HCIZ INTEGRAL FORMULA AS UNITARITY OF A CANONICAL MAP BETWEEN REPRODUCING KERNEL SPACES

MARTIN MIGLIOLI

Abstract. In this article we prove that the Harish-Chandra-Itzykson-Zuber (HCIZ) integral is equivalent to the unitarity of a canonical map between invariant subspaces of Segal-Bargmann spaces. As a consequence, we provide alternative proofs of the HCIZ integral formula and other results.

Keywords. HCIZ integral, Segal Bargmann space, reproducing kernel, unitary map, Schur functions.

(Martin Miglioli) Instituto Argentino de Matemática-CONICET. Saavedra 15, Piso 3, (1083) Buenos Aires, Argentina

E-mail address: martin.miglioli@gmail.com.

Date: February 15, 2024.

The author was supported by IAM-CONICET, grants PIP 2010-0757 (CONICET) and PICT 2010-2478 (ANPCyT).

1. Introduction

In [H57] Harish-Chandra proved a formula for orbital integrals, see the expository article [McS21]. The importance of such integrals for mathematical physics was first noted by Itzykson and Zuber [IZ80], the unitary integral is now known as the Harish-Chandra-Itzykson-Zuber (HCIZ) integral and has become an important identity in quantum field theory, random matrix theory, and algebraic combinatorics. It is usually written

$$\int_U e^{\mathrm{Tr}(uAu^{-1}B)}du = \left(\prod_{p=1}^{n-1} p!\right) \frac{\det\left[e^{a_i b_j}\right]_{i,j=1}^n}{\Delta(A)\Delta(B)} \tag{1}$$

where U is the group of n-by-n unitary matrices, A and B are fixed n-by-n diagonal matrices with eigenvalues $a_1 < \cdots < a_n$ and $b_1 < \cdots < b_n$ respectively, and

$$\Delta(A) = \prod_{i<j}(a_j - a_i)$$

is the Vandermonde determinant. In this article we link the HCIZ integral formula to the theory of Segal-Bargmann spaces. In these spaces holomorphic and reproducing kernel techniques are available. Also, the orthonormal bases and annihilation and creation operators have a simple form, see [H00].

The paper is organized as follows. In Section 2 we derive the reproducing kernels of invariant subspaces of two Segal-Bargmann spaces. We prove that the canonical map between these two spaces is unitary if and only if the HCIZ integral formula holds in Section 3. Finally, we provide alternative proofs of known results in Section 4.

2. Invariant subspaces of Segal-Bargmann spaces

In this section we recall results about Segal-Bargmann spaces and show properties of invariant subspaces. References are Section 2, 3.2 and 6.1 of the lecture notes [H00] and Chapter 4 of [N11]. As in these references we adopt the physics convention about the conjugate linearity in hermitian forms.

For $n \in \mathbb{N}$ let $\mathbb{C}^{n\times n} = M_n(\mathbb{C})$ be the space of $n \times n$ complex matrices with inner product $\langle x, y\rangle = \mathrm{Tr}(xy^*)$, where Tr is the trace. Let $U \subseteq M_n(\mathbb{C})$ be the group of unitary matrices, and $\mathrm{GL}_n(\mathbb{C}) \subseteq M_n(\mathbb{C})$ the group of invertible matrices . The Segal-Bargmann space $\mathcal{F}(\mathbb{C}^{n\times n})$ is the space of holomorphic functions on $\mathbb{C}^{n\times n}$ such that

$$\pi^{-n^2}\int_{\mathbb{C}^{n\times n}} \overline{F(z)}F(z)e^{-\mathrm{Tr}(z^*z)}dz < \infty$$

with inner product

$$\langle F, G\rangle = \pi^{-n^2}\int_{\mathbb{C}^{n\times n}} \overline{F(z)}G(z)e^{-\mathrm{Tr}(z^*z)}dz.$$

For $a \in \mathbb{C}^{n\times n}$ the function

$$K_a(z) = e^{\mathrm{Tr}(za^*)}$$

is called the coherent state with parameter a and it satisfies

$$F(a) = \langle K_a, F\rangle$$

for all $F \in \mathcal{F}(\mathbb{C}^{n\times n})$. The function

$$K(z, a) = K_a(z) = e^{\mathrm{Tr}(za^*)}$$

is the reproducing kernel for the Segal-Bargmann space. We note that in $\mathcal{F}(\mathbb{C}^{n\times n})$ multiplication by a variable is the adjoint of derivation by the same variable. These are the creation and annihilation operators, see section 6.1 in [H00].

Remark 2.1. *The Gaussian measure $\pi^{-n^2}e^{-\operatorname{Tr}(z^*z)}dz$ is the measure of the complex Ginibre ensemble, see* [M04, Chapter 15]. *In this ensemble the expectation of the modulus squared of the sum of eigenvalues is*

$$\mathbb{E}\,|\lambda_1(z)+\cdots+\lambda_n(z)|^2 = \langle \operatorname{Tr}(z), \operatorname{Tr}(z)\rangle = \left(\operatorname{Tr}(\frac{\partial}{\partial z})\operatorname{Tr}(z)\right)\Big|_{z=0} = n,$$

where $\operatorname{Tr}(\frac{\partial}{\partial z}) = \frac{\partial}{\partial z_{11}}+\cdots+\frac{\partial}{\partial z_{nn}}$. One can compute the inner product directly knowing that scaled monomials form an orthonormal basis, but we used the fact that multiplication by z_{ij} is the adjoint of $\frac{\partial}{\partial z_{ij}}$. The expectation of the modulus squared of the product of eigenvalues is

$$\mathbb{E}\,|\lambda_1(z)\ldots\lambda_n(z)|^2 = \langle \det(z), \det(z)\rangle = \left(\det(\frac{\partial}{\partial z})\det(z)\right)\Big|_{z=0} = n!.$$

We define a unitary representation of U on $\mathcal{F}(\mathbb{C}^{n\times n})$ by

$$u \mapsto (F(z)\mapsto F(u^{-1}zu))$$

for $u\in U$ and $F\in\mathcal{F}(\mathbb{C}^{n\times n})$. This representation is unitary since the Gaussian measure $\pi^{-n^2}e^{-\operatorname{Tr}(z^*z)}dz$ is invariant by conjugation by unitary matrices. The subspace of fixed points are the F such that $F(z)=F(u^{-1}zu)$ for all $u\in U$:

$$\mathcal{F}(\mathbb{C}^{n\times n})^U = \{F\in\mathcal{F}(\mathbb{C}^{n\times n}) : F(z)=F(u^{-1}zu) \text{ for all } u\in U\}.$$

We apply the "unitarian trick": for fixed $F\in\mathcal{F}(\mathbb{C}^{n\times n})^U$ and fixed $z\in\mathbb{C}^{n\times n}$ the map

$$x\mapsto F(z)-F(e^x z e^{-x})$$

is holomorphic and is equal to 0 for skew-Hermitian x, so it is 0 for all $x\in M_n(\mathbb{C})$. Therefore the functions in $\mathcal{F}(\mathbb{C}^{n\times n})^U$ are actually the functions invariant under conjugation by all $g\in \mathrm{GL}_n(\mathbb{C})$, so they are functions of the spectrum. The orthogonal projection P onto $\mathcal{F}(\mathbb{C}^{n\times n})^U$ is given by

$$PF(x) = \int_U F(u^{-1}xu)du,$$

where the integral is over the Haar measure on U. The coherent states on $\mathcal{F}(\mathbb{C}^{n\times n})^U$ are given by

$$Q_a = PK_a$$

since for $F\in\mathcal{F}(\mathbb{C}^{n\times n})^U$ and $a\in\mathbb{C}^{n\times n}$ we have

$$F(a) = \langle K_a, F\rangle = \langle K_a, PF\rangle = \langle PK_a, F\rangle.$$

Proposition 2.2. *The reproducing kernel on $\mathcal{F}(\mathbb{C}^{n\times n})^U$ is given by*

$$Q(z,a) = \int_U e^{\operatorname{Tr}(u^{-1}zua^*)}du.$$

Proof. Note that for $z,a\in C^{n\times n}$

$$Q(z,a) = PK_a(z) = \int_U e^{\operatorname{Tr}(u^{-1}zua^*)}du.$$

□

We now apply analogous arguments for another Segal-Bargmann space and another unitary representation. Let $\mathbb{C}^n$ be endowed with the dot product $x \cdot y = \sum_{i=1}^n x_i y_i$ and let S_n be the symmetric group. The Segal-Bargmann space $\mathcal{F}(\mathbb{C}^n)$ is the space of holomorphic functions on $\mathbb{C}^n$ such that

$$\pi^{-n} \int_{\mathbb{C}^n} \overline{F(z)} F(z) e^{-|z|^2} dz < \infty$$

where $|z|^2 = z \cdot \overline{z}$. The inner product is

$$\langle F, G \rangle = \pi^{-n} \int_{\mathbb{C}^n} \overline{F(z)} G(z) e^{-|z|^2} dz.$$

The function

$$K(z, a) = K_a(z) = e^{z \cdot \overline{a}}$$

is the reproducing kernel for the Segal-Bargmann space. We define a unitary representation of S_n on $\mathcal{F}(\mathbb{C}^n)$ as

$$\sigma F(z_i) = \operatorname{sgn}(\sigma) F(z_{\sigma(i)}).$$

The fixed point space is the space $\mathcal{F}(\mathbb{C}^n)_{\text{alt}}$ of alternating holomorphic functions:

$$\mathcal{F}(\mathbb{C}^n)_{\text{alt}} = \{F \in \mathcal{F}(\mathbb{C}^n)) : F(z_1, \dots, z_n) = \operatorname{sgn}(\sigma) F(z_{\sigma(1)}, \dots, z_{\sigma(n)}) \text{ for all } \sigma \in S_n\}.$$

The projection P onto this space is given by

$$PF = \frac{1}{n!} \sum_{\sigma \in S_n} \sigma F.$$

Proposition 2.3. *The reproducing kernel on $\mathcal{F}(\mathbb{C}^n)_{\text{alt}}$ is given by*

$$R(z, a) = \frac{1}{n!} \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) e^{\sigma(z) \cdot \overline{a}}.$$

Proof. Note that for $z, a \in C^n$

$$R(z, a) = PK_a(z) = \frac{1}{n!} \sum_{\sigma \in S_n} \sigma K_a(z) = \frac{1}{n!} \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) e^{\sigma(z) \cdot \overline{a}}.$$

□

3. HCIZ integral formula as unitarity of a canonical map

In this section we prove the main result. The next proposition gives a formulation of the HCIZ integral formula in terms of inner products in $\mathcal{F}(\mathbb{C}^{n \times n})^U$ and in $\mathcal{F}(\mathbb{C}^n)_{\text{alt}}$. We denote with $D \subseteq \mathbb{C}^{n \times n}$ the set of complex diagonal matrices, with D_{reg} the set of complex diagonal matrices with distinct eigenvalues, and with $D_{\mathbb{R}}$ the set of diagonal matrices with real entries. As before Δ denotes the Vandermonde determinant. We set the constant

$$c = \left(\prod_{p=1}^n p! \right)^{-\frac{1}{2}}.$$

Proposition 3.1. *The HCIZ integral formula is equivalent to*

$$\langle Q_x, Q_y \rangle_{\mathcal{F}(\mathbb{C}^{n \times n})^U} = \left\langle \frac{R_x}{c\Delta(\overline{x})}, \frac{R_y}{c\Delta(\overline{y})} \right\rangle_{\mathcal{F}(\mathbb{C}^n)_{\text{alt}}} \tag{2}$$

for all $x, y \in D_{\text{reg}}$.

Proof. The left hand side is

$$\langle Q_x, Q_y\rangle = Q_y(x) = \int_U e^{\mathrm{Tr}(uxu^{-1}y^*)}du,$$

where we used the reproducing property and the formula for the kernel. The right hand side is

$$\begin{aligned}\left\langle \frac{R_x}{c\Delta(\overline{x})}, \frac{R_y}{c\Delta(\overline{y})}\right\rangle &= \frac{1}{c^2\Delta(x)\Delta(\overline{y})}R_y(x)\\ &= \left(\prod_{p=1}^{n} p!\right)\frac{1}{\Delta(x)\Delta(\overline{y})}R_y(x)\\ &= \left(\prod_{p=1}^{n} p!\right)\frac{1}{\Delta(x)\Delta(\overline{y})}\frac{1}{n!}\sum_{\sigma\in S_n}\mathrm{sgn}(\sigma)e^{\sigma(x)\cdot\overline{y}}.\end{aligned}$$

The HCIZ formula (1) states that the left hand side and the right hand side agree when $x, y \in D_{\mathrm{reg}} \cap D_{\mathbb{R}}$. Since both sides are holomorphic in x and anti-holomorphic in y we get equality for all $x, y \in D_{\mathrm{reg}}$. □

Theorem 3.2. *The HCIZ integral formula implies that the map $\psi : \mathcal{F}(\mathbb{C}^{n\times n})^U \to \mathcal{F}(\mathbb{C}^n)_{\mathrm{alt}}$ given by*

$$\psi(F)(x) = c\Delta(x)F|_D(x)$$

for $F \in \mathcal{F}(\mathbb{C}^{n\times n})^U$ and $x \in D$ is well defined and unitary. Conversely, if this map is well defined and unitary, then the HCIZ integral formula holds. The map ψ satisfies

$$\psi(Q_x) = \frac{R_x}{c\Delta(\overline{x})}$$

for any $x \in D_{\mathrm{reg}}$.

Proof. Define the map

$$\phi : \mathrm{span}\{Q_x\}_{x\in D_{\mathrm{reg}}} \to \mathrm{span}\left\{\frac{R_x}{c\Delta(\overline{x})}\right\}_{x\in D_{\mathrm{reg}}}$$

by

$$\sum_{i=1}^{m}\alpha_i Q_{x_i} \mapsto \sum_{i=1}^{m}\alpha_i\frac{R_{x_i}}{c\Delta(\overline{x_i})},$$

where $\alpha_i \in \mathbb{C}$. Equation (2) in the previous proposition implies that the map is well defined and an isometry, therefore it extends to an isometry between

$$\overline{\mathrm{span}}\{Q_x\}_{x\in D_{\mathrm{reg}}} \quad \text{and} \quad \overline{\mathrm{span}}\left\{\frac{R_x}{c\Delta(\overline{x})}\right\}_{x\in D_{\mathrm{reg}}},$$

where $\overline{\mathrm{span}}(A)$ denotes the closure of $\mathrm{span}(A)$.

We now prove that $\overline{\mathrm{span}}\{Q_x\}_{x\in D_{\mathrm{reg}}} = \mathcal{F}(\mathbb{C}^{n\times n})^U$. If this does not hold take an $F \in \mathcal{F}(\mathbb{C}^{n\times n})^U$ which is orthogonal to $\mathrm{span}\{Q_x\}_{x\in D_{\mathrm{reg}}}$. This means that

$$F(x) = \langle Q_x, F\rangle = 0$$

for all $x \in D_{\mathrm{reg}}$. Since F is invariant by conjugation of the variables it vanishes on all $g \in \mathrm{GL}_n(\mathbb{C})$ with n distinct eigenvalues. Since these matrices are dense in $\mathbb{C}^{n\times n}$ we conclude that $F = 0$. The fact that

$$\overline{\mathrm{span}}\left\{\frac{R_x}{c\Delta(\overline{x})}\right\}_{x\in D_{\mathrm{reg}}} = \mathcal{F}(\mathbb{C}^n)_{\mathrm{alt}}$$

is proved similarly. Therefore ϕ defines a unitary map from $\mathcal{F}(\mathbb{C}^{n\times n})^U$ to $\mathcal{F}(\mathbb{C}^n)_{\text{alt}}$. For $F \in \mathcal{F}(\mathbb{C}^{n\times n})^U$ and $x \in D_{\text{reg}}$ we get

$$\begin{aligned} F(x) =& \langle Q_x, F\rangle = \langle \phi(Q_x), \phi(F)\rangle = \left\langle \frac{R_x}{c\Delta(\overline{x})}, \phi(F)\right\rangle \\ =& \frac{1}{c\Delta(x)}\langle R_x, \phi(F)\rangle = \frac{1}{c\Delta(x)}\phi(F)(x), \end{aligned}$$

where the first and last equalities follow from the reproducing property, the second from the unitarity of ϕ, and the third from the definition of ϕ. Therefore

$$\phi(F)(x) = c\Delta(x)F(x)$$

for $x \in D_{\text{reg}}$. Hence, the map ϕ is equal to the map ψ and the first assertion of the theorem is proved.

To prove the second assertion assume that ψ is well defined and unitary. Then for $F \in \mathcal{F}(\mathbb{C}^{n\times n})^U$ and $x \in D_{\text{reg}}$ we have

$$F(x) = \langle Q_x, F\rangle = \langle \psi(Q_x), \psi(F)\rangle = \langle \psi(Q_x), c\Delta F|_D\rangle.$$

Also

$$\langle R_x, \psi(F)\rangle = \langle R_x, c\Delta F|_D\rangle = c\Delta(x)F(x),$$

hence

$$F(x) = \left\langle \frac{R_x}{c\Delta(\overline{x})}, \psi(F)\right\rangle.$$

Therefore for fixed $x \in D_{\text{reg}}$

$$\langle \psi(Q_x), \psi(F)\rangle = \left\langle \frac{R_x}{c\Delta(\overline{x})}, \psi(F)\right\rangle$$

for all $F \in \mathcal{F}(\mathbb{C}^{n\times n})^U$, and since ψ is unitary this implies that

$$\psi(Q_x) = \frac{R_x}{c\Delta(\overline{x})}.$$

From this property and Proposition 3.1 the HCIZ formula follows. ☐

4. Differentiation of polynomials and orthonormal bases

In this section we give alternative proofs of known results. For a polynomial F we define $F^*(z) = \overline{F(\overline{z})}$, that is, the coefficients of F^* are the complex conjugates of the coefficients of F.

Proposition 4.1. *The unitarity of the map ψ in Theorem 3.2 implies that for polynomials $F, G \in F \in \mathcal{F}(\mathbb{C}^{n\times n})^U$ we have*

$$F\left(\frac{\partial}{\partial z}\right)G(x) = \frac{1}{\Delta(x)}\left(F\Big|_D\left(\frac{\partial}{\partial z_1},\ldots,\frac{\partial}{\partial z_n}\right)(\Delta.G|_D)\right)(x) \tag{3}$$

for all $x \in D_{\text{reg}}$. Conversely, if formula (3) holds for all polynomials $F, G \in F \in \mathcal{F}(\mathbb{C}^{n\times n})^U$, then the map ψ is unitary.

Proof. We denote by $M_F G = F.G$ the multiplication operator. For a polynomial $F \in \mathcal{F}(\mathbb{C}^{n\times n})^U$ it is easy to check that

$$M_F = \psi^{-1} \circ M_{F|_D} \circ \psi.$$

Therefore, by applying adjoints

$$F^*\left(\frac{\partial}{\partial z}\right) = (M_F)^* = \psi^{-1}\circ (M_{F|_D})^*\circ\psi = \psi^{-1}\circ F^*\Big|_D\left(\frac{\partial}{\partial z_1},\dots,\frac{\partial}{\partial z_n}\right)\circ\psi.$$

Here we used the fact that ψ is unitary and the fact that multiplication by a variable is the adjoint of derivation by the same variable. We evaluate

$$\left(F^*\Big|_D\left(\frac{\partial}{\partial z_1},\dots,\frac{\partial}{\partial z_n}\right)\circ\psi\right)G = cF^*\Big|_D\left(\frac{\partial}{\partial z_1},\dots,\frac{\partial}{\partial z_n}\right)(\Delta.G|_D).$$

Applying ψ^{-1} and evaluating at $x\in D_{\mathrm{reg}}$ is the same as multiplying by

$$\frac{1}{c\Delta(x)},$$

so we get the formula for F^*. Since $F=(F^*)^*$ the first claim follows.

We denote for simplicity $\partial = \frac{\partial}{\partial z}$. To prove the second claim we need to verify that for all polynomials $F,G\in F\in\mathcal{F}(\mathbb{C}^{n\times n})^U$ the equality

$$\langle F^*, G\rangle = \langle\psi(F^*),\psi(G)\rangle = \langle c\Delta F^*|_D, c\Delta G|_D\rangle$$

holds. By the definition of the inner product in terms of differentiation at $z=0$ we have

$$\langle F^*, G\rangle = F(\partial)G|_{z=0}$$

and

$$\langle c\Delta F^*|_D, c\Delta G|_D\rangle = c^2\Delta(\partial)F|_D(\partial)(\Delta G|_D)|_{z=0}.$$

Therefore, we need to verify that

$$F(\partial)G|_{z=0} = c^2\Delta(\partial)F|_D(\partial)(\Delta G|_D)|_{z=0},$$

which by formula (3) is equivalent to

$$\frac{1}{\Delta}F|_D(\partial)(\Delta G|_D)\Big|_{z=0} = c^2\Delta(\partial)F|_D(\partial)(\Delta G|_D)|_{z=0}.$$

We set

$$F|_D(\partial)(\Delta G|_D) = \Delta(d+H),$$

where $d\in\mathbb{C}$ and H is a symmetric polynomial on $\mathbb{C}^n$ without constant term.

Hence, we have to show that

$$\frac{1}{\Delta}\Delta(d+H)\Big|_{z=0} = c^2\Delta(\partial)(\Delta(d+H))|_{z=0}.$$

The left hand side is d and the right hand side is

$$c^2 d\Delta(\partial)\Delta|_{z=0} + c^2\Delta(\partial)(\Delta.H)|_{z=0}.$$

We have

$$\Delta(\partial)\Delta = \left(\prod_{p=1}^{n} p!\right) = \frac{1}{c^2}.$$

Since $\Delta.H$ are monomials of of higher order that Δ we get

$$\Delta(\partial)(\Delta.H)|_{z=0} = 0.$$

Therefore the right hand side is also d. □

Formula (3) was obtained by Harish-Chandra in [H57] for semi-simple Lie groups. His proof of (1) is based on this result, see Theorem 3.8 in [McS21].

Remark 4.2. *If $F \in \mathcal{F}(\mathbb{C}^{n\times n})^U$ is a polynomial then*

$$F\left(\frac{\partial}{\partial z}\right) Q_a = F(\overline{a})Q_a$$

for any $a \in \mathbb{C}^{n\times n}$. This follows from

$$\left\langle F\left(\frac{\partial}{\partial z}\right) Q_a, G\right\rangle = \langle Q_a, F^*.G\rangle = F^*(a)G(a) = F^*(a)\langle Q_a, G\rangle = \langle \overline{F^*(a)}Q_a, G\rangle$$

for any polynomial $G \in \mathcal{F}(\mathbb{C}^{n\times n})^U$. By the same argument

$$F\left(\frac{\partial}{\partial z}\right) R_a = F(\overline{a})R_a$$

for any $a \in \mathbb{C}^n$ and any symmetric polynomial F on $\mathbb{C}^n$.

Let $\mathbb{N}_0$ stand for the non negative integers. For $\mu \in \mathbb{N}_0^n$ we denote the monomials as usual with $z^\mu = z_1^{\mu_1} \dots z_n^{\mu_n}$. We use the notation $\mu! = \mu_1! \dots \mu_n!$. In the Segal-Bargmann space $\mathcal{F}(\mathbb{C}^n)$ an orthonormal basis of the space is

$$\left(\frac{1}{\sqrt{\mu!}} z^\mu\right)_{\mu\in\mathbb{N}_0^n},$$

see [H00, Section 3.2].

The set Π of partitions is defined as

$$\Pi = \{\lambda \in \mathbb{N}_0^n : \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n\}.$$

We set $\delta = (n-1, n-2, \dots, 0)$ and $\mathbf{1} = (1, 1, \dots, 1)$. For $\lambda \in \Pi$, if $(x_1, \dots, x_n)$ are the eigenvalues of $x \in \mathrm{GL}_n(\mathbb{C})$ we define $\chi_\lambda(x) = s_\lambda(x_1, \dots, x_n)$, where s_λ is a Schur polynomial. These polynomials are defined by

$$s_\lambda(x_1, \dots, x_n) = \frac{a_{\lambda+\delta}(x_1, \dots, x_n)}{a_\delta(x_1, \dots, x_n)},$$

where

$$a_\mu(x_1, \dots, x_n) = \det \left[x_i^{\mu_j}\right]_{i,j=1}^n .$$

Note that $a_\delta(x_1, \dots, x_n) = \Delta(x_1, \dots, x_n)$ is the Vandermonde determinant.

Remark 4.3. *The irreducible polynomial representations of the general linear group $\mathrm{GL}_n(\mathbb{C})$ are labelled by Young diagrams, which we think of as vectors $\lambda \in \Pi$. The character of the λ-representation is given by χ_λ.*

For $\lambda \in \Pi$ we denote

$$d_\lambda(z) = \frac{1}{\sqrt{n!(\lambda+\delta)!}} a_{\lambda+\delta}(z) \in \mathcal{F}(\mathbb{C}^n)_{\mathrm{alt}}.$$

Proposition 4.4. *An orthonormal basis of the space $\mathcal{F}(\mathbb{C}^n)_{\mathrm{alt}}$ is given by*

$$(d_\lambda)_{\lambda\in\Pi} .$$

Proof. It is straightforward to check that this set is an orthonormal set. To check that it is a basis assume that $F \in \mathcal{F}(\mathbb{C}^n)_{\mathrm{alt}}$ and that $\langle F, a_{\lambda+\delta}\rangle = 0$ for all $\lambda \in \Pi$. Take a $\mu \in \mathbb{N}_0^n$ and assume that all the μ_i are different, then $z^\mu = \sigma(z^{\lambda+\delta})$ for a $\sigma \in S_n$ and a $\lambda \in \Pi$. We consider as in Section 2 the orthogonal projection P onto $\mathcal{F}(\mathbb{C}^n)_{\mathrm{alt}}$. Then

$$\langle F, z^\mu\rangle = \langle PF, z^\mu\rangle = \langle F, Pz^\mu\rangle = \left\langle F, \mathrm{sgn}(\sigma)\frac{1}{n!}a_{\lambda+\delta}\right\rangle = 0.$$

If there are $i \neq j$ such that $\mu_i = \mu_j$ take as σ the transposition of i and j. Then

$$\langle F, z^\mu \rangle = \langle \sigma(F), z^\mu \rangle = \langle F, \sigma(z^\mu) \rangle = -\langle F, z^\mu \rangle,$$

so that $\langle F, z^\mu \rangle = 0$. We proved that the inner product of F with all the elements of the orthonormal basis of $\mathcal{F}(\mathbb{C}^n)$ vanish, so $F = 0$. □

For $\lambda \in \Pi$ we denote

$$e_\lambda(z) = \sqrt{\frac{\delta!}{(\lambda+\delta)!}}\chi_\lambda(z) \in \mathcal{F}(\mathbb{C}^{n\times n})^U.$$

Proposition 4.5. *The unitarity of the map ψ in Theorem 3.2 implies that $(e_\lambda)_\lambda$ is an orthonormal basis. Conversely, the fact that $(e_\lambda)_\lambda$ is an orthonormal basis implies the unitarity of ψ.*

Proof. We check that for $\lambda \in \Pi$

$$\psi(e_\lambda) = ca_\delta e_\lambda|_D = d_\lambda.$$

The proposition follows. □

The fact that $(e_\lambda)_{\lambda\in\Pi}$ is orthonormal was proved in [FR09, Proposition 2]. Since $\mathcal{F}(\mathbb{C}^{n\times n})^U$ and $\mathcal{F}(\mathbb{C}^n)_{\mathrm{alt}}$ are reproducing kernel spaces we have the formulas for the kernels

$$R(x,y) = \sum_{\lambda\in\Pi} d_\lambda(x)\overline{d_\lambda(y)},$$

and

$$Q(x,y) = \sum_{\lambda\in\Pi} e_\lambda(x)\overline{e_\lambda(y)},$$

so that

$$Q(x,y) = \int_U e^{\mathrm{Tr}(uxu^{-1}y^*)}du = \sum_{\lambda\in\Pi} \frac{\delta!}{(\lambda+\delta)!}\chi_\lambda(x)\overline{\chi_\lambda(y)}.$$

Conversely, we have

Proposition 4.6. *If the expansion*

$$Q(x,y) = \sum_{\lambda\in\Pi} e_\lambda(x)\overline{e_\lambda(y)}$$

holds, then $(e_\lambda)_{\lambda\in\Pi}$ is an orthonormal basis of $\mathcal{F}(\mathbb{C}^{n\times n})^U$.

Proof. The e_λ with $|\lambda| = n$ are polynomials of degree n and are orthogonal to $e_{\lambda'}$ with $|\lambda'| = m \neq n$. Therefore, for λ' such that $|\lambda| = n \in \mathbb{N}$ we have

$$e_{\lambda'}(a) = \langle Q(z,a), e_{\lambda'}(z)\rangle = e_{\lambda'}(a)\langle e_{\lambda'}, e_{\lambda'}\rangle + \sum_{\lambda:|\lambda|=n,\lambda\neq\lambda'} e_\lambda(a)\langle e_\lambda, e_{\lambda'}\rangle.$$

For a λ'' such that $\lambda'' \neq \lambda'$ choose an $a \in D$ such that $e_{\lambda''}(a) \neq 0$ and $e_\lambda(a) = 0$ for all other λ with $|\lambda| = n$, we conclude that

$$e_{\lambda''}(a) = e_{\lambda''}(a)\langle e_{\lambda''}, e_{\lambda'}\rangle = 0$$

so the orthonormality of $(e_\lambda)_{\lambda\in\Pi}$ follows. To prove that this set is a basis let $F \in \mathcal{F}(\mathbb{C}^{n\times n})^U$ be written as a sum $F = \sum_{n\in\mathbb{N}_0} P_n$ of polynomials P_n of degree n. If $\langle e_\lambda, F\rangle = 0$ for all

$\lambda \in \Pi$, then for $n \in \mathbb{N}_0$ we have $\langle e_\lambda, P_n \rangle = 0$ for all λ with $|\lambda| = n$. Therefore, for $a \in D$ we have

$$P_n(a) = \langle Q_a, P_n \rangle = \left\langle \sum_{\lambda : |\lambda| = n} \overline{e_\lambda(a)} e_\lambda(z), P_n(z) \right\rangle = 0,$$

so all the P_n vanish and $F = 0$. □

The next proposition computes the coefficients of the expansion of an invariant holomorphic function in terms of the characters χ_λ.

Proposition 4.7. *For an $F \in \mathcal{F}(\mathbb{C}^{n\times n})^U$ written as*

$$F = \sum_{\lambda \in \Pi} f_\lambda \chi_\lambda$$

the coefficients f_λ are given by

$$f_\lambda = \left(\textit{coefficient of } z^{\lambda+\delta} \textit{ in } \Delta . F|_D \right)$$

for $\lambda \in \Pi$.

Proof. We have the expansion of F in terms of the orthonormal basis

$$F = \sum_{\lambda \in \Pi} \langle e_\lambda, F \rangle e_\lambda .$$

By the unitarity of ψ the Fourier coefficients are

$$\langle e_\lambda, F \rangle = \langle \psi(e_\lambda), \psi(F) \rangle = \langle d_\lambda, c\Delta . F|_D \rangle$$

for $\lambda \in \Pi$. Note that

$$d_\lambda(z) = \frac{1}{\sqrt{n!(\lambda+\delta)!}} a_{\lambda+\delta}(z) = \frac{1}{\sqrt{n!(\lambda+\delta)!}} n! P(z^{\lambda+\delta}) = \sqrt{\frac{n!}{(\lambda+\delta)!}} P(z^{\lambda+\delta}),$$

where P is the orthogonal projection to the alternating functions as in Section 2. Therefore

$$\langle d_\lambda, c\Delta . F|_D \rangle = \sqrt{\frac{n!}{(\lambda+\delta)!}} \left\langle P(z^{\lambda+\delta}), c\Delta . F|_D \right\rangle = \sqrt{\frac{n!}{(\lambda+\delta)!}} \left\langle z^{\lambda+\delta}, c\Delta . F|_D \right\rangle .$$

Also

$$\langle z^{\lambda+\delta}, c\Delta . F|_D \rangle = (\lambda+\delta)! (\text{ coefficient of } z^{\lambda+\delta} \text{ in } c\Delta . F|_D).$$

Some calculations yield

$$f_\lambda = \sqrt{\prod_{p=1}^{n} p!} \left(\text{ coefficient of } z^{\lambda+\delta} \text{ in } c\Delta . F|_D \right),$$

and since

$$\sqrt{\prod_{p=1}^{n} p!} = \frac{1}{c}$$

the proposition follows. □

Acknowledgements

I thank Colin McSwiggen for several discussions related to the Harish-Chandra-Itzykson-Zuber integral.

References

[FR09] P. J. Forrester, E. M. Rains, *Matrix averages relating to Ginibre ensembles.* J. Phys. A 42 (2009), no. 38, 385205, 13 pp. MR2540392

[H00] B. C. Hall, *Holomorphic methods in analysis and mathematical physics.* First Summer School in Analysis and Mathematical Physics (Cuernavaca Morelos, 1998), 1–59, Contemp. Math., 260, Aportaciones Mat., Amer. Math. Soc., Providence, RI, 2000.

[H57] Harish-Chandra. *Differential operators on a semisimple Lie algebra.* American Journal of Mathematics, 79:87-120, 1957.

[IZ80] C. Itzykson, J.-B. Zuber, *The planar approximation. II.* Journal of Mathematical Physics, 21:411-421, 1980.

[McS21] C. McSwiggen, *The Harish-Chandra integral: an introduction with examples.* Enseign. Math. 67 (2021), no. 3-4, 229–299.

[M04] Mehta, Madan Lal. Random matrices. Third edition. Pure and Applied Mathematics (Amsterdam), 142. Elsevier/Academic Press, Amsterdam, 2004.

[N11] Y. A. Neretin, *Lectures on Gaussian integral operators and classical groups.* EMS Series of Lectures in Mathematics. European Mathematical Society (EMS), Zürich, 2011.