The case of equality in Hölder's inequality for matrices and operators*

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Abstract

Let $p > 1$ and $1/p + 1/q = 1$. Consider Hölder's inequality

$$\|ab^*\|_1 \leq \|a\|_p \|b\|_q$$

for the p-norms of $n \times n$ matrices. This note contains a simple proof (based on the case $p = 2$) of the fact that equality holds if and only if $|a|^p = \lambda |b|^q$ for some $\lambda \geq 0$. Without modification, the method of proof holds if a, b are matrices, compact operators, elements of a finite C^*-algebra or a semi-finite von Neumann algebra.

1 Introduction

The purpose of this note is to give a simple proof of the fact that equality holds in Hölder's inequality for the p-norms of matrices a, b if and only if $|a|^p = \lambda |b|^q$ for a precise $\lambda \geq 0$.

A first proof of this result for positive matrices $a, b \geq 0$ goes back to Dixmier [Dix53] where he reduces the problem to the equality in the classical Hölder inequality for functions. Yet another different proof is based on the s-numbers of operators and majorization theory, i.e. the proof given by M. Manjegani in [Ma07]; that proof depends on the solution of the case of equality in Young's inequality for nuclear operators which was given in [AF03] (for the case of equality for the singular values of compact operators, see [La16]).

1.1 Notation and the Cauchy-Schwarz inequality

Let $M_n(\mathbb{C})$ denote the $n \times n$ matrices with complex entries, let τ denote the usual trace of a matrix $\tau(a) = \sum_i a_{ii}$. We will denote the p-norms of matrices with $\|x\|_p = (\tau|x|^p)^{1/p}$ for $p \geq 1$, and $\|x\|$ will denote the norm of $x \in M_n(\mathbb{C})$, that is the supremum norm of matrices. What follows is the well-known Hölder inequality for $a, b \in M_n(\mathbb{C})$:

$$\|ab^*\|_1 \leq \|a\|_p \|b\|_q.$$

For a proof of this inequality for $a, b \in M_n(\mathbb{C})$, see Bhatia's book [Bha97].

Since the main result of this note is based on it, let us start by recalling the well-known Cauchy-Schwarz inequality with precision; for a proof see Proposition 2.1.3 in Kadison and Ringrose's book [KR83].

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Lemma 1.1. *Let $x, y \in M_n(\mathbb{C})$ and set $\langle x, y\rangle = \tau(xy^*)$, $\|x\|_2 = \sqrt{\langle x, x\rangle}$. Then*

$$|\langle x, y\rangle| \leq \|x\|_2\|y\|_2.$$

Moreover, if $\tau(xy^*) = \|x\|_2\|y\|_2$, then $x = \lambda y$ for some $\lambda \geq 0$.

Remark 1.2. *Let $a = u|a|$ be the polar decomposition of $a \in M_n(\mathbb{C})$, then $u^*u|a| = |a|$. Write $a = xy^*$ with $x = u|a|^{1/2}$ and $y = |a|^{1/2}$, and use Cauchy-Schwarz to obtain $|\tau(a)| \leq \tau(|a|)$ with equality $\tau(a) = \tau|a| = 1$ if and only if $a = |a| \geq 0$.*

2 Hölder

We are now ready to consider the case of equality in Hölder's inequality.

Theorem 2.1. *If $p > 1$ and $a, b \neq 0$ are in $M_n(\mathbb{C})$, equality holds in Hölder inequality*

$$\|ab^*\|_1 = \|a\|_p\|b\|_q$$

if and only if $\dfrac{|a|^p}{\|a\|_p^p} = \dfrac{|b|^q}{\|b\|_q^q}$.

Proof. Let $b = \nu|b|$ be the polar decomposition of b, then $|ab^*| = \nu||a||b||\nu^*$ and $\nu^*|ab^*|\nu = ||a||b||$ therefore $\|ab^*\|_1 = \||a||b|\|_1$ for all a, b. We will write p_a for the projection onto the range subspace of a. Using the homogeneity, it suffices to consider the case $\|a\|_p = \|b\|_q = 1$.
If $|a|^p = |b|^q$, then $\|a\|_p^p = \|b\|_q^q$, $\|a\|_p\|b\|_q = \|a\|_p^p$ and $|a||b| = |a|^p$. Hence

$$\|ab^*\|_1 = \||a||b|\|_1 = \||a|^p\|_1 = \|a\|_p^p = \|a\|_p\|b\|_q.$$

To prove the converse, write $||a||b|| = w^*|a||b|$. Without loss of generality we can assume that $p \in (1, 2]$ (if $p = 2$, then $|a|^0$ denotes p_a and this proof can be considerably shortened). Then

$$1 = \|ab^*\|_1 = \||a||b|\|_1 = \tau(w^*|a||b|) = \tau(w^*|a|^{p/2}\,|a|^{1-p/2}|b|) = \tau(xy^*)$$

with $x = w^*|a|^{p/2}, y = |b||a|^{1-p/2}$, which by the Cauchy-Schwarz inequality is less than or equal to

$$\|x\|_2\|y\|_2 \leq \tau(|a|^p)^{1/2}\||b||a|^{1-p/2}\|_2 = \||b||a|^{1-p/2}\|_2 = \tau(|b|^2|a|^{2-p})^{1/2}$$

since $\|a\|_p = 1$. Now, pick $r = q/2 \geq 1$, r' its conjugate exponent, then by Hölder's inequality,

$$\tau(|b|^2|a|^{2-p}) \leq \||b|^2|a|^{2-p}\|_1 \leq (\tau|b|^q)^{1/r}(\tau(|a|^{(2-p)r'}))^{1/r'} = (\tau(|a|^{(2-p)r'}))^{1/r'}$$

since $\|b\|_q = 1$. But $(2-p)r' = p$, hence the expresssion is less or equal than 1, thus all the expressions are equal. Now, note first that

$$1 = \tau(|b|^2|a|^{2-p}) = \||b|^2|a|^{2-p}\|_1 = \tau(||b|^2|a|^{2-p}|)$$

and this is only possible (Remark 1.2) if $|b|^2|a|^{2-p} \geq 0$, which can only happen if $|b|^2$ commutes with $|a|^{2-p}$, or equivalently, if $|a|$ commutes with $|b|$; in particular $w = 1$ and $\tau(|a||b|) = \|ab^*\|_1 = 1$.
On the other hand we have also shown that $0 \leq \tau(xy^*) = \|x\|_2\|y\|_2$ and by Lemma 1.1, it follows that $x = \lambda y$ for some $\lambda \geq 0$, in our case $|a|^{p/2} = \lambda|b||a|^{1-p/2}$ which implies $|a|^p = \lambda|b||a|$. Taking traces we get

$$1 = \|a\|_p^p = \lambda\tau(|a||b|) = \lambda\|ab^*\|_1 = \lambda.$$

Then $\lambda = 1$ and we can also assert that $|a|^{p-1} = |b|p_a$. But then $|b|^q p_a = p_a|b|^q = |a|^{q(p-1)} = |a|^p$ and

$$\tau(p_a|b|^q) = \tau(|a|^p) = 1 = \tau(|b|^q),$$

or equivalently $\tau((1-p_a)|b|^q) = 0$, which is only possible if $|b|^q = p_a|b|^q = |a|^p$ by the faithfulness of the trace. □

Remark 2.2. *For the case of $p = 1$, assume $\|ab^*\|_1 = \|a\|_1\|b\|_\infty$. If one goes through the previous proof (take $r = \infty$, $r' = 1$), arrives to $p_a\|b\|_\infty = p_a|b| = |b|p_a$, which is the necessary and sufficent condition to obtain the equality just mentioned.*

Remark 2.3. *Throughout, the matrix algebra $M_n(\mathbb{C})$ can be replaced with any matrix algebra $\mathcal{A}$, or with more generality, any semi-finite von Neumann algebra with semi-finite faithful normal trace τ. We can also include C^*-algebras with a finite trace, because they can be embedded into its double commutant which is a finite von Neumann algebra by a classical result of Takesaki [Ta79, Proposition V.3.19]. Moreover, in the semi-finite case, the argument works without modification for unbounded $a, b \in \widetilde{\mathcal{A}}$, the algebra of τ-measurable operators affiliated with $\mathcal{A}$ (see Nelson's paper [Ne74]). For compact operators the standard reference on Hölder's inequality is the book of Simon [Si05], and for the continuous case, see Nelson's paper [Ne74] on noncommutative integration.*

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