Optimal frame designs for multitasking devices with energy restrictions

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Abstract

Let $d = (d_j)_{j\in\mathbb{I}_m} \in \mathbb{N}^m$ be a finite sequence (of dimensions) and $\alpha = (\alpha_i)_{i\in\mathbb{I}_n}$ be a sequence of positive numbers (weights), where $\mathbb{I}_k = \{1, \ldots, k\}$ for $k \in \mathbb{N}$. We introduce the $(\alpha\,,\,d)$-designs i.e., families $\Phi = (\mathcal{F}_j)_{j\in\mathbb{I}_m}$ such that $\mathcal{F}_j = \{f_{ij}\}_{i\in\mathbb{I}_n}$ is a frame for $\mathbb{C}^{d_j}$, $j \in \mathbb{I}_m$, and such that the sequence of non-negative numbers $(\|f_{ij}\|^2)_{j\in\mathbb{I}_m}$ forms a partition of α_i, $i \in \mathbb{I}_n$. We show, by means of a recursive finite-step algorithm, that there exist $(\alpha\,,\,d)$-designs $\Phi^{\rm op} = (\mathcal{F}_j^{\rm op})$ that are universally optimal; that is, for every convex function $\varphi : [0,\infty) \to [0,\infty)$ then $\Phi^{\rm op}$ minimizes the joint convex potential induced by φ among $(\alpha\,,\,d)$-designs, namely

$$\sum_{j\in\mathbb{I}_m} \mathrm{P}_\varphi(\mathcal{F}_j^{\rm op}) \leq \sum_{j\in\mathbb{I}_m} \mathrm{P}_\varphi(\mathcal{F}_j)$$

for every $(\alpha\,,\,d)$-design $\Phi = (\mathcal{F}_j)_{j\in\mathbb{I}_m}$; in particular, $\Phi^{\rm op}$ minimizes both the joint frame potential and the joint mean square error among $(\alpha\,,\,d)$-designs. This corresponds to the existence of optimal encoding-decoding schemes for multitasking devices with energy restrictions.

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1 Introduction

A finite sequence $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n}$ of vectors in $\mathbb{C}^d$ is a frame for $\mathbb{C}^d$ if $\mathcal{F}$ is a (possibly redundant) system of generators for $\mathbb{C}^d$. In this case, it is well known that there exist finite sequences $\mathcal{G} = \{g_i\}_{i\in\mathbb{I}_n}$ in $\mathbb{C}^d$ - the so called duals of $\mathcal{F}$ - such that

$$f = \sum_{i\in\mathbb{I}_n} \langle f\,,\,f_i\rangle\; g_i = \sum_{i\in\mathbb{I}_n} \langle f\,,\,g_i\rangle\; f_i \quad \text{for} \quad f \in \mathbb{C}^d\,. \tag{1}$$

Thus, we can encode/decode the vector f in terms of the inner products $(\langle f\,,\,f_i\rangle)_{i\in\mathbb{I}_n} \in \mathbb{C}^n$: (see [5, 10, 11] and the references therein). These redundant linear encoding-decoding schemes are of special interest in applied situations, in which there might be noise in the transmission channel : in this context, the linear relations between the frame elements can be used to produce simple linear tests to verify whether the sequence of received coefficients has been corrupted by the noise of the channel. In case the received coefficients are corrupted we can attempt to correct the sequence and obtain a reasonable (in some cases perfect) reconstruction of f (see [4, 17]).

Given a finite sequence $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n}$ in $\mathbb{C}^d$, the frame operator $S_\mathcal{F} \in \mathcal{M}_d(\mathbb{C})^+$ is given by

$$S_\mathcal{F} f = \sum_{i\in\mathbb{I}_n} \langle f\,,\,f_i\rangle\; f_i \quad \text{for} \quad f \in \mathbb{C}^d\,. \tag{2}$$

If $S_\mathcal{F}$ is invertible (i.e. if $\mathcal{F}$ is a frame) the canonical dual of $\mathcal{F}$ is given by $g_i = S_\mathcal{F}^{-1} f_i$ for $i \in \mathbb{I}_n$; this dual plays a central role in applications since it has several optimal (minimal) properties within the set of duals of $\mathcal{F}$. Unfortunately, the computation of the canonical dual depends on finding $S_\mathcal{F}^{-1}$, which is a challenging task from the numerical point of view. A way out of this problem is to consider those frames $\mathcal{F}$ for which $S_\mathcal{F}^{-1}$ is easy to compute (e.g. tight frames). In general, the numerical stability of the computation of $S_\mathcal{F}^{-1}$ depends on the spread of the eigenvalues of $S_\mathcal{F}$. In [3] Benedetto and Fickus introduced a convex functional called the frame potential of a sequence $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n}$ given by

$$\mathrm{FP}\,(\mathcal{F}) = \sum_{i\,,\,j\in\mathbb{I}_n} |\langle f_i\,,\,f_j\rangle|^2 \geq 0\,. \tag{3}$$

In [3] the authors showed that under some normalization conditions, $\mathrm{FP}\,(\mathcal{F})$ provides an scalar measure of the spread of the eigenvalues of $\mathcal{F}$. More explicitly, the authors showed that the minimizers of FP among sequences $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n}$ for which $\|f_i\| = 1$, $i \in \mathbb{I}_n$, are exactly the n/d-tight frames. It is worth pointing out that these minimizers are also optimal for transmission through noisy channels (in which erasures of the frame coefficients may occur, see [4, 17]).

In some applications of frame theory, we are drawn to consider frames $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n}$ such that $\|f_i\|^2 = \alpha_i$, $i \in \mathbb{I}_n$, for some prescribed sequence $\alpha = (\alpha_i)_{i\in\mathbb{I}_n} \in (\mathbb{R}_{>0})^n$. In practice, we can think of frames with prescribed norms as designs for encoding-decoding schemes to be applied by a device with some sort of energy restrictions (e.g. a device with limited access to energy power): in this case, control of the norms of the frame elements amounts to control the energy needed to apply the linear scheme.

It is then natural to wonder whether there are tight frames with norms prescribed by α. This question has motivated the study of the frame design problem (see [1, 6, 8, 9, 12, 13, 14, 18] and [15, 16, 20, 19, 22, 23, 24] for the more general frame completion problem with prescribed norms). It is well known that in some cases there are no tight frames in the class of sequences in $\mathbb{C}^d$ with norms prescribed by α; in these cases, it is natural to consider minimizers of the frame potential within this class, since the eigenvalues of the frame operator of such minimizers have minimal spread (thus, inducing more stable linear reconstruction processes). These considerations lead to the study of optimal designs with prescribed structure. In [7], the authors compute the structure of such minimizers and show it resembles that of tight frames.

It is worth pointing out that there are other measures of the spread of the spectra of frame operators (e.g. the mean squared error (MSE)). It turns out that both the MSE and the FP lie within the class of convex potentials introduced in [21]. It is shown in [21] that there are solutions $\mathcal{F}^{\rm op}$ to the frame design problem which are **structural** in the sense that they are minimizers of every convex potential (e.g. MSE and FP) among frames with squared norms prescribed by α. A fundamental tool to show the existence of such structural optimal frame designs is the so-called majorization in $\mathbb{R}^n$, which is a partial order used in matrix analysis (see [2]).

In the present paper we consider an extension of the optimal frame design problem as follows: given a finite sequence (of dimensions) $d = (d_j)_{j\in\mathbb{I}_m} \in \mathbb{N}^m$ and a sequence of positive numbers (weights) $\alpha = (\alpha_i)_{i\in\mathbb{I}_n} \in \mathbb{R}^n_{>0}$, we consider the set $\mathcal{D}(\alpha\,,\,d)$ of $(\alpha\,,\,d)$-designs. i.e. sequences $\Phi = (\mathcal{F}_j)_{j\in\mathbb{I}_m}$ such that each $\mathcal{F}_j = \{f_{ij}\}_{i\in\mathbb{I}_n}$ is a frame for $\mathbb{C}^{d_j}$, for $j \in \mathbb{I}_m$ and such that

$$\sum_{j\in\mathbb{I}_m} \|f_{ij}\|^2 = \alpha_i \quad \text{for} \quad i \in \mathbb{I}_n\,. \tag{4}$$

Notice that the restrictions on the norms above involve vectors in the (possibly different) spaces $f_{ij} \in \mathbb{C}^{d_j}$ for $j \in \mathbb{I}_m$. As in the case of frames with prescribed norms, $(\alpha\,,\,d)$-designs can be considered as encoding-decoding schemes to be applied by a multitasking device with some sort of energy restriction; in this context, the frames $\Phi = (\mathcal{F}_j)_{j\in\mathbb{I}_m}$ induce linear schemes in the spaces $(\mathbb{C}^{d_j})_{j\in\mathbb{I}_m}$ that run in parallel. In this case, we want to control the overall energy needed (in each step of the encoding-decoding scheme) to apply simultaneously the m linear schemes, through the restrictions in Eq.(4). It is natural to consider those $(\alpha\,,\,d)$-designs that give rise to the more stable multitasking processes. In order the measure the overall stability of the family $\Phi = (\mathcal{F}_j)_{j\in\mathbb{I}_m}$ we can consider the joint frame potential of Φ or the joint MSE of Φ given by

$$\mathrm{FP}\,(\Phi) = \sum_{j\in\mathbb{I}_m} \mathrm{FP}\,(\mathcal{F}_j) \quad , \quad \mathrm{MSE}(\Phi) = \sum_{j\in\mathbb{I}_m} \mathrm{MSE}(\mathcal{F}_j) \quad \text{respectively .}$$

More generally, given a convex function $\varphi : [0,\infty) \to [0,\infty)$ we introduce the joint convex potential $\mathrm{P}_\varphi(\Phi)$ induced by φ (see Section 3.1 for details); this family of convex potentials (that contains the joint frame potential and joint MSE) provides with natural measures of numerical stability of the family $\Phi = (\mathcal{F}_j)_{j\in\mathbb{I}_m}$.

The main problem that we study in this paper is the construction of $(\alpha\,,\,d)$-designs that are optimal in $\mathcal{D}(\alpha\,,\,d)$ with respect to every joint convex potential. The kernel of this problem is the computation of optimal weight partitions, in the following sense: Consider the set of $(\alpha\,,\,m)$-weight partitions given by

$$P_{\alpha\,,\,m} = \{A \in \mathcal{M}_{n,m}(\mathbb{R}_{\geq 0}) : \ A\,\mathbb{1}_m = \alpha\}\ ,$$

where $\mathbb{1}_m = (1\,,\,\dots\,,\,1) \in \mathbb{R}^m$. Given $A \in P_{\alpha\,,\,m}$, consider the set of A-designs, given by

$$\mathcal{D}(A) = \{\Psi = (\mathcal{F}_j)_{j\in\mathbb{I}_m} \in \mathcal{D}(\alpha\,,\,d) : (\|f_{ij}\|^2)_{i\in\mathbb{I}_n, j\in\mathbb{I}_m} = A\} \subseteq \mathcal{D}(\alpha\,,\,d)\ ,$$

which can be considered as a slice of $\mathcal{D}(\alpha\,,\,d)$. For each slice, a water-filling process works, and it produces the spectral structure (defined in Remark 4.1) of $(\alpha\,,\,d)$-designs that are minimizers in $\mathcal{D}(A)$ of every joint convex potential (see [21] or Theorem 2.6). These frames can be computed by a finite-step algorithm (see Remark 2.7).

Our main result is the computation of an optimal weight partition $A_0 \in P_{\alpha\,,\,m}$ in terms of a iterative multi-water-filling process. Within the slice $\mathcal{D}(A_0)$, the previously mentioned minimizers are structural solutions to the optimal $(\alpha\,,\,d)$-designs, in the sense that their spectral structure is majorized by those of sequences in the whole set $\mathcal{D}(\alpha\,,\,d)$. We further obtain the uniqueness of the spectral structure of these universally optimal $(\alpha\,,\,d)$-designs (while the optimal $(\alpha\,,\,m)$-weight partitions $A_0 \in P_{\alpha\,,\,m}$ are not necessarily unique), and some monotonicity properties of the spectra of this optimal $(\alpha\,,\,d)$-designs with respect to the initial weights $\alpha=(\alpha_i)_{i\in\mathbb{I}_n}$; thus, our results generalize the results in [3, 7, 21].

Our approach to the existence of optimal $(\alpha\,,\,m)$-weight partitions and $(\alpha\,,\,d)$-designs is constructive. Indeed, we introduce a recursive finite-step algorithm that produces an optimal $(\alpha\,,\,m)$-weight partition, based on the existence of an associated optimal $(\alpha'\,,\,m')$-weight partition of smaller order. Along the way we (inductively) show that the output of this algorithm has certain specific features, so that the recursive process is well defined. Moreover, we include several numerical examples of optimal $(\alpha\,,\,d)$-designs obtained with the implementation of our algorithm in MATLAB.

The paper is organized as follows. In Section 2 we recall the notion of majorization together with some fundamental results about this pre-order. We also include some notions and results related with frame theory and convex potentials. In Section 3 we formalize the notion of $(\alpha\,,\,m)$-weight partitions, $(\alpha\,,\,d)$-designs and describe in detail our main goals. After establishing some properties of the water-filling construction for vectors, we describe a recursive algorithm (based on the water-filling technique) that computes a special $(\alpha\,,\,m)$-weight partition and show that the process is well defined. In Section 4 we show that the output of our algorithm corresponds to an optimal $(\alpha\,,\,m)$-weight partition. We show that the spectral structure of optimal $(\alpha\,,\,d)$-designs is unique and show a monotonicity property of the spectra of optimal $(\alpha\,,\,d)$-designs with respect to the initial weights. The paper ends with Section 5, in which we present several numerical examples that exhibit the properties of the optimal $(\alpha\,,\,d)$-designs computed with the algorithm.

2 Preliminaries

In this section we introduce the notation, terminology and results from matrix analysis and frame theory that we will use throughout the paper. General references for these results are the texts [2] and [5, 10, 11].

2.1 Majorization

In what follows we adopt the following

Notation and terminology. We let $\mathcal{M}_{k,d}(\mathcal{S})$ be the set of $k\times d$ matrices with coefficients in $\mathcal{S}\subset\mathbb{C}$ and write $\mathcal{M}_{d,d}(\mathbb{C})=\mathcal{M}_d(\mathbb{C})$ for the algebra of $d\times d$ complex matrices. We denote by $\mathcal{H}(d)\subset\mathcal{M}_d(\mathbb{C})$ the real subspace of selfadjoint matrices and by $\mathcal{M}_d(\mathbb{C})^+\subset\mathcal{H}(d)$ the cone of positive semidefinite matrices. We let $\mathcal{U}(d)\subset\mathcal{M}_d(\mathbb{C})$ denote the group of unitary matrices. For $d\in\mathbb{N}$, let $\mathbb{I}_d=\{1,\ldots,d\}$ and let $\mathbb{1}_d=(1)_{i\in\mathbb{I}_d}\in\mathbb{R}^d$ be the vector with all its entries equal to 1. Given $x=(x_i)_{i\in\mathbb{I}_d}\in\mathbb{R}^d$ we denote by $x^\downarrow=(x_i^\downarrow)_{i\in\mathbb{I}_d}$ (respectively $x^\uparrow=(x_i^\uparrow)_{i\in\mathbb{I}_d}$) the vector obtained by rearranging the entries of x in non-increasing (respectively non-decreasing) order. We denote by $(\mathbb{R}^d)^\downarrow=\{x^\downarrow:\ x\in\mathbb{R}^d\}$, $(\mathbb{R}^d_{\geq 0})^\downarrow=\{x^\downarrow:\ x\in\mathbb{R}^d_{\geq 0}\}$ and analogously for $(\mathbb{R}^d)^\uparrow$ and $(\mathbb{R}^d_{\geq 0})^\uparrow$. Given a matrix $A\in\mathcal{H}(d)$ we denote by $\lambda(A)=\lambda^\downarrow(A)=(\lambda_i(A))_{i\in\mathbb{I}_d}\in(\mathbb{R}^d)^\downarrow$ the eigenvalues of A counting multiplicities and arranged in non-increasing order, and by $\lambda^\uparrow(A)$ the same vector but ordered in non-decreasing order. If $x,\,y\in\mathbb{C}^d$ we denote by $x\otimes y\in\mathcal{M}_d(\mathbb{C})$ the rank-one matrix given by $(x\otimes y)\,z=\langle z\,,\,y\rangle\,x$, for $z\in\mathbb{C}^d$.

Next we recall the notion of majorization between vectors, that will play a central role throughout our work.

Definition 2.1. Let $x\in\mathbb{R}^n$ and $y\in\mathbb{R}^d$. We say that x is *submajorized* by y, and write $x\prec_w y$, if

$$\sum_{i\in\mathbb{I}_j}x_i^\downarrow\leq\sum_{i\in\mathbb{I}_j}y_i^\downarrow\quad\text{ for every }\quad 1\leq j\leq\min\{n\,,\,d\}\,.$$

If $x\prec_w y$ and $\operatorname{tr}x=\sum_{i\in\mathbb{I}_n}x_i=\sum_{i\in\mathbb{I}_d}y_i=\operatorname{tr}y$, then x is *majorized* by y, and write $x\prec y$. △

Given $x,\, y \in \mathbb{R}^d$ we write $x \leqslant y$ if $x_i \le y_i$ for every $i \in \mathbb{I}_d$. It is a standard exercise to show that $x \leqslant y \implies x^\downarrow \leqslant y^\downarrow \implies x \prec_w y$.

Remark 2.2. Let $\gamma_1 \geq \ldots \geq \gamma_p \in \mathbb{R}$ and consider $\alpha = (\gamma_1 \mathbb{1}_{r_1}, \ldots, \gamma_p \mathbb{1}_{r_p}) = (\alpha_i)_{i\in\mathbb{I}_r} \in (\mathbb{R}^r)^\downarrow$, where $r \stackrel{\text{def}}{=} \sum_{i\in\mathbb{I}_p} r_i$. Set $s_k = \sum_{j\in\mathbb{I}_k} r_j$, for $k \in \mathbb{I}_p$. Given $\beta \in (\mathbb{R}^r)^\downarrow$ such that $\operatorname{tr}(\alpha) = \operatorname{tr}(\beta)$ then

$$\alpha \prec \beta \iff \sum_{i\in\mathbb{I}_k} \gamma_i\, r_i \le \sum_{j\in\mathbb{I}_{s_k}} \beta_j \ , \qquad \text{for} \qquad k \in \mathbb{I}_{p-1} \ . \tag{5}$$

Indeed, if the right conditions hold and there exists $0 \le k \le p-1$ with $s_k < t < s_{k+1}$ ($s_0 = 0$) and such that $\sum\limits_{j\in\mathbb{I}_t} \alpha_j > \sum\limits_{j\in\mathbb{I}_t} \beta_j$, it is easy to see that

$$\sum_{j=s_k+1}^{t} \beta_j < \sum_{j=s_k+1}^{t} \alpha_j = (t - s_k)\,\gamma_{k+1} \implies \beta_t < \gamma_{k+1} \implies \sum_{j\in\mathbb{I}_{s_{k+1}}} \beta_j < \sum_{i\in\mathbb{I}_{k+1}} \gamma_i\, r_i \ ,$$

which contradicts our assumption (5). Therefore $\alpha \prec \beta$. △

It is well known that majorization is intimately related with tracial inequalities of convex functions. The following result summarizes these relations (see for example [2]):

Theorem 2.3. *Let $x,\, y \in \mathbb{R}^d$. If $\varphi : I \to \mathbb{R}$ is a convex function defined on an interval $I \subseteq \mathbb{R}$ such that $x,\, y \in I^d$ then:*

1. *If $x \prec y$, then $\operatorname{tr}\varphi(x) \stackrel{def}{=} \sum\limits_{i\in\mathbb{I}_d} \varphi(x_i) \le \sum\limits_{i\in\mathbb{I}_d} \varphi(y_i) = \operatorname{tr}\varphi(y)$.*
2. *If only $x \prec_w y$, but φ is an increasing function, then still $\operatorname{tr}\varphi(x) \le \operatorname{tr}\varphi(y)$.*
3. *If $x \prec y$ and φ is a strictly convex function such that $\operatorname{tr}\,\varphi(x) = \operatorname{tr}\,\varphi(y)$ then there exists a permutation σ of $\mathbb{I}_d$ such that $y_i = x_{\sigma(i)}$ for $i \in \mathbb{I}_d$, i.e. $x^\downarrow = y^\downarrow$.*

□

2.2 Frames and convex potentials

In what follows we adopt the following

Notation and terminology: let $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n}$ be a finite sequence in $\mathbb{C}^d$. Then,

1. $T_\mathcal{F} \in \mathcal{M}_{d,n}(\mathbb{C})$ denotes the synthesis operator of $\mathcal{F}$ given by $T_\mathcal{F} \cdot (\alpha_i)_{i\in\mathbb{I}_n} = \sum_{i\in\mathbb{I}_n} \alpha_i\, f_i$.
2. $T_\mathcal{F}^* \in \mathcal{M}_{n,d}(\mathbb{C})$ denotes the analysis operator of $\mathcal{F}$ and it is given by $T_\mathcal{F}^* \cdot f = (\langle f, f_i\rangle)_{i\in\mathbb{I}_n}$.
3. $S_\mathcal{F} \in \mathcal{M}_d(\mathbb{C})^+$ denotes the frame operator of $\mathcal{F}$ and it is given by $S_\mathcal{F} = T_\mathcal{F}\, T_\mathcal{F}^*$. Hence,
$$S_\mathcal{F} f = \sum_{i\in\mathbb{I}_n} \langle f, f_i\rangle f_i = \sum_{i\in\mathbb{I}_n} f_i \otimes f_i(f) \qquad \text{for} \qquad f \in \mathbb{C}^d \,.$$
4. We say that $\mathcal{F}$ is a frame for $\mathbb{C}^d$ if it spans $\mathbb{C}^d$; equivalently, $\mathcal{F}$ is a frame for $\mathbb{C}^d$ if $S_\mathcal{F}$ is a positive invertible operator acting on $\mathbb{C}^d$. In this case we have the canonical reconstruction formula
$$f = \sum_{i\in\mathbb{I}_n} \langle f \,,\, S_\mathcal{F}^{-1}\, f_i\rangle\ f_i = \sum_{i\in\mathbb{I}_n} \langle f \,,\, f_i\rangle\ S_\mathcal{F}^{-1}\, f_i \qquad \text{for} \qquad f \in \mathbb{C}^d$$
in terms of the so-called canonical dual frame $\{S_\mathcal{F}^{-1}\, f_i\}_{i\in\mathbb{I}_n}$.

In several applied situations it is desired to construct a finite sequence $\mathcal{G} = \{g_i\}_{i\in\mathbb{I}_n} \in (\mathbb{C}^d)^n$, in such a way that the spectra of the frame operator of $\mathcal{G}$ is given by some $\lambda \in (\mathbb{R}^d_{\geq 0})^\downarrow$ and the squared norms of the frame elements are prescribed by a sequence of positive numbers $\alpha = (\alpha_i)_{i\in\mathbb{I}_n}$. This is known as the (classical) frame design problem and it has been studied by several research groups (see for example [1, 6, 8, 9, 12, 13, 14, 18]). The following result characterizes the existence of such frame designs in terms of majorization relations.

Theorem 2.4 ([1, 20])**.** Let $\lambda \in (\mathbb{R}^d_{\geq 0})^\downarrow$ and consider $\alpha = (\alpha_i)_{i\in\mathbb{I}_n} \in (\mathbb{R}^n_{>0})^\downarrow$. Then there exists a sequence $\mathcal{G} = \{g_i\}_{i\in\mathbb{I}_n}$ in $\mathbb{C}^d$ with $\|g_i\|^2 = \alpha_i$ for $i \in \mathbb{I}_n$ and such that $\lambda(S_\mathcal{G}) = \lambda$ if and only if $\alpha \prec \lambda$.

The previous result shows the flexibility of structured frame designs, which is important in applied situations. Also, numerical stability of the encoding-decoding scheme induced by a frame plays a role in applications; hence, a central problem in this area is to described the structured frame designs that maximize the stability of their encoding-decoding scheme. One of the most important (scalar) measures of stability is the so-called frame potential introduced by Benedetto and Fickus in [3] given by

$$\mathrm{FP}\,(\mathcal{F}) = \sum_{i\,,\,j\in\mathbb{I}_n} |\langle f_i\,,\,f_j\rangle|^2 = \mathrm{tr}(S_\mathcal{F}^2) \quad \text{for} \quad \mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n} \in (\mathbb{C}^d)^n\,.$$

Benedetto and Fickus have shown that (under certain normalization conditions) minimizers of the frame potential induce the most stable encoding-decoding schemes. More generally, we can measure the stability of the scheme induced by the sequence $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n} \in (\mathbb{C}^d)^n$ in terms of convex potentials. In order to introduce these potentials we consider the sets

$$\mathrm{Conv}(\mathbb{R}_{\geq 0}) = \{\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \ : \ \varphi \ \text{ is a convex function }\}$$

and $\mathrm{Conv_s}(\mathbb{R}_{\geq 0}) = \{\varphi \in \mathrm{Conv}(\mathbb{R}_{\geq 0}) : \varphi \text{ is strictly convex }\}$.

Definition 2.5. Following [21] we consider the convex potential P_φ associated to $\varphi \in \mathrm{Conv}(\mathbb{R}_{\geq 0})$, given by

$$\mathrm{P}_\varphi(\mathcal{F}) \ = \mathrm{tr}\ \varphi(S_\mathcal{F}) = \textstyle\sum_{i\in\mathbb{I}_d} \varphi(\lambda_i(S_\mathcal{F})\,) \quad \text{for} \quad \mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n} \in (\mathbb{C}^d)^n\ ,$$

where the matrix $\varphi(S_\mathcal{F})$ is defined by means of the usual functional calculus. △

Convex potentials allow us to model several well known measures of stability considered in frame theory. For example, in case $\varphi(x) = x^2$ for $x \in \mathbb{R}_{\geq 0}$ then P_φ is the Benedetto-Fickus frame potential; in case $\varphi(x) = x^{-1}$ for $x \in \mathbb{R}_{>0}$ then P_φ is known as the mean squared error (MSE).

Going back to the problem of stable designs, it is worth pointing out the existence of structured designs that are optimal with respect to every convex potential. Indeed, given $\alpha = (\alpha_i)_{i\in\mathbb{I}_n} \in \mathbb{R}^n_{\geq 0}$ and $d \in \mathbb{N}$ with $d \leq n$, let

$$\mathcal{B}_{\alpha\,,\,d} = \{\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n} \in (\mathbb{C}^d)^n : \ \|f_i\|^2 = \alpha_i\,,\ i \in \mathbb{I}_n\}\,. \tag{6}$$

We endow $\mathcal{B}_{\alpha\,,\,d}$ (which is a product space) with the product metric. The structure of (local) minimizers of convex potentials in $\mathcal{B}_{\alpha\,,\,d}$ has been extensively studied. The first results were obtained for the frame potential in [3] and in a more general context in [7]. The case of general convex potentials was studied in [15, 16, 19, 20, 21, 22, 23, 24] (in some cases in the more general setting of frame completion problems with prescribed norms).

Theorem 2.6 ([7, 21, 22, 23])**.** *Let $\alpha = (\alpha_i)_{i\in\mathbb{I}_n} \in (\mathbb{R}^n_{\geq 0})^\downarrow$ and let $d \in \mathbb{N}$ be such that $d \leq n$. Then, there exists $\gamma^{\mathrm{op}}_{\alpha\,,\,d} = \gamma^{\mathrm{op}} = (\gamma^{\mathrm{op}}_i)_{i\in\mathbb{I}_d} \in (\mathbb{R}^d_{\geq 0})^\downarrow$ such that:*

1. *There exist* $\mathcal{F}^{\rm op} \in \mathcal{B}_{\alpha\,,\,d}$ *such that* $\lambda(S_{\mathcal{F}^{\rm op}}) = \gamma^{\rm op}$.

2. *If* $\#\{i \in \mathbb{I}_n : \ \alpha_i > 0\} \geq d$ *then* $\gamma^{\rm op} \in (\mathbb{R}^d_{>0})^\downarrow$ *(so* $\mathcal{F}^{\rm op}$ *is a frame for* $\mathbb{C}^d$*).*

3. *If* $\mathcal{F}^{\rm op} \in \mathcal{B}_{\alpha\,,\,d}$ *is such that* $\lambda(S_{\mathcal{F}^{\rm op}}) = \gamma^{\rm op}$ *then for every* $\varphi \in \text{Conv}(\mathbb{R}_{\geq 0})$ *we have that*

$$\mathrm{P}_\varphi(\mathcal{F}^{\rm op}) \leq \mathrm{P}_\varphi(\mathcal{F}) \quad \text{for every} \quad \mathcal{F} \in \mathcal{B}_{\alpha\,,\,d}\,. \tag{7}$$

4. *If we assume further that* $\varphi \in \text{Conv}_{\rm s}(\mathbb{R}_{\geq 0})$ *and* $\mathcal{F} \in \mathcal{B}_{\alpha\,,\,d}$ *is a local minimizer of* $\mathrm{P}_\varphi : \mathcal{B}_{\alpha\,,\,d} \to \mathbb{R}_{\geq 0}$ *(*$\mathcal{B}_{\alpha\,,\,d}$ *endowed with the product metric) then* $\lambda(S_{\mathcal{F}}) = \gamma^{\rm op}$. □

Remark 2.7. The vector $\gamma^{\rm op}_{\alpha\,,\,d}$ of Theorem 2.6 can be described and computed by means of the so called water-filling construction of the vector α in dimension d (see Definition 3.5). We shall study this construction with detail in subsection 3.2. In particular, we shall give a short proof of almost all items of Theorem 2.6 using the majorization properties of the water-filling construction (see Remark 3.8).

Once the vector $\gamma^{\rm op}_{\alpha\,,\,d}$ is computed, we can apply the one-sided Bendel-Mickey algorithm (see [8, 9, 12, 14]) to compute $\mathcal{F}^{\rm op} \in \mathcal{B}_{\alpha\,,\,d}$ i.e. a finite sequence of vectors in $\mathbb{C}^d$ with prescribed norms and prescribed spectra of its frame operator. △

3 On the weight partition problem: a constructive approach

We begin this section by introducing notation and terminology that allow us to model the optimal design problem. Then, we develop some properties of the water-filling construction that we will need in the sequel. We end this section with the introduction and study of a recursive and finite-step algorithm that constructs a distinguished weight partition. We will make use of this algorithm in the next section, where we prove the existence of optimal designs.

3.1 Modeling the problem

Recall that given a finite sequence of non-negative real numbers $\alpha = (\alpha_i)_{i\in\mathbb{I}_n} \in \mathbb{R}^n_{\geq 0}$ and $d \in \mathbb{N}$ with $d \leq n$, we consider

$$\mathcal{B}_{\alpha\,,\,d} = \{\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n} \in (\mathbb{C}^d)^n : \ \|f_i\|^2 = \alpha_i\,, \ i \in \mathbb{I}_n\}\,.$$

We now introduce some new notions

Definition 3.1. Let $\alpha = (\alpha_i)_{i\in\mathbb{I}_n} \in \mathbb{R}^n_{>0}$, $m \in \mathbb{N}$ and let $d = (d_j)_{j\in\mathbb{I}_m} \in (\mathbb{N}^m)^\downarrow$ be such that $d_1 \leq n$. We consider

1. the set of $(\alpha\,,\,m)$-weight partitions given by

$$P_{\alpha\,,\,m} = \{A \in \mathcal{M}_{n,m}(\mathbb{R}_{\geq 0}) : \ A\mathbb{1}_m = \alpha\}\,.$$

2. the set of $(\alpha\,,\,d)$-designs given by

$$\mathcal{D}(\alpha, d) = \bigcup_{A\in P_{\alpha,m}} \prod_{j=1}^m \mathcal{B}_{c_j(A)\,,\,d_j}$$

where $c_j(A) = (a_{ij})_{i\in\mathbb{I}_n} \in \mathbb{R}^n_{\geq 0}$ denotes the j-th column of $A = (a_{ij})_{i\in\mathbb{I}_n, j\in\mathbb{I}_m}$, for $j \in \mathbb{I}_m$.

Remark 3.2. Consider the notation and terminology of Definition 3.1. Notice that

1. Given $A = (a_{ij})_{i\in\mathbb{I}_n,\, j\in\mathbb{I}_m} \in \mathcal{M}_{n\,,m}(\mathbb{C})$ then
$$A \in P_{\alpha\,,m} \quad \Leftrightarrow \quad a_{ij} \geq 0 \quad \text{and} \quad \sum_{j\in\mathbb{I}_m} a_{ij} = \alpha_i \quad \text{for} \quad i \in \mathbb{I}_n\,.$$

2. $\mathcal{D}(\alpha, d)$ is the set of all finite sequences $(\mathcal{F}_j)_{j\in\mathbb{I}_m}$, where $\mathcal{F}_j = \{f_{ij}\}_{i\in\mathbb{I}_n} \in (\mathbb{C}^{d_j})^n$ for $j \in \mathbb{I}_m$ are such that $(\|f_{ij}\|^2)_{i\in\mathbb{I}_n,\, j\in\mathbb{I}_m} \in P_{\alpha\,,m}$, i.e.
$$\sum_{j\in\mathbb{I}_m} \|f_{ij}\|^2 = \alpha_i \quad \text{for} \quad i \in \mathbb{I}_n\,.$$
We point out that (in order to simplify our description of the model) we consider $(\alpha\,,\, d)$-designs in a broad sense; namely, if $\Phi = (\mathcal{F}_j)_{j\in\mathbb{I}_m} \in \mathcal{D}(\alpha, d)$ then $\mathcal{F}_j$ is not necessarily a frame for $\mathbb{C}^{d_j}$, for $j \in \mathbb{I}_m$.

△

In order to compare the overall stability of the linear encoding-decoding schemes induced by an $(\alpha\,,\, d)$-design we introduce the following

Definition 3.3. Let $\alpha = (\alpha_i)_{i\in\mathbb{I}_n} \in \mathbb{R}^n_{>0}$, $m \in \mathbb{N}$ and let $d = (d_j)_{j\in\mathbb{I}_m} \in (\mathbb{N}^m)^\downarrow$ be such that $d_1 \leq n$. Given $\varphi \in \text{Conv}(\mathbb{R}_{\geq 0})$ we consider the joint potential induced by φ on $\Phi = (\mathcal{F}_j)_{j\in\mathbb{I}_m} \in \mathcal{D}(\alpha\,,\, d)$ given by
$$\mathrm{P}_\varphi(\Phi) = \sum_{j\in\mathbb{I}_m} \mathrm{P}_\varphi(\mathcal{F}_j) \;(= \sum_{j\in\mathbb{I}_m} \mathrm{tr}\;\; \varphi(S_{\mathcal{F}_j}) = \sum_{j\in\mathbb{I}_m} \sum_{i\in d_j} \varphi(\lambda_i(S_{\mathcal{F}_j})\,)\,.$$

△

Consider the notation and terminology of Definitions 3.1 and 3.3. We can now describe the main problems that we consider in this work as follows:

P1. Show that there exist $(\alpha\,,\, d)$-designs $\Phi^{\text{op}} = (\mathcal{F}_j^{\text{op}})_{j\in\mathbb{I}_m} \in \mathcal{D}(\alpha, d)$ that are optimal in the following structural sense: for every $\varphi \in \text{Conv}(\mathbb{R}_{\geq 0})$ then Φ^{op} minimizes the joint convex potential P_φ in $\mathcal{D}(\alpha\,,\, d)$, that is
$$\mathrm{P}_\varphi(\Phi^{\text{op}}) = \min\{\mathrm{P}_\varphi(\Phi) :\; \Phi \in \mathcal{D}(\alpha\,,\, d)\}\,. \tag{8}$$
In this case we say that Φ^{op} is an *optimal* $(\alpha\,,\, d)$*-design*.

P2. Describe an algorithmic procedure that computes optimal $(\alpha\,,\, d)$-designs.

P3. Characterize the optimal $(\alpha\,,\, d)$-designs in terms of some structural properties.

P4. Study further properties of optimal $(\alpha\,,\, d)$-designs.

We will solve problems P1.-P3. and study some (monotone) dependence of optimal $(\alpha\,,\, d)$-designs on the initial weights α. In particular, we will show that is $\Phi^{\text{op}} = (\mathcal{F}_j^{\text{op}})_{j\in\mathbb{I}_m}$ is an optimal $(\alpha\,,\, d)$-design then, $\mathcal{F}_j^{\text{op}}$ is a frame for $\mathbb{C}^{d_j}$ for each $j \in \mathbb{I}_m$ (see Section 4).

Remark 3.4. There is a reformulation of our problem in a more concise model. Let α and d be as in Definition 3.1. Set $|d| = \mathrm{tr}\; d$ and assume that $\mathcal{H} = \mathbb{C}^{|d|} = \bigoplus_{j\in\mathbb{I}_m} \mathcal{H}_j$ for some subspaces with $\dim \mathcal{H}_j = d_j$, for $j \in \mathbb{I}_m$. Let us denote by $P_j : \mathcal{H} \to \mathcal{H}_j \subseteq \mathcal{H}$ the corresponding projections.

Notice that a sequence $\mathcal{G} = \{g_i\}_{i\in\mathbb{I}_n} \in \mathcal{B}_{\alpha\,,\,|d|} \subseteq \mathcal{H}^n \iff$ the sequence $\Phi = (\mathcal{F}_j)_{j\in\mathbb{I}_m}$ given by $\mathcal{F}_j = P_j(\mathcal{G})$ (i.e. $f_{ij} = P_j(g_i) \in \mathcal{H}_j \cong \mathbb{C}^{d_j}$, $i \in \mathbb{I}_n$) for $j \in \mathbb{I}_m$, satisfies that $\Phi \in \mathcal{D}(\alpha\,,\, d)$.

Consider the pinching map $\mathcal{C}_d : \mathcal{M}_{|d|}(\mathbb{C}) \to \mathcal{M}_{|d|}(\mathbb{C})$ given by $\mathcal{C}_d(A) = \sum_{j\in\mathbb{I}_m} P_j\, A\, P_j$, for every $A \in \mathcal{M}_{|d|}(\mathbb{C})$. Then, for each $\varphi \in \text{Conv}(\mathbb{R}_{\geq 0})$ we can define a d-pinched potential

$$\text{P}_{\varphi\,,\,\text{d}}(\mathcal{G}) \overset{\text{def}}{=} \text{tr}\,\varphi(\mathcal{C}_d(S_\mathcal{G})) \quad \text{for every} \quad \mathcal{G} \in \mathcal{H}^n \ ,$$

which describes simultaneously the behavior of the projections of $\mathcal{G}$ to each subspace $\mathcal{H}_j$. Actually, with the previous notations,

$$\text{P}_{\varphi\,,\,\text{d}}(\mathcal{G}) = \sum_{j\in\mathbb{I}_m} \text{tr}\,\varphi(P_j\, S_\mathcal{G}\, P_j) = \sum_{j\in\mathbb{I}_m} \text{P}_\varphi(\mathcal{F}_j) = \text{P}_\varphi(\Phi)\ .$$

Therefore the problem of finding optimal $(\alpha\,,\,d)$-designs (and studying their properties) translates to the study of sequences $\mathcal{G} \in \mathcal{B}_{\alpha\,,\,|d|}$ which minimize the d-pinched potentials $\text{P}_{\varphi\,,\,\text{d}}$.

We point out that for $\varphi \in \text{Conv}(\mathbb{R}_{\geq 0})$ and $\mathcal{G} \in \mathcal{H}^n$

$$\text{P}_{\varphi\,,\,\text{d}}(\mathcal{G}) \neq \text{P}_\varphi(\mathcal{G})$$

in general, where $\text{P}_\varphi(\mathcal{G}) = \text{tr}\ \varphi(S_\mathcal{G})$ (see Definition 2.5). Indeed, previous results related with the structure of minimizers of convex potentials in $\mathcal{B}_{\alpha\,,\,|d|}$ (e.g. [21]) do not apply to the d-pinched potential and we require a new approach to study this problem. △

3.2 The water-filling construction

Consider the optimal $(\alpha\,,\,d)$-design problem in case $m = 1$. Hence, we let $\alpha = (\alpha_i)_{i\in\mathbb{I}_n} \in (\mathbb{R}^n_{>0})^\downarrow$ and let $d = (d_1) \in \mathbb{N}$ be such that $d_1 \leq n$. In this case

$$\mathcal{D}(\alpha\,,\,d) = \mathcal{B}_{\alpha\,,\,d_1} \subset (\mathbb{C}^{d_1})^n\ ,$$

and the existence of optimal $(\alpha\,,\,d_1)$-designs is a consequence of Theorem 2.6. In order to give an explicit description of the vector $\gamma^{\text{op}}_{\alpha\,,\,d}$ in Theorem 2.6 we introduce the following construction, that will also play a central role in our present work.

Definition 3.5 (The water-filling construction)**.** Let $\alpha = (\alpha_i)_{i\in\mathbb{I}_n} \in (\mathbb{R}^n_{\geq 0})^\downarrow$ and let $d \in \mathbb{N}$ be such that $d \leq n$. We define the water-filling of α in dimension d as the vector

$$\gamma = (\max\{\alpha_i\,,\,c\})_{i\in\mathbb{I}_d} \in (\mathbb{R}^d_{\geq 0})^\downarrow \quad \text{where } c \geq \alpha_d \text{ is uniquely determined by} \quad \text{tr}\,\gamma = \text{tr}\,\alpha\,.$$

In this case we say that c is the *water-level* of γ; notice that c is determined by the equation $\sum_{i\in\mathbb{I}_d} \max\{\alpha_i\,,\,c\} = \sum_{i\in\mathbb{I}_n} \alpha_i$ or equivalently by the equation

$$\sum_{i\in\mathbb{I}_d} (c-\alpha_i)^+ = \sum_{i=d+1}^{n} \alpha_i\ . \tag{9}$$

The spectra of optimal $(\alpha\,,\,d_1)$-designs is computed in terms of the water-filling construction (see Remark 3.8). Hence, it is not surprising that the water-filling construction plays a key role in our construction of optimal $(\alpha\,,\,d)$-designs for $m > 1$ (and general $d = (d_j)_{j\in\mathbb{I}_m} \in (\mathbb{N}^m)^\downarrow$). Thus, in this section we explore some properties of this construction that we will use in the next section. One of the main motivation for considering the water-filling comes from its relation with majorization.

Theorem 3.6 ([21, 22])**.** *Let $\alpha = (\alpha_i)_{i\in\mathbb{I}_n} \in (\mathbb{R}^n_{\geq 0})^\downarrow$, let $d \in \mathbb{N}$ be such that $n \geq d$, and let $\gamma \in (\mathbb{R}^d_{\geq 0})^\downarrow$ denote the water-filling of α in dimension d. Then,*

1. *$\alpha \prec \gamma$;*

2. *If $\beta \in \mathbb{R}^d_{\geq 0}$ is such that $\alpha \prec \beta$ then $\gamma \prec \beta$.* □

Remark 3.7. Theorem 3.6 can be deduced from Definition 3.5 and Remark 2.2. Indeed, notice that the water-filling of α in dimension d is, by construction, a particular case of a vector with a structure as described in Remark 2.2, with $d_1 = d_2 = \ldots = d_{p-1} = 1$ and $d_p = d - p + 1$ (here, $p-1$ would be $p - 1 = \max\{i : \alpha_i \geq c\}$). Thus, it is easy to see that any vector $\beta \in \mathbb{R}^d_{\geq 0}$ such that $\alpha \prec \beta$ satisfies the corresponding inequalities of Eq. (5). Therefore, item 2 follows by Remark 2.2. On the other hand, item 1. follows from the definition of majorization. △

Remark 3.8. In order to show how the water-filling/majorization interacts with the optimal frame design problems, we give a short proof of items 1. and 3. in Theorem 2.6 in terms of Theorems 2.3, 2.4 and 3.6 (notice that the proof of item 2. in Theorem 2.6 is a direct consequence of the water-filling construction). Indeed, consider the notation of Theorem 2.6: in this case, if we let $\gamma^{\rm op}$ denote the water-filling of α in dimension d then the first item in Theorem 3.6 together with Theorem 2.4 show that there exists $\mathcal{F}^{\rm op} \in \mathcal{B}_{\alpha\,,\,d}$ such that $\lambda(S_{\mathcal{F}^{\rm op}}) = \gamma^{\rm op}$. Moreover, if $\mathcal{F} \in \mathcal{B}_{\alpha\,,\,d}$ then Theorem 2.4 shows that $\alpha \prec \lambda(S_{\mathcal{F}})$ so $\lambda(S_{\mathcal{F}^{\rm op}}) \prec \lambda(S_{\mathcal{F}})$ by Theorem 3.6; now we see that Eq. (7) follows from Theorem 2.3 (and Definition 2.5). The spectral structure of local minimizers of strictly convex potentials is a more delicate issue (see [23]). Nevertheless, we can show the uniqueness of the spectral structure of global minimizers of strictly convex potentials as follows: assume that $\mathcal{F} \in \mathcal{B}_{\alpha\,,\,d}$ is such that $\mathrm{P}_\varphi(\mathcal{F}^{\rm op}) = \mathrm{P}_\varphi(\mathcal{F})$. Then the equality $\lambda(S_{\mathcal{F}}) = \gamma^{\rm op}$ is a consequence of the majorization relation $\lambda(S_{\mathcal{F}^{\rm op}}) \prec \lambda(S_{\mathcal{F}})$ and Theorem 2.3. △

In what follows we state and prove several properties of the water-filling construction that we will need in the next subsection.

Proposition 3.9. *Let $\alpha = (\alpha_i)_{i\in\mathbb{I}_n} \in (\mathbb{R}^n_{\geq 0})^\downarrow$ and let $d \in \mathbb{N}$ be such that $d \leq n$. Then*

1. *Let $t \geq 0$ and $\gamma(t) \in (\mathbb{R}^n_{\geq 0})^\downarrow$ denote the water-filling of $t \cdot \alpha$ in dimension d. Then, we have that $\gamma(t) = t \cdot \gamma(1)$.*
2. *Let $\beta = (\beta_i)_{i\in\mathbb{I}_n} \in (\mathbb{R}^n_{\geq 0})^\downarrow$ be such that $\alpha_i \geq \beta_i$ for $i \in \mathbb{I}_n$. If $\gamma = (\gamma_i)_{i\in\mathbb{I}_d}$ and $\delta = (\delta_i)_{i\in\mathbb{I}_d}$ denote the water-fillings in dimension d of α and β respectively, then $\gamma_i \geq \delta_i$ for $i \in \mathbb{I}_d$.*
3. *Assume that $d' \in \mathbb{N}$ is such that $d' \leq d$. If there exists $c \in \mathbb{R}_{>0}$ such that the water-filling of α in dimension d is $c \cdot \mathbb{1}_d$ then there exists $c' \geq c$ such that the water-filling of α in dimension d' is $c' \cdot \mathbb{1}_{d'}$.*

Proof. 1. The case $n = d$ is trivial so we assume that $d < n$. Let $\gamma = \gamma(1)$ be the water-filling of α in dimension d. Hence there exists a unique $c \geq \alpha_d$ such that $\gamma = (\max\{\alpha_i\,,\,c\})_{i\in\mathbb{I}_d}$ where

$$\sum_{i=d+1}^{n} \alpha_i = \sum_{i\in\mathbb{I}_d} (c - \alpha_i)^+ = \sum_{i=r}^{d} c - \alpha_i\,,$$

for $r = \min\{j \in \mathbb{I}_d : \ c \geq \alpha_j\} \in \mathbb{I}_d$. Hence, for $t > 0$ we see that

$$\sum_{i=d+1}^{n} t\,\alpha_i = \sum_{i=r}^{d} t\,c - t\,\alpha_i = \sum_{i\in\mathbb{I}_d} (t\,c - t\,\alpha_i)^+\,,$$

since $r = \min\{j \in \mathbb{I}_d : \ t\,c \geq t\,\alpha_j\} \in \mathbb{I}_d$. Therefore, $\gamma(t) = (\max\{t\,\alpha_i\,,\,t\,c\})_{i\in\mathbb{I}_d} = t\,\gamma$. Notice that in case $t = 0$ the result is trivial.

2. The case $n = d$ is trivial so we assume that $d < n$. By construction, there exists $c \geq \alpha_d$ and $e \geq \beta_d$ such that $\gamma = (\max\{\alpha_i\,,\,c\})_{i\in\mathbb{I}_d}$ and $\delta = (\max\{\beta_i\,,\,e\})_{i\in\mathbb{I}_d}$ where

$$\sum_{i\in\mathbb{I}_d} (c - \alpha_i)^+ = \sum_{i=d+1}^{n} \alpha_i \quad \text{and} \quad \sum_{i\in\mathbb{I}_d} (e - \beta_i)^+ = \sum_{i=d+1}^{n} \beta_i\,.$$

Assume that $e > c$.

If we assume that $e > \beta_d$ then

$$\sum_{i=d+1}^{n} \beta_i = \sum_{i\in\mathbb{I}_d} (e-\beta_i)^+ > \sum_{i\in\mathbb{I}_d} (c-\beta_i)^+ \geq \sum_{i\in\mathbb{I}_d} (c-\alpha_i)^+ = \sum_{i=d+1}^{n} \alpha_i \geq \sum_{i=d+1}^{n} \beta_i \,,$$

which is a contradiction.

If we assume that $e = \beta_d$ then

$$0 = \sum_{i=d+1}^{n} \beta_i = \sum_{i\in\mathbb{I}_d} (e-\beta_i)^+ \geq \sum_{i\in\mathbb{I}_d} (c-\beta_i)^+ \geq \sum_{i\in\mathbb{I}_d} (c-\alpha_i)^+ = \sum_{i=d+1}^{n} \alpha_i \geq 0 \,.$$

Hence $c = \alpha_d$ and $e = \beta_d \leq \alpha_d = c$, which contradicts our previous assumption.

Hence, we conclude that $e \leq c$ and therefore $\delta_i = \max\{\beta_i \,,\, e\} \leq \max\{\alpha_i \,,\, c\} = \gamma_i$, for $i \in \mathbb{I}_d$.

3. Notice that $d' \leq d \leq n$. On the other hand,

$$\sum_{i\in\mathbb{I}_d} \max\{\alpha_i \,,\, c\} = \sum_{i=d+1}^{n} \alpha_i \implies \sum_{i\in\mathbb{I}_{d'}} \max\{\alpha_i \,,\, c\} \leq \sum_{i=d'+1}^{n} \alpha_i \,.$$

Therefore, if $\delta = (\max\{\alpha_i \,,\, c'\})_{i\in\mathbb{I}_{d'}}$ is the water-filling of α in dimension d' we see that $c' \geq c$ and hence $\delta = c' \, \mathbb{1}_{d'}$, since $c' \geq c \geq \alpha_i$ for $i \in \mathbb{I}_d$. □

Definition 3.10. Let $\mathbf{a}' = (a'_i)_{i\in\mathbb{I}_n} \in (\mathbb{R}^n_{\geq 0})^\downarrow$ and let $\gamma' = (\gamma'_i)_{i\in\mathbb{I}_d}$ be its water-filling in dimension $d \leq n$, with water-level c'. We define the functions $a_i(t) : [0 \,,\, \gamma'_1] \to [0, \gamma'_1]$ for $i \in \mathbb{I}_n$ as follows:

$$a_i(t) = \frac{\min\{t \,,\, c'\}}{c'} \; \min\{a'_i \,,\, \max\{t \,,\, c'\}\} \,.$$

Notice that $\mathbf{a}(t) := (a_i(t))_{i\in\mathbb{I}_n} \in (\mathbb{R}^n_{\geq 0})^\downarrow$, for $t \in [0 \,,\, \gamma'_1]$. △

In the next section, we will make use of the functions $a_i(t)$ introduced in Definition 3.10 above to build an algorithm that constructs $(\alpha \,,\, m)$-weight partitions (see Algorithm 3.14). Thus, we study some of the elementary properties of these functions.

Lemma 3.11. *Consider the notation of Definition 3.10. If we let $\gamma(t) \in (\mathbb{R}^d_{\geq 0})^\downarrow$ denote the water-filling of $\mathbf{a}(t) \in (\mathbb{R}^n_{\geq 0})^\downarrow$ in dimension d, then*

$$\gamma(t) = (\min\{\gamma'_i \,,\, t\})_{i\in\mathbb{I}_d} \quad \textit{for} \quad t \in [0 \,,\, \gamma'_1] \,.$$

Proof. Since γ' is the water-filling of $\mathbf{a}'$ in dimension d, then

$$\gamma' = (\max\{a'_i \,,\, c'\})_{i\in\mathbb{I}_d} \in (\mathbb{R}^d_{\geq 0})^\downarrow \quad \text{for} \quad c' \geq a'_d \quad \text{such that} \quad \sum_{i\in\mathbb{I}_d} (c' - a'_i)^+ = \sum_{i=d+1}^{n} a'_i.$$

On the other hand, if we let $\gamma(t) \in (\mathbb{R}^d_{\geq 0})^\downarrow$ denote the water-filling of $\mathbf{a}(t)$ in dimension d then

$$\gamma(t) = (\max\{a_i(t) \,,\, c(t)\})_{i\in\mathbb{I}_d} \quad \text{for} \quad c(t) \geq a_d(t) \quad \text{such that} \quad \sum_{i\in\mathbb{I}_d} (c(t) - a_i(t))^+ = \sum_{i=d+1}^{n} a_i(t).$$

Then, considering Definition 3.10:

1. If $c' \le t \le \gamma_1'$, then $a_i(t) = \min\{a_i'\,,\, t\}$, for $i \in \mathbb{I}_n$. Hence $c(t) = c'$ and

$$\gamma(t) = (\max\{a_i(t)\,,\, c'\})_{i\in\mathbb{I}_d} = (\max\{\min\{a_i'\,,\, t\}\,,\, c'\})_{i\in\mathbb{I}_d} = (\min\{\gamma_i'\,,\, t\})_{i\in\mathbb{I}_d}\,.$$

2. If $0 \le t < c'$, then $a_i(t) = \frac{t}{c'}\min\{a_i'\,,\, c'\}$, for $i \in \mathbb{I}_n$. If $\delta \in \mathbb{R}^d$ denotes the water-filling of $\mathbf{b} = (\min\{a_i'\,,\, c'\})_{i\in\mathbb{I}_n}$ then it is clear that $\delta = c'\,\mathbb{1}_d$. Since $\gamma(t)$ coincides with the water-filling of $\frac{t}{c'}\,\mathbf{b}$ then, by Proposition 3.9, we see that

$$\gamma(t) = \frac{t}{c'}\,(c'\,\mathbb{1}_d) = t\,\mathbb{1}_d = (\min\{\gamma_i'\,,\, t\})_{i\in\mathbb{I}_d}\,.$$

□

Lemma 3.12. *Let $\mathbf{a}' = (a_i')_{i\in\mathbb{I}_n} \in (\mathbb{R}^n_{\ge 0})^\downarrow$ and let $\gamma' = (\gamma_i')_{i\in\mathbb{I}_d}$ be its water-filling in dimension d, with water-level c'. Let $a_i(t) : [0\,,\,\gamma_1'] \to [0\,,\,\gamma_1']$, for $i \in \mathbb{I}_n$, be as in Definition 3.10. Assume that $a_1' \ge c'$ and set $\mathbf{a}'^{(2)} := (a_i')_{i=2}^n \in (\mathbb{R}^{n-1}_{\ge 0})^\downarrow$. Then*

1. *$\gamma'^{(2)} := (\gamma_i')_{i=2}^d \in (\mathbb{R}^{d-1}_{\ge 0})^\downarrow$ is the water-filling of $\mathbf{a}'^{(2)}$ in dimension $d-1$.*

2. *If we let $a_i^{(2)}(t) : [0\,,\,\gamma_2'] \to [0\,,\,\gamma_2']$ for $2 \le i \le n$, be constructed as in Definition 3.10 with respect to $\mathbf{a}'^{(2)}$ and $d' = d-1$ then*

$$a_i(t) = a_i^{(2)}(t) \quad \textit{for} \quad t \in [0\,,\,\gamma_2'] \quad \textit{and} \quad 2 \le i \le n\,.$$

3. *If $\gamma(t) = (\gamma_i(t))_{i\in\mathbb{I}_d}$ denotes the water-filling of $a(t) = (a_i(t))_{i\in\mathbb{I}_n}$ in dimension d then*

$$t = \gamma_1(t) = a_1(t) \quad \textit{for} \quad t \in [0, \gamma_1']\,.$$

Proof. Notice that the first claim is straightforward. In order to prove the second claim, notice that the water-level of $\gamma'^{(2)}$ is c'. We consider the following two cases:

Case 1: $a_2' \ge c'$, so that $\gamma_2' = a_2' \ge c'$. In this case, for $2 \le i \le n$:

- If $c' \le t \le \gamma_2'$: $a_i^{(2)}(t) = \min\{a_i'\,,\, t\} = a_i(t)$;
- If $0 \le t \le c'$: $a_i^{(2)}(t) = \frac{t}{c'}\,\min\{a_i'\,,\, c'\} = a_i(t)$.

Case 2: $a_2' < c' = \gamma_2'$ and hence $\gamma_2' = c'$. In this case, if $0 \le t \le \gamma_2'$ and $2 \le i \le n$:

$$a_i^{(2)}(t) = \frac{t}{\gamma_2'}\,a_i = \frac{t}{c'}\,\min\{a_i'\,,\, c'\} = a_i(t)\,,$$

since $c' = \gamma_2' \ge a_2' \ge a_i'$, for $2 \le i \le n$.

The proof of the third follows by the fact that $a_1' \ge c'$ implies $a_1' = \gamma_1'$. Therefore, using Definition 3.10 and Lemma 3.11, we have $a_1(t)' = t = \min\{\gamma_1', t\} = \gamma_1(t)$ for $t \in [0, \gamma_1']$. □

3.3 An algorithmic construction of a weight partition

We begin this section by explaining the key role played by weight-partitions

Remark 3.13 (A reduction to the problem of computing optimal $(\alpha\,,\,d)$-designs)**.** Let $\alpha = (\alpha_i)_{i\in\mathbb{I}_n} \in (\mathbb{R}^n_{>0})^\downarrow$ and $m \in \mathbb{N}$ with $m \geq 1$ be given; consider $d = (d_j)_{j\in\mathbb{I}_m} \in (\mathbb{N}^m)^\downarrow$ such that $d_1 \leq n$. Let $A = (a_{ij})_{i\in\mathbb{I}_n\,,\,j\in\mathbb{I}_m} \in P_{\alpha\,,\,m}$ be a fixed $(\alpha\,,\,m)$-weight partition. We can consider the set of A-designs given by

$$\mathcal{D}(A) = \prod_{j=1}^{m} \mathcal{B}_{c_j(A)\,,\,d_j}$$

where $c_j(A) = (a_{ij})_{i\in\mathbb{I}_n} \in \mathbb{R}^n_{\geq 0}$ denotes the j-th column of A, for $j \in \mathbb{I}_m$. Notice that $\mathcal{D}(A) \subset \mathcal{D}(\alpha\,,\,d)$ can be considered as a slice of $\mathcal{D}(\alpha\,,\,d)$. By Theorem 2.6, for each $j \in \mathbb{I}_m$ there exists $\mathcal{F}_j^0 \in \mathcal{B}_{c_j(A)\,,\,d_j}$ that can be computed by a finite-step algorithm (see Remark 2.7), such that for every $\varphi \in \text{Conv}(\mathbb{R}_{\geq 0})$,

$$\text{P}_\varphi(\mathcal{F}_j^0) \leq \text{P}_\varphi(\mathcal{F}_j) \quad \text{for every} \quad \mathcal{F}_j \in \mathcal{B}_{c_j(A)\,,\,d_j}\,. \tag{10}$$

Hence, $\Phi^0 := (\mathcal{F}_j^0)_{j\in\mathbb{I}_m} \in \mathcal{D}(A)$ is such that for every $\varphi \in \text{Conv}(\mathbb{R}_{\geq 0})$,

$$\text{P}_\varphi(\Phi^0) = \sum_{j\in\mathbb{I}_m} \text{P}_\varphi(\mathcal{F}_j^0) \overset{(10)}{\leq} \sum_{j\in\mathbb{I}_m} \text{P}_\varphi(\mathcal{F}_j) = \text{P}_\varphi(\Phi) \quad \text{for every} \quad \Phi = (\mathcal{F}_j)_{j\in\mathbb{I}_m} \in \mathcal{D}(A)\,. \tag{11}$$

That is, once we fix $A \in P_{\alpha\,,\,m}$ then there is an structural solution Φ^0 for the optimization of convex potentials in (the slice) $\mathcal{D}(A)$; moreover, Φ^0 can be algorithmically computed.

The previous remarks show that the problem of computing optimal structural $(\alpha\,,\,d)$-designs Φ^{op} (as in Eq. (8)) reduces to the problem of finding optimal $(\alpha\,,\,m)$-weight partitions $A^{\text{op}} \in P_{\alpha\,,\,m}$ - in terms of a finite-step algorithm - in the sense that the structural optimal solution in the slice $\mathcal{D}(A^{\text{op}})$ (as described above) is an structural solution for the optimization of convex potentials in the set $\mathcal{D}(\alpha\,,\,d)$ of all $(\alpha\,,\,d)$-designs. Therefore, in what follows we shall focus on constructing such optimal $(\alpha\,,\,m)$-weight partitions. △

Let $\alpha = (\alpha_i)_{i\in\mathbb{I}_n} \in (\mathbb{R}^n_{>0})^\downarrow$ and $m \in \mathbb{N}$ with $m \geq 1$ be given and consider $d = (d_j)_{j\in\mathbb{I}_m} \in (\mathbb{N}^m)^\downarrow$. In this section we describe a finite-step algorithm whose input are α and d and whose output are the sequences $\mathbf{a}_j^{\text{op}} = (a_{ij}^{\text{op}})_{i\in\mathbb{I}_n} \in (\mathbb{R}^n_{\geq 0})^\downarrow$ for $j \in \mathbb{I}_m$, such that $(a_{ij}^{\text{op}})_{i\in\mathbb{I}_n\,,\,j\in\mathbb{I}_m} \in P_{\alpha\,,\,m}$. We further construct $\gamma_j^{\text{op}} = (\gamma_{ij}^{\text{op}})_{i\in\mathbb{I}_{d_j}} \in (\mathbb{R}^{d_j}_{\geq 0})^\downarrow$ that is the water-filling of $\mathbf{a}_j^{\text{op}}$ in dimension d_j, for $j \in \mathbb{I}_m$. The procedure is recursive in m i.e. assuming that we have applied the algorithm to α and the dimensions $d_1 \geq \ldots \geq d_{m-1}$ with a certain output, we use it to construct the output for α and $d_1 \geq \ldots \geq d_{m-1} \geq d_m \geq 1$. Along the way, we (inductively) assume some specific features of the output; we will show that the recursive process is well defined. The output of this algorithm will play a central role in the construction of optimal $(\alpha\,,\,d)$-designs in the next section.

Algorithm 3.14.

INPUT:

- $\alpha = (\alpha_i)_{i\in\mathbb{I}_n} \in (\mathbb{R}^n_{>0})^\downarrow$;
- $m \in \mathbb{N}$ with $m \geq 1$ and $d_1 \geq \ldots \geq d_m \geq 1$, with $n \geq d_1$.

ALGORITHM:

- In case $m = 1$ we set $a_{i1}^{\text{op}} = \alpha_i$ for $i \in \mathbb{I}_n$, so $\gamma_1^{\text{op}} \in (\mathbb{R}^{d_1}_{>0})^\downarrow$.

- In case $m > 1$: assume that we have constructed $\mathbf{a}'_j = (a'_{ij})_{i\in\mathbb{I}_n}$ for $j \in \mathbb{I}_{m-1}$ according to the algorithm, using the input $\alpha = (\alpha_i)_{i\in\mathbb{I}_n}$ and the dimensions $d_1 \geq \ldots \geq d_{m-1}$. Thus, $(a'_{ij})_{i\in\mathbb{I}_n\,,\,j\in\mathbb{I}_{m-1}}$ is an $(\alpha\,,\,m-1)$-weight partition. Let $\gamma'_j = (\gamma'_{ij})_{i\in\mathbb{I}_{d_j}} \in (\mathbb{R}^{d_j}_{\geq 0})^\downarrow$ be the vector obtained by water-filling of $\mathbf{a}'_j$ in dimension d_j, for $j \in \mathbb{I}_{m-1}$. We denote by c'_j the water-level of γ'_j.

  We assume (Inductive Hypothesis) that $\mathbf{a}'_j \in (\mathbb{R}^n_{\geq 0})^\downarrow$, and that for $1 \leq r \leq s \leq m-1$

$$\gamma'_{ir} = \gamma'_{is} \quad \text{for} \quad i \in \mathbb{I}_{d_s}\,.$$

  1. For $0 \leq t \leq \gamma'_{11}$ we introduce the partitions $A^{(1)}(t) = (a^{(1)}_{ij}(t))_{i\in\mathbb{I}_n\,,\,j\in\mathbb{I}_m} \in P_{\alpha\,,\,m}$ as follows:

     For $j \in \mathbb{I}_{m-1}$, $a^{(1)}_{ij}(t)$ is as in Definition 3.10, applied to $\mathbf{a}'_j$, i.e.:

$$a^{(1)}_{ij}(t) = \frac{\min\{t\,,\,c'_j\}}{c'_j}\min\{a'_{ij}\,,\,\max\{t\,,\,c'_j\}\} \quad \text{for} \quad i \in \mathbb{I}_n\,.$$

     Notice that $\mathbf{a}^{(1)}_j(t) := (a^{(1)}_{ij}(t))_{i\in\mathbb{I}_n} \in (\mathbb{R}^n_{\geq 0})^\downarrow$, for $0 \leq t \leq \gamma'_{11}$ and $j \in \mathbb{I}_{m-1}$.

     We set:
$$a^{(1)}_{im}(t) = \alpha_i - \sum_{j\in\mathbb{I}_{m-1}} a^{(1)}_{ij}(t) \quad \text{for} \quad 0 \leq t \leq \gamma'_{11} \quad \text{and} \quad i \in \mathbb{I}_n\,.$$

     Claim 1 (see Remark 3.15). $\mathbf{a}^{(1)}_m(t) = (a^{(1)}_{im}(t))_{i\in\mathbb{I}_n} \in (\mathbb{R}^n_{\geq 0})^\downarrow$.
  2. For $0 \leq t \leq \gamma'_{11}$ and $j \in \mathbb{I}_m$ we also set:

$$\gamma^{(1)}_j(t) = (\gamma^{(1)}_{ij}(t))_{i\in\mathbb{I}_{d_j}} \in (\mathbb{R}^{d_j}_{\geq 0})^\downarrow$$

     that is obtained by water-filling of $\mathbf{a}^{(1)}_j(t)$ in dimension d_j.
  3. Claim 2 (see Remark 3.15): with the previous definitions:

$$\gamma^{(1)}_{1j}(t) = t \quad \text{for} \quad 0 \leq t \leq \gamma'_{11} \quad \text{and} \quad j \in \mathbb{I}_{m-1}\,. \tag{12}$$

     Claim 3 (see Remark 3.15): the functions $a^{(1)}_{im}(t)$ are non-increasing and $\gamma^{(1)}_{1m}(t)$ is strictly decreasing in $[0, \gamma'_{11}]$. Moreover, $\gamma^{(1)}_{1m}(0) \geq \gamma'_{11}$ and $\gamma^{(1)}_{1m}(\gamma'_{11}) = 0$.

     Therefore, there exists a unique value $t_1 \in [0, \gamma'_{11}]$ such that

$$\gamma^{(1)}_{1m}(t_1) = t_1 \overset{(12)}{=} \gamma^{(1)}_{1j}(t_1) \quad \text{for} \quad j \in \mathbb{I}_{m-1}\,.$$

     For this t_1 we consider two cases:

     Case 1: assume that $\gamma^{(1)}_m(t_1) \neq \gamma^{(1)}_{1m}(t_1)\,\mathbb{1}_{d_m}$ so that $d_m > 1$. In this case:
     (a) We set $a^{\text{op}}_{1j} := a^{(1)}_{1j}(t_1)$ for $j \in \mathbb{I}_m$;
     (b) We re-initialize the algorithm by setting $d_j := d_j - 1$, for $j \in \mathbb{I}_m$ and considering $(a'_{ij})_{i=2}^n \in (\mathbb{R}^{n-1}_{\geq 0})^\downarrow$ for $j \in \mathbb{I}_{m-1}$, which forms an $((\alpha_i)_{i=2}^n\,,\,m-1)$-weight partition (see Remark 3.17).

(c) Hence, we apply the construction of step 1 to $(a'_{ij})_{i=2}^n$, for $j \in \mathbb{I}_{m-1}$, together with the new dimensions and compute:

$$\mathbf{a}_j^{(2)}(t) = (a_{ij}^{(2)}(t))_{i=2}^n \in (\mathbb{R}_{\geq 0}^{n-1})^\downarrow \quad \text{and} \quad \gamma_j^{(2)}(t) = (\gamma_{ij}^{(2)}(t))_{i=2}^{d_j} \in (\mathbb{R}_{\geq 0}^{d_j-1})^\downarrow .$$

Thus, there exists a unique $t_2 \in [0\,,\,\gamma'_{21}]$ such that

$$\gamma_{2m}^{(2)}(t_2) = t_2 = \gamma_{2j}^{(2)}(t_2) \quad \text{for} \quad j \in \mathbb{I}_{m-1} .$$

In particular, we define (at least)

$$a_{2j}^{\rm op} = a_{ij}^{(2)}(t_2) \quad \text{for} \quad j \in \mathbb{I}_m .$$

Case 2: assume that $\gamma_m^{(1)}(t_1) = \gamma_{1m}^{(1)}(t_1) \cdot \mathbb{1}_{d_m}$.

In this case we set $a_{ij}^{\rm op} = a_{ij}^{(1)}(t_1)$, for $i \in \mathbb{I}_n$ and $j \in \mathbb{I}_m$. Thus we compute the optimal weights $\mathbf{a}_j^{\rm op} = (a_{ij}^{\rm op})_{i\in\mathbb{I}_n}$, for $j \in \mathbb{I}_m$. The algorithm stops. This case shall be subsequently referred to as "the Algorithm stops in the first iteration" assuming that the process starts computing the weight $a_{1m}^{\rm op}$.

OUTPUT:

Notice that the algorithm stops at some point, having defined a partition $A^{\rm op} = (a_{ij}^{\rm op})_{i\in\mathbb{I}_n\,,j\in\mathbb{I}_m}$. In this case we set $\mathbf{a}_j^{\rm op} = (a_{ij}^{\rm op})_{i\in\mathbb{I}_n}$ and $\gamma_j^{\rm op} = (\gamma_{ij}^{\rm op})_{i\in\mathbb{I}_{d_j}}$ that is obtained by water-filling of $\mathbf{a}_j^{\rm op} \in (\mathbb{R}_{\geq 0}^n)^\downarrow$ in dimension d_j, for $j \in \mathbb{I}_m$. △

In the next remark we prove Claims 1,2 and 3 stated in Algorithm 3.14. After that, we consider some results in order to show that the inductive hypothesis, assumed in Algorithm 3.14, holds for m groups (see Remark 3.22) so that the recursive process is well defined.

Remark 3.15. Consider the notation and definitions of Algorithm 3.14.

Proof of Claim 1. For $j \in \mathbb{I}_{m-1}$ we consider $b_{ij}(t) = a'_{ij} - a_{ij}^{(1)}(t)$, for $0 \leq t \leq \gamma'_{11}$ and $i \in \mathbb{I}_n$. Thus, for $j \in \mathbb{I}_{m-1}$ we have that:

(a1) If $c'_j \leq t \leq \gamma'_{11}$ then $b_{ij}(t) = (a'_{ij} - t)^+$ for $i \in \mathbb{I}_n$.

(a2) If $0 \leq t < c'_j$ then $b_{i1}(t) = (a'_{ij} - c'_j)^+ + \frac{c'_j - t}{c'_j} \min\{a'_{ij},\, c'_j\}$ for $i \in \mathbb{I}_n$.

Hence, we see that $\mathbf{b}_j(t) := (b_{ij}(t))_{i\in\mathbb{I}_n} \in (\mathbb{R}_{\geq 0}^n)^\downarrow$, for $0 \leq t \leq \gamma'_{11}$ and $j \in \mathbb{I}_{m-1}$. Also, for $j \in \mathbb{I}_{m-1}$ and $i \in \mathbb{I}_n$ we have that the function $b_{ij}(t)$ is non-increasing for $t \in [0, \gamma'_{11}]$.

By definition, we have that for $0 \leq t \leq \gamma'_{11}$ and $i \in \mathbb{I}_n$

$$a_{im}^{(1)}(t) = \alpha_i - \sum_{j\in\mathbb{I}_{m-1}} a_{ij}^{(1)}(t) = \sum_{j\in\mathbb{I}_{m-1}} (a'_{ij} - a_{ij}^{(1)}(t)) = \sum_{j\in\mathbb{I}_{m-1}} b_{ij}(t)\,,$$

which shows that $\mathbf{a}_m^{(1)}(t) = (a_{im}^{(1)}(t))_{i\in\mathbb{I}_n} \in (\mathbb{R}_{\geq 0}^n)^\downarrow$.

Proof of Claim 2. Notice that the functions $a_i(t)$ introduced in Definition 3.10 allows to describe a sub-routine that computes the vectors $\mathbf{a}_j^{(1)}(t) = (a_{ij}^{(1)}(t))_{i\in\mathbb{I}_n}$ for $j \in \mathbb{I}_{m-1}$ - as described in Algorithm 3.14 - in terms of the vectors $\mathbf{a}'_j = (a'_{ij})_{i\in\mathbb{I}_n}$ for $j \in \mathbb{I}_{m-1}$. Hence, the claim follows from Lemma 3.11.

Proof of Claim 3. An analysis similar to that considered in the proof of Claim 1 above shows that: $a^{(1)}_{im}(t)$ are non-increasing while $a^{(1)}_{1m}(t)$ and $\mathrm{tr}(\mathbf{a}^{(1)}_m(t))$ are strictly decreasing functions in $[0,\gamma'_{11}]$. Notice that by construction, $a^{(1)}_j(0)=0_n\in\mathbb{R}^n$ is the zero vector, for $j\in\mathbb{I}_{m-1}$ and hence $a^{(1)}_{im}(0)=\alpha_i$ for $i\in\mathbb{I}_n$. Therefore, since $\mathbf{a}'_j\leq\alpha$ and $d_1\geq d_m$ it is straightforward to check that

$$\gamma^{(1)}_{im}(0)\geq\gamma'_{i1}\quad\text{for}\quad i\in\mathbb{I}_{d_m}\implies\gamma^{(1)}_{1m}(0)\geq\gamma'_{11}\,.$$

On the other hand, using (the inductive hypothesis) we see that

$$\gamma'_{1j}=\gamma'_{11}\quad\text{for}\quad j\in\mathbb{I}_{m-1}\implies\mathbf{a}^{(1)}_j(\gamma'_{11})=\mathbf{a}'_j\quad\text{for}\quad j\in\mathbb{I}_{m-1}\,.$$

This last fact shows that $\mathbf{a}^{(1)}_m(\gamma'_{11})=0_n\in\mathbb{R}^n$, so that $\gamma^{(1)}_{1m}(\gamma'_{11})=0$. △

Remark 3.16. With the notation, notions and constructions of Algorithm 3.14, assume that there exists $\ell\in\mathbb{I}_{m-1}$ such that $c'_\ell\geq a'_{1\ell}$. Then, the algorithm stops in the first iteration. Indeed, notice that in this case $\gamma'_\ell=c'_\ell\,\mathbb{1}_{d_\ell}$. Using the inductive hypothesis we conclude that

$$\gamma'_{ij}=c'_\ell\quad\text{for}\quad i\in\mathbb{I}_{d_m}\quad\text{and}\quad j\in\mathbb{I}_{m-1}$$

where we are using that $d_m\leq d_j$ for $j\in\mathbb{I}_{m-1}$. Therefore, the water-filling of $\mathbf{b}_j(t)=(b_{ij}(t))_{i\in\mathbb{I}_n}$ in dimension d_m is a multiple of $\mathbb{1}_{d_m}$ for $t\geq0$; indeed, let $j\in\mathbb{I}_{m-1}$ and notice that by inductive hypothesis

$$\gamma'_{1j}=\gamma'_{1\ell}=c'_\ell\geq c'_j\,.$$

Hence, we consider the following cases (see the proof of Claim 1 in Remark 3.15):

1. In case $\gamma'_{1j}=c'_\ell>c'_j$: we have that $a'_{ij}=\gamma'_{ij}=c'_\ell$ for $i\in\mathbb{I}_{d_\ell}$. We now consider the following subcases:

   (a1) If $c'_j\leq t\leq\gamma'_{1j}$: then $b_{ij}(t)=(a'_{ij}-t)^+$ for $i\in\mathbb{I}_n$. Hence, in particular, $b_{ij}(t)=(c'_\ell-t)^+$ for $i\in\mathbb{I}_{d_m}$. This last fact shows that the water-filling of $\mathbf{b}_j(t)\in(\mathbb{R}^n_{\geq0})^\downarrow$ in dimension d_m is a multiple of $\mathbb{1}_{d_m}$.

   (a2) If $0\leq t<c'_j$: then

$$b_{ij}(t)=(a'_{ij}-c'_j)^++\frac{c'_j-t}{c'_j}\,\min\{a'_{ij}\,,\,c'_j\}\quad\text{for}\quad i\in\mathbb{I}_n\,.$$

   In particular, $b_{ij}(t)=(c'_\ell-c'_j)^++c'_j-t=c'_\ell-t$ for $i\in\mathbb{I}_{d_m}$. Again, this shows that the water-filling of $\mathbf{b}_j(t)\in(\mathbb{R}^n_{\geq0})^\downarrow$ in dimension d_m is a multiple of $\mathbb{1}_{d_m}$.

2. Case 2: $\gamma'_{1j}=c'_\ell=c'_j$; then, $\gamma'_{ij}=c'_\ell$ for $i\in\mathbb{I}_{d_j}$. Hence, if $0\leq t\leq c'_j=\gamma'_{11}$ then

$$b_{ij}(t)=\frac{c'_j-t}{c'_j}\,a'_{ij}\quad\text{for}\quad i\in\mathbb{I}_n\,.$$

   Then, by item 1 in Proposition 3.9 we see that the water-filling of $\mathbf{b}_j(t)$ in dimension d_j is a multiple of $\mathbb{1}_{d_j}$. Finally, by item 3 in Proposition 3.9 we see that the water-filling of $\mathbf{b}_j(t)$ in dimension d_m is a multiple of $\mathbb{1}_{d_m}$.

In this case it is straightforward to check that the water-filling of $\mathbf{a}^{(1)}_m(t)=\sum_{j\in\mathbb{I}_{m-1}}\mathbf{b}_j(t)$ in dimension d_m is a multiple of $\mathbb{1}_{d_m}$ for $t\in[0\,,\,\gamma'_{11}]$. Therefore, (according to case 2 in Algorithm 3.14) the algorithm stops in the first step. △

Remark 3.17. Consider the notation, notions and constructions of Algorithm 3.14. Assume that Algorithm 3.14 does not stop in the first iteration (notice that in this case $d_m \geq 2$). We show (inductively on m) that $(a_{ij}^{\rm op})_{i=2}^n$ for $j \in \mathbb{I}_m$ coincides with the output of the algorithm for the weights $(\alpha_i)_{i=2}^n$ and the dimensions $d_1 - 1 \geq \ldots \geq d_m - 1$.

We first point out that the case $m = 1$ is straightforward. Indeed, in this case the output of the algorithm for α and d_1 is α. Analogously, the output of the algorithm for $(\alpha_i)_{i=2}^n$ and $d_1 - 1$ is $(\alpha_i)_{i=2}^n$.

We now assume that $m \geq 2$. Recall that the algorithm is based on $\mathbf{a}'_j = (a'_{ij})_{i\in\mathbb{I}_n}$ for $j \in \mathbb{I}_{m-1}$, which is the output of the algorithm for α and the $m-1$ dimensions $d_1 \geq \ldots \geq d_{m-1}$. If the algorithm for computing $\mathbf{a}'_j = (a'_{ij})_{i\in\mathbb{I}_n}$ for $j \in \mathbb{I}_{m-1}$ stops in the first iteration then we have that

$$\gamma'_{m-1} = (\gamma'_{i\,(m-1)})_{i\in\mathbb{I}_{d_{m-1}}} = c'_{m-1}\,\mathbb{1}_{d_{m-1}}\,.$$

But in this case $c'_{m-1} \geq a'_{1(m-1)}$, so Remark 3.16 shows that the algorithm based on $\mathbf{a}'_j = (a'_{ij})_{i\in\mathbb{I}_n}$ for $j \in \mathbb{I}_{m-1}$ - that computes $\mathbf{a}_j^{\rm op}$ for $j \in \mathbb{I}_m$ - stops in the first iteration; this last fact contradicts our initial assumption. Therefore, we can apply the inductive hypothesis and conclude that the output of the algorithm with initial data $(\alpha_i)_{i=2}^n$ and dimensions $d_1 - 1 \geq \ldots \geq d_{m-1} - 1$ is $(a'_{ij})_{i=2}^n$ for $j \in \mathbb{I}_{m-1}$.

After the first iteration of the algorithm with initial data α and $d_1 \geq \ldots \geq d_m$, the algorithm defines $(a_{1j}^{\rm op})_{j\in\mathbb{I}_m}$ and re-initializes (case 1 (b) in Algorithm 3.14) using $(a'_{ij})_{i=2}^n$ for $j \in \mathbb{I}_{m-1}$ and the dimensions $d_1 - 1 \geq \ldots d_m - 1 \geq 1$, and iterates until computing $(a_{ij}^{\rm op})_{i=2}^n$ for $j \in \mathbb{I}_m$ with this data. But, by the comments above, we see that $(a_{ij}^{\rm op})_{i=2}^n$ for $j \in \mathbb{I}_m$ is actually being computed by applying the algorithm to the output of the algorithm with initial data $(\alpha_i)_{i=2}^n$ and $(m-1)$ blocks with dimensions $d_1 - 1 \geq \ldots \geq d_{m-1} - 1$; the claim now follows from this last fact.

$\triangle$

Lemma 3.18. *Consider the notation, notions and constructions of Algorithm 3.14. Assume that the algorithm does not stop in the first iteration. Therefore, there exists t_1 and t_2 such that*

$$t_1 = \gamma_{1j}^{(1)}(t_1) \quad \text{and} \quad t_2 = \gamma_{2j}^{(2)}(t_2) \quad \text{for} \quad j \in \mathbb{I}_m$$

where $\gamma_j^{(2)}(t_2) = (\gamma_{ij}^{(2)}(t_2))_{i=2}^{d_j} \in (\mathbb{R}_{\geq 0}^{d_j-1})^{\downarrow}$ denotes the water-filling of $\mathbf{a}_j^{(2)}(t_2)$ obtained in the second iteration of the algorithm. In this case:

1. *$a_{ij}^{(1)}(t) = a_{ij}^{(2)}(t)$ for $2 \leq i \leq n$, $j \in \mathbb{I}_m$ and $t \in [0\,,\,\gamma'_{21}]$; hence,*
2. *$a_{ij}^{\rm op} = a_{ij}^{(1)}(t_i)$ for $i = 1,\,2$ and $j \in \mathbb{I}_m$.*
3. *$a_{1j}^{\rm op} = \gamma_{1j}^{(1)}(t_1) = \frac{\alpha_1}{m}$ for $j \in \mathbb{I}_m$.*
4. *$t_1 \geq t_2$.*

Proof. We first notice that in our case $d_m > 1$ and $c'_\ell < a'_{1\ell}$ for $\ell \in \mathbb{I}_{m-1}$; otherwise, Remark 3.16 shows that the algorithm stops in the first step. Notice that the functions $\mathbf{a}_j(t)$ are actually computed using the construction in Definition 3.10 based on $\mathbf{a}'_j = (a'_{ij})_{i\in\mathbb{I}_n}$, for $j \in \mathbb{I}_{m-1}$. The previous comments also show that Lemma 3.12 applies and therefore, if $\mathbf{a}_j^{(2)}(t) = (a_{ij}^{(2)}(t))_{i=2}^n$ then

$$a_{ij}^{(1)}(t) = a_{ij}^{(2)}(t) \quad \text{for} \quad 2 \leq i \leq n \quad , \quad j \in \mathbb{I}_m \quad \text{and} \quad t \in [0\,,\,\gamma'_{21}]\,.$$

Since the algorithm does not stop in the first step we see that

$$t_1 = \gamma_{1j}^{(1)}(t_1) = a_{1m}^{(1)}(t_1) \quad \text{for} \quad j \in \mathbb{I}_m\,.$$

The previous comments together with the third item in Lemma 3.12 show that

$$a_{1j}^{(1)}(t_1) = \gamma_{1j}^{(1)}(t_1) = t_1 \quad , \quad j \in \mathbb{I}_m \implies \alpha_1 = \sum_{j \in \mathbb{I}_m} a_{1j}^{(1)}(t_1) = m\, t_1 \quad \text{and} \quad a_{1j}^{\rm op} = a_{1j}^{(1)}(t_1) = \frac{\alpha_1}{m} \, .$$

We now assume that $t_1 < t_2$ and reach a contradiction. We consider the following two cases:

Case 1: the algorithm stops in the second step. In this case,

$$t_2 = \gamma_{2m}^{(2)}(t_2) \implies \gamma_m^{(2)}(t_2) = t_2 \, \mathbb{1}_{d_m - 1} \quad \text{and} \quad \sum_{i=2}^{n} a_{im}^{(2)}(t_2) = (d_m - 1)\, t_2 \, .$$

Recall that the functions $a_{im}(t)$ are non-increasing in $[0\,,\,\gamma_{11}']$: hence, using that $t_1 < t_2$ then

$$a_{im}^{(1)}(t_1) \geq a_{im}^{(1)}(t_2) = a_{im}^{(2)}(t_2) \quad \text{for} \quad 2 \leq i \leq n \, .$$

On the other hand, $a_{1m}^{(1)}(t_1) = t_1 < t_2$. Then,

$$\frac{1}{d_m} \sum_{i \in \mathbb{I}_n} a_{im}^{(1)}(t_1) \geq \frac{1}{d_m} \left(t_1 + \sum_{i=2}^{n} a_{im}^{(2)}(t_2) \right) = \frac{1}{d_m} (t_1 + (d_m - 1)\, t_2) > t_1 \, .$$

These facts show that $\gamma_m^{(1)}(t_1)$ should be a multiple of $\mathbb{1}_{d_m}$ (since $\gamma_m^{(1)}(t_1)$ is the water-filling of $a_m^{(1)}(t_1)$ in dimension d_m) which contradicts the assumption that the algorithm does not stop in the first step.

Case 2: the algorithm does not stop in the second step. In this case, if $t_1 < t_2$ then

$$t_2 = \gamma_{2m}^{(2)}(t_2) = a_{2m}^{(2)}(t_2) = a_{2m}^{(1)}(t_2) \leq a_{1m}^{(1)}(t_2) \leq a_{1m}^{(1)}(t_1) = t_1$$

since $a_m^{(1)}(t) \in (\mathbb{R}_{\geq 0}^n)^\downarrow$ and $a_{1m}^{(1)}(t)$ is a non-increasing function; therefore $t_2 \leq t_1$ which is a contradiction.

□

Proposition 3.19. *Consider the notation, notions and constructions of Algorithm 3.14 for* $m \geq 2$. *Assume that the algorithm computing* $\mathbf{a}_j^{\rm op} \in (\mathbb{R}_{\geq 0}^n)$, *for* $j \in \mathbb{I}_m$, *stops in the* k*-th iteration (so* $k \leq d_m$*). Let* $t_1, \ldots, t_k$ *be constructed in each iteration of Algorithm 3.14. Then*

1. $a_{\ell j}^{(i)}(t) = a_{\ell j}^{(1)}(t)$ *for* $t \in [0, \gamma_{i1}']$, $i \leq \ell \leq n$, $i \in \mathbb{I}_n$ *and for* $j \in \mathbb{I}_m$;

2. *Let* $\gamma_j^{(i)}(t_i) = (\gamma_{\ell j}^{(i)}(t_i))_{\ell=i}^{d_j}$ *be the water-filling of* $(a_{\ell j}^{(1)}(t_i))_{\ell=i}^{n}$ *in dimension* $d_j - i + 1$, *for* $i \in \mathbb{I}_{k-1}$. *Then, for* $j \in \mathbb{I}_m$ *we have that*

$$a_{ij}^{\rm op} = a_{ij}^{(1)}(t_i) = \gamma_{ij}^{(i)}(t_i) = \frac{\alpha_i}{m} \quad , \quad i \in \mathbb{I}_{k-1} \quad \textit{and} \quad a_{ij}^{\rm op} = a_{ij}^{(1)}(t_k) \quad , \quad k \leq i \leq n \, . \tag{13}$$

3. $t_1 \geq t_2 \ldots \geq t_k \geq 0$.

Moreover, we have that $\mathbf{a}_j^{\rm op} \in (\mathbb{R}_{\geq 0}^n)^\downarrow$, *for* $j \in \mathbb{I}_m$.

Proof. In case the algorithm stops in the first step (i.e. $k = 1$) then $\mathbf{a}_j^{\rm op} = \mathbf{a}_j^{(1)}(t_1) \in (\mathbb{R}_{\geq 0}^n)^\downarrow$, (according to Claim 1 in Algorithm 3.14 for $j = m$, while this property clearly holds by construction for $j \in \mathbb{I}_{m-1}$).

In case $k > 1$ then, with the notation and terminology from Lemma 3.18, we have that $t_2 \leq t_1$ and $a_{ij}^{(1)}(t) = a_{ij}^{(2)}(t)$ for $2 \leq i \leq n$, $j \in \mathbb{I}_m$. In particular, $a_{ij}^{\rm op} = a_{ij}^{(1)}(t_i)$ for $i = 1, 2$ and $j \in \mathbb{I}_m$. If $j \in \mathbb{I}_{m-1}$ then $a_{1j}^{(1)}(t)$ is a non-decreasing function, and hence

$$a_{2j}^{\rm op} = a_{2j}^{(1)}(t_2) \leq a_{1j}^{(1)}(t_2) \leq a_{1j}^{(1)}(t_1) = \gamma_{1j}^{(1)}(t_1) = a_{1j}^{\rm op} ,$$

where we have also used that $\mathbf{a}_j^{(1)}(t_2) \in (\mathbb{R}_{\geq 0}^n)^\downarrow$. On the other hand,

$$a_{1m}^{\rm op} = \gamma_{1m}^{(1)}(t_1) = t_1 \geq t_2 = \gamma_{2m}^{(2)}(t_2) \geq a_{2m}^{(2)}(t_2) = a_{2m}^{\rm op} .$$

Using Remark 3.17 we can repeat the previous argument together with Lemma 3.18 $(k-1)$ times (applied to subsequent truncations of the initial weights and dimensions) and conclude that $t_1 \geq t_2 \geq \ldots \geq t_{k-1}$; hence, for $j \in \mathbb{I}_m$ we have that

$$a_{\ell j}^{(i)}(t) = a_{\ell j}^{(1)}(t) \quad \text{for} \quad i \leq \ell \leq n \quad , \quad i \in \mathbb{I}_k \quad \text{and} \tag{14}$$

$$a_{(i-1)j}^{\rm op} \geq a_{ij}^{\rm op} = a_{ij}^{(1)}(t_i) = \gamma_{ij}^{(i)}(t_i) = \frac{\alpha_i}{m} \quad \text{for} \quad 2 \leq i \leq k-1 . \tag{15}$$

Since the algorithm stops in the k-th step we see that

$$a_{ij}^{\rm op} = a_{ij}^{(k)}(t_k) = a_{ij}^{(1)}(t_k) \quad \text{for} \quad k \leq i \leq n \quad \text{and} \quad j \in \mathbb{I}_m . \tag{16}$$

Eq. (13) together with the fact that $\mathbf{a}_j^{\rm op} \in (\mathbb{R}_{\geq 0}^n)^\downarrow$ is a consequence of Eqs. (15) and (16). □

Remark 3.20. Consider the notation, notions and constructions of Algorithm 3.14 for $m \geq 2$. Assume that the algorithm stops in the k-th iteration, for $k \geq 2$. In this case, in the i-th iteration (for $i \in \mathbb{I}_k$) the algorithm defines the functions $a_{\ell j}^{(i)}(t)$ for $i \leq \ell \leq n$ and $j \in \mathbb{I}_m$. Proposition 3.19 now shows that we only need to define the functions $a_{ij}^{(1)}(t)$ for $i \in \mathbb{I}_n$ and $j \in \mathbb{I}_m$. This simplifies considerably the complexity of the algorithm. △

Proposition 3.21. *Consider the notation, notions and constructions of Algorithm 3.14 for $m \geq 2$. Assume that the algorithm computing $\mathbf{a}_j^{\rm op} \in (\mathbb{R}_{\geq 0}^n)$, for $j \in \mathbb{I}_m$, stops in the k-th iteration. If $t_1, \ldots, t_k$ are constructed in each iteration of the Algorithm 3.14 then:*

1. *For $j \in \mathbb{I}_m$, $\gamma_{ij}^{\rm op} = \gamma_{ij}^{(i)}(t_i) = \frac{\alpha_i}{m}$ for $i \in \mathbb{I}_{k-1}$.*
2. *For $j \in \mathbb{I}_{m-1}$ then $\gamma_{ij}^{\rm op} = \min\{\gamma'_{ij} , t_k\}$ for $k \leq i \leq d_j$; $\gamma_{im}^{\rm op} = t_k$ for $k \leq i \leq d_m$.*

Moreover, for $1 \leq r \leq s \leq m$ we have that

$$\gamma_{ir}^{\rm op} = \gamma_{is}^{\rm op} \quad \textit{for} \quad i \in \mathbb{I}_{d_s} . \tag{17}$$

Proof. In case the algorithm stops in the first iteration (i.e. $k = 1$), then let $0 < t_1 \leq \gamma'_{11}$ be such that

$$t_1 = \gamma_{1j}^{(1)}(t_1) \quad \text{for} \quad j \in \mathbb{I}_m \quad \text{and} \quad \gamma_m^{(1)}(t_1) = t_1 \, \mathbb{1}_{d_m} . \tag{18}$$

Recall that $\mathbf{a}_j^{(1)}(t) = (a_{ij}^{(1)}(t))_{i \in \mathbb{I}_n}$ is constructed as in Definition 3.10, based on $\mathbf{a}'_j = (a'_{ij})_{i \in \mathbb{I}_n}$ and that $\gamma'_j = (\gamma'_{ij})_{i \in \mathbb{I}_{d_j}}$ is the water-filling of $\mathbf{a}'_j$ in dimension d_j, for $j \in \mathbb{I}_{m-1}$. Hence, by Lemma 3.11 applied to $\mathbf{a}_j^{(1)}(t_1)$ for $j \in \mathbb{I}_{m-1}$, we conclude that

$$\gamma_j^{\rm op} = \gamma_j^{(1)}(t_1) = (\min\{\gamma'_{ij} , t_1\})_{i \in \mathbb{I}_{d_j}} \quad \text{for} \quad j \in \mathbb{I}_{m-1} . \tag{19}$$

If we assume that $t_1 > \gamma'_{d_m 1}$ then, notice that $\gamma'_{1j} = \gamma'_{11} \geq t_1 > \gamma'_{d_m 1} = \gamma'_{d_m j}$ for $j \in \mathbb{I}_{m-1}$. Thus, if c'_j denotes the water-level of γ'_j then

$$a'_{1j} = \gamma'_{1j} \geq t_1 > \max\{a'_{d_m j}\,,\, c'_j\} \quad \text{for} \quad j \in \mathbb{I}_{m-1} \implies a^{(1)}_{d_m m}(t_1) = 0\,.$$

But in this case, $\gamma^{(1)}_{d_m m} = 0$ contradicting Eq. (18). Therefore, $t_1 \leq \gamma'_{d_m\,1}$ and using Eq. (19)

$$\gamma^{\rm op}_{ij} = t_1 = \gamma^{\rm op}_{im} \quad \text{for} \quad j \in \mathbb{I}_{m-1} \quad \text{and} \quad i \in \mathbb{I}_{d_m}\,.$$

In case $k = 1$ the result follows from these remarks.

In case $k > 1$ then, by Proposition 3.19, we get that

$$a^{\rm op}_{ij} = \frac{\alpha_i}{m} = \gamma^{(i)}_{ij}(t_i) = a^{(1)}_{ij}(t_i) \quad \text{for} \quad i \in \mathbb{I}_{k-1} \quad \text{and} \quad a^{\rm op}_{ij} = a^{(1)}_{ij}(t_k) \quad \text{for} \quad k \leq i \leq n\,,$$

for $t_1 \geq \ldots \geq t_k > 0$.

Claim: for $j \in \mathbb{I}_m$ we have that

$$\gamma^{\rm op}_j = (\frac{\alpha_1}{m}\,,\,\ldots\,,\,\frac{\alpha_{k-1}}{m}\,,\,\gamma^{(k)}_{kj},\ldots,\gamma^{(k)}_{d_j j}) \in (\mathbb{R}^{d_j}_{\geq 0})^{\downarrow}\,,$$

where $\gamma^{(k)}_j = (\gamma^{(k)}_{ij})_{i=k}^{d_j}$ is the water-filling of $(a^{(1)}_{ij}(t_k))_{i=k}^n$.

Consider first $j \in \mathbb{I}_{m-1}$. Let $c_j(t_i)$ denote the water-level of the water-filling $\gamma^{(i)}_j(t_i) \in \mathbb{R}^{d_j - i + 1}$ of $\mathbf{a}^{(i)}_j(t_i) = (a^{(1)}_{\ell j}(t_i))_{\ell=i}^n$ in dimension $d_j - i + 1$, for $i \in \mathbb{I}_k$ (notice that $\gamma^{(k)}_j = \gamma^{(k)}_j(t_k)$). Since $a^{(1)}_{(k-1)j}(t_{(k-1)}) = \gamma^{(k-1)}_{(k-1)j}(t_{(k-1)}) \geq c_j(t_{(k-1)})$ then, by Lemma 3.12, we get that $(\gamma^{(k-1)}_{\ell j}(t_{(k-1)}))_{\ell=k}^{d_j}$ is the water-filling of $(a^{(1)}_{\ell j}(t_{(k-1)}))_{\ell=k}^n$. On the other hand, using that $t_{(k-1)} \geq t_k \geq 0$ and that $a^{(1)}_{\ell j}(t)$ is a non-decreasing function of t, for $k \leq \ell \leq n$ and $j \in \mathbb{I}_{m-1}$, then Proposition 3.9 (item 2) shows that

$$\frac{\alpha_{(k-1)}}{m} = a^{(1)}_{(k-1)j}(t_{(k-1)}) = \gamma^{(k-1)}_{(k-1)j}(t_{(k-1)}) \geq \gamma^{(k-1)}_{kj}(t_{(k-1)}) \geq \gamma^{(k)}_{kj}(t_k) \geq c_j(t_k)\,.$$

This last fact shows that the water-filling of $(a^{(1)}_{(k-1)j}(t_{(k-1)})\,,\, a^{(1)}_{kj}(t_k),\ldots,a^{(1)}_{nj}(t_k)) = (a^{\rm op}_{ij})_{i=k-1}^n$ in dimension $d_j - k + 2$ is $(\frac{\alpha_{(k-1)}}{m}\,,\,\gamma^{(k)}_{kj}\,,\,\ldots\,,\,\gamma^{(k)}_{d_j j})$, which proves the claim above.

We now consider the case $j = m$. In this case, since the algorithm stops in the k-th iteration we have that

$$\mathbf{a}^{\rm op}_m = (\frac{\alpha_1}{m}\,,\,\ldots\,,\,\frac{\alpha_{(k-1)}}{m}\,,\, a^{(1)}_{km}(t_k)\,,\,\ldots\,,\, a^{(1)}_{nm}(t_k)) \quad \text{and} \quad \gamma^{(k)}_m = t_k\,\mathbb{1}_{d_m - k + 1}\,.$$

The claim now follows from the facts that $\frac{\alpha_{(k-1)}}{m} = t_{(k-1)} \geq t_k$ and that $\gamma^{(k)}_m$ is the water-filling of $(a^{(1)}_{im}(t_k))_{i=k}^n$ in dimension $d_m - k + 1$. □

Remark 3.22 (Algorithm 3.14 is well defined)**.** Notice that Propositions 3.19 and 3.21 show that the inductive hypothesis (assumed during the construction of the algorithm for m blocks in terms of the output of the algorithm for $m-1$ blocks) in Algorithm 3.14 is also verified for the output of the algorithm for m blocks. Hence, the recursive process is well defined and always constructs an output. △

4 Main results

In this section we prove our main results, namely that the $(\alpha\,,\,m)$-weight partition obtained with the process described in Algorithm 3.14 give rise to optimal $(\alpha\,,\,d)$-designs (see Theorems 4.4 and 4.5 below). Our proof is based on the existence of majorization relations between vectors Γ_Φ associated to $(\alpha\,,\,d)$-designs (as described in Remark 4.1). We further show the uniqueness of the spectral structure of optimal $(\alpha\,,\,d)$-designs. We end this section by showing an interesting monotonicity property of the spectra of optimal $(\alpha\,,\,d)$-designs with respect to the initial weights.

4.1 Existence of optimal $(\alpha\,,\,d)$-designs

We begin by considering the following remark, which motivates our approach to prove the existence of optimal weight partitions in terms of majorization relations (see Theorem 4.4 below).

Remark 4.1. Let $\alpha = (\alpha_i)_{i\in\mathbb{I}_n} \in (\mathbb{R}^n_{>0})^\downarrow$, $m \in \mathbb{N}$ and let $d = (d_j)_{j\in\mathbb{I}_m} \in (\mathbb{N}^m)^\downarrow$ with $d_1 \le n$. Let $\Phi = (\mathcal{F}_j)_{j\in\mathbb{I}_m} \in \mathcal{D}(\alpha\,,\,d)$ and let $S_j = S_{\mathcal{F}_j} \in \mathcal{M}_{d_j}(\mathbb{C})^+$ denote the frame operators of $\mathcal{F}_j$, for $j \in \mathbb{I}_m$. We construct the vector

$$\Lambda_\Phi = (\lambda(S_1)\,,\,\ldots\,,\,\lambda(S_m)) \in \mathbb{R}^{|d|}_{\ge 0}$$

where $|d| = \sum_{j\in\mathbb{I}_m} d_j$ and $\lambda(S_j) \in (\mathbb{R}^{d_j}_{\ge 0})^\downarrow$ denotes the vector of eigenvalues of S_j, for $j \in \mathbb{I}_m$. If $\varphi \in \operatorname{Conv}(\mathbb{R}_{\ge 0})$ and P_φ denotes the joint convex potential induced by φ then,

$$\mathrm{P}_\varphi(\Phi) = \sum_{j\in\mathbb{I}_m} \mathrm{P}_\varphi(\mathcal{F}_j) = \sum_{j\in\mathbb{I}_m} \operatorname{tr}(\varphi(\lambda(S_j))) = \sum_{\ell\in\mathbb{I}_{|d|}} \varphi((\Lambda_\Phi)_\ell) =: \operatorname{tr}(\varphi(\Lambda_\Phi))\,. \tag{20}$$

Therefore, by Theorem 2.3 and Eq. (20), the existence of an (optimal) $(\alpha\,,\,d)$-design satisfying Eq. (8) for every $\varphi \in \operatorname{Conv}(\mathbb{R}_{\ge 0})$ is equivalent to the existence of $\Phi^{\rm op} = (\mathcal{F}_j^{\rm op})_{j\in\mathbb{I}_m} \in \mathcal{D}(\alpha\,,\,d)$ such that

$$\Lambda_{\Phi^{\rm op}} \prec \Lambda_\Phi \quad \text{for every} \quad \Phi = (\mathcal{F}_j)_{j\in\mathbb{I}_m} \in \mathcal{D}(\alpha\,,\,d)\,.$$

△

Definition 4.2. Let $\alpha = (\alpha_i)_{i\in\mathbb{I}_n} \in (\mathbb{R}^n_{>0})^\downarrow$, $m \in \mathbb{N}$ and let $d = (d_j)_{j\in\mathbb{I}_m} \in (\mathbb{N}^m)^\downarrow$ with $d_1 \le n$. Let $\mathbf{a}_j^{\rm op} = (a_{ij}^{\rm op})_{i\in\mathbb{I}_n}$, for $j \in \mathbb{I}_m$, be the output of Algorithm 3.14 for the input α and d. Let $\gamma_j^{\rm op} = (\gamma_{ij}^{\rm op})_{i\in\mathbb{I}_{d_j}} \in (\mathbb{R}^{d_j}_{\ge 0})^\downarrow$ be the water-filling of $\mathbf{a}_j^{\rm op}$ in dimension d_j, for $j \in \mathbb{I}_m$. We then consider the vector $\Gamma^{\rm op}(\alpha\,,\,d) = \Gamma^{\rm op}$ given by

$$\Gamma^{\rm op} = (\gamma_j^{\rm op})_{j\in\mathbb{I}_m} \in \mathbb{R}^{|d|}_{\ge 0} \quad \text{where} \quad |d| = \sum_{j\in\mathbb{I}_m} d_j\,.$$

△

Remark 4.3. Consider the notation from Definition 4.2 above. It follows from Eq. (17) in the proof of the inductive hypothesis that $\Gamma^{\rm op} = (\gamma_j^{\rm op})_{j\in\mathbb{I}_m}$ can be described using only the vector $\gamma_1^{\rm op}$. To be more precise, suppose first that $\gamma_1^{\rm op} \in (\mathbb{R}^{d_1}_{\ge 0})^\downarrow$ is

$$\gamma_1^{\rm op} = (a_{11}^{\rm op}\,,\,\ldots\,,\,a_{s1}^{\rm op}\,,\,c_1\,,\,\ldots\,,\,c_1).$$

Then, since for every $j \in \mathbb{I}_m$, $\gamma_{ij}^{\rm op} = \gamma_{i1}^{\rm op}$ for $i \in \mathbb{I}_{d_j}$, each $a_{\ell 1}^{\rm op}$ appears r_ℓ times in $\Gamma^{\rm op}$, where $r_\ell = \#\{j \in \mathbb{I}_m \,:\, d_j \ge \ell\}$, for $\ell \in \mathbb{I}_s$. Moreover, by inspection of the routine described in Algorithm 3.14, if we let $r_{s+1} = |d| - \sum_{\ell\in\mathbb{I}_s} r_\ell$ then

$$r_\ell\; a_{\ell 1}^{\rm op} = \alpha_\ell \quad \text{for} \quad \ell \in \mathbb{I}_s \quad \text{and} \quad r_{s+1}\; c_1 = \sum_{i=s+1}^n \alpha_i \implies \Gamma^{\rm op} = (\frac{\alpha_1}{r_1}\,\mathbb{1}_{r_1}\,,\,\ldots\,,\,\frac{\alpha_s}{r_s}\,\mathbb{1}_{r_s}\,,\,c_1\mathbb{1}_{r_{s+1}})\,.$$

Finally, it is clear that if $\gamma_1^{\rm op} = c_1\mathbb{1}_{d_1}$, then $\Gamma_1^{\rm op} = c_1\mathbb{1}_{|d|}$. It is worth to mention that, $d_1 \le n$ implies that $c_1 > 0$, since otherwise we would have $\alpha_n = 0$. △

The following is our first main result.

Theorem 4.4. *Let* $\alpha = (\alpha_i)_{i\in\mathbb{I}_n} \in (\mathbb{R}^n_{>0})^\downarrow$, $m \in \mathbb{N}$ *and let* $d = (d_j)_{j\in\mathbb{I}_m} \in (\mathbb{N}^m)^\downarrow$ *with* $d_1 \le n$. *Let* $\mathbf{a}_j^{\rm op} = (a_{ij}^{\rm op})_{i\in\mathbb{I}_n}$, *for* $j \in \mathbb{I}_m$, *be the output of the Algorithm 3.14. Let* $\gamma_j^{\rm op} = (\gamma_{ij}^{\rm op})_{i\in\mathbb{I}_{d_j}}$ *be the water-filling of* $\mathbf{a}_j^{\rm op}$ *in dimension* d_j, *for* $j \in \mathbb{I}_m$. *If* $\Phi = (\mathcal{F}_j)_{j\in\mathbb{I}_m} \in \mathcal{D}(\alpha\,,\,d)$ *and we let*

$$\Lambda_\Phi = (\lambda(S_{\mathcal{F}_j}))_{j\in\mathbb{I}_m} \in \mathbb{R}^{|d|}_{\ge 0} \quad \text{then} \quad \Gamma^{\rm op} \prec \Lambda_\Phi\,,$$

where $\Gamma^{\rm op} = (\gamma_j^{\rm op})_{j\in\mathbb{I}_m} \in \mathbb{R}^{|d|}_{\ge 0}$ *is as in Definition 4.2.*

Proof. Let $\Phi = (\mathcal{F}_j)_{j\in\mathbb{I}_m} \in \mathcal{D}(\alpha\,,\,d)$ be such that $\mathcal{F}_j = \{f_{ij}\}_{i\in\mathbb{I}_n}$, for $j \in \mathbb{I}_m$. Hence, by construction we have that

$$\sum_{j\in\mathbb{I}_m} \|f_{ij}\|^2 = \alpha_i \quad \text{for} \quad i \in \mathbb{I}_n\,. \tag{21}$$

On the other hand, we also have that

$$(\|f_{ij}\|^2)_{i\in\mathbb{I}_n} \prec \lambda(S_{\mathcal{F}_j}) =: (\lambda_{ij})_{i\in\mathbb{I}_{d_j}} \quad \text{for} \quad j \in \mathbb{I}_m\,.$$

Hence, we conclude that

$$\sum_{i\in\mathbb{I}_t} \|f_{ij}\|^2 \le \sum_{i=1}^{\min\{t\,,\,d_j\}} \lambda_{ij} \quad \text{for} \quad t \in \mathbb{I}_n \quad \text{and} \quad j \in \mathbb{I}_m\,. \tag{22}$$

It follows immediately that $\Gamma^{\rm op} \prec \Lambda_\Phi$ if $\gamma_{11}^{\rm op} = \gamma_{d_1 1}^{\rm op}$ (since we have $\Gamma^{\rm op} = \gamma_{11}^{\rm op}\mathbb{1}_{|d|}$).

Suppose that $\gamma_{11}^{\rm op} > \gamma_{d_1 1}^{\rm op}$. Consider the set $\{r_\ell\}_{\ell\in\mathbb{I}_{s+1}}$ defined in previous Remark. That is, the set constructed using $d_1 \ge d_2 \ge \ldots \ge d_m$ and the water level s of $\gamma_1^{\rm op}$ by $r_\ell = \#\{i\,:\,d_i \ge \ell\}$, for $\ell \in \mathbb{I}_s$ and $r_{s+1} = |d| - \sum_{\ell\in\mathbb{I}_s} r_\ell$. Consider the following arrangement of $\Lambda_\Phi = (\lambda(S_{\mathcal{F}_j}))_{j\in\mathbb{I}_m}$:

$$\Lambda'_\Phi = (\lambda_{11}\,,\,\lambda_{12}\,,\,\ldots\,,\,\lambda_{1r_1}\,,\,\lambda_{21}\,,\,\ldots\,,\,\lambda_{2r_2}\,,\,\ldots\,,\,\lambda_{s1}\,,\,\ldots\,,\,\lambda_{sr_s}\,,\,\beta_1\,,\,\ldots\,,\,\beta_{r_{s+1}}),$$

where the entries β_i are $(\lambda_{ij})_{i=s+1}^{d_j}$, $j \in \{i \in \mathbb{I}_m\,:\,d_i \ge s+1\}$ arranged in any order. Let $s_k = \sum_{\ell\in\mathbb{I}_k} r_\ell$, $k \in \mathbb{I}_s$, then,

$$\sum_{j\in\mathbb{I}_{s_k}} (\Lambda_\Phi)^\downarrow_j \ge \sum_{j\in\mathbb{I}_{s_k}} (\Lambda'_\Phi)_j = \sum_{j\in\mathbb{I}_m} \sum_{i=1}^{\min\{k\,,\,d_j\}} \lambda_{ij} \ge \sum_{j\in\mathbb{I}_m}\sum_{i\in\mathbb{I}_k} \|f_{ij}\|^2 = \sum_{i\in\mathbb{I}_k} \alpha_i = \sum_{j\in\mathbb{I}_{s_k}} \Gamma^{\rm op}_j\,.$$

Hence, by Remark 2.2, we have $\Gamma^{\rm op} \prec \Lambda_\Phi$. □

Theorem 4.4 together with the argument in Remark 4.1 allow us to obtain our second main result.

Theorem 4.5. *Let* $\alpha = (\alpha_i)_{i\in\mathbb{I}_n} \in (\mathbb{R}^n_{>0})^\downarrow$, $m \in \mathbb{N}$ *and let* $d = (d_j)_{j\in\mathbb{I}_m} \in (\mathbb{N}^m)^\downarrow$ *with* $d_1 \le n$. *Let* $\mathbf{a}_j^{\rm op} = (a_{ij}^{\rm op})_{i\in\mathbb{I}_n}$, *for* $j \in \mathbb{I}_m$, *be the output of Algorithm 3.14. Let* $\gamma_j^{\rm op} = (\gamma_{ij}^{\rm op})_{i\in\mathbb{I}_{d_j}}$ *be the water-filling of* $\mathbf{a}_j^{\rm op}$ *in dimension* d_j, *for* $j \in \mathbb{I}_m$. *Then,*

1. *There exists* $\Phi^{\rm op} = (\mathcal{F}_j^{\rm op}) \in \mathcal{D}(\alpha\,,\,d)$ *with* $\mathcal{F}_j^{\rm op} = \{f_{ij}^{\rm op}\}_{i\in\mathbb{I}_n}$ *and such that:*

   1.a) $\|f_{ij}^{\rm op}\|^2 = a_{ij}^{\rm op}$, *for* $i \in \mathbb{I}_n$ *and* $j \in \mathbb{I}_m$.

   1.b) $\lambda(S_{\mathcal{F}_j^{\rm op}}) = \gamma_j^{\rm op} \in (\mathbb{R}^{d_j}_{>0})^\downarrow$, *for* $j \in \mathbb{I}_m$.

   1.c) $\mathcal{F}_j^{\rm op}$ *is a frame for* $\mathbb{C}^{d_j}$, *for* $j \in \mathbb{I}_m$.

2. *If $\varphi \in \operatorname{Conv}(\mathbb{R}_{\geq 0})$ then we have that*

$$P_\varphi(\Phi^{\rm op}) = \sum_{j\in\mathbb{I}_m} \mathrm{P}_\varphi(\mathcal{F}_j^{\rm op}) \leq \sum_{j\in\mathbb{I}_m} \mathrm{P}_\varphi(\mathcal{F}_j) = \mathrm{P}_\varphi(\Phi) \quad \text{for every} \quad \Phi = (\mathcal{F}_j)_{j\in\mathbb{I}_m} \in \mathcal{D}(\alpha\,,\,d)\,. \tag{23}$$

Moreover, if $\Phi = (\mathcal{F}_j)_{j\in\mathbb{I}_m} \in \mathcal{D}(\alpha\,,\,d)$ is such that there exists $\varphi \in \operatorname{Conv_s}(\mathbb{R}_{\geq 0})$ for which equality holds in Eq. (23), then $\lambda(S_{\mathcal{F}_j}) = \gamma_j^{\rm op}$, for $j \in \mathbb{I}_m$.

Proof. With the previous notation, we have that $A^{\rm op} = (a_{ij}^{\rm op})_{i\in\mathbb{I}_n\,,\,j\in\mathbb{I}_m} \in P_{\alpha\,,\,m}$ is a weight partition of α. On the other hand, by Theorem 3.6 we see that $\mathbf{a}_j^{\rm op} \prec \gamma_j^{\rm op}$, for $j \in \mathbb{I}_m$. Hence Theorem 2.4 shows that there exist $\mathcal{F}_j^{\rm op} = \{f_{ij}^{\rm op}\}_{i\in\mathbb{I}_n} \in (\mathbb{C}^{d_j})^n$ with $\|f_{ij}^{\rm op}\|^2 = a_{ij}^{\rm op}$ for $i \in \mathbb{I}_n$ and $\lambda(S_{\mathcal{F}_j^{\rm op}}) = \gamma_j^{\rm op}$, for $j \in \mathbb{I}_m$. On the other hand, consider $\Gamma^{\rm op} = (\gamma_j^{\rm op})_{j\in\mathbb{I}_m}$ as in Definition 4.2; then, according to Remark 4.3 $(\Gamma^{\rm op}) \in \mathbb{R}_{>0}^{|d|}$ and hence, $\gamma_j^{\rm op} \in (\mathbb{R}_{>0}^{d_j})^\downarrow$, for $j \in \mathbb{I}_m$. In particular, $\mathcal{F}_j^{\rm op}$ is a frame for $\mathbb{C}^{d_j}$, for $j \in \mathbb{I}_m$, and $\Phi^{\rm op} := (\mathcal{F}_j^{\rm op})_{j\in\mathbb{I}_m} \in \mathcal{D}(\alpha\,,\,d)$ satisfies item 1.

Let $\Phi = (\mathcal{F}_j)_{j\in\mathbb{I}_m} \in \mathcal{D}(\alpha\,,\,d)$ and let $\Lambda_\Phi = (\lambda(S_{\mathcal{F}_j}))_{j\in\mathbb{I}_m} \in \mathbb{R}_{\geq 0}^{|d|}$, where $|d| = \sum_{j\in\mathbb{I}_m} d_j$. If we let $\Lambda_{\Phi^{\rm op}} = (\lambda(S_{\mathcal{F}_j^{\rm op}}))_{j\in\mathbb{I}_m} \in \mathbb{R}_{\geq 0}^{|d|}$ then, by construction, we have that $\Lambda_{\Phi^{\rm op}} = \Gamma^{\rm op}$. By Theorem 4.4 and Remark 4.1 we get that for every $\varphi \in \operatorname{Conv}(\mathbb{R}_{\geq 0})$ then

$$\Lambda_{\Phi^{\rm op}} \prec \Lambda_\Phi \implies \mathrm{P}_\varphi(\Phi^{\rm op}) = \sum_{j\in\mathbb{I}_m} \mathrm{P}_\varphi(\mathcal{F}_j^{\rm op}) = \operatorname{tr}(\varphi(\Lambda_{\Phi^{\rm op}})) \leq \operatorname{tr}(\varphi(\Lambda_\Phi)) = \sum_{j\in\mathbb{I}_m} \mathrm{P}_\varphi(\mathcal{F}_j) = \mathrm{P}_\varphi(\Phi)\,.$$

Assume further that $\varphi \in \operatorname{Conv_s}(\mathbb{R}_{\geq 0})$ and $\Phi = (\mathcal{F}_j)_{j\in\mathbb{I}_m} \in \mathcal{D}(\alpha\,,\,d)$ is such that equality holds in Eq. (23). We introduce the set

$$\mathcal{M} = \{\Lambda_\Psi : \ \Psi = (\mathcal{G}_j)_{j\in\mathbb{I}_m} \in \mathcal{D}(\alpha\,,\,d)\} \subset \mathbb{R}_{\geq 0}^{|d|}\,.$$

We claim that $\mathcal{M}$ is a convex set: indeed, let $t \in [0,1]$ and $\Psi = (G_j)_{j\in\mathbb{I}_m}\,,\ \Theta = (H_j)_{j\in\mathbb{I}_m} \in \mathcal{D}(\alpha\,,\,d)$ be such that $G_j = (g_{ij})_{i\in\mathbb{I}_n}$ and $H_j = (h_{ij})_{i\in\mathbb{I}_n}$. Set

$$a_{ij} = \|g_{ij}\|^2 \quad \text{and} \quad b_{ij} = \|h_{ij}\|^2 \quad \text{for} \quad i \in \mathbb{I}_n \quad \text{and} \quad j \in \mathbb{I}_m\,.$$

Further, set $\mathbf{a}_j = (a_{ij})_{i\in\mathbb{I}_n}$, $\mathbf{b}_j = (b_{ij})_{i\in\mathbb{I}_n}$ and $\mathbf{c}_j = (c_{ij})_{i\in\mathbb{I}_n} = t\,\mathbf{a}_j + (1-t)\,\mathbf{b}_j \in \mathbb{R}^n$, for $j \in \mathbb{I}_m$. Using the convexity of $P_{\alpha\,,\,m}$ we see that $(c_{ij})_{i\in\mathbb{I}_n\,,\,j\in\mathbb{I}_m} \in P_{\alpha\,,\,m}$. On the other hand, if $j \in \mathbb{I}_m$ and $S \subset \{1\,,\,\ldots\,,\,n\}$ is such that $\#S = k$ then, if we let $\mathbf{a}_j^\downarrow$ and $\mathbf{b}_j^\downarrow$ denote the re-arrangements of $\mathbf{a}_j$ and $\mathbf{b}_j$ in non-increasing order, we get that

$$\begin{aligned}\sum_{i\in S} c_{ij} &= \sum_{i\in S} t\,a_{ij} + (1-t)\,b_{ij} \leq \sum_{i\in\mathbb{I}_k} t\,(\mathbf{a}_j^\downarrow)_i + (1-t)\,(\mathbf{b}_j^\downarrow)_i \leq \sum_{i\in\mathbb{I}_k} t\,\lambda_i(S_{G_j}) + (1-t)\,\lambda_i(S_{H_j}) \\ &= \sum_{i\in\mathbb{I}_k} (t\,\lambda(S_{G_j}) + (1-t)\,\lambda(S_{H_j}))_i^\downarrow\,.\end{aligned}$$

This last fact shows that $\mathbf{c}_j \prec t\,\lambda(S_{G_j}) + (1-t)\,\lambda(S_{H_j})$ for $j \in \mathbb{I}_m$. Hence, Theorem 2.4 shows that for each $j \in \mathbb{I}_m$ there exit $\mathcal{K}_j = (k_{ij})_{i\in\mathbb{I}_n} \in (\mathbb{C}^{d_j})^n$ such that $\|k_{ij}\|^2 = c_{ij}$, for $i \in \mathbb{I}_n$, and $\lambda(S_{\mathcal{K}_j}) = t\,\lambda(S_{G_j}) + (1-t)\,\lambda(S_{H_j})$. Therefore, $\Pi = (\mathcal{K}_j)_{j\in\mathbb{I}_m} \in \mathcal{D}(\alpha\,,\,d)$ and $t\,\Lambda_\Psi + (1-t)\,\Lambda_\Theta = (t\,\lambda(S_{G_j}) + (1-t)\,\lambda(S_{H_j}))_{j\in\mathbb{I}_m} = \Lambda_\Pi$ and the claim follows.

We finally introduce $F_\varphi : \mathcal{M} \to \mathbb{R}_{\geq 0}$, $F_\varphi(\Lambda) = \operatorname{tr}(\varphi(\Psi))$ for $\Lambda \in \mathcal{M}$. Since φ is strictly convex we immediately see that F - which is defined on the convex set $\mathcal{M}$ - is strictly convex as well. Hence, there exists a unique $\Lambda^{(\varphi)} \in \mathcal{M}$ such that

$$F(\Lambda^{(\varphi)}) = \min\{F(\Lambda) : \ \Lambda \in \mathcal{M}\}\,.$$

Notice that by hypothesis, we have that

$$F(\Lambda_\Phi) = F(\Lambda_{\Phi^{\rm op}}) = \min\{F(\Lambda) : \ \Lambda \in \mathcal{M}\} \implies (\lambda(S_{\mathcal{F}_j}))_{j\in\mathbb{I}_m} = \Lambda_\Phi = \Lambda_{\Phi^{\rm op}} = (\lambda(S_{\mathcal{F}_j^{\rm op}}))_{j\in\mathbb{I}_m}\,.$$

□

Remark 4.6 (Finite-step algorithm for constructing optimal $(\alpha\,,\,d)$-designs)**.** Consider the notation in Theorem 4.5. We apply Algorithm 3.14 and obtain $\mathbf{a}_j^{\rm op}$ for $j\in\mathbb{I}_m$. Then, we compute the optimal spectra $\gamma_j^{\rm op}$ for $j\in\mathbb{I}_m$ by water-filling, in terms of a simple finite step-algorithm. Finally, we can apply well known algorithms (see [8, 9, 12, 14]) to compute $\mathcal{F}_j^{\rm op}=\{f_{ij}^{\rm op}\}_{i\in\mathbb{I}_n}$ such that $\lambda(S_{\mathcal{F}_j^{\rm op}})=\gamma_j^{\rm op}$ and such that $(\|f_{ij}^{\rm op}\|^2)_{i\in\mathbb{I}_n}=\mathbf{a}_j^{\rm op}$ for $j\in\mathbb{I}_m$. In this case, $\Phi^{\rm op}=(\mathcal{F}_j^{\rm op})_{j\in\mathbb{I}_m}$ is an optimal $(\alpha\,,\,d)$-design, as in Theorem 4.5. Thus, the conjunction of these routines allows us to effectively compute $\Phi^{\rm op}$ in a finite number of steps. $\triangle$

Remark 4.7. Consider the notation in Theorem 4.5; as a consequence of this result, we see that the spectral structures of $(\alpha\,,\,d)$-designs that minimize a convex potential (induced by a strictly convex function) on $\mathcal{D}(\alpha\,,\,d)$ coincide with that of $\Phi^{\rm op}$, so this spectral structure is unique. It is natural to wonder whether the $(\alpha\,,\,m)$-weight partitions corresponding to such minimizers also coincide. It turns out that this is not the case; indeed, consider the following example: let $\alpha=\mathbb{1}_6\in(\mathbb{R}_{>0}^6)^\downarrow$, $m=2$ and let $d=(4,2)\in\mathbb{N}^2$. If we apply Algorithm 3.14 we obtain the output

$$\mathbf{a}_1^{\rm op}=\frac{4}{6}\,\mathbb{1}_6\in(\mathbb{R}_{>0}^6)^\downarrow \quad \text{and} \quad \mathbf{a}_2^{\rm op}=\frac{2}{6}\,\mathbb{1}_6\in(\mathbb{R}_{>0}^6)^\downarrow \implies \gamma_1^{\rm op}=\mathbb{1}_4 \quad \text{and} \quad \gamma_2^{\rm op}=\mathbb{1}_2\,.$$

Thus, if $\Phi^{\rm op}=(\mathcal{F}_1^{\rm op}\,,\,\mathcal{F}_1^{\rm op})$ then $\mathcal{F}_j^{\rm op}$ is a Parseval frame for $\mathbb{C}^{d_j}$, $j=1,2$. Alternatively, we can consider the weights $\mathbf{a}_1^0=(1,1,1,1,0,0)$ and $\mathbf{a}_2^0=(0,0,0,0,1,1)$: moreover, if we let $\{e_\ell^{(k)}\}_{\ell\in\mathbb{I}_k}$ denote the canonical basis of $\mathbb{C}^k$ for $k\in\mathbb{N}$ and let

$$\mathcal{F}_1^0=\{e_1^{(4)},\ldots,e_4^{(4)},0,0\}\in(\mathbb{C}^4)^6 \quad \text{and} \quad \mathcal{F}_2^0=\{0,0,0,0,e_1^{(2)},e_2^{(2)}\}\in(\mathbb{C}^2)^6$$

then $\|f_{ij}^0\|^2=(\mathbf{a}_j^0)_i$ for $i\in\mathbb{I}_6$, $j=1,2$. Hence, $\Phi^0:=(\mathcal{F}_1^0,\mathcal{F}_2^0)\in\mathcal{D}(\alpha\,,\,2)$ and is such that $\lambda(S_{\mathcal{F}_j^0})=\lambda(S_{\mathcal{F}_j^{\rm op}})$ for $j=1,2$. That is, the spectral structure of Φ^0 coincides with the spectral structure of $\Phi^{\rm op}$ and therefore, for every $\varphi\in\operatorname{Conv}(\mathbb{R}_{\geq 0})$ we have that

$$\mathrm{P}_\varphi(\Phi^0)\leq\mathrm{P}_\varphi(\Phi) \quad \text{for every} \quad \Phi\in\mathcal{D}(\alpha\,,\,2)\,.$$

Nevertheless, $\mathbf{a}_j^{\rm op}\neq\mathbf{a}_j^0$ for $j=1,2$. That is, these two different weight partitions generate (see Remark 3.13) optimal $(\alpha\,,\,2)$-designs. Thus, weight partitions inducing optimal $(\alpha\,,\,2)$-designs are not unique. As a final comment, let us mention that the $(\alpha\,,\,2)$-designs $\Phi^{\rm op}$ and Φ^0 are qualitatively different: indeed, notice that $\mathcal{F}_j^0$ is obtained from the canonical basis of $\mathbb{C}^{d_j}$ by appending zero vectors and thus, it is essentially an orthonormal basis, $j=1,2$. We can quantify this feature by considering

$$\mathrm{re}_{eff}(\mathcal{F}_j^0)=\#\{i\in\mathbb{I}_6:\ \|f_{ij}^0\|\neq 0\}-d_j=0 \quad \text{for} \quad j=1,2\,.$$

We consider the quantity $\mathrm{re}_{eff}(\mathcal{F}_j^0)$ as an effective redundancy of $\mathcal{F}_j^0$, for $j=1,2$. In contrast, notice that

$$\mathrm{re}_{eff}(\mathcal{F}_j^{\rm op})=\#\{i\in\mathbb{I}_6:\ \|f_{ij}^{\rm op}\|\neq 0\}-d_j=6-d_j>0 \quad \text{for} \quad j=1,2\,.$$

Moreover, for general α and d, an heuristic analysis of Algorithm 3.14 reveals another optimal feature of the $(\alpha\,,\,d)$-design $\Phi^{\rm op}=(\mathcal{F}_j^{\rm op})_{j\in\mathbb{I}_m}$ obtained from the weight partition induced by $\mathbf{a}_j^{\rm op}$ for $j\in\mathbb{I}_m$: namely, that

$$\mathrm{re}_{eff}(\mathcal{F}_j^{\rm op})\geq\mathrm{re}_{eff}(\mathcal{F}_j^0) \quad \text{for} \quad j\in\mathbb{I}_m$$

for every (structural) optimal $(\alpha\,,\,d)$-design $\Phi^0=(\mathcal{F}_j^0)_{j\in\mathbb{I}_m}$. $\triangle$

In the following result we show that there is a monotonic dependence of the (unique) spectra of optimal $(\alpha\,,\,d)$-designs with respect to the initial weights.

Theorem 4.8. *Let $\alpha_i \geq \beta_i > 0$ for $i \in \mathbb{I}_n$ and let $d = (d_j)_{j\in\mathbb{I}_m} \in (\mathbb{N}^m)^{\downarrow}$. Let $\mathbf{a}_j^{\rm op}$ (respectively $\mathbf{b}_j^{\rm op}$) for $j \in \mathbb{I}_m$ be of the output of the algorithm with the input $\alpha = (\alpha_i)_{i\in\mathbb{I}_n}$ (respectively with the input $\beta = (\beta_i)_{i\in\mathbb{I}_n}$) and d. Let $\gamma_j^{\rm op} = (\gamma_{ij}^{\rm op})$ (respectively $\delta_j^{\rm op} = (\delta_{ij}^{\rm op})$) be the water-filling of $\mathbf{a}_j^{\rm op}$ (respectively $\mathbf{b}_j^{\rm op}$) in dimension d_j, for $j \in \mathbb{I}_m$. Then*

$$\gamma_{ij}^{\rm op} \geq \delta_{ij}^{\rm op} \quad \textit{for} \quad i \in \mathbb{I}_{d_j} \quad \textit{and} \quad j \in \mathbb{I}_m\,. \tag{24}$$

Proof. By Remark 4.3 it suffices to prove that $\gamma_{i1}^{\rm op} \geq \delta_{i1}^{\rm op}$ for $i \in \mathbb{I}_{d_1}$. Suppose first that $\delta_1^{\rm op} = c_2 \mathbb{1}_{d_1}$. Then, if $\gamma_i^{\rm op} = c_1 \mathbb{1}_{d_1}$, clearly $c_1 = \frac{\sum_{i\in\mathbb{I}_n} \alpha_i}{|d|} \geq \frac{\sum_{i\in\mathbb{I}_n} \beta_i}{|d|} = c_2$ and we are done. Assume now that

$$\gamma_1^{\rm op} = (a_{11}^{\rm op}, \ldots, a_{s1}^{\rm op}, c_1, \ldots, c_1) \quad \text{and} \quad c_2 > c_1\,.$$

Notice that the hypothesis on $\delta_1^{\rm op}$ implies that $(b_{ij}^{\rm op})_{i\in\mathbb{I}_n} \prec c_2 \mathbb{1}_{d_j}$, for each $j \in \mathbb{I}_m$. Hence, in particular,

$$\sum_{i=s+1}^{n} b_{ij}^{\rm op} \geq (d_j - s)^+ c_2 \implies \sum_{i=s+1}^{n} \beta_i = \sum_{j\in\mathbb{I}_m} \sum_{i=s+1}^{n} b_{ij}^{\rm op} \geq \sum_{j\in\mathbb{I}_m} (d_j - s)^+ c_2 = \left(\sum_{i=s+1}^{d_1} r_i \right) c_2\,.$$

On the other hand, using the previous inequality and the hypothesis

$$\sum_{i=s+1}^{n} \beta_i \geq \left(\sum_{i=s+1}^{d_1} r_i \right) c_2 > \left(\sum_{i=s+1}^{d_1} r_i \right) c_1 = r_{s+1}\, c_1 = \sum_{i=s+1}^{n} \alpha_i\,,$$

where $r_{s+1} = |d| - \sum_{i\in\mathbb{I}_s} r_i$ (see Remark 4.3). This last inequality contradicts the assumption that $\alpha_i \geq \beta_i$ for $i \in \mathbb{I}_n$.

Assume now that $\gamma_1^{\rm op} = c_1\, \mathbb{1}_{d_1}$ and that $\delta_1^{\rm op} \neq c_2\, \mathbb{1}_{d_1}$. Then, notice that $c_1 \geq a_{1j}^{\rm op}$ for $j \in \mathbb{I}_m$ and therefore,

$$c_1 \geq \frac{1}{m} \sum_{j\in\mathbb{I}_m} a_{1j}^{\rm op} = \frac{\alpha_1}{m} \geq \frac{\beta_1}{m} = \delta_{11}^{\rm op} \geq \delta_{i1}^{\rm op} \quad \text{for} \quad i \in \mathbb{I}_{d_1}\,.$$

Finally, assume that $\gamma_1^{\rm op} \neq c_1\, \mathbb{1}_{d_1}$ and $\delta_1^{\rm op} \neq c_2\, \mathbb{1}_{d_1}$; hence,

$$\gamma_1^{\rm op} = (a_{11}^{\rm op}, \ldots, a_{s1}^{\rm op}, c_1, \ldots, c_1) \quad \text{and} \quad \delta_1^{\rm op} = (b_{11}^{\rm op}, \ldots, b_{t1}^{\rm op}, c_2, \ldots, c_2)$$

and set $u = \min\{s\,,\, t\}$. Then, $\gamma_{i1}^{\rm op} = \frac{\alpha_i}{r_i} \geq \frac{\beta_i}{r_i} = \delta_{i1}^{\rm op}$ for $i \in \mathbb{I}_u$. On the other hand, by Remark 3.17, $\gamma' = (\gamma_{i1})_{i=u+1}^{d_1}$ and $\delta' = (\delta_{i1})_{i=u+1}^{d_1}$ coincide with $\rho_1^{\rm op}$ and $\zeta_1^{\rm op}$, where $(\rho_j^{\rm op})_j$ (respectively $(\zeta_1^{\rm op})_j$) are the optimal spectra corresponding to the reduced problem with weights $(\alpha_i)_{i=u+1}^{n}$ (respectively $(\beta_i)_{i=u+1}^{n}$) and dimensions $(d_j - u)^+$, for $j \in \mathbb{I}_m$. Now, by construction, we have that $\rho_1^{\rm op} = c_1\, \mathbb{1}_{d_1-u}$ or $\zeta_1^{\rm op} = c_2\, \mathbb{1}_{d_1-u}$, so we can apply the arguments in the first part of the proof to deduce that $\rho_1^{\rm op} \geq \zeta_1^{\rm op}$ (entry-wise) from which the result follows. □

5 Numerical examples

The following examples were obtained via an implementation of Algorithm 3.14 using MATLAB.

Example 5.1. Consider the family of weights given by $\alpha = \{9, 8, 7, 5, 4, 2.5, 2, 2, 1.5, 0.6, 0.5\}$ and suppose that the dimensions to be considered are $d_1 = 7$, $d_2 = 5$, $d_3 = 3$. Then, the optimal partition of α is

$$A^{\text{op}} = \begin{bmatrix} 3 & 3 & 3 \\ 2.7583 & 2.7583 & 2.4833 \\ 2.7583 & 2.7583 & 1.4833 \\ 2.7583 & 1.8135 & 0.4282 \\ 2.5267 & 1.1307 & 0.3425 \\ 1.5792 & 0.7067 & 0.2141 \\ 1.2634 & 0.5654 & 0.1713 \\ 1.2634 & 0.5654 & 0.1713 \\ 0.9475 & 0.4240 & 0.1285 \\ 0.3790 & 0.1696 & 0.0514 \\ 0.3158 & 0.1413 & 0.0428 \end{bmatrix}$$

In this case, the optimal spectra related to this partition are:

$$\gamma_1^{\text{op}} = (3, 2.7583, 2.7583, 2.7583, 2.7583, 2.7583, 2.7583)$$
$$\gamma_2^{\text{op}} = (3, 2.7583, 2.7583, 2.7583, 2.7583)$$
$$\gamma_3^{\text{op}} = (3, 2.7583, 2.7583)$$

Once we have the optimal partitions and optimal spectra, we can construct examples of frames using these data, applying known algorithms like one-sided Bendel-Mickey algorithm (see [8, 9, 12, 14]):

$$\mathcal{F}_1 = \begin{bmatrix} 0.0705 & 0.1956 & -0.0616 & -0.6865 & -0.6865 & 0.3994 & -0.0845 & -0.3230 & -1.1553 & 0.2649 & -0.3180 \\ 0.2804 & -0.2311 & -0.2142 & 0.2434 & 0.2434 & -0.4716 & 1.2808 & 0.2534 & -0.4197 & 0.3309 & -0.5206 \\ 0.0380 & -0.1106 & -0.5728 & -0.8134 & -0.8134 & -0.2257 & 0.3005 & 0.0342 & 0.9482 & 0.1009 & -0.2125 \\ -0.0004 & -0.1760 & -0.3643 & -0.2125 & -0.2125 & -0.3592 & 0.2804 & 0.2989 & -0.4753 & -1.0345 & 0.9956 \\ -0.4655 & -0.4260 & -0.5815 & 0.0796 & 0.0796 & -0.8695 & -0.8294 & 0.3134 & -0.3310 & 0.0127 & -0.6235 \\ 0.1120 & 0.2501 & 0.3128 & -0.1034 & -0.1034 & 0.5106 & -0.0368 & 1.2019 & 0.0419 & -0.6107 & -0.7246 \\ 0.0391 & -0.0112 & 0.0316 & 0.0949 & 0.0949 & -0.0229 & 0.1448 & -0.9781 & 0.1061 & -1.0607 & -0.8232 \end{bmatrix}$$

$$\mathcal{F}_2 = \begin{bmatrix} 0.1841 & 0.2017 & 0.3189 & 0.0682 & -0.2093 & -0.2340 & -0.2960 & 0.6595 & 0.5432 & 0.0340 & 1.3437 \\ 0.0249 & 0.0273 & 0.0432 & 0.6049 & 0.6893 & 0.7707 & 0.9748 & 0.0893 & 0.4598 & 0.3451 & 0.1842 \\ -0.1947 & -0.2132 & -0.3372 & -0.3744 & -0.1430 & -0.1599 & -0.2022 & -0.6973 & 1.2517 & 0.5169 & -0.1238 \\ -0.2625 & -0.2876 & -0.4547 & -0.2253 & 0.1440 & 0.1610 & 0.2037 & -0.9404 & -0.4997 & -0.3351 & 1.0506 \\ -0.0015 & -0.0016 & -0.0025 & -0.0619 & -0.0723 & -0.0808 & -0.1022 & -0.0053 & -0.6598 & 1.5029 & 0.2041 \end{bmatrix}$$

$$\mathcal{F}_3 = \begin{bmatrix} -0.0342 & -0.0375 & -0.0593 & -0.3714 & -0.3952 & -0.4419 & -0.5590 & -0.6249 & -1.1632 & -0.2605 & -0.3888 \\ -0.1953 & -0.2139 & -0.3383 & -0.1805 & 0.0479 & 0.0536 & 0.0678 & 0.0758 & 0.1410 & -1.4873 & 0.5519 \\ 0.0592 & 0.0649 & 0.1026 & -0.0281 & -0.1129 & -0.1263 & -0.1597 & -0.1786 & -0.3324 & 0.4511 & 1.5950 \end{bmatrix}$$

Example 5.2. Using the same set of dimensions, take now $\alpha = \{8.5, 7, 6, 4, 3.8, 2, 1.6, 1.4, 1, 0.5, 0.4\}$. Notice that we are considering weights that are term by term smaller than previous α. In this case, the Algorithm provides the following optimal spectra:

$$\gamma_1^{\text{op}} = (2.8333, 2.3333, 2.3, 2.3, 2.3, 2.3, 2.3)$$
$$\gamma_2^{\text{op}} = (2.8333, 2.3333, 2.3, 2.3, 2.3)$$
$$\gamma_3^{\text{op}} = (2.8333, 2.3333, 2.3)$$

illustrating the monotonicity proved in previous section.

Example 5.3. When $\alpha = \{20, 19.5, 10, 5, 4.5, 3, 2.4, 2\}$ and $d = \{5, 4, 4, 3, 2\}$, Algorithm 3.14 provides the following optimal partition:

$$A^{\text{op}} = \begin{bmatrix} 4 & 4 & 4 & 4 & 4 \\ 3.9 & 3.9 & 3.9 & 3.9 & 3.9 \\ 3.3625 & 2.8875 & 2.5 & 1.25 & 0 \\ 1.9896 & 1.1354 & 1.25 & 0.625 & 0 \\ 1.7907 & 1.0218 & 1.125 & 0.5625 & 0 \\ 1.1938 & 0.6812 & 0.75 & 0.375 & 0 \\ 0.955 & 0.545 & 0.6 & 0.3 & 0 \\ 0.7959 & 0.4541 & 0.5 & 0.25 & 0 \end{bmatrix}$$

Notice that first two weights α_1 and α_2 are considerably bigger than the rest, this forces the concentration in the first two weights in the last column of the partition A^{op}, i.e. the smaller subspace requires only two vectors. Related to this behavior of the optimal partitions, one can see that the optimal spectra is

$$\begin{aligned}
\gamma_1^{\mathrm{op}} &= (4, 3.9, 3.3625, 3.3625, 3.3625)\\
\gamma_2^{\mathrm{op}} &= (4, 3.9, 3.3625, 3.3625)\\
\gamma_3^{\mathrm{op}} &= (4, 3.9, 3.3625, 3.3625)\\
\gamma_4^{\mathrm{op}} &= (4, 3.9, 3.3625)\\
\gamma_5^{\mathrm{op}} &= (4, 3.9)
\end{aligned}$$

where the smaller spectrum does not have the water-filling constant 3.3625.

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