
$$\int_0^{2x_nR}\cos(s)s^{4m-1}\varphi(\frac{s}{x_nR})\frac{Q(s)-Q(-s)}{(|a(x)|^2s^{4m}+Q(s))(|a(x)|^2s^{4m}+Q(-s))}ds$$
Hence, we have
$$|A_1| \leq \int_0^\infty s^{4m-1}\frac{|Q(s)|+|Q(-s)|}{s^{8m}+x_n^{8m}(1+|\xi'|)^{8m}}ds \leq$$
$$\sum_{k=1}^{2m}\int_0^\infty \frac{s^{8m-k-1}x_n^k(1+|\xi'|)^k}{s^{8m}+x_n^{8m}(1+|\xi'|)^{8m}}ds + \sum_{k=1}^{2m}\int_0^\infty \frac{s^{8m-2k-1}x_n^{2k}(1+|\xi'|)^{2k}}{s^{8m}+x_n^{8m}(1+|\xi'|)^{8m}}ds \leq$$
$$c\sum_{k=1}^{2m}\int_0^\infty \frac{t^{8m-2k-1}}{1+t^{8m}}ds \leq C$$
$$A_2 = \int_{-1}^{1}\frac{\sin(s)s^{4m-1}}{|a(x)|^2s^{4m}+Q}\varphi(\frac{s}{x_nR})ds + \int_{|s|\geq 1}\frac{\sin(s)s^{4m-1}}{|a(x)|^2s^{4m}+Q}\varphi(\frac{s}{x_nR})ds = B_1 + B_2$$
$|B_1| \leq 1$

For, B_2, we assume $2x_nR \geq 1$, since otherwise $B_2 = 0$.

Write
$$B_2 = \int_1^{+\infty}\frac{\sin(s)s^{4m-1}}{|a(x)|^2s^{4m}+Q}\varphi(\frac{s}{x_nR})ds + \int_{-\infty}^{-1}\frac{\sin(s)s^{4m-1}}{|a(x)|^2s^{4m}+Q}\varphi(\frac{s}{x_nR})ds = C_1 + C_2$$
$$C_1 = \frac{\cos(1)}{|a(x)|^2s^{4m}+Q(1)} + \int_1^{+\infty}\cos(s)\frac{d}{ds}(\frac{s^{4m-1}\varphi(\frac{s}{x_nR})}{|a(x)|^2s^{4m}+Q(s)})ds.$$

And it follows easily that $|C_1| \leq C$; and exactly the same $|C_2| \leq C$. Also,

$$B = \sum_{k=1}^{2m} \sum_{|\alpha|\leq k} \bar{a}_\alpha(x)(\xi')^\alpha x_n^k \int \frac{e^{is} s^{4m-1-k}}{|a(x)|^2 s^{4m} + Q(s)} \varphi(\frac{s}{x_n R}) ds.$$

Change variables, $t = \frac{s}{x_n(1+|\xi'|)}$ to get

$$B = \sum_{k=1}^{2m} \sum_{|\alpha|\leq k} \bar{a}_\alpha(x)(\xi')^\alpha x_n^k (x_n(1+|\xi'|))^{4m-k} \int \frac{e^{itx_n(1+|\xi'|)} t^{4m-1-k} \varphi(\frac{t(1+|\xi'|)}{R}) dt}{|a(x)|^2 t^{4m} x_n^{4m}(1+|\xi'|)^{4m} + Q(tx_n(1+|\xi'|))} =$$

$$\sum_{k=1}^{2m} \sum_{|\alpha|\leq k} \bar{a}_\alpha(x) \frac{(\xi')^\alpha}{(1+|\xi'|)^k} \int \frac{e^{itx_n(1+|\xi'|)} t^{4m-1-k} \varphi(\frac{t(1+|\xi'|)}{R}) dt}{|a(x)|^2 t^{4m} + Q(tx_n(1+|\xi'|))(x_n(1+|\xi'|))^{-4m}}.$$

Note that $|a(x)|^2 t^{4m} + Q(tx_n(1+|\xi'|))(x_n(1+|\xi'|))^{-4m} \geq C(1+t^{4m})$.

And hence

$$|B| \leq \sum_{k=1}^{2m} \sum_{|\alpha|\leq k} |a_\alpha(x)| \int_{-\infty}^{+\infty} \frac{|t|^{4m-k-1}}{1+t^{4m}} dt \leq C.$$

We use the estimate above, to estimate the integral

$$I_R = \int \frac{e^{i(x'-y').\xi'}}{1+|\xi'|^2} \int \frac{e^{ix_n\xi_n} \xi_n^{2m-1}}{p(x,\xi)} \varphi(\frac{\xi}{R}) d\xi_n \, d\xi'.$$

We claim that $|I_R| \leq \frac{C}{|x'-y'|^{n-3}}$.

We prove this claim.

Write

$$I_R = \frac{1}{|x'-y'|^{2k}} \int (-\Delta_{\xi'})^k e^{i(x'-y').\xi'} \int \frac{e^{ix_n\xi_n} \xi_n^{2m-1}}{(1+|\xi'|^2) p(x,i\xi)} \varphi(\frac{\xi}{R}) d\xi_n \, d\xi' =$$

$$\frac{1}{|x'-y'|^{2k}} \int e^{i(x'-y').\xi'} \int e^{ix_n\xi_n} (-\Delta_{\xi'})^k (\frac{\xi_n^{2m-1}\varphi(\frac{\xi}{R})}{(1+|\xi'|^2)p(x,\xi)}) d\xi_n \, d\xi' =$$

$$\frac{1}{|x'-y'|^{2k}} \int e^{i(x'-y').\xi'} \int e^{ix_n\xi_n} (-\Delta_{\xi'})^k (\frac{\xi_n^{2m-1}\varphi(\frac{\xi}{R})}{(1+|\xi'|^2)p(x,i\xi)}) \varphi(|x'-y'|\xi') d\xi_n \, d\xi' +$$

$$\frac{1}{|x'-y'|^{2k}} \int e^{i(x'-y').\xi'} \int e^{ix_n\xi_n} (-\Delta_{\xi'})^k (\frac{\xi_n^{2m-1}\varphi(\frac{\xi}{R})}{(1+|\xi'|^2)p(x,\xi)}) (1-\varphi(|x'-y'|\xi')) d\xi_n \, d\xi' = A+B.$$

Rewrite

$$A=\frac{1}{|x'-y'|^{2k}}\int_{|\xi'|\leq\frac{2}{|x'-y'|}}(-\Delta_{\xi'})^k(e^{i(x'-y').\xi'}\varphi(|x'-y'|\xi'))\int\frac{e^{ix_n\xi_n}\xi_n^{2m-1}\varphi(\frac{\xi}{R})}{(1+|\xi'|^2)p(x,i\xi)})d\xi_n\,d\xi'.$$

And estimate

$$|A|\leq\int_{|\xi'|\leq\frac{2}{|x'-y'|}}\left|\int\frac{e^{ix_n\xi_n}\xi_n^{2m-1}\varphi(\frac{\xi}{R})}{p(x,i\xi)}d\xi_n\right|\frac{1}{1+|\xi'|^2}d\xi'\leq\int_{|\xi'|\leq\frac{2}{|x'-y'|}}\frac{1}{1+|\xi'|^2}d\xi'\leq\frac{C}{|x'-y'|^{n-3}}$$

by previous estimate.

To estimate B, first note that

$$\left|(-\Delta_{\xi'})^k(\int\frac{e^{ix_n\xi_n}\xi_n^{2m-1}\varphi(\frac{\xi}{R})}{p(x,i\xi)}d\xi_n\frac{1}{1+|\xi'|^2})\right|\leq\frac{C}{(1+|\xi'|)^{2+2k}}$$

again by previous estimate. And, hence we have

$$|B|\leq\frac{C}{|x'-y'|^{2k}}\int_{|\xi'|\geq\frac{1}{|x'-y'|}}\frac{1}{(1+|\xi'|)^{2+2k}}d\xi'\leq\frac{C}{|x'-y'|^{n-3}}$$

Finishing the prove of the claim.

We use the estimate above to prove two auxiliary integrals in half space.

Let

$$I_R=\int_{R^n_+}\eta_\epsilon(y)\int\frac{e^{i(x-y).\xi}\xi_n^{2m}\varphi(\frac{\xi}{R})}{p(x,i\xi)}d\xi\,dy.$$

We claim $|I_R|\leq C$.

Write,

$$I_R=\int_{R^n_+}\eta_\epsilon(y)\frac{\partial}{\partial y_n}(\int\frac{e^{i(x-y).\xi}\xi_n^{2m-1}\varphi(\frac{\xi}{R})}{p(x,i\xi)}d\xi)\,dy=$$

$$=-\int_{R^n_+}\frac{\partial}{\partial y_n}(\eta_\epsilon(y))\int\frac{e^{i(x-y).\xi}\xi_n^{2m-1}\varphi(\frac{\xi}{R})}{p(x,i\xi)}d\xi\,dy+\int_{R^{n-1}}\eta_\epsilon(y',0)\int\frac{e^{i(x'-y').\xi'}e^{ix_n\xi_n}\xi_n^{2m-1}\varphi(\frac{\xi}{R})}{p(x,i\xi)}d\xi\,dy'=$$
A_R+B_R

$$|A_R|\leq\frac{C}{\epsilon}\int_{B^+_\epsilon}\frac{1}{|x-y|^{n-1}}dy\leq C$$

and

$$B_R=\int_{R^{n-1}}\eta_\epsilon(y',0)(1-\Delta_{y'})(\int\frac{e^{i(x'-y').\xi'}e^{ix_n\xi_n}\xi_n^{2m-1}\varphi(\frac{\xi}{R})}{(1+|\xi'|^2)p(x,i\xi)}d\xi)\,dy'=$$

$$\int_{R^{n-1}} (1-\Delta_{y'})\eta_\epsilon(y',0) \int \frac{e^{i(x'-y').\xi'} e^{ix_n\xi_n} \xi_n^{2m-1} \varphi(\frac{\xi}{R})}{(1+|\xi'|^2)p(x,i\xi)} d\xi\, dy'.$$

Hence

$$|B_R| \le \frac{C}{\epsilon^2} \int_{B'_\epsilon} \frac{1}{|x'-y'|^{n-3}} dy' \le C$$

The first inequality following by previous estimate.

This proves the claim.

We now consider a similar integral over a ball.

Let $x, \bar{x} \in \mathbf{R}^n_+$, let $\rho = 2|x-\bar{x}|$ and $\hat{x} = \frac{x+\bar{x}}{2}$ and $B^+_\rho(\hat{x}) = B_\rho(\hat{x}) \cap \mathbf{R}^n_+$.

Let

$$I_R = \int_{B^+_\rho(\hat{x})} \eta_\epsilon(y) \int \frac{e^{i(x-y).\xi} \xi_n^{2m} \varphi(\frac{\xi}{R})}{p(x,i\xi)} d\xi\, dy.$$

We claim $|I_R| \le C$.

Write

$$I_R = \int_{B^+_\rho(\hat{x})} \eta_\epsilon(y) \frac{\partial}{\partial y_n} (\int \frac{e^{i(x-y).\xi} \xi_n^{2m-1} \varphi(\frac{\xi}{R})}{p(x,i\xi)} d\xi)\, dy =$$

$$-\int_{B^+_\rho(\hat{x})} \frac{\partial}{\partial y_n}(\eta_\epsilon(y)) \int \frac{e^{i(x-y).\xi} \xi_n^{2m-1} \varphi(\frac{\xi}{R})}{p(x,i\xi)} d\xi\, dy + \int_{\partial B^+_\rho(\hat{x})} \eta_\epsilon(y) \int \frac{e^{i(x'-y').\xi'} e^{ix_n\xi_n} \xi_n^{2m-1} \varphi(\frac{\xi}{R})}{p(x,i\xi)} d\xi\, dS_y =$$

$A_R + B_R$.

$$|A_R| \le \frac{C}{\epsilon} \int_{B_\epsilon} \frac{1}{|x-y|^{n-1}} dy \le C$$

and

$$B_R = \int_{\partial B^+_\rho(\hat{x}) \cap R^n_+} \eta_\epsilon(y) \int \frac{e^{i(x'-y').\xi'} e^{ix_n\xi_n} \xi_n^{2m-1} \varphi(\frac{\xi}{R})}{p(x,i\xi)} d\xi\, dS_y +$$

$$\int_{B^+_\rho(\hat{x}) \cap \{y_n=0\}} \eta_\epsilon(y) \int \frac{e^{i(x'-y').\xi'} e^{ix_n\xi_n} \xi_n^{2m-1} \varphi(\frac{\xi}{R})}{p(x,i\xi)} d\xi\, dy' = C_R + D_R.$$

$$|C_R| \le \int_{\partial B^+_\rho(\hat{x}) \cap R^n_+} \frac{1}{|x-y|^{n-1}} dS_y \le C.$$

To estimate D_R, let $B^+_\rho(\hat{x}) \cap \{y_n = 0\} = B'_r(\hat{x}')$.

Write

$$D_R = \int_{B'_r(\hat{x}')} \eta_\epsilon(y',0) \int \frac{e^{i(x'-y').\xi'} e^{ix_n\xi_n} \xi_n^{2m-1} \varphi(\frac{\xi}{R})}{p(x,i\xi)} d\xi\, dy' =$$

$$\int_{B_r'(\hat{x}')} (\eta_\epsilon(y',0) - \eta_\epsilon(x',0)) \int \frac{e^{i(x'-y').\xi'} e^{ix_n\xi_n} \xi_n^{2m-1} \varphi(\frac{\xi}{R})}{p(x,i\xi)} d\xi\, dy' +$$

$$\eta_\epsilon(x',0) \int_{B_r'(\hat{x}')} \int \frac{e^{i(x'-y').\xi'} e^{ix_n\xi_n} \xi_n^{2m-1} \varphi(\frac{\xi}{R})}{p(x,i\xi)} d\xi\, dy' = E_R + \eta_\epsilon(x',0) F_R.$$

$$|E_R| \le \frac{C}{\epsilon} \int_{B_r'(\hat{x}')} |x'-y'|^{2-n} dy' \le C$$

since $r \le \rho \le 2\epsilon$.

To estimate F_R, consider first the case $\hat{x}_n \ge \frac{\rho}{2}$. Note that $\hat{x}_n - x_n \le |\hat{x}_n - x_n| \le |x - \hat{x}| = \frac{\rho}{4}$. Therefore $x_n \ge \hat{x}_n - \frac{\rho}{4} \ge \frac{\rho}{4}$.

In this case

$$|F_R| \le \int_{B_r'(\hat{x}')} \frac{1}{(|x'-y'|^2 + x_n^2)^{\frac{n-1}{2}}} dy' \le \frac{1}{x_n^{n-1}} \int_{B_r'(\hat{x}')} 1 dy' \le C(\frac{r}{x_n})^{n-1} \le C(\frac{\rho}{x_n})^{n-1} \le C.$$

For the case $\hat{x}_n \le \frac{\rho}{2}$, note that since $r^2 + x_n^2 = \rho^2$, we get $r^2 \ge \rho^2 - \frac{\rho^2}{4} = \frac{3}{4}\rho^2$, so $r \ge \frac{\sqrt{3}}{2}\rho$ and $|x' - \hat{x}'| \le |x - \hat{x}| = \frac{\rho}{4} \le \frac{1}{2\sqrt{3}} r$ and hence, $\mathrm{dist}(x', \partial B_r'(\hat{x}')) \ge (1 - \frac{1}{2\sqrt{3}})r$.

Hence, in this case we write

$$F_R = \int_{B_r'(\hat{x}')} (1 - \Delta_{y'}) \int \frac{e^{i((x'-y').\xi'} e^{ix_n\xi_n} \xi_n^{2m-1} \varphi(\frac{\xi}{R})}{(1+|\xi'|^2) p(x,i\xi)} d\xi\, dy' =$$

$$\int_{B_r'(\hat{x}')} \int \frac{e^{i(x'-y').\xi'} e^{ix_n\xi_n} \xi_n^{2m-1} \varphi(\frac{\xi}{R})}{(1+|\xi'|^2) p(x,i\xi)} d\xi\, dy' - \int_{\partial B_r'(\hat{x}')} \left\langle \nabla_{y'} \int \frac{e^{i((x'-y').\xi'} e^{ix_n\xi_n} \xi_n^{2m-1} \varphi(\frac{\xi}{R})}{(1+|\xi'|^2) p(x,i\xi)} d\xi\, , \frac{y' - \hat{x}'}{r} \right\rangle dS_{y'}.$$

$= G_R + H_R$

$$|G_R| \le \int_{B_r'(\hat{x}')} \frac{1}{|x'-y'|^{n-3}} dy' \le C.$$

and

$$|H_R| \le C \int_{\partial B_r'(\hat{x}')} \frac{1}{|x'-y'|^{n-2}} dS_{y'} \le \frac{C}{r^{n-2}} \sigma(\partial B_r'(\hat{x}')) \le C.$$

This finishes the estimation of the auxiliary integrals in half space for the operator N.

4.2. **Auxiliary integrals for half space for operator H.** We now proceed to prove estimates for the operator H.

First, let us be more precise about the contours $\gamma^{-+}(\xi')$.

We take into account that the m roots of $p^-(y, i\xi', z)$ satisfy $Re(\lambda^-(y, \xi')) \leq -\lambda(1+|\xi'|)$ and $|\lambda^-(y, \xi')| \leq \Lambda(1+|\xi'|)$ so we take $\gamma^-(\xi')$ a piecewise smooth contour parametrized by an angle θ which is the arc of the circle centered at 0 of radius $2\Lambda(1+|\xi'|)$ joining the points with real part equal to $\frac{-\lambda}{2}(1+|\xi'|)$ in counterclockwise sense, followed by the vertical segment joining these two points. Denote by $w(\theta, \xi')$ the points on this contour.

Similarly, since the m roots of $p^+(y, i\xi', z)$ satisfy $Re(\lambda^+(y, \xi')) \geq \lambda(1+|\xi'|)$ and $|\lambda^+(y, \xi)| \leq \Lambda(1+|\xi'|)$ we take $\gamma^+(\xi')$ piecewise smooth contour parametrized by an angle ϕ which is the arc of the circle centered at 0 of radius $2\Lambda(1+|\xi'|)$ joining the points with real part equal to $\frac{\lambda}{2}(1+|\xi'|)$ in counterclockwise sense, followed by the vertical segment joining these two points. Denote by $z(\phi, \xi')$ the points on this contour.

The following estimates follow directly from the definition.

We have for all ξ',y, θ and ϕ, that

$$\frac{\lambda}{2}(1+|\xi'|) \leq |w(\theta,\xi')-\lambda^-(y,\xi')| \leq 2\Lambda(1+|\xi'|) \qquad \frac{\lambda}{2}(1+|\xi'|) \leq |z(\phi,\xi')-\lambda^+(y,\xi')| \leq 2\Lambda(1+|\xi'|).$$

And

$$\frac{\lambda}{2}(1+|\xi'|) \leq |w(\theta, \xi') - z(\phi, \xi')| \leq 4\Lambda(1+|\xi'|).$$

And for all ξ',y, θ and ϕ,and any multi index α, we have

$$|D^\alpha_{\xi'} w(\theta, \xi') \leq C(1+|\xi'|)^{1-|\alpha|} \qquad |D^\alpha_{\xi'} z(\phi, \xi') \leq C(1+|\xi'|)^{1-|\alpha|}.$$

$$|D^\alpha_{\xi'} D_\theta w(\theta, \xi') \leq C(1+|\xi'|)^{1-|\alpha|} \qquad |D^\alpha_{\xi'} D_\phi z(\phi, \xi') \leq C(1+|\xi'|)^{1-|\alpha|}.$$

To continue,

let $\Phi_\beta(\xi', x, y)$ denote either one of the functions

$$\Phi_\beta(\xi', x, y) = \int_{\gamma^+(\xi')} \int_{\gamma^-(\xi')} \frac{e^{-y_n z} e^{x_n w} w^{|\beta|}}{p^+(y, i\xi', z) p^-(y, i\xi', w)(w-z)} dw\, dz$$

or

$$\Phi_\beta(\xi', x, y) = \int_{\gamma^+(\xi')} \int_{\gamma^-(\xi')} \frac{e^{-y_n z} e^{x_n w} (\xi')^\beta}{p^+(y, i\xi', z) p^-(y, i\xi', w)(w-z)} dw\, dz.$$

We claim that

$$|D^{\alpha}_{\xi'}\Phi_{\beta}(\xi', x, y)| \leq C\frac{e^{-\frac{(x_n+y_n)(1+|\xi'|)}{2}}}{(1+|\xi'|)^{2m-1+|\alpha|-|\beta|}}.$$

We prove the claim for the first function. The other is exactly the same.

To prove this claim, note that we can write $\Phi_{\beta}(\xi', x, y)$ as a sum of four terms of the form

$$\int_{\theta}\int_{\phi}\frac{e^{-y_n z}e^{x_n w}w^{|\beta|}}{p^{+}(y, i\xi', z)p^{-}(y, i\xi', w)(w-z)}z_{\phi}w_{\theta}d\phi\, d\theta.$$

for appropriate limits of integration in ϕ and θ that do not depend on ξ' and where $z = z(\phi, \xi')$ and $w = w(\theta, \xi')$.

Write the integrand as $\frac{g_1 g_2}{g_3}$ where $g_1 = e^{-y_n z}e^{x_n w}w^{|\beta|}$, $g_2 = z_{\phi}w_{\theta}$ and $g_3 = p^{+}(y, i\xi', z)p^{-}(y, i\xi', w)(w-z)$. By direct computation we obtain that

$$|D^{\gamma}_{\xi'}g_1| \leq C\frac{e^{-\frac{(x_n+y_n)(1+|\xi'|)}{2}}}{(1+|\xi'|)^{|\gamma|-|\beta|}}. \qquad |D^{\gamma}_{\xi'}g_2| \leq C(1+|\xi'|)^{2-|\gamma|}.$$

Also note that we can write

$$g_3 = \prod_{j=1}^{2m+1}(\phi_j(\xi') - \psi_j(\xi'))$$

where $|\phi_j(\xi') - \psi_j(\xi'| \approx 1 + |\xi|$ and $|D^{\alpha}_{\xi'}(\phi_j(\xi') - \psi_j(\xi'))| \leq C(1+|\xi'|)^{1-|\alpha|}$. It follows that

$$|D^{\gamma}_{\xi'}g_3| \leq C(1+|\xi'|)^{2m+1-|\gamma|}$$

and hence

$$|D^{\gamma}_{\xi'}(g_3)^{-1}| \leq C\frac{1}{(1+|\xi'|)^{2m+1+|\gamma|}}.$$

The claim is a direct consequence of the estimates above.

We use this estimate to prove that for any multi index β such that $n-2m+|\beta| > 0$ we have

$$|\int e^{i(x'-y').\xi'}\Phi_{\beta}(\xi', x, y)d\xi'| \leq C\frac{1}{(|x'-y'| + (x_n+y_n))^{n-2m+|\beta|}}. \tag{4.9}$$

To prove this, write

$$\int e^{i(x'-y').\xi'}\Phi_{\beta}(\xi', x, y)d\xi' = \frac{1}{|x'-y'|^{2k}}\int e^{i(x'-y').\xi'}(-\Delta_{\xi'})^{k}\Phi_{\beta}(\xi', x, y)d\xi' =$$

$$\frac{1}{|x'-y'|^{2k}}\int(-\Delta_{\xi'})^k(e^{i(x'-y').\xi'}\varphi(|x'-y'|\xi'))\Phi_\beta(\xi',x,y)d\xi'+$$
$$\frac{1}{|x'-y'|^{2k}}\int e^{i(x'-y').\xi'}(-\Delta_{\xi'})^k(\Phi_\beta(\xi',x,y))(1-\varphi(|x'-y'|\xi'))d\xi'=A+B.$$
$$|A|\leq C\int_{|\xi'|\leq\frac{2}{|x'-y'|}}\Phi_\beta(\xi',x,y))d\xi'\leq C\int_{|\xi'|\leq\frac{2}{|x'-y'|}}\frac{e^{-\frac{(x_n+y_n)(1+|\xi'|)}{2}}}{(1+|\xi'|)^{2m-1-|\beta|}}d\xi':=Q$$

Now
$$Q\leq C\int_{|\xi'|\leq\frac{2}{|x'-y'|}}\frac{1}{(1+|\xi'|)^{2m-1+-|\beta|}}d\xi'\leq\frac{C}{|x'-y'|^{n-2m+|\beta|}}.$$

And also setting $\zeta'=(x_n+y_n)\xi'$, we get
$$Q\leq\frac{(x_n+y_n)^{2m-1-|\beta|}}{(x_n+y_n)^{n-1}}\int_{|\zeta'|\leq\frac{2(x_n+y_n)}{|x'-y'|}}\frac{e^{-\frac{|\zeta'|}{2}}}{(x_n+y_n+|\zeta'|)^{2m-1-|\beta|}}d\zeta'.$$

Since $2m-1-|\beta|<n-1$, we get
$$Q\leq C\frac{1}{(x_n+y_n)^{n-2m+|\beta|}}\int_{R^{n-1}}\frac{e^{-\frac{|\zeta'|}{2}}}{|\zeta'|^{2m-1-|\beta|}}d\zeta'\leq C\frac{1}{(x_n+y_n)^{n-2m+|\beta|}}.$$

So
$$|A|\leq C\min\{\frac{1}{|x'-y'|^{n-2m+|\beta|}};\frac{1}{(x_n+y_n)^{n-2m+|\beta|}}\}\leq C\frac{1}{(|x'-y'|+x_n+y_n)^{n-2m+|\beta|}}.$$

And
$$|B|\leq\frac{1}{|x'-y'|^{2k}}\int_{|\xi'|\geq\frac{1}{|x'-y'|}}\frac{e^{\frac{-(x_n+y_n)(1+|\xi'|)}{2}}}{(1+|\xi'|)^{2m-1-|\beta|+2k}}d\xi':=Q$$
$$Q\leq\frac{1}{|x'-y'|^{2k}}\int_{|\xi'|\geq\frac{1}{|x'-y'|}}\frac{1}{(1+|\xi'|)^{2m-1+-|\beta|+2k}}d\xi'\leq C\frac{1}{|x'-y'|^{n-2m+|\beta|}}.$$

And also if $\frac{1}{x_n+y_n}\leq\frac{1}{|x'-y'|}$, setting $\zeta'=(x_n+y_n)\xi'$, we get
$$Q\leq C\frac{1}{(x_n+y_n)^{n-2m+|\beta|}}(\frac{x_n+y_n}{|x'-y'|})^{2k}\int_{|\zeta'|\geq\frac{x_n+y_n}{|x'-y'|}}\frac{e^{-\frac{|\zeta'|}{2}}}{(x_n+y_n+|\zeta'|)^{2m-1-|\beta|+2k}}d\zeta'.$$

Noting that $\rho:=\frac{x_n+y_n}{|x'-y'|}\geq 1$ we get
$$Q\leq C\frac{1}{(x_n+y_n)^{n-2m+|\beta|}}\rho^{2k}\int_{|\zeta'|\geq\rho}e^{-\frac{|\zeta'|}{2}}d\zeta'\leq C\frac{1}{(x_n+y_n)^{n-2m+|\beta|}}.$$

And hence also

$$|B| \leq C\frac{1}{(|x'-y'|+x_n+y_n)^{n-2m+|\beta|}}.$$

And (4.9) is proved.

We will apply the above estimates to the function

$$\Psi(\xi',x,y)=\int_{\gamma^+(\xi')}\int_{\gamma^-(\xi')}\frac{e^{-y_nz}e^{x_nw}p(x,i\xi',w)}{a(y)p^+(y,i\xi',z)p^-(y,i\xi',w)(w-z)}dw\,dz.$$

Note that Ψ can also be written as

$$\Psi(\xi',x,y)=\int_{\gamma^+(\xi')}\int_{\gamma^-(\xi')}\frac{e^{-y_nz}e^{x_nw}(p(x,i\xi',w)-p(y,\xi',w))}{a(y)p^+(y,i\xi',z)p^-(y,i\xi',w)(w-z)}dw\,dz=$$

$$\sum_{k=0}^{2m}\sum_{|\alpha|\leq k}(a_\alpha(x)-a_\alpha(y))(i\xi')^\alpha\int_{\gamma^+(\xi')}\int_{\gamma^-(\xi')}\frac{e^{-y_nz}e^{x_nw}w^{2m-k}}{a(y)p^+(y,i\xi',z)p^-(y,i\xi',w)(w-z)}dw\,dz.$$

And we have

$$|D^\alpha_{\xi'}\Psi(\xi',x,y)|\leq CK_{\epsilon;\delta}|x-y|^\delta\frac{e^{-\frac{(x_n+y_n)(1+|\xi'|)}{2}}}{(1+|\xi'|)^{-1+|\alpha|}}.$$

And hence,

$$|\int e^{i(x'-y').\xi'}\Psi(\xi',x,y)d\xi'|\leq CK_{\epsilon;\delta}\frac{|x-y|^\delta}{(|x'-y'|+x_n+y_n)^n}. \tag{4.10}$$

just as in (4.9) Another function which we need to consider is

$$\Theta(\xi',x,y,\bar{x})=\int_{\gamma^+(\xi')}\int_{\gamma^-(\xi')}\frac{e^{-y_nz}e^{x_nw}p(x,i\xi',w)(p^-(\bar{x},i\xi',w)-p^-(y,\xi',w))}{a(y)(w-z)p^+(y,i\xi',z)p^-(y,i\xi',w)p^-(\bar{x},i\xi',w)}dw\,dz$$

To estimate Θ, first notice that we can write

$$p^-(\bar{x},i\xi',w)-p^-(y,i\xi',w)=\sum_{k=0}^{m}(a_k(\bar{x},\xi')-a_k(y,\xi')w^{m-k}$$

where $a_k(\bar{x},\xi')$ are the symmetric functions of the m roots $\lambda(\bar{x},\xi')$ of $p^-(\bar{x},i\xi',w)$. This means that $a_k(\bar{x},\xi')$ is a sum of products of k of the roots.

Using (2.4), it follows that for $k=1,...,m$, we have

$$|D^\alpha_{\xi'}(a_k(\bar{x},\xi')-a_k(y,\xi'))|\leq CK_{\epsilon;\delta}\frac{|\bar{x}-y|^\delta}{(1+|\xi'|)^{-k+|\alpha|}}.$$

We now claim that

$$|D^{\alpha}_{\xi'}\Theta(\xi',x,y,\bar{x})| \leq CK_{\epsilon;\delta}\frac{|\bar{x}-y|^{\delta}e^{-\frac{(x_n+y_n)(1+|\xi'|)}{2}}}{(1+|\xi'|)^{-1+|\alpha|}}.$$

Note

$$D^{\alpha}_{\xi'}\left((a_k(\bar{x},\xi')-a_k(y,\xi'))\int_{\gamma^+(\xi')}\int_{\gamma^-(\xi')}\frac{e^{-y_nz}e^{x_nw}p(x,i\xi',w)w^{m-k}}{a(y)(w-z)p^+(y,i\xi',z)p^-(y,i\xi',w)p^-(\bar{x},i\xi',w)}dw\,dz\right)=$$

$$\sum_{|\gamma|+|\beta|=|\alpha|}D^{\gamma}_{\xi'}\left((a_k(\bar{x},\xi')-a_k(y,\xi'))\right)D^{\beta}_{\xi'}\left(\int_{\gamma^+(\xi')}\int_{\gamma^-(\xi')}\frac{e^{-y_nz}e^{x_nw}p(x,i\xi',w)w^{m-k}}{a(y)(w-z)p^+(y,i\xi',z)p^-(y,i\xi',w)p^-(\bar{x},i\xi',w)}dw\,dz\right)$$

Hence

$$|D^{\alpha}_{\xi'}\Theta(\xi',x,y,\bar{x})| \leq \sum_{|\gamma|+|\beta|=|\alpha|}CK_{\epsilon;\delta}\frac{|\bar{x}-y|^{\delta}}{(1+|\xi'|)^{-k+|\gamma|}}\,\frac{e^{-\frac{(x_n+y_n)(1+|\xi'|)}{2}}}{(1+|\xi'|)^{-1+k+|\beta|}} \leq$$

$$CK_{\epsilon;\delta}\frac{|\bar{x}-y|^{\delta}e^{-\frac{(x_n+y_n)(1+|\xi'|)}{2}}}{(1+|\xi'|)^{-1+|\alpha|}}.$$

And exactly as (4.9) we get the estimate,

$$|\int e^{i(\bar{x}'-y').\xi'}\Theta(\xi',x,y,\bar{x})d\xi'| \leq CK_{\epsilon;\delta}\frac{|\bar{x}-y|^{\delta}}{(|\bar{x}'-y'|+x_n+y_n)^n}. \tag{4.11}$$

We now prove two auxiliary integrals in half space for the operator H.

For $x,\bar{x}\in B^+_{\epsilon}$, and $x_n\geq\bar{x}_n$, consider

$$I=\int_{R^n_+}\eta_{\epsilon}(y)\int e^{i(\bar{x}'-y').\xi'}\int_{\gamma^+(\xi')}\int_{\gamma^-(\xi')}\frac{e^{-y_nz}e^{x_nw}p(x,i\xi',w)}{a(y)p^+(y,i\xi',z)p^-(y,i\xi',w)(w-z)}\,dw\,dz\,d\xi'\,dy.$$

We show using (4.10)and (4.11) that $|I|\leq CK_{\epsilon;\delta}$.

Write

$$I=\int_{R^n_+}\eta_{\epsilon}(y)\int e^{i(\bar{x}'-y').\xi'}\int_{\gamma^+(\xi')}\int_{\gamma^-(\xi')}\frac{e^{-y_nz}e^{x_nw}p(x,i\xi',w)}{a(x)p^+(x,i\xi',z)p^-(x,i\xi',w)(w-z)}\,dw\,dz\,d\xi'\,dy+$$

$$\int_{R^n_+}\eta_{\epsilon}(y)\int e^{i(\bar{x}'-y').\xi'}\int_{\gamma^+(\xi')}\int_{\gamma^-(\xi')}\frac{e^{-y_nz}e^{x_nw}p(x,i\xi',w)}{(w-z)}$$

$$\left(\frac{1}{a(y)p^+(y,i\xi',z)p^-(y,i\xi',w)}-\frac{1}{a(x)p^+(x,i\xi',z)p^-(x,i\xi',w)}\right)dw\,dz\,d\xi'\,dy.$$

$=A+B$

Notice that

$$A = \int_{R^n_+} \eta_\epsilon(y) \int e^{i(\bar{x}'-y').\xi'} \int_{\gamma^+(\xi')} \int_{\gamma^-(\xi')} \frac{e^{-y_n z} e^{x_n w}(p(x,i\xi',w)-p(\bar{x},i\xi',w)}{a(x)p^+(x,i\xi',z)p^-(x,i\xi',w)(w-z)}\, dw\, dz\, d\xi'\, dy =$$

$$\int_{R^n_+} \eta_\epsilon(y)(1-\Delta_{y'}) \int \frac{e^{i(\bar{x}'-y').\xi'}}{1+|\xi'|^2} \int_{\gamma^+(\xi')} \int_{\gamma^-(\xi')} \frac{e^{-y_n z} e^{x_n w}(p(x,i\xi',w)-p(\bar{x},i\xi',w)}{a(x)p^+(x,i\xi',z)p^-(x,i\xi',w)(w-z)}\, dw\, dz\, d\xi'\, dy =$$

$$= \int_{R^n_+} (1-\Delta_{y'})\eta_\epsilon(y) \int \frac{e^{i(\bar{x}'-y').\xi'}}{1+|\xi'|^2} \int_{\gamma^+(\xi')} \int_{\gamma^-(\xi')} \frac{e^{-y_n z} e^{x_n w}(p(x,i\xi',w)-p(\bar{x},i\xi',w)}{a(x)p^+(x,i\xi',z)p^-(x,i\xi',w)(w-z)}\, dw\, dz\, d\xi'\, dy.$$

Hence,

$$|A| \le CK_{\epsilon;\delta}|x-\bar{x}|^\delta \frac{1}{\epsilon^2} \int_{B^+_\epsilon} \frac{1}{(|\bar{x}'-y'|+x_n+y_n)^{n-2}} dy \le$$

$$CK_{\epsilon;\delta}|x-\bar{x}|^\delta \frac{1}{\epsilon^2} \int_{B^+_\epsilon} \frac{1}{(|\bar{x}'-y'|+\bar{x}_n+y_n)^{n-2}} dy \le CK_{\epsilon;\delta}|x-\bar{x}|^\delta \le CK_{\epsilon;\delta}\epsilon^\delta.$$

$B = C_1 + C_2 + C_3$.

where

$$C_1 = \int_{R^n_+} \eta_\epsilon(y) \int e^{i(\bar{x}'-y').\xi'} \int_{\gamma^+(\xi')} \int_{\gamma^-(\xi')} \frac{e^{-y_n z} e^{x_n w} p(x,i\xi',w)}{p^-(\bar{x},i\xi',w)(w-z)} \frac{(p^-(\bar{x},i\xi',w)-p^-(y,i\xi',w))}{a(y)p^+(y,i\xi',z)p^-(y,i\xi',w)}\, dw\, dz\, d\xi'\, dy$$

and C_2 and C_3 are similar terms.

$$|C_1| \le CK_{\epsilon;\delta} \int_{R^n_+} \eta_\epsilon(y) \frac{|\bar{x}-y|^\delta}{(|\bar{x}'-y'|+x_n+y_n)^n}\, dy \le CK_{\epsilon;\delta} \int_{R^n_+} \eta_\epsilon(y) \frac{|\bar{x}-y|^\delta}{(|\bar{x}'-y'|+\bar{x}_n+y_n)^n}\, dy \le CK_{\epsilon;\delta}\epsilon^\delta.$$

This finishes the estimation of I.

Finally, we consider an auxiliary integral over a ball in half space.

Let

$$I_B = \int_{B^+_\rho(\hat{x})} \eta_\epsilon(y) \int e^{i(\bar{x}'-y').\xi'} \int_{\gamma^+(\xi')} \int_{\gamma^-(\xi')} \frac{e^{-y_n z} e^{x_n w}(p(x,i\xi',w)-p(\bar{x},i\xi',w))}{(w-z)a(\bar{x})p^+(\bar{x},i\xi',z)p^-(\bar{x},i\xi',w)}\, dw\, dz\, d\xi'\, dy.$$

We show that $|I_B| \le CK_{\epsilon;\delta}|x-\bar{x}|^\delta$.

$$I_B = \int_{B^+_\rho(\hat{x})} (\eta_\epsilon(y)-\eta_\epsilon(\bar{x})) \int e^{i(\bar{x}'-y').\xi'} \int_{\gamma^+(\xi')} \int_{\gamma^-(\xi')} \frac{e^{-y_n z} e^{x_n w}(p(x,i\xi',w)-p(\bar{x},i\xi',w))}{(w-z)a(\bar{x})p^+(\bar{x},i\xi',z)p^-(\bar{x},i\xi',w)}\, dw\, dz\, d\xi'\, dy+$$

$$\eta_\epsilon(\bar{x}) \int_{B^+_\rho(\hat{x})} \int e^{i(\bar{x}'-y').\xi'} \int_{\gamma^+(\xi')} \int_{\gamma^-(\xi')} \frac{e^{-y_n z} e^{x_n w}(p(x,i\xi',w)-p(\bar{x},i\xi',w))}{(w-z)a(\bar{x})p^+(\bar{x},i\xi',z)p^-(\bar{x},i\xi',w)}\, dw\, dz\, d\xi'\, dy =$$

$A + \eta_\epsilon(\bar{x})B$.

$$|A| \leq CK_{\epsilon;\delta}|x-\bar{x}|^\delta \frac{1}{\epsilon}\int_{B^+_\rho(\hat{x})} |y-\bar{x}| \frac{1}{(|\bar{x}'-y'|+x_n+y_n)^n} dy$$

$$\leq CK_{\epsilon;\delta}|x-\bar{x}|^\delta \frac{1}{\epsilon}\int_{B^+_\rho(\hat{x})} |y-\bar{x}| \frac{1}{(|\bar{x}'-y'|+\bar{x}_n+y_n)^n} dy$$

$$\leq CK_{\epsilon;\delta}|x-\bar{x}|^\delta \frac{\rho}{\epsilon} \leq CK_{\epsilon;\delta}|x-\bar{x}|^\delta.$$

$$B = \int_{B^+_\rho(\hat{x})} (1-\Delta_{y'}) \int \frac{e^{i(\bar{x}'-y').\xi'}}{1+|\xi'|^2} \int_{\gamma^+(\xi')} \int_{\gamma^-(\xi')} \frac{e^{-y_n z} e^{x_n w}(p(x,i\xi',w)-p(\bar{x},i\xi',w))}{(w-z)a(\bar{x})p^+(\bar{x},i\xi',z)p^-(\bar{x},i\xi',w)}\, dw\, dz\, d\xi'\, dy =$$

$$\int_{B^+_\rho(\hat{x})} \int \frac{e^{i(\bar{x}'-y').\xi'}}{1+|\xi'|^2} \int_{\gamma^+(\xi')} \int_{\gamma^-(\xi')} \frac{e^{-y_n z} e^{x_n w}(p(x,i\xi',w)-p(\bar{x},i\xi',w))}{(w-z)a(\bar{x})p^+(\bar{x},i\xi',z)p^-(\bar{x},i\xi',w)}\, dw\, dz\, d\xi'\, dy -$$

$$\int_{\partial B^+_\rho(\hat{x})\cap R^n_+} \left\langle \nabla_{y'}\left(\int \frac{e^{i(\bar{x}'-y').\xi'}}{1+|\xi'|^2} \int_{\gamma^+(\xi')} \int_{\gamma^-(\xi')} \frac{e^{-y_n z} e^{x_n w}(p(x,i\xi',w)-p(\bar{x},i\xi',w))}{(w-z)a(\bar{x})p^+(\bar{x},i\xi',z)p^-(\bar{x},i\xi',w)}\, dw\, dz\, d\xi' \right), \frac{y'-\hat{x}'}{\rho} \right\rangle dS_y =$$

$C_1 + C_2$. We have

$$|C_1| \leq CK_{\epsilon;\delta}|x-\bar{x}|^\delta \int_{B^+_\rho(\hat{x})} \frac{1}{(|\bar{x}'-y'|+x_n+y_n)^{n-2}}\, dy$$

$$\leq CK_{\epsilon;\delta}|x-\bar{x}|^\delta \int_{B^+_\rho(\hat{x})} \frac{1}{(|\bar{x}'-y'|+\bar{x}_n+y_n)^{n-2}}\, dy \leq CK_{\epsilon;\delta}|x-\bar{x}|^\delta.$$

and

$$C_2 \leq CK_{\epsilon;\delta}|x-\bar{x}|^\delta \int_{\partial B^+_\rho(\hat{x})\cap R^n_+} \frac{1}{(|\bar{x}'-y'|+x_n+y_n)^{n-1}}\, dS_y$$

$$\leq CK_{\epsilon;\delta}|x-\bar{x}|^\delta \int_{\partial B^+_\rho(\hat{x})\cap R^n_+} \frac{1}{(|\bar{x}'-y'|+\bar{x}_n+y_n)^{n-1}}\, dS_y$$

$$\leq CK_{\epsilon;\delta}|x-\bar{x}|^\delta \int_{\partial B^+_\rho(\hat{x})\cap R^n_+} \frac{1}{|\bar{x}-y|^{n-1}}\, dS_y \leq CK_{\epsilon;\delta}|x-\bar{x}|^\delta.$$

This finishes the auxiliary integrals in half space.

We record the to auxiliary estimates we just proved:

$$|I| \leq CK_{\epsilon;\delta} \qquad |I_B| \leq CK_{\epsilon;\delta}|x-\bar{x}|^\delta \tag{4.12}$$

4.3. **Main results in half space.** Just as before, by using a cutoff function one shows that for $|\alpha| \leq 2m$, we have for $x \in R^n_+$

$$D^\alpha(N(g\eta_\epsilon))(x) = \lim_{R\to\infty} \int_{R^n_+} g(y)\eta_\epsilon(y) \int \frac{e^{i(x-y).\xi}(i\xi)^\alpha \varphi(\frac{\xi}{R})}{p(y,i\xi)} d\xi\, dy.$$

And hence,

$$L(N(g\eta_\epsilon))(x) = \lim_{R\to\infty} \int_{R^n_+} g(y)\eta_\epsilon(y) \int \frac{e^{i(x-y).\xi} p(x,i\xi)\varphi(\frac{\xi}{R})}{p(y,i\xi)} d\xi\, dy.$$

Therefore, we have

$$L(N(g\eta_\epsilon))(x) - g(x)\eta_\epsilon(x) = \lim_{R\to\infty} \int_{R^n_+} g(y)\eta_\epsilon(y) \int \frac{e^{i(x-y).\xi}(p(x,i\xi)-p(y,i\xi))\varphi(\frac{\xi}{R})}{p(y,i\xi)} d\xi\, dy.$$

Set $T(g\eta_\epsilon)(x) = L(N(g\eta_\epsilon))(x) - g(x)\eta_\epsilon(x)$

By repeating the same proof for free space, this time using the auxiliary integrals in half space we can prove the following theorem.

Theorem 4.1. $|T(g\eta_\epsilon)|_{\delta;B^+_\epsilon} \leq CK_{\delta;\epsilon}|g|_{\delta;B^+_\epsilon}$.

Proof. The proof is exactly the same as the one in free space using this time the auxiliary integrals in half space. □

By arguments like the one in free space one also shows that for any $|\alpha| \leq 2m$ except $\alpha = 2me_n$, we have

$$D^\alpha N(g\eta_\epsilon) \in C^\delta(B^+_\epsilon).$$

To show that $\frac{\partial^{2m} N(g\eta_\epsilon)}{\partial x_n^{2m}} \in C^\delta(B^+_\epsilon)$, we note that

$$a(x)\frac{\partial^{2m} N(g\eta_\epsilon)}{\partial x_n^{2m}} = -\sum_{\alpha\neq 2me_n} a_\alpha(x) D^\alpha N(g\eta_\epsilon)(x) + L(N(g\eta_\epsilon))(x)$$

which is a sum of two functions in $C^\delta(B^+_\epsilon)$.

Since $a \geq \lambda$ and $a \in C^\delta(B^+_\epsilon)$ the result follows.

We now proceed to prove estimates for the operator H.

We have that

$$L(H(g\eta_\epsilon))(x) = \int_{R^n_+} g(y)\eta_\epsilon(y) \int e^{i(x'-y').\xi'} \int_{\gamma^+(\xi')} \int_{\gamma^-(\xi')} \frac{e^{-y_n z} e^{x_n w} p(x,i\xi',w)}{a(y)p^+(y,i\xi',z)p^-(y,i\xi',w)(w-z)} dw\, dz\, d\xi'\, dy =$$

$$\int_{R^n_+} g(y)\eta_\epsilon(y)\int e^{i(x'-y').\xi'}\int_{\gamma^+(\xi')}\int_{\gamma^-(\xi')}\frac{e^{-y_nz}e^{x_nw}(p(x,i\xi',w)-p(y,i\xi',w))}{a(y)p^+(y,i\xi',z)p^-(y,i\xi',w)(w-z)}dw\,dz\,d\xi'\,dy.$$

where the last equality follows since $f(w) := \frac{e^{x_nw}p^+(y,i\xi',w)}{w-z}$ is analytic inside $\gamma^-(\xi')$ for each fixed $z \in \gamma^+(\xi')$.

We now prove the main estimate for the operator H. Set $F(g\eta_\epsilon) = L(H(g\eta_\epsilon))$.

Theorem 4.2. $|F(g\eta_\epsilon)|_{\delta;B^+_\epsilon} \le CK_{\delta;\epsilon}|g|_{\delta;B^+_\epsilon}$

Proof. Set

$$\Psi(x,y,\xi') = \int_{\gamma^+(\xi')}\int_{\gamma^-(\xi')}\frac{e^{-y_nz}e^{x_nw}(p(x,i\xi',w)-p(y,i\xi',w))}{a(y)p^+(y,i\xi',z)p^-(y,i\xi',w)(w-z)}dw\,dz.$$

Let $x,\bar{x} \in B^+_\epsilon$ and may assume $x_n \ge \bar{x}_n$.

Write $F(g\eta_\epsilon)(x) - F(g\eta_\epsilon)(\bar{x}) =$

$$\int_{R^n_+} g(y)\eta_\epsilon(y)\int (e^{i(x'-y').\xi'} - e^{i(\bar{x}'-y').\xi'})\Psi(x,y,\xi')d\xi'\,dy +$$

$$\int_{R^n_+} g(y)\eta_\epsilon(y)\int e^{i(\bar{x}'-y').\xi'}(\Psi(x,y,\xi') - \Psi(\bar{x},y,\xi'))d\xi'\,dy =$$

$A+B$. Estimation of the term A: Write

$$A = \int_{R^n_+}(g(y)-g(\bar{x}))\eta_\epsilon(y)\int (e^{i(x'-y').\xi'} - e^{i(\bar{x}'-y').\xi'})\Psi(x,y,\xi')d\xi'\,dy +$$

$$g(\bar{x})\int_{R^n_+}\eta_\epsilon(y)\int (e^{i(x'-y').\xi'} - e^{i(\bar{x}'-y').\xi'})\Psi(x,y,\xi')d\xi'\,dy =$$

$A_1 + g(\bar{x})A_2$. Now,

$$A_1 = \int_{B^+_\rho(\hat{x})}(g(y)-g(\bar{x}))\eta_\epsilon(y)\int e^{i(x'-y').\xi'}\Psi(x,y,\xi')d\xi'\,dy -$$

$$\int_{B^+_\rho(\hat{x})}(g(y)-g(\bar{x}))\eta_\epsilon(y)\int e^{i(\bar{x}'-y').\xi'}\Psi(x,y,\xi')d\xi'\,dy -$$

$$\int_{R^n_+\setminus B^+_\rho(\hat{x})}(g(y)-g(\bar{x}))\eta_\epsilon(y)\int (e^{i(x'-y').\xi'} - e^{i(\bar{x}'-y').\xi'})\Psi(x,y,\xi')d\xi'\,dy = B_1 - B_2 + B_3$$

$$|B_1| \le CK_{\delta;\epsilon}[g]\int_{B^+_\rho(\hat{x})}|y-\bar{x}|^\delta\frac{|y-x|^\delta}{(|x'-y'|+x_n+y_n)^n}dy \le CK_{\delta;\epsilon}[g]|x-\bar{x}|^\delta.$$

$$|B_2| \leq CK_{\delta;\epsilon}[g] \int_{B^+_\rho(\hat{x})} |y-\bar{x}|^\delta \frac{|y-x|^\delta}{(|\bar{x}'-y'|+x_n+y_n)^n}\, dy \leq CK_{\delta;\epsilon}[g]|x-\bar{x}|^\delta$$

where we have used that $x_n \geq \bar{x}_n$.

And

$$|B_3| \leq CK_{\delta;\epsilon}[g]|x-\bar{x}| \int_0^1 \int_{R^n_+\setminus B^+_\rho(\hat{x})} |y-\bar{x}|^\delta \frac{1}{(|x'_s-y'|+x_n+y_n)^{n+1}}\, dy\, ds$$
$$\leq CK_{\delta;\epsilon}[g]|x-\bar{x}| \int_{R^n_+\setminus B^+_\rho(\hat{x})} \frac{|\bar{x}-y|^\delta}{|\bar{x}-y|^{n+1}}\, dy$$
$$\leq CK_{\delta;\epsilon}[g]|x-\bar{x}|^\delta.$$

Next,

$$A_2 = \int_{B^+_\rho(\hat{x})} \eta_\epsilon(y) \int e^{i(x'-y').\xi'} \Psi(x,y,\xi') d\xi'\, dy -$$
$$\int_{B^+_\rho(\hat{x})} \eta_\epsilon(y) \int e^{i(\bar{x}'-y').\xi'} \Psi(x,y,\xi') d\xi'\, dy +$$
$$\int_{R^n_+\setminus B^+_\rho(\hat{x})} \eta_\epsilon(y) \int (e^{i(x'-y').\xi'} - e^{i(\bar{x}'-y').\xi'}) \Psi(x,y,\xi') d\xi'\, dy = C_1 - C_2 + C_3.$$

$$|C_1| \leq CK_{\delta;\epsilon} \int_{B^+_\rho(\hat{x})} \frac{|x-y|^\delta}{|y-x|^n}\, dy \leq CK_{\delta;\epsilon}|x-\bar{x}|^\delta$$

$$C_2 = \int_{B^+_\rho(\hat{x})} \eta_\epsilon(y) \int e^{i(\bar{x}'-y').\xi'} \int_{\gamma^+(\xi')} \int_{\gamma^-(\xi')} \frac{e^{-y_n z} e^{x_n w} p(x,i\xi',w)}{a(\bar{x})(w-z)p^+(\bar{x},i\xi',z)p^-(\bar{x},i\xi',w)}\, dw\, dz\, d\xi'\, dy +$$
$$\int_{B^+_\rho(\hat{x})} \eta_\epsilon(y) \int e^{i(\bar{x}'-y').\xi'} \int_{\gamma^+(\xi')} \int_{\gamma^-(\xi')} \frac{e^{-y_n z} e^{x_n w} p(x,i\xi',w)}{w-z}$$
$$\left(\frac{1}{a(y)p^+(y,i\xi',z)p^-(y,i\xi',w)} - \frac{1}{a(\bar{x})p^+(\bar{x},i\xi',z)p^-(\bar{x},i\xi',w)}\right) dw\, dz\, d\xi'\, dy =$$
$$= D_1 + D_2$$

We can write

$$D_1 = \int_{B^+_\rho(\hat{x})} \eta_\epsilon(y) \int e^{i(\bar{x}'-y').\xi'} \int_{\gamma^+(\xi')} \int_{\gamma^-(\xi')} \frac{e^{-y_n z} e^{x_n w}(p(x,i\xi',w) - p(\bar{x},i\xi',w))}{a(\bar{x})(w-z)p^+(\bar{x},i\xi',z)p^-(\bar{x},i\xi',w)}\, dw\, dz\, d\xi'\, dy.$$

, and estimate

$$|D_1| \leq CK_{\delta;\epsilon}|x-\bar{x}|^\delta.$$

by the estimations (4.12).

And

$$|D_2| \leq CK_{\delta;\epsilon} \int_{B^+_\rho(\hat{x})} \frac{|\bar{x}-y|^\delta}{(|\bar{x}'-y'|+x_n+y_n)^n}\,dy \leq CK_{\delta;\epsilon}|x-\bar{x}|^\delta.$$

again by (4.12) and using that $x_n \geq \bar{x}_n$.

This finishes the estimation of the term A.

Estimation of the term B:

We write

$$B = \int_{R^n_+} (g(y)-g(\bar{x}))\eta_\epsilon(y) \int e^{i(\bar{x}'-y').\xi'}(\Psi(x,y,\xi')-\Psi(\bar{x},y,\xi'))d\xi'\,dy +$$

$$g(\bar{x}) \int_{R^n_+} \eta_\epsilon(y) \int e^{i(\bar{x}'-y').\xi'}(\Psi(x,y,\xi')-\Psi(\bar{x},y,\xi'))d\xi'\,dy = E_1 + g(\bar{x})E_2.$$

And

$$E_1 = \int_{R^n_+} (g(y)-g(\bar{x}))\eta_\epsilon(y) \int e^{i(\bar{x}'-y').\xi'} \int_{\gamma^+(\xi')} \int_{\gamma^-(\xi')} \frac{e^{-y_n z}e^{x_n w}(p(x,i\xi',w)-p(\bar{x},i\xi',w))}{a(y)(w-z)p^+(y,i\xi',z)p^-(y,i\xi',w)}\,dw\,dz\,d\xi'\,dy +$$

$$\int_{R^n_+} (g(y)-g(\bar{x}))\eta_\epsilon(y) \int e^{i(\bar{x}'-y').\xi'} \int_{\gamma^+(\xi')} \int_{\gamma^-(\xi')} \frac{e^{-y_n z}(e^{x_n w}-e^{\bar{x}_n w})p(\bar{x},i\xi',w)}{a(y)(w-z)p^+(y,i\xi',z)p^-(y,i\xi',w)}\,dw\,dz\,d\xi'\,dy = F_1 + F_2$$

We have

$$|F_1| \leq CK_{\delta;\epsilon}[g]|\bar{x}-x|^\delta \int \eta_\epsilon(y) \frac{|\bar{x}-y|^\delta}{(|\bar{x}'-y'|+x_n+y_n)^n}\,dy \leq CK_{\delta;\epsilon}[g]|\bar{x}-x|^\delta$$

since $x_n \geq \bar{x}_n$. Next, write

$$F_2 = \int_{B^+_\rho(\hat{x})} (g(y)-g(\bar{x}))\eta_\epsilon(y) \int e^{i(\bar{x}'-y').\xi'} \int_{\gamma^+(\xi')} \int_{\gamma^-(\xi')} \frac{e^{-y_n z}e^{x_n w}(p(\bar{x},i\xi',w)-p(y,i\xi',w))}{a(y)(w-z)p^+(y,i\xi',z)p^-(y,i\xi',w)}\,dw\,dz\,d\xi'\,dy -$$

$$\int_{B^+_\rho(\hat{x})} (g(y)-g(\bar{x}))\eta_\epsilon(y) \int e^{i(\bar{x}'-y').\xi'} \int_{\gamma^+(\xi')} \int_{\gamma^-(\xi')} \frac{e^{-y_n z}e^{\bar{x}_n w}(p(\bar{x},i\xi',w)-p(y,i\xi',w))}{a(y)(w-z)p^+(y,i\xi',z)p^-(y,i\xi',w)}\,dw\,dz\,d\xi'\,dy +$$

$$\int_{R^n_+\setminus B^+_\rho(\hat{x})} (g(y)-g(\bar{x}))\eta_\epsilon(y) \int e^{i(\bar{x}'-y').\xi'} \int_{\gamma^+(\xi')} \int_{\gamma^-(\xi')} \frac{e^{-y_n z}(e^{x_n w}-e^{\bar{x}_n w})p(\bar{x},i\xi',w)}{a(y)(w-z)p^+(y,i\xi',z)p^-(y,i\xi',w)}\,dw\,dz\,d\xi'\,dy$$

$$= G_1 - G_2 + G_3.$$

And estimate,

$$|G_1| \leq CK_{\delta;\epsilon}[g] \int_{B^+_\rho(\hat{x})} \frac{|\bar{x}-y|^\delta}{(|\bar{x}'-y'|+x_n+y_n)^n}\,dy \leq CK_{\delta;\epsilon}[g]|\bar{x}-x|^\delta$$

since $x_n \geq \bar{x}_n$. In exactly the same way we get

$$|G_2| \leq CK_{\delta;\epsilon}[g]|\bar{x}-x|^{\delta}.$$

And

$$|G_3| \leq CK_{\delta;\epsilon}[g]|\bar{x}-x| \int_{R^n_+ \setminus B^+_\rho(\hat{x})} \frac{|\bar{x}-y|^{\delta}}{(|\bar{x}'-y'|+\bar{x}_n+y_n)^{n+1}}\, dy \leq CK_{\delta;\epsilon}[g]|\bar{x}-x|^{\delta}.$$

As for E_2 we proceed in a similar fashion,

$$E_2 = \int_{R^n_+} \eta_\epsilon(y) \int e^{i(\bar{x}'-y').\xi'} \int_{\gamma^+(\xi')} \int_{\gamma^-(\xi')} \frac{e^{-y_n z} e^{x_n w}(p(x,i\xi',w)-p(\bar{x},i\xi',w)}{a(y)(w-z)p^+(y,i\xi',z)p^-(y,i\xi',w)}\, dw\, dz\, d\xi'\, dy +$$

$$\int_{R^n_+} \eta_\epsilon(y) \int e^{i(\bar{x}'-y').\xi'} \int_{\gamma^+(\xi')} \int_{\gamma^-(\xi')} \frac{e^{-y_n z}(e^{x_n w}-e^{\bar{x}_n w})p(\bar{x},i\xi',w)}{a(y)(w-z)p^+(y,i\xi',z)p^-(y,i\xi',w)}\, dw\, dz\, d\xi'\, dy = H_1+H_2$$

$$H_1 = \int_{R^n_+} \eta_\epsilon(y) \int e^{i(\bar{x}'-y').\xi'} \int_{\gamma^+(\xi')} \int_{\gamma^-(\xi')} \frac{e^{-y_n z} e^{x_n w}(p(x,i\xi',w)-p(\bar{x},i\xi',w)}{a(\bar{x})(w-z)p^+(\bar{x},i\xi',z)p^-(\bar{x},i\xi',w)}\, dw\, dz\, d\xi'\, dy +$$

$$\int_{R^n_+} \eta_\epsilon(y) \int e^{i(\bar{x}'-y').\xi'} X\, d\xi'\, dy$$

where

$$X = \int_{\gamma^+(\xi')} \int_{\gamma^-(\xi')} \frac{e^{-y_n z} e^{x_n w}(p(x,i\xi',w)-p(\bar{x},i\xi',w))}{w-z}$$

$$\left(\frac{1}{a(y)p^+(y,i\xi',z)p^-(y,i\xi',w)} - \frac{1}{a(\bar{x})p^+(\bar{x},i\xi',z)p^-(\bar{x},i\xi',w)}\right) dw\, dz\, d\xi'\, dy.$$

So, $H_1 = J_1 + J_2$ with

$|J_1| \leq CK_{\delta;\epsilon}|\bar{x}-x|^{\delta}$ and $|J_2| \leq CK_{\delta;\epsilon}|\bar{x}-x|^{\delta}$, both by (4.12).

And

$$H_2 = \int_{B^+_\rho(\hat{x})} \eta_\epsilon(y) \int e^{i(\bar{x}'-y').\xi'} \int_{\gamma^+(\xi')} \int_{\gamma^-(\xi')} \frac{e^{-y_n z} e^{x_n w}(p(\bar{x},i\xi',w)-p(y,i\xi',w))}{a(y)(w-z)p^+(y,i\xi',z)p^-(y,i\xi',w)}\, dw\, dz\, d\xi'\, dy +$$

$$\int_{B^+_\rho(\hat{x})} \eta_\epsilon(y) \int e^{i(\bar{x}'-y').\xi'} \int_{\gamma^+(\xi')} \int_{\gamma^-(\xi')} \frac{e^{-y_n z} e^{\bar{x}_n w}(p(\bar{x},i\xi',w)-p(y,i\xi',w))}{a(y)(w-z)p^+(y,i\xi',z)p^-(y,i\xi',w)}\, dw\, dz\, d\xi'\, dy +$$

$$\int_{R^n_+ \setminus B^+_\rho(\hat{x})} \eta_\epsilon(y) \int e^{i(\bar{x}'-y').\xi'} \int_{\gamma^+(\xi')} \int_{\gamma^-(\xi')} \frac{e^{-y_n z}(e^{\bar{x}_n w}-e^{x_n w})(p(\bar{x},i\xi',w)-p(y,i\xi',w))}{a(y)(w-z)p^+(y,i\xi',z)p^-(y,i\xi',w)}\, dw\, dz\, d\xi'\, dy =$$

$$K_1 + K_2 + K_3$$

where

$$|K_1|, |K_2| \leq CK_{\delta;\epsilon} \int_{B^+_\rho(\hat{x})} \frac{|\bar{x} - y|^\delta}{(|\bar{x}' - y'| + \bar{x}_n + y_n)^n} \leq CK_{\delta;\epsilon}|\bar{x} - x|^\delta$$

and

$$|K_3| \leq CK_{\delta;\epsilon}|\bar{x} - x| \int_{R^n_+ \setminus B^+_\rho(\hat{x})} \frac{|\bar{x} - y|^\delta}{(|\bar{x}' - y'| + \bar{x}_n + y_n)^{n+1}} \leq CK_{\delta;\epsilon}|\bar{x} - x|^\delta.$$

This finishes the prove that

$$[F(g\eta_\epsilon)]_{\delta;B^+_\epsilon} \leq CK_{\delta;\epsilon}|g|_{\delta;B^+_\epsilon}.$$

The proof that $|F(g\eta_\epsilon)|_{0;B^+_\epsilon} \leq CK_{\delta;\epsilon}|g|_{0;B^+_\epsilon}$, is straight forward and we omit it.

□

.

We claim that $H(g\eta_\epsilon) \in C^{2m+\delta}(B^+_\epsilon)$.

By the same arguments as above one shows that for $|\alpha| \leq 2m$ and $\alpha \neq 2me_n$ that

$$D^\alpha(H(g\eta_\epsilon)) \in C^\delta(B^+_\epsilon).$$

Moreover,

$$a(x)\frac{\partial^{2m}H(g\eta_\epsilon)}{\partial x_n^{2m}} = -\sum_{\alpha \neq 2me_n} a_\alpha(x)D^\alpha H(g\eta_\epsilon)(x) + L(H(g\eta_\epsilon))(x)$$

which is a sum of two functions in $C^\delta(B^+_\epsilon)$.

Since $a \geq \lambda$ and $a \in C^\delta(B^+_\epsilon)$ the result follows.

In order to analyze the boundary values it is convenient to have equivalent expressions for the function $N(g\eta_\epsilon)$.

Let

$$h_R(y) = \int e^{iy.\xi} \frac{\varphi(\frac{\xi}{R})}{p(y, i\xi)}\, d\xi$$

and $h(y) = \lim_{R\to\infty} h_R(y)$.

The convergence is point wise in $\mathbf{R}^n \setminus \{0\}$, and in $L^1(\mathbf{R}^n)$.

We claim that h has the following equivalent definitions.

$$h(y) = \frac{1}{|y|^{2k}} \int e^{iy.\xi}(-\Delta_\xi)^k(\frac{1}{p(y, i\xi)})\, d\xi$$

for any positive integer k such that $2m + 2k > n$

And
$$h(y) = \int e^{iy'.\xi'} \Lambda(y)\, d\xi'$$
where
$$\Lambda(y) = \int_{\gamma^-(\xi')} \frac{e^{y_n z}}{p(y, i\xi', z)}\, dz$$
for $y_n > 0$ and
$$\Lambda(y) = \int_{\gamma^+(\xi')} \frac{-e^{y_n z}}{p(y, i\xi', z)}\, dz$$
for $y_n < 0$

and where $\gamma^{+-}(\xi')$ are the contours described previously.

The proof follows by integration by parts and shifting contours of integration.

We now define our approximate solution with the correct boundary values.

Given $g \in C^\delta(B_\epsilon^+)$, we define for $x \in B_\epsilon^+$ the function
$$u(x) = N(g\eta_\epsilon)(x) - H(g\eta_\epsilon)(x)$$
We have shown that $u \in C^{2m+\delta}(B_\epsilon^+)$.

We also have shown that
$$|Lu - g\eta_\epsilon|_{\delta;B_\epsilon^+} \le CK_{\delta;\epsilon}|g|_{\delta;B_\epsilon^+}.$$

We now claim that $u(x', 0) = 0, \frac{\partial u(x',0)}{\partial x_n} = 0, ..., \frac{\partial^{m-1} u(x',0)}{\partial x_n^{m-1}} = 0.$

To prove this claim, let $x \in B_\epsilon^+$ $r > 0$ small and $0 < x_n < \frac{r}{2}$ and write
$$u(x) = \int_{B_\epsilon^+ \cap \{y_n \ge r\}} g(y)\eta_\epsilon(y) \int e^{i(x'-y').\xi'} \int_{\gamma^+(\xi')} \frac{-e^{(x_n-y_n)z}}{p(y, i\xi', z)}\, dz\, d\xi'\, dy$$
$$-\int_{B_\epsilon^+ \cap \{y_n \ge r\}} g(y)\eta_\epsilon(y) \int e^{i(x'-y').\xi'} \int_{\gamma^+(\xi')} \frac{e^{-y_n z}}{a(y)p^+(y, i\xi', z)} \int_{\gamma^-(\xi')} \frac{e^{x_n w}}{(w-z)p(y, i\xi', w)}\, dw\, dz\, d\xi'\, dy + E(r)$$
where $E(r) \to 0$ as $r \to 0$.

Since
$$\int_{\gamma^-(\xi')} \frac{e^{x_n w}}{(w-z)p^-(y, i\xi', w)}\, dw \to \int_{\gamma^-(\xi')} \frac{1}{(w-z)p^-(y, i\xi', w)}\, dw$$
and
$$\int_{\gamma^-(\xi')} \frac{1}{(w-z)p^-(y, i\xi', w)}\, dw = -\frac{1}{p^-(y, i\xi', z)}$$
we get that $u(x', 0) = 0$.

The argument to show $\frac{\partial u(x',0)}{\partial x_n} = 0$ is the same.

Note that

$$\frac{\partial u(x',x_n)}{\partial x_n} = \int_{B_\epsilon^+\cap\{y_n\geq r\}} g(y)\eta_\epsilon(y)\int e^{i(x'-y').\xi'}\int_{\gamma^+(\xi')}\frac{-ze^{(x_n-y_n)z}}{p(y,i\xi',z)}\,dz\,d\xi'\,dy$$

$$-\int_{B_\epsilon^+\cap\{y_n\geq r\}} g(y)\eta_\epsilon(y)\int e^{i(x'-y').\xi'}\int_{\gamma^+(\xi')}\frac{e^{-y_nz}}{a(y)p^+(y,i\xi',z)}\int_{\gamma^-(\xi')}\frac{we^{x_nw}}{(w-z)p(y,i\xi',w)}\,dw\,dz\,d\xi'\,dy+E(r)$$

where $E(r)\to 0$ as $r\to 0$. And

$$\int_{\gamma^-(\xi')}\frac{we^{x_nw}}{(w-z)p^-(y,i\xi',w)}\,dw \to \int_{\gamma^-(\xi')}\frac{w}{(w-z)p^-(y,i\xi',w)}\,dw$$

as $x_n\to 0$.

And

$$\int_{\gamma^-(\xi')}\frac{w}{(w-z)p^-(y,i\xi',w)}\,dw = -\frac{z}{p^-(y,i\xi',z)}.$$

The same argument works up to the $m-1$ derivative. This last case using that

$$\int_{\gamma^-(\xi')}\frac{w^{m-1}}{(w-z)p^-(y,i\xi',w)}\,dw = -\frac{z^{m-1}}{p^-(y,i\xi',z)}.$$

We are now ready to prove the main theorem of this section.

Theorem 4.3. *There exists $\epsilon > 0$ depending on structure so that given $f\in C^\delta(B_\epsilon^+)$, there exists $u\in C^{2m+\delta}(B_\epsilon^+)$, such that $Lu = f$ in $B^+_{\frac{\epsilon}{2}}$ and u satisfies the m boundary conditions, $u(x',0) = 0$, $\frac{\partial u(x',0)}{\partial x_n} = 0, ..., \frac{\partial^{m-1}u(x',0)}{\partial x_n^{m-1}} = 0$.*

Proof. Define

$$T(g)(x) = L(N(g)(x) - H(g)(x)) - g(x)$$

For $g\in C^\delta(B_\epsilon^+))$, it follows by theorem(4.2) and theorem(4.1) that

$$|T(g\eta_\epsilon)|_{\delta;B_\epsilon^+} \leq CK_{\delta,\epsilon}|g|_{\delta;B_\epsilon^+}.$$

For ϵ to be chosen, define the sequence $g_k\in C^\delta(B_\epsilon^+))$ by

$g_0 = f$ and $g_{k+1} = f - T(g_k\eta_\epsilon)$

Note that

$$|g_{k+1}-g_k|_{\delta;B_\epsilon^+} = |T((g_k-g_{k-1})\eta_\epsilon)|_{\delta;B_\epsilon^+} \leq CK_{\delta,\epsilon}|g_k-g_{k-1}||_{\delta;B_\epsilon^+}$$

$$\leq C\epsilon^{\beta-\delta}K_{\beta,\epsilon}|g_k-g_{k-1}||_{\delta;B_\epsilon^+}.$$

Choosing ϵ small enough so that $C\epsilon^{\beta-\delta}K_{\beta,\epsilon} < 1$, we can conclude that there exists $g \in C^{\delta}(B^+_\epsilon))$ such that $g = f - T(g\eta_\epsilon)$

This means that $g(1-\eta_\epsilon) = f - (L(N(g\eta_\epsilon) - H(g\eta_\epsilon)))$ in B^+_ϵ. And hence,

$$f = L(N(g\eta_\epsilon) - H(g\eta_\epsilon))$$

in $B^+_{\frac{\epsilon}{2}}$.

Let $u = N(g\eta_\epsilon) - H(g\eta_\epsilon)$ in $B^+_{\frac{\epsilon}{2}}$.

u satisfies the conclusions of the theorem.

□

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