Symmetries and reflections from composition operators in the disk

E. Andruchow, G. Corach, L. Recht

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Abstract

We study the composition operators C_a acting on the Hardy space H^2 of the unit disk, given by $C_a f = f \circ \varphi_a$, where

$$\varphi_a(z) = \frac{a - z}{1 - \bar{a}z},$$

for $|a| < 1$. These operators are reflections: $C_a^2 = 1$. We study their eigenspaces $N(C_a \pm 1)$, their relative position (i.e., the intersections between these spaces and their orthogonal complementes for $a \neq b$ in the unit disk) and the symmetries induced by C_a and these eigenspaces.

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1 Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$ be the unit disk and $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ the unit circle. Consider the analytic automorphisms φ_a which map $\mathbb{D}$ onto $\mathbb{D}$ of the form

$$\varphi_a(z) = \frac{a - z}{1 - \bar{a}z},$$

for $a \in \mathbb{D}$. Save for a constant of module one, all automorphisms of the disk are of this form. Note the fact that $\varphi_a(\varphi_a(z)) = z$. This implies that the composition operators induced by these automorphisms are *reflections* (i.e., operators C such that $C^2 = 1$). Namely, let $H^2 = H^2(\mathbb{D})$ be the Hardy space of the disk, i.e.

$$H^2 = \{f : \mathbb{D} \to \mathbb{C} : f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ with } \sum_{n=0}^{\infty} |a_n|^2 < \infty\}.$$

Then, an analytic map $\varphi : \mathbb{D} \to \mathbb{D}$ induces the (bounded linear, see [6]) operator $C_\varphi : H^2 \to H^2$,

$$C_\varphi f = f \circ \varphi.$$

In particular, for $a \in \mathbb{D}$, the operator $C_a := C_{\varphi_a}$ satisfies $(C_a)^2 = 1$, the identity operator. The eigenspaces of C_a are

$$N(C_a - 1) = \{f \in H^2 : f \circ \varphi_a = f\}$$

and

$$N(C_a + 1) = \{g \in H^2 : g \circ \varphi_a = -g\},$$

which verify that $N(C_a - 1) \dot{+} N(C_a + 1) = H^2$. Here $\dot{+}$ means direct (non necessarily orthogonal) sum, we reserve the symbol $\oplus$ for orthogonal sums.

Reflections which additionally are selfadoint are called *symmetries*: S is a symmetry if $S = S^* = S^{-1}$. Associated to a reflection C, there are three natural symmetries: $\mathbf{r}(C)$, $\mathbf{n}(C)$ and $\rho(C)$. The first two correspond to the decompositions $H^2 = N(C-1) \oplus N(C-1)^\perp$ and $H^2 = N(C+1) \oplus N(C+1)^\perp$ respectively. The third is of differential geometric nature, and is described below. The aim of this paper is the study of the operators C_a for $a \in \mathbb{D}$, the description of their eigenspaces, their relative position, and the induced symmetries. In this task, it will be important the role of the unique fixed point ω_a of φ_a inside the disk. Namely,

$$\omega_a := \frac{1}{\bar{a}}\{1 - \sqrt{1 - |a|^2}\} \text{ if } a \neq 0, \quad \text{and } \omega_0 = 0.$$

The contents of the paper are the following. In Section 2 we recall basic facts on the manifolds of reflections and symmetries, in particular the condition for existence of geodesics between points in the latter space. In Section 3 we state basic formulas concerning the operators C_a. In Section 4 we characterize the symmetries ρ_a, obtained as the unitary part of the polar decomposition of C_a. For this task, we use Rosenblum's computation for the spectral measure of a selfadjoint Toeplitz operator [11]. Using a result by E. Berkson [3], we show that locally, the map $a \mapsto \rho_a$ ($a \in \mathbb{D}$) is injective (it remains unanswered wether it is globally injective in the disk $\mathbb{D}$). We also obtain formulas for the range an nullspace symmetries of C_a, and a power series expansion for ρ_a. The rest of the paper is devoted to the study of the eigenspaces of C_a, and their relative position. If $a = 0$, then the fixed point of φ_0 is $\omega_0 = 0$ and C_0 is the reflection (and symmetry) $f(z) \mapsto f(-z)$. Thus the eigenspaces of C_0 are the subspaces $\mathcal{E}$ and $\mathcal{O}$ of even and odd functions of H^2. It is elemenatry to see that for arbitrary $a \in \mathbb{D}$, the eigenspaces of C_a are

$$N(C_a - 1) = C_{\omega_a}(\mathcal{E}) \quad \text{and} \quad N(C_a + 1) = C_{\omega_a}(\mathcal{O}).$$

We then analyze the position of these eigenspaces for $a \neq b$. For instance (Theorem 5.4),

$$N(C_a - 1) \cap N(C_b - 1) = \mathbb{C}1 \text{ and } N(C_a + 1) \cap N(C_b + 1) = \{0\}.$$

The computations of the intersections involving the orthogonal of these spaces is more cumbersome, and we are only able to do it in the special case when $b = 0$ (Theorem 5.7). These facts, which are stated in Section 5, are used in Section 6 to show which of these eigenspaces are conjugate with the exponential of the Grassmann manifold of H^2.

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2 Preliminaries, on reflections and symmetries

Denote the set of reflections by

$$\mathcal{Q} = \{T \in \mathcal{B}(H^2) : T^2 = 1\}.$$

The set $\mathcal{Q}$ has rich geometric structure (see for instance [4]): is it an homogeneous C^∞ submanifold of $\mathcal{B}(H^2)$, carrying the action of the group $Gl(H^2)$ of invertible operators in H^2:

$$G \cdot T = GTG^{-1}, \ T \in \mathcal{Q}, G \in Gl(H^2).$$

The set $\mathcal{P}$ of *selfadjoint* reflections, or *symmetries*, is

$$\mathcal{P} = \{V \in \mathcal{Q} : V^* = V\}.$$

Note that a symmetry V is a selfadjoint unitary operator. Reflections and symmetries can be viewed alternatively as oblique and orthogonal projections, respectively. A reflection T gives rise to an idempotent (or oblique projection) with range equal to the eigenspace $\{f \in H^2 : Tf = f\}$: $Q_+ = \frac{1}{2}(1+T)$ (and another with range equal to the other eigenspace $\{g \in H^2 : Tg = -g\}$ of T: $Q_- = \frac{1}{2}(1 - T)$). If S is a symmetry, the corresponding idempotents P_+ and P_- are orthogonal projections.

The set $\mathcal{P}$, in turn, can be regarded as the Grassmann manifold of H^2: to each reflection V corresponds a unique projection $P_+ = \frac{1}{2}(1+V)$ and a unique subspace $\mathcal{S}$ such that $R(P_+) = \mathcal{S}$. The geometry of the Grassmann manifold in this operator theoretic context was developed in [4], [10]: $\mathcal{P}$ is presented as a homogeneous space of the unitary group (as in the classical finite dimensional setting), with a linear reductive connection and a Finsler metric. In [2] the necessary and sufficient condition for the existence of a geodesic of this connection between two subspaces $\mathcal{S}$ and $\mathcal{T}$ was stated: namely, that

$$\dim(\mathcal{S} \cap \mathcal{T}^\perp) = \dim(\mathcal{S}^\perp \cap \mathcal{T}). \qquad (1)$$

Moreover, the geodesic is of the form $\delta(t) = e^{itX}\mathcal{S}$, for $X^* = X$ codiagonal with respect to both $\mathcal{S}$ and $\mathcal{T}$:

$$X(\mathcal{S}) \subset \mathcal{S}^\perp \quad \text{and} \quad X(\mathcal{T}) \subset \mathcal{T}^\perp.$$

The geodesic can be chosen of minimal length for the Finsler metric (see [10], [4], [2]). This latter condition amounts to finding X such that $\|X\| \leq \pi/2$.

The condition for the existence of a *unique* minimal geodesic (up to reparametrization) was given:

$$\mathcal{S} \cap \mathcal{T}^\perp = \{0\} = \mathcal{S}^\perp \cap \mathcal{T}. \tag{2}$$

In this case, the exponent $X = X_{\mathcal{S},\mathcal{T}}$ is unique with the above mentioned conditions ($X_{\mathcal{S},\mathcal{T}}$ selfadjoint, codiagonal with respect to $\mathcal{S}$ and $\mathcal{T}$, with norm less or equal then $\pi/2$, satisfying $e^{iX_{\mathcal{S},\mathcal{T}}}\mathcal{S} = \mathcal{T}$.).

In this paper we shall examine existence and uniqueness of geodesics of the Grassmann manifold of H^2, for the eigenspaces of C_a.

One of the remarkable features of the space $\mathcal{Q}$ is the several natural projection maps that it has onto $\mathcal{P}$. The natural projection maps $\mathcal{Q} \to \mathcal{P}$ are the range, nullspace and unitary part in the polar decomposition:

1. The *range map* $\mathbf{r}$, which maps $T \in \mathcal{Q}$ to the symmetry $\mathbf{r}(T) = 2P_{R(Q_+)} - 1$, i.e. the symmetry which is the identity on $R(Q_+) = \{f \in H^2 : Tf = f\}$. We recall the formula for the orthogonal projection $P_{R(Q)}$ onto the range $R(Q)$ of an idempotent Q (see for instance [1]):
$$P_{R(Q)} = Q(Q + Q^* - 1)^{-1}.$$
Then
$$P_{R(Q_+)} = \frac{1}{2}(1+T)\{\frac{1}{2}(1+T)+\frac{1}{2}(1+T^*)-1\}^{-1} = (1+T)\{T+T^*\}^{-1},$$
and therefore
$$\mathbf{r}(T) = 2(1+T)\{T+T^*\}^{-1} - 1 = (2 + T - T^*)\{T + T^*\}^{-1}. \tag{3}$$

2. The *null-space* map $\mathbf{n}$, which maps $T \in \mathcal{Q}$ to the symmetry which is the identity on $R(Q_-) = \{g \in H^2 : Tg = -g\}$, which by similar computations is given by
$$\mathbf{n}(T) = 2(T-1)\{T+T^*\}^{-1} - 1 = (T - T^* - 2)\{T + T^*\}^{-1}. \tag{4}$$
.

3. The *unitary part ρ in the polar decomposition*, which maps T to
$$\rho(T) = T(T^*T)^{-1/2}, \tag{5}$$

the unitary part in the polar decomposition $T = \rho(T)(T^*T)^{1/2}$. We refer the reader to [4] for the properties of this element $\rho(T)$. Among them, the most remarkable, that $\rho(T)$ is a symmetry. We shall recall the other properties of $\rho(T)$ in due course. Note, for instance, that $(T^*T)^{-1} = TT^*$, so that

$$(T^*T)^{-1/2} = (TT^*)^{1/2}.$$

Notice the following formulas:

Proposition 2.1. *Let $T \in \mathcal{Q}$ then*

$$\mathbf{r}(T) = 2(1+T)(T^*T+1)^{-1} \quad \textit{and} \quad \mathbf{n}(T) = 2(1-T)(T^*T+1)^{-1}.$$

Proof. Let $T = \rho(T)|T|$ be the polar decomposition. It is a straightforward computation (or see [4]) that $|T|\rho(T) = \rho(T)|T^*|$. Also it is eassy to see that since $T^2 = 1$, $|T^*| = |T|^{-1}$. Then

$$T + T^* = \rho(T)|T| + |T|\rho(T) = \rho(T)(|T| + |T^*|) = \rho(T)(|T| + |T|^{-1}).$$

Using again that $|T|\rho(T) = \rho(T)|T|^{-1}$ (and therefore also that $\rho(T)|T| = |T|^{-1}\rho(T)$), we have that $\rho(T)$ commutes with $|T| + |T|^{-1}$. Then

$$(T + T^*)^{-1} = \rho(T)(|T| + |T|^{-1})^{-1} = (|T| + |T|^{-1})^{-1}\rho(T).$$

By an elementary functional calculus argument, we have that $(|T| + |T|^{-1})^{-1} = |T|(|T|^2 + 1)^{-1}$. Then

$$(T + T^*)^{-1} = \rho(T)|T|(|T|^2 + 1)^{-1} = T(|T|^2 + 1)^{-1}.$$

Thus,

$$\mathbf{r}(T) = 2(T+1)T(|T|^2 + 1)^{-1} = 2(1+T)(|T|^2 + 1)^{-1},$$

and similarly

$$\mathbf{n}(T) = 2(T-1)T(|T|^2 + 1)^{-1} = 2(1-T)(|T|^2 + 1)^{-1}.$$

□

We shall return to these formulas for the case $T = C_a$ later, after we further characterize $|C_a|$.

3 The operators C_a

It is not a trivial task to compute the adjoint of a composition operator, however, for the special case of automorphisms of the disk, it was shown by Cowen [5] (see also [6]) that

$$C_a^* = (C_{\varphi_a})^* = M_{\frac{1}{1-\bar{a}z}} C_a (M_{1-\bar{a}z})^*,$$

where, for a bounded analytic function g in $\mathbb{D}$, M_g denotes the multiplication operator. Equivalently,

$$C_a^* = M_{\frac{1}{1-\bar{a}z}} C_a - a M_{\frac{1}{1-\bar{a}z}} C_a (M_z)^*, \tag{6}$$

where $(M_z)^*$ (or co-shift) is the adjoint of the shift operator $S = M_z$.

In order to characterize the polar decomposition of C_a, it will be useful to compute $C_a C_a^*$. Note that, for $f \in H^2$, after straightforward computations,

$$C_a C_a^* f(z) = \frac{1-\bar{a}z}{1-|a|^2}\{f(z) - a\frac{f(z)-f(0)}{z}\}. \tag{7}$$

Also note how C_a relates to the shift operator

$$S: H^2 \to H^2,\ Sf(z) = zf(z), \text{ with adjoint } S^*f(z) = \frac{f(z)-f(0)}{z}:$$

$$C_a C_a^* = \frac{1}{1-|a|^2}(1-\bar{a}S)(1-aS^*). \tag{8}$$

For $a \in \mathbb{D}$, denote by k_a the Szego kernel: for $f \in H^2$, $\langle f, k_a \rangle = f(a)$, i.e.,

$$k_a(z) = \frac{1}{1-\bar{a}z}. \tag{9}$$

Remark 3.1. Note the fact that

$$C_a C_a^*(k_a) = 1.$$

Indeed, this follows after a straightforward computation. Therefore, we have also that

$$C_a^* C_a(1) = k_a.$$

For $a \in \mathbb{D}$, denote by

$$\rho_a = \rho(C_a). \tag{10}$$

Note that if $a = 0$, $\varphi_0(z) = -z$ and $C_0 f(z) = f(-z)$ is a symmetry, thus $C_0^* = C_0$, $|C_0| = 1$ and $\rho_0 = C_0$.

Returning to the characterization of the modulus of C_a, we have that

Lemma 3.2. *With the current notations,*

$$|C_a^*| = \frac{1}{\sqrt{1-|a|^2}}|1-aS^*|.$$

and

$$\rho_a = \frac{1}{\sqrt{1-|a|^2}}C_a|1-aS^*|.$$

Remark 3.3. There is another symmetry related to C_a. In the book [6] (Exercise 2.1.9:), it is stated that for $a \in \mathbb{D}$, if we put

$$\psi_a(z) = \frac{\sqrt{1-|a|^2}}{1-\bar{a}z} = \sqrt{1-|a|^2}\ k_a = \frac{k_a}{\|k_a\|_2},$$

then the operator $W_a \in \mathcal{B}(H^2)$, $W_a = M_{\psi_a}C_a$. i.e.,

$$W_a f(z) = \psi_a(z) f(\varphi_a(z))$$

is a unitary operator. In fact, it is straightforward to verify that $W_a^2 = 1$, i.e., W_a is a symmetry.

Note the relationship between ρ_a and W_a:

$$C_a = \frac{1}{\sqrt{1-|a|^2}}M_{1-\bar{a}z}W_a = \frac{1}{\sqrt{1-|a|^2}}(1-\bar{a}S)W_a. \tag{11}$$

It follows that the symmetry W_a intertwines $C_aC_a^*$ and $C_a^*C_a$:

$$C_a^*C_a = \frac{1}{1-|a|^2}W_a(1-aS^*)(1-\bar{a}S)W_a = W_a(C_a^*C_a)W_a,$$

thus

$$|C_a| = \frac{1}{\sqrt{1-|a|^2}}W_a|1-\bar{a}S|W_a = W_a|C_a^*|W_a, \tag{12}$$

and $|C_a|^{-1} = \sqrt{1-|a|^2}W_a|1-\bar{a}S|^{-1}W_a$.

Remark 3.4. Note that $C_a = \rho_a|C_a|$ implies that

$$C_aC_a^* = \rho_a|C_a|^2\rho_a = \rho_aC_a^*C_a\rho_a.$$

Then

$$W_a\rho_aC_a^*C_a(W_a\rho_a)^* = W_a\rho_aC_a^*C_a\rho_aW_a = W_aC_aC_a^*W_a = C_a^*C_a,$$

i.e., $W_a\rho_a$ commutes with $C_a^*C_a$ (and therefore also with its inverse $C_aC_a^*$).

4 The symmetry ρ_a

If $\psi \in L^\infty(\mathbb{T})$, as is usual notation, let $T_\psi \in \mathcal{B}(H^2)$ be the Toeplitz operator with symbol ψ: $T_\psi f = P_{H^2}(\psi f)$.

The following remark is certainly well known:

Lemma 4.1. *For $a \in \mathbb{D}$,*

$$W_a S W_a = T_{\varphi_a} = M_{\varphi_a}.$$

Proof. Straightforward computation:

$$W_a S W_a f(z) = \sqrt{1-|a|^2}\; W_a\left(\frac{z}{1-\bar{a}z} f\left(\frac{a-z}{1-\bar{a}z}\right)\right) = \frac{a-z}{1-\bar{a}z} f(z).$$

□

Therefore:

Theorem 4.2.

$$|Ca| = \sqrt{1-|a|^2}\Big(T_{|1-\bar{a}z|^{-2}}\Big)^{1/2} = |T_{\psi_a}|.$$

Proof. As remarked above,

$$|C_a|^2 = C_a^* C_a = \frac{1}{1-|a|^2} W_a(1-aS^*)(1-\bar{a}S)W_a$$

$$= \frac{1}{1-|a|^2} W_a(1-aS^*)W_a W_a(1-\bar{a}S)W_a$$

which by Lemma 4.1 equals

$$\frac{1}{1-|a|^2}(1-a(W_aSW_a)^*)(1-\bar{a}W_aSW_a) = \frac{1}{1-|a|^2}(1-aT_{\varphi_a}^*)(1-\bar{a}T_{\varphi_a})$$

$$= \frac{1}{1-|a|^2} T_{1-\bar{a}\varphi_a}^* T_{1-\bar{a}\varphi_a}.$$

Now we use the fact that $T_\psi^* = T_{\bar{\psi}}$ and that if $\psi, \bar{\theta} \in H^\infty$, then $T_\theta T_\psi = T_{\theta\psi}$ (see chapter 7 of Douglas' book [8], specifically Prop. 7.5 for the second assertion). Then

$$C_a^* C_a = \frac{1}{1-|a|^2} T_{(1-a\bar{\varphi}_a)(1-\bar{a}\varphi_a)}.$$

Since $1-\bar{a}\varphi_a(z) = \frac{1-|a|^2}{1-\bar{a}z}$, it follows that this expression above equals

$$(1-|a|^2) T_{\frac{1}{|1-\bar{a}z|^2}},$$

and the proof follows. □

As a consequence, we may use the remarkable description of the spectral decomposition of selfadjoint Toeplitz operators obtained by M. Rosenblum un [11]. Let us quote in the next remark this description:

Remark 4.3. Suppose that $\omega : \mathbb{T} \to \mathbb{R}$ is a measurable function that satifies the following conditions:

1. ω is bounded from below: $\omega(\theta) > -\infty$.
2. For each $\lambda \in \mathbb{R}$, the set

$$\Gamma_\lambda := \{e^{i\theta} \in \mathbb{T} : \omega(\theta) \geq \lambda\}$$

is a.e. an arc.

Then Rosenblum's **Theorem 3** in [11] states that: if T_ω is the Toeplitz operator with symbol ω, $\Lambda \subset \mathbb{R}$ is a Borel subset and $E(\Lambda)$ is the spectral measure (of T_ω) associated to Λ, $u, v \in \mathbb{D}$, one has that

$$\langle E(\Lambda)k_u, k_v\rangle = \int_\Lambda \Phi(\bar{u};\lambda)\overline{\Phi(\bar{v};\lambda)}dm(\lambda), \tag{13}$$

where

$$\Phi(u;\lambda) = \Psi(u;\lambda)\left(1 - ue^{i\alpha(\lambda)}\right)^{-1/2}\left(1 - ue^{i\beta(\lambda)}\right)^{-1/2},$$

$$\Psi(u;\lambda) = \exp\Big(-\int_{-\pi}^{\pi} \log|\omega(\theta) - \lambda| P(u,\theta)d\theta\Big),$$

$$P(u,\theta) = \frac{1}{4\pi}\frac{1 + ue^{i\theta}}{1 - ue^{i\theta}},$$

$\alpha(\lambda) \leq \beta(\lambda) \in [-\pi, \pi]$ are such that

$$\Gamma_\lambda = \{e^{i\theta} : \alpha(\lambda) \leq \theta \leq \beta(\lambda)\},$$

and

$$dm(\lambda) = \frac{1}{\pi}\sin(\frac{1}{2}(\beta(\lambda) - \alpha(\lambda)))d\lambda.$$

In particular, note that the spectral measure of T_ω is absolutely continuous with respect to the Lebesgue measure.

In our case, we must analize $\omega(\theta) = \frac{1}{|1-\bar{a}e^{i\theta}|^2} = |k_a(e^{i\theta})|^2$. We consider the case $a \neq 0$ (for $a = 0$ recall that $\rho_0 = C_0$). The function ω is continuous, so condition 1. is fulfilled. With respect to condition 2., note that, for $\lambda \leq 0$, Γ_λ is empty, and for $\lambda > 0$

$$\Gamma_\lambda = \{e^{i\theta} : |\frac{a}{|a|^2} - e^{i\theta}| \leq \frac{1}{|a|\sqrt{\lambda}}\}.$$

Consider the following figure

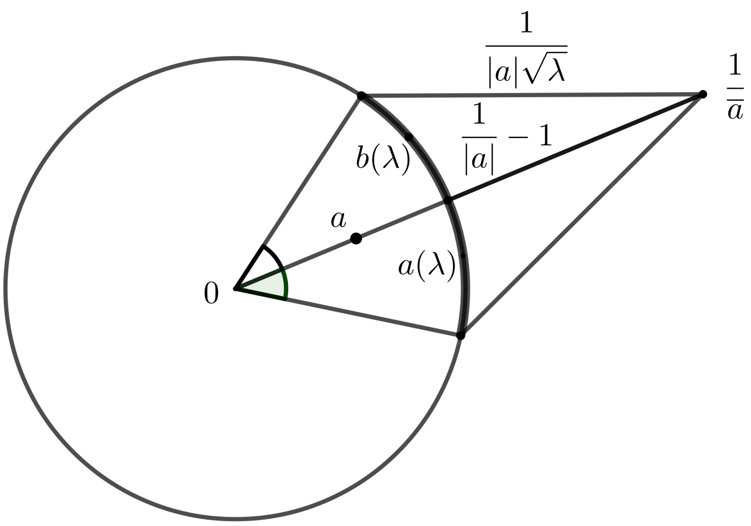

Figure 1

Then clearly the spectral measure is zero if

- $\lambda > \dfrac{1}{(1-|a|)^2}$ (here $\alpha(\lambda) = \beta(\lambda)$ and Γ_λ has measure zero), or if
- $\lambda < \dfrac{1}{(1+|a|)^2}$ (here $\alpha(\lambda) = -\pi$, $\beta(\lambda) = \pi$ and $\Gamma_\lambda = \mathbb{T}$).

For $\lambda \in \left[\dfrac{1}{(1+|a|)^2}, \dfrac{1}{(1-|a|)^2}\right]$ we have the following figure:

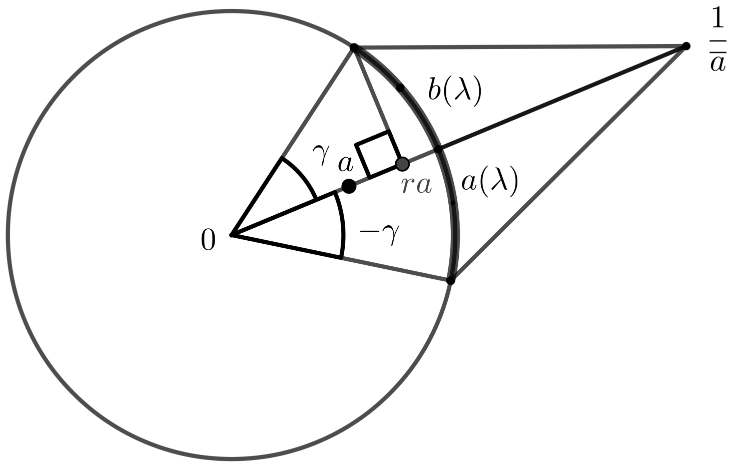

Figure 2

Therefore, after elementary computations, on has that $\beta(\lambda) = \arcsin(\gamma)$, $\alpha(\lambda) = -\arcsin(\gamma)$ and

$$\sin\left(\frac{1}{2}(\beta(\lambda)-\alpha(\lambda))\right) = \sin(\gamma) = \sqrt{1-\frac{1}{4}\left(1+\frac{1}{|a|}(1-\frac{1}{\lambda})\right)^2}.$$

Thus, we may characterize the function $\rho_a 1$ (the symmetry ρ_a at the element $1 \in H^2$). To this effect, recall that the set $\{k_u : u \in \mathbb{D}\}$ is total in H^2.

Proposition 4.4. *With the current notations, for $v \in \mathbb{D}$, we have that $\langle \rho_a 1, k_v\rangle$ equals*

$$\frac{\sqrt{1-|a|^1}}{\pi}\int_{\frac{1}{(1+|a|)^2}}^{\frac{1}{(1-|a|)^2}} \lambda^{1/2}\Phi(0;\lambda)\overline{\Phi(\bar{v};\lambda)}\sqrt{1-\frac{1}{4}\left(1+\frac{1}{|a|}(1-\frac{1}{\lambda})\right)^2}\, d\lambda.$$

Proof. Recall that

$$\rho_a = C_a(C_a^* C_a)^{-1/2} = (C_a C_a^*)^{-1/2} C_a = (C_a^* C_a)^{1/2} C_a,$$

so that (since $1 = k_0$)

$$\rho_a 1 = |C_a|^{1/2} C_a 1 = |C_a|^{1/2} 1 = |C_a|^{1/2} k_0,$$

and then

$$\langle \rho_a 1, k_v\rangle = \sqrt{1-|a|^2}\langle T^{1/2}_{|1-\bar{a}e^{i\theta}|^{-2}} k_0, k_v\rangle,$$

and the formula folllows applying Rosenblum's result and the above elementary computations. □

Remark 4.5. In particular, we have that $\rho_a 1(0) = \langle \rho_a 1, 1\rangle$ equals

$$\frac{\sqrt{1-|a|^2}}{\pi}\int_{\frac{1}{(1+|a|)^2}}^{\frac{1}{(1-|a|)^2}} \lambda^{1/2}|\Phi(0,\lambda)|^2\sqrt{1-\frac{1}{4}\left(1+\frac{1}{|a|}(1-\frac{1}{\lambda})\right)^2}\,d\lambda,$$

with

$$|\Phi(0,\lambda)|^2 = \exp\left(-\frac{1}{2\pi}\int_{-\pi}^{\pi}\log||1-\bar{a}e^{i\theta}|^{-2}-\lambda|d\theta\right).$$

Clearly, if $A\subset\mathbb{D}$ is a finite set, then $\{k_a : a\in\mathbb{D}\setminus A\}$ is also total in H^2. Therefore we may characterize ρ_a as follows:

Theorem 4.6. *With the current notations, for $a,u,v\in\mathbb{D}$, with $u\neq a$, we have that $\langle\rho_a k_u, k_v\rangle$ equals*

$$\frac{\bar{u}(|a|^2-1)^{3/2}}{2\pi(\bar{u}-\bar{a})}\int_{\frac{1}{(1+|a|)^2}}^{\frac{1}{(1-|a|)^2}} \lambda^{1/2}\Phi(\varphi_a(u);\lambda)\overline{\Phi(\bar{v};\lambda)}\sqrt{4-\left(1+\frac{1}{|a|}(1-\frac{1}{\lambda})\right)^2}\,d\lambda$$

$$+\frac{\bar{a}}{\bar{a}-\bar{u}}\frac{\sqrt{1-|a|^2}}{2\pi}\int_{\frac{1}{(1+|a|)^2}}^{\frac{1}{(1-|a|)^2}} \lambda^{1/2}\Phi(0;\lambda)\overline{\Phi(\bar{v};\lambda)}\sqrt{4-\left(1+\frac{1}{|a|}(1-\frac{1}{\lambda})\right)^2}\,d\lambda.$$

These inner products characterize ρ_a, because $\{k_u : u\in\mathbb{D}, u\neq a\}$ is a total set in H^2.

Proof. The last assertion is clear.

Recall that

$$\rho_a = C_a(C_a^*C_a)^{-1/2} = (C_aC_a^*)^{-1/2}C_a = (C_a^*C_a)^{1/2}C_a.$$

Note that

$$C_ak_u(z) = \frac{1-\bar{a}z}{1-\bar{u}a-z(\bar{a}-\bar{u})} = \frac{1}{1-\bar{u}a}\frac{1-\bar{a}z}{1-\overline{\varphi_a(u)}z},$$

which after routine computations (using that $a\neq u$, and $1=k_0$) yields

$$C_ak_u = \frac{\bar{u}(1-|a|^2)}{\bar{u}-\bar{a}}k_{\overline{\varphi_a(u)}} + \frac{\bar{a}}{\bar{a}-\bar{u}}k_0.$$

Therefore,

$$\rho_ak_u = (C_a^*C_a)^{1/2}C_ak_u = (C_a^*C_a)^{1/2}(\frac{\bar{u}(1-|a|^2)}{\bar{u}-\bar{a}}k_{\overline{\varphi_a(u)}} + \frac{\bar{a}}{\bar{a}-\bar{u}}k_0),$$

and thus

$$\langle\rho_ak_u,k_v\rangle = \sqrt{1-|a|^2}\langle T^{1/2}_{|1-\bar{a}e^{i\theta}|^{-2}}(\frac{\bar{u}(1-|a|^2)}{\bar{u}-\bar{a}}k_{\overline{\varphi_a(u)}} + \frac{\bar{a}}{\bar{a}-\bar{u}}k_0),k_v\rangle$$

$$= \sqrt{1-|a|^2}\{\frac{\bar{u}(1-|a|^2)}{\bar{u}-\bar{a}}\langle T^{1/2}_{|1-\bar{a}e^{i\theta}|^{-2}}k_{\overline{\varphi_a(u)}},k_v\rangle+\frac{\bar{a}}{\bar{a}-\bar{u}}\langle T^{1/2}_{|1-\bar{a}e^{i\theta}|^{-2}}k_0,k_v\rangle\}.$$

The formula folllows applying Rosenblum's result and the above elementary computations. □

4.1 A result by E. Berkson

We are indebted to Daniel Suárez for pointing us the result below. In [3], E. Berkson proved the following Theorem:

Theorem 4.7. [3] *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be a bounded analytic map, $\tilde{\varphi}$ its boundary function, and $A = \tilde{\varphi}^{-1}(\mathbb{T})$. Suppose that $|A| > 0$ (= normalized Lebesgue measure in $\mathbb{T}$). Let $\psi : \mathbb{D} \to \mathbb{D}$ be another analytic map, and C_φ and C_ψ denote the composition operators on $H^p(\mathbb{D})$, $1 \le p < \infty$. If*

$$\|C_\psi - C_\varphi\| < \left(\frac{|A|}{2}\right)^{1/p},$$

then $\psi = \varphi$.

As a consequence, for $a \neq b \in \mathbb{D}$ we have that ($p = 2$):

$$\|C_a - C_b\| \ge \frac{1}{\sqrt{2}} \tag{14}$$

On the other hand, it is a consequence of Theorem 4.2 that

$$C_a^* C_a - C_b^* C_b = T_{\frac{1-|a|^2}{1-\bar{a}z}} - T_{\frac{1-|b|^2}{1-\bar{b}z}} = T_{\delta_{a,b}},$$

where $\delta_{a,b}(z) = \dfrac{1-|a|^2}{1-\bar{a}z} - \dfrac{1-|b|^2}{1-\bar{b}z}$. Thus

$$\|C_a^* C_a - C_b^* C_b\| = \|\delta_{a,b}\|_\infty = \sup\{|\delta_{a,b}(z)| : z \in \mathbb{T}\}.$$

Remark 4.8. Note that, after an elementary computation, $\delta_{a,b}$ also equals

$$\delta_{a,b}(z) = \bar{a}\varphi_a(z) - \bar{b}\varphi_b(z).$$

So that we have

$$||a| - |b|| \le ||\delta_{a,b}||_\infty \le |a| + |b|.$$

Moreover,

$$|\delta_{a,b}(z)| = \Big|\frac{1-|a|^2}{1-\bar{a}z} - \frac{1-|b|^2}{1-\bar{b}z}\Big| \le \Big|\frac{1-|a|^2}{1-\bar{a}z} - \frac{1-|b|^2}{1-\bar{a}z}\Big| + \Big|\frac{1-|b|^2}{1-\bar{a}z} - \frac{1-|b|^2}{1-\bar{b}z}\Big|$$

$$= \frac{1}{|1-\bar{a}z|}||a|^2 - |b|^2| + (1-|b|^2)\frac{|z||a-b|}{|1-\bar{a}z||1-\bar{b}z|}$$

$$\le \frac{1}{1-|a|}(|a|+|b|)|a-b| + (1+|b|)\frac{|a-b|}{1-|a|} \le \frac{4}{1-|a|}|a-b|.$$

In particular, contrary to what happens to C_b and C_a, if $b \to a$, then both $C_b^* C_b \to C_a^* C_a$ and $|C_b| \to |C_a|$. Therefore, we have the following:

Proposition 4.9. *Fix $a \in \mathbb{D}$ and $r < \frac{1}{\sqrt{2}}$, consider the open neighbourhood $\mathcal{B}_r(a)$ of a in $\mathbb{D}$ given by*

$$\mathcal{B}_r(a) := \{b \in \mathbb{D} : \||C_b| - |C_a|\| < r\}.$$

Then, if $b \in \mathcal{B}_r(a)$, $b \neq a$, we have that

$$\|\rho_b - \rho_a\| \geq \left(\frac{1}{\sqrt{2}} - r\right) \frac{1 + |a|}{\sqrt{1 - |a|^2}}.$$

Proof. By Berkson's Theorem, if $a \neq b$

$$\frac{1}{\sqrt{2}} \leq \|C_a - C_b\| = \|\rho_a|C_a| - \rho_b|C_b|\| \leq \|\rho_a|C_a| - \rho_b|C_a|\| + \|\rho_b|C_a| - \rho_b|C_b|\|$$

$$\leq \||C_a|\|\|\rho_a - \rho_b\| + \||C_a| - |C_b|\|,$$

because ρ_b is a unitary operator. If $b \in \mathcal{B}_r(a)$,

$$\frac{1}{\sqrt{2}} \leq \|C_a\|\|\rho_a - \rho_b\| + r.$$

The proof follows recalling that $\||C_a|\| = \|C_a\| = \frac{\sqrt{1-|a|^2}}{1+|a|}$. □

4.2 Formulas for $\mathbf{r}(C_a)$ and $\mathbf{n}(C_a)$.

Using Theorem 4.2 we can refine the formulas for $\mathbf{r}(T)$ and $\mathbf{n}(T)$ obtained in Proposition 2.1, the range and nullspace symmetries induced by a reflection T, to the case when $T = C_a$:

Corollary 4.10. *We have*

$$\mathbf{r}(C_a) = 2(1 + C_a)T_{\mathbf{g}_a}^{-1} \quad \textit{and} \quad \mathbf{r}(C_a) = 2(1 - C_a)T_{\mathbf{g}_a}^{-1},$$

where $T_{\mathbf{g}_a}$ is the Toeplitz operator with symbol

$$\mathbf{g}_a(z) = 1 + \frac{1 - |a|^2}{|1 - \bar{a}z|^2}.$$

Proof. Note that for $T = C_a$ we have $\mathbf{n}(C_a) = 2(1 + C_a)(|C_a|^2 + 1)^{-1}$, and from Theorem 4.2 we know that

$$|C_a|^2 = (1 - |a|^2)T_{\frac{1}{|1-\bar{a}z|^2}}.$$

Then

$$|C_a|^2 + 1 = (1 - |a|^2)T_{\frac{1}{|1-\bar{a}z|^2}} + 1 = T_{1+\frac{1-|a|^2}{|1-\bar{a}z|^2}} = T_{\mathbf{g}_a}.$$

The computation of $\mathbf{r}(C_a)$ is similar.

□

4.3 A power series expansion for ρ_a

Let us further consider $|1-\bar{a}S|^{-1}$. Note that

$$|1-\bar{a}S|^2 = (1-aS^*)(1-\bar{a}S) = 1+|a|^2 - 2\mathrm{Re}(\bar{a}S),$$

where $\mathrm{Re}T = \frac{1}{2}(T+T^*)$, for $T \in \mathcal{B}(H^2)$, as is usual notation. Then

$$|1-\bar{a}S|^2 = (1+|a|^2)\left(1-\frac{2}{1+|a|^2}\mathrm{Re}(\bar{a}S)\right) = (1+|a|^2)\left(1-\frac{2|a|}{1+|a|^2}T\right),$$

where $a = |a|\omega$ and $T = \mathrm{Re}(\bar{\omega}S)$ is a contraction. Using the power series expansion $(1-kt)^{-1/2} = 1+\sum_{n=1}^{\infty}(2n-1)(2n-3)\dots 1(\frac{k}{2})^n t^n$, we get

Lemma 4.11. *With the current notations, i.e.* $T = \mathrm{Re}(\bar{\omega}S)$*,* $a = |a|\omega$*, we have that*

1.

$$|1-\bar{a}S|^{-1} = \frac{1}{\sqrt{1+|a|^2}}\left(1+\sum_{n=1}^{\infty}(2n-1)(2n-3)\dots 1(\frac{|a|}{1+|a|^2})^n T^n\right),$$

where $T = \frac{1}{2}(\bar{\omega}S + \omega S^*)$ *and* $a = |a|\omega$*.*

2.

$$|C_a|^{-1} = \sqrt{1-|a|^2}W_a\{1+\sum_{n=1}^{\infty}(2n-1)(2n-3)\dots 1(\frac{|a|}{1+|a|^2})^n T^n\}W_a$$

$$= \sqrt{1-|a|^2}\left(1+\sum_{n=1}^{\infty}(2n-1)(2n-3)\dots 1(\frac{|a|}{1+|a|^2})^n (W_aTW_a)^n\right).$$

3.

$$\rho_a = (1-\bar{a}S)\{1+\sum_{n=1}^{\infty}(2n-1)(2n-3)\dots 1(\frac{|a|}{1+|a|^2})^n T^n\}W_a$$

$$= \Big(\mu(1-\bar{a}S)\Big)W_a,$$

where $\mu(A)$ *= unitary part in the polar decomposition of* A*:* $A = \mu(A)|A|$*.*

Proof. Straightforward computations. □

Next we see that the map $\mathbb{D} \ni a \mapsto |C_a|$ is one to one:

Proposition 4.12. *Let* $a, b \in \mathbb{D}$*. Then* $|C_a| = |C_b|$ *if and only if* $|C_a^*| = |C_b^*|$ *if and only if* $a = b$

Proof. Recall that $(C_a^* C_a)^{-1} = C_a C_a^*$, and thus $|C_a|^{-1} = |C_a^*|$. By uniqueness of the positive square root of operators, clearly $|C_a^*| = |C_b^*|$ if and only if $C_a C_a^* = C_b C_b^*$. Next note that at the constant function $1 \in H^2$, we have (since $S^* 1 = 0$)

$$C_a C_a^*(1) = \frac{1}{1-|a|^2}(1-\bar{a}S)(1-aS^*)(1) = \frac{1}{1-|a|^2}(1-\bar{a}S)(1) = \frac{1-\bar{a}z}{1-|a|^2}.$$

Evaluating at $z = 0$, we get that $C_a C_a^* = C_b C_b^*$ implies that $|a| = |b|$, and thus $1 - \bar{a}z = 1 - \bar{b}z$ for all $z \in \mathbb{D}$, i.e., $a = b$. □

Question 4.13. Proposition 4.9 states that given $a \in \mathbb{D}$, there is an open neighbourhood of a such that for b in this neighbourhood, $\rho_a = \rho_b$ implies $a = b$. We do not now though if globally the map $\mathbb{D} \ni a \mapsto \rho_a \in \mathcal{B}(H^2)$ is injective.

5 The eigenspaces of C_a

Denote by $\mathcal{E}$ and $\mathcal{O}$ the closed subspaces of even and odd functions in H^2. Note that they are, respectively, $\mathcal{E} = N(C_0 - 1)$ and $\mathcal{O} = N(C_0 + 1)$. For general $a \in \mathbb{D}$, the eigenspaces of C_a are

$$N(C_a - 1) = \{f \in H^2 : f \circ \varphi_a = f\} \quad \text{and} \quad N(C_a + 1) = \{g \in H^2 : g \circ \varphi_a = g\}.$$

For $a \in \mathbb{D}$, denote by ω_a the fixed point of φ_a inside $\mathbb{D}$. Explicitely:

$$\omega_a = \frac{1}{\bar{a}}\{1 - \sqrt{1-|a|^2}\}. \tag{15}$$

Elementary computations shows that

$$\varphi_{\omega_a} \circ \varphi_a = -\varphi_{\omega_a} \tag{16}$$

which at $z = 0$ gives

$$\varphi_{\omega_a}(a) = -\omega_a. \tag{17}$$

Theorem 5.1. *For $a \in \mathbb{D}$, the eigenspaces of C_a are*

$$N(C_a - 1) = \{f = \sum_{n=0}^{\infty} \alpha_n (\varphi_{\omega_a})^{2n} : (\alpha_n) \in \ell^2\} = C_{\omega_a}(\mathcal{E}), \tag{18}$$

and

$$N(C_a + 1) = \{g = \sum_{n=0}^{\infty} \alpha_n (\varphi_{\omega_a})^{2n+1} : (\alpha_n) \in \ell^2\} = C_{\omega_a}(\mathcal{O}). \tag{19}$$

Proof. It follows from (16) that the **even** powers of φ_{ω_a} belong to $N(C_a - 1)$:

$$(\varphi_{\omega_a})^{2n} \circ \varphi_a = (\varphi_{\omega_a})^{2n},$$

and the **odd** powers belong to $N(C_a + 1)$:

$$(\varphi_{\omega_a})^{2n+1} \circ \varphi_a = -(\varphi_{\omega_a})^{2n+1}.$$

Therefore, any sequence of coefficients $(\alpha_n) \in \ell^2$ gives an element

$$f = \sum_{n=0}^{\infty} \alpha_n (\varphi_{\omega_a})^{2n} \in N(C_a - 1),$$

and an element

$$g = \sum_{n=0}^{\infty} \alpha_n (\varphi_{\omega_a})^{2n+1} \in N(C_a + 1).$$

Conversely, suppose that $f \in N(C_a - 1)$. Using (17)

$$f \circ \varphi_{\omega_a} = f \circ \varphi_a \circ \varphi_{\omega_a},$$

and since $\varphi_a \circ \varphi_{\omega_a} = \frac{a\bar{\omega}_a - 1}{1 - \bar{a}\omega_a} \varphi_{\varphi_{\omega_a}(a)} = -\varphi_{-\omega_a}$, we get

$$f \circ \varphi_{\omega_a}(z) = f \circ \varphi_{\omega_a}(-z),$$

i.e., $f \circ \varphi_{\omega_a} \in \mathcal{E}$. The fact for odd functions is similar. □

Note that if we denote $h(z) = \sum_{n=0}^{\infty} \alpha_n z^{2n}$, which is an arbitrary even function in H^2, we have that $f = h \circ \varphi_{\omega_a} = C_{\omega_a} h$. And similarly if $k(z) = \sum_{n=0}^{\infty} \alpha_n z^{2n+1}$ is an arbitrary odd function in H^2, $g = C_{\omega_a} k$. Then

$$C_{\omega_a}\big|_{\mathcal{E}} : \mathcal{E} \to N(C_a - 1) \quad \text{and} \quad C_{\omega_a}\big|_{\mathcal{O}} : \mathcal{O} \to N(C_a + 1).$$

Theorem 5.2. *The restrictions $C_{\omega_a}\big|_{\mathcal{E}}$ and $C_{\omega_a}\big|_{\mathcal{O}}$ are bounded linear isomorphisms. Their inverses are, respectively, $C_{\omega_a}\big|_{N(C_a - 1)}$ and $C_{\omega_a}\big|_{N(C_a + 1)}$.*

Proof. Note that

$$H^2 = C_{\omega_a}(\mathcal{E} \oplus \mathcal{O}) = C_{\omega_a}(\mathcal{E}) \dot{+} C_{\omega_a}(\mathcal{O}) \subset N(C_a - 1) \dot{+} N(C_a + 1),$$

were $\dot{+}$ denotes direct (non necessarily orthogonal) sum. It follows that $C_{\omega_a}(\mathcal{E}) = N(C_a - 1)$ and $C_{\omega_a}(\mathcal{O}) = N(C_a + 1)$. This completes the proof, since C_a is its own inverse. □

Remark 5.3. Clearly, if $p, g \in H^2$ are, respectively, inner and outer functions, then $C_a p = p \circ \varphi_a$ and $C_a g = g \circ \varphi_a$ are also, respectively, inner and outer. Therefore, if $f \in N(C_a - 1)$, and $f = pg$ is the inner/outer factorization of f, then $f = C_a p \cdot C_a g$ is another inner/outer factorization. By uniqueness, it must be $C_a p = \omega p$ for some $\omega \in \mathbb{T}$. But then p is an eigenfunction of C_a, and so it must be $\omega = \pm 1$. Therefore, if $f \in N(C_a - 1)$, then either **a)** $p, g \in N(C_a - 1)$ or **b)** $p, g \in N(C_a + 1)$. The latter case cannot happen: the outer function g verifies that $C_{\omega_a} g$ is odd, and therefore it vanishes at $z = 0$,

$$0 = C_{\omega_a} g(0) = g(\omega_a).$$

A similar consideration can be done for $N(C_a + 1)$. If $f = pg$ is the inner/outer factorization of $f \in N(C_a + 1)$, then again $C_a p = \pm p$. If $C_a p = p$, then

$$-f = -pg = f \circ \varphi_a = (p \circ \varphi_a)(g \circ \varphi_a)$$

implies $p \circ \varphi_a = \pm p$. If $p \circ \varphi_a = p$, then $g \circ \varphi_a = -g$, and therefore the outer function g vanishes, a contradiction. Thus $p \in N(C_a + 1)$ and $g \in N(C_a - 1)$.

Let us examine the position of the subspaces $N(C_a \pm 1)$ and their orthogonal complements.

Theorem 5.4. *Let $a \neq b$ in $\mathbb{D}$. Then*

1.
$$N(C_a - 1) \cap N(C_b - 1) = \mathbb{C}1,$$
where $1 \in H^2$ is the constant function.

2.
$$N(C_a + 1) \cap N(C_b + 1) = \{0\}.$$

Proof. Let us first prove 1. As seen above, the reflection C_{ω_a} carries $N(C_a - 1)$ onto the space $\mathcal{E}$ of even functions. Another way of putting this, is that

$$C_{\omega_a} C_a C_{\omega_a} = C_0.$$

Note that since $C_b^{-1} = C_b$, this product is in fact a conjugation. A straigtforward computation shows that in general, for $b, d \in \mathbb{D}$

$$\varphi_d \circ \varphi_b \circ \varphi_d = \varphi_{d \bullet b}, \text{ where } \; d \bullet b := \frac{2d - b - \bar{b} d^2}{1 + |d|^2 - \bar{b} d - b \bar{d}}. \tag{20}$$

Note that

$$C_{\omega_a} \left(N(C_b - 1) \right) = N(C_{\omega_a} C_b C_{\omega_a} - 1) = N(C_{\omega \bullet b} - 1).$$

Then $N(C_a - 1) \cap N(C_b - 1) = \mathbb{C}1$ if and only if

$$\mathbb{C}1 = C_{\omega_a}(N(C_a - 1) \cap N(C_b - 1)) = \mathcal{E} \cap N(C_{\omega_a \bullet b} - 1),$$

i.e., we have reduced to the case when one of the to points is the origin. Let us prove that $N(C_a - 1) \cap \mathcal{E} = \mathbb{C}1$. Let $f \in N(C_a - 1) \cap \mathcal{E}$, for $a \in \mathbb{D}$, $a \neq 0$. Then

$$f = f \circ \varphi_0 = f \circ \varphi_a.$$

In particular, this implies that

$$f = f \circ \varphi_a \circ \varphi_0 \circ \ldots \circ \varphi_a = f \circ (\varphi_a \circ \varphi_0)^{(n)} \circ \varphi_a,$$

for all $n \geq 1$ (here $(\varphi_a \circ \varphi_0)^{(n)}$ denotes the composition of $\varphi_a \circ \varphi_0$ with itself n times). We shall need the following computation:

Lemma 5.5.

$$(\varphi_a \circ \varphi_0)^{(n)} \varphi_a = \varphi_{a_n},$$

where

$$a_n = \frac{a}{|a|} \frac{1 - \left(\frac{1-|a|}{1+|a|}\right)^{n+1}}{1 + \left(\frac{1-|a|}{1+|a|}\right)^{n+1}}.$$

Proof. Our claim is equivalent to

$$a_n = \frac{a}{|a|} \frac{(1 + |a|)^{n+1} - (1 - |a|)^{n+1}}{(1 + |a|)^{n+1} + (1 - |a|)^{n+1}}.$$

The proof is by induction in n. It is an elementary computation. For $n = 1$, we have that

$$\varphi_a \circ \varphi_0 \circ \varphi_a(z) = \varphi_a(-\frac{a - z}{1 - \bar{a}z}) = \frac{a + \frac{a-z}{1-\bar{a}z}}{1 + \bar{a}\frac{a-z}{1-\bar{a}z}} = \frac{2a - (1 + |a|^2)z}{1 + |a|^2 - 2az}$$

$$= \frac{\frac{2a}{1+|a|^2} - z}{1 - \frac{2\bar{a}}{1+|a|^2}z} = \varphi_{\frac{2a}{1+|a|^1}}(z).$$

On the other hand,

$$a_1 = \frac{a}{|a|} \frac{(1 + |a|)^2 - (1 + |a|)^2}{(1 + |a|)^2 + (1 - |a|)^2} = \frac{2a}{1 + |a|^2}.$$

Suppose the formula valid for n. Then

$$(\varphi \circ \varphi_0)^{n+1} \circ \varphi_a(z) = (\varphi \circ \varphi_0)^n \circ \varphi_a \circ (\varphi_a \circ \varphi_0)(z) = \varphi_{a_n}(-\frac{a - z}{1 - \bar{a}z})$$

$$= \frac{a_n + \frac{a-z}{1-\bar{a}z}}{1 + \bar{a}_n \frac{a-z}{1-\bar{a}z}} = \frac{a(\frac{\mathbf{f}_n}{|a|} + 1) - (|a|\mathbf{f}_n + 1)z}{|a|\mathbf{f}_n + 1 - \bar{a}(\frac{\mathbf{f}_n}{|a|} + 1)z} = \frac{\beta_n - z}{1 - \bar{\beta}_n z} = \varphi_{\beta_n}(z),$$

where

$$\beta_n = a \, \frac{(\frac{\mathbf{f}_n}{|a|} + 1)}{|a|\mathbf{f}_n + 1} \quad \text{and} \quad \mathbf{f}_n = \frac{(1+|a|)^{n+1} - (1-|a|)^{n+1}}{(1+|a|)^{n+1} + (1-|a|)^{n+1}}.$$

Thus, we have to show that $\beta_n = a_n$. Note that

$$\beta_n = \frac{a}{|a|} \, \frac{\mathbf{f}_n + |a|}{|a|\mathbf{f}_n + 1}$$

and that

$$\frac{\mathbf{f}_n + |a|}{|a|\mathbf{f}_n + 1} = \frac{(1+|a|)^{n+2} - (1-|a|)^{n+2}}{(1+|a|)^{n+2} + (1-|a|)^{n+2}},$$

which completes the proof of the lemma. □

Returning to the proof of the theorem, it is clear that constant functions belong to $\mathcal{E} \cap N(C_a - 1)$. Suppose that there is a non constant $f \in \mathcal{E} \cap N(C_a - 1)$. Then $f_0 = f - f(0) \in \mathcal{E} \cap N(C_a - 1)$ as well. As remarked above, $f_0 = f_0 \circ \varphi_{a_n}$ for all $n \geq 0$ (for $n = 0$, $a_0 = a$). It follows that 0 and a_n, $n \geq 0$ are zeros of f_0. Since f_0 is also even, also $-a_n$, ≥ 0 occur as zeros of f_0. Consider $f_0 = BSg$ the factorization of f_0 with B a Blaschke product, S singular inner and g outer. Then the pairs of factors

$$\varphi_{a_n} \cdot \varphi_{-a_n}$$

appear in the expression of B. Since $f_0 = f_0 \circ \varphi_a$, and $S \circ \varphi_a$ and $g \circ \varphi_a$ are non vanishing in $\mathbb{D}$, it follows that

$$(\varphi_{a_n} \circ \varphi_a) \cdot (\varphi_{-a_n} \circ \varphi_a)$$

must also appear in the expression of B. Note that

$$\varphi_{a_n} \circ \varphi_a = (\varphi_a \circ \varphi_0)^{(n)} \circ \varphi_a \circ \varphi_a = ((\varphi_a \circ \varphi_0)^{(n)} = (\varphi_a \circ \varphi_0)^{(n-1)} \circ \varphi_a \circ \varphi_0$$

$$= \varphi_{a_{n-1}} \circ \varphi_0.$$

Also

$$\varphi_{-a_n}(z) = -\frac{a_n + z}{1 + \bar{a}_n z} = -\varphi_{a_n}(-z) = \varphi_0 \circ \varphi_{a_n} \circ \varphi_0 = \varphi_0 \circ (\varphi_a \circ \varphi_0)^{(n)} \circ \varphi_a \circ \varphi_0$$

$$= \varphi_0 \circ (\varphi_a \circ \varphi_0)^{(n+1)}.$$

Then

$$\varphi_{-a_n} \circ \varphi_a = \varphi_0 \circ (\varphi_a \circ \varphi_0)^{(n+1)} \circ \varphi_a = \varphi_0 \circ \varphi_{a_{n+1}}.$$

Note the effect of C_a on the following pairs of factors of B:

$$z \cdot z = z^2 = \varphi_0^2 \xrightarrow{C_a} (\varphi_0 \circ \varphi_a)^2 = \varphi_a^2,$$

$$\varphi_a \cdot \varphi_{-a} \xrightarrow{C_a} (\varphi_a \circ \varphi_a) \cdot (\varphi_{-a} \circ \varphi_a) = z \cdot (-\varphi_{a_1}) = -z\varphi_{a_1},$$

and

$$\varphi_{a_1} \cdot \varphi_{a_{-1}} \xrightarrow{C_a} (\varphi_{a_1} \circ \varphi_a) \cdot (\varphi_{a_{-1}} \circ \varphi_a) = (\varphi_a \circ \varphi_0) \cdot \varphi_{a_2} = -\varphi_{-a} \cdot \varphi_{a_2}.$$

Other pairs of factors in the expression of B, after applying C_a, do not involve φ_a or φ_0, due to the spreading of the indices. Summarizing, after applying C_a, we get the products

$$(\varphi_a)^2, \ -z\varphi_{a_1} \text{ and } -\varphi_{-a} \cdot \varphi_{a_2},$$

i.e., we do not recover the original factors z^2 and $\varphi_a \cdot \varphi_{-a}$. It follows that f is contant.

To prove 2., a similar trick as above allows us to reduce to the case of $a \neq 0$ and $b = 0$, i.e., we must prove that there are no nontrivial odd functions in $N(C_a + 1)$. Let $f \in H^2$ be odd such that $f \circ \varphi_a = -f$. Then $f^2 = f \cdot f$ is even and $(f(\varphi_a(z)))^2 = (-f(z))^2 = (f(z))^2$, i.e., $f^2 \in N(C_a - 1)$. Therefore, by the previous case, f^2 is constant, and therefore $f \equiv 0$. $\square$

Corollary 5.6. *The maps $\mathbb{D} \to \mathcal{P}$ given by*

$$a \mapsto \mathbf{r}(C_a) \quad \textit{and} \quad a \mapsto \mathbf{n}(C_a)$$

re one to one.

Let us further proceed with the study of the position of the subspaces $N(C_a \pm 1)$ and $N(C_b \pm 1)$ for $a \neq b$, considering now their orthogonal complements. We shall restrict to the case $b = 0$. The conditions look similar, but as far as we could figure it out, some of the proofs may be quite different.

Theorem 5.7. *Let $a \in \mathbb{D}$, $a \neq 0$.*

1.

$$N(C_0 - 1)^\perp \cap N(C_a - 1) = \{0\} = N(C_0 - 1) \cap N(C_a - 1)^\perp,$$

2.

$$N(C_0 + 1)^\perp \cap N(C_a + 1) = \{0\} = N(C_0 + 1) \cap N(C_a + 1)^\perp,$$

3.

$$N(C_0 - 1)^\perp \cap N(C_a - 1)^\perp = \{0\} = N(C_0 + 1)^\perp \cap N(C_a + 1)^\perp.$$

Proof. Assertion 1.: for the left hand equality, let $f \in N(C_0 - 1)^\perp = \mathcal{O}$ such that $f \circ \varphi_a = f$. Then, by the above results, $f^2 \in \mathcal{E} \cap N(C_a - 1)$, and therefore f^2 is constant. Then f, being constant and odd, is zero.

The right hand equality: suppose $f \in N(C_0 - 1) \cap N(C_a - 1)^\perp$ is $\neq 0$, i.e., f is even and $\langle f, C_{\omega_a}(z^{2k})\rangle = 0$ for $k \geq 0$ (in particular, when $n = 0$ we get $f(0) = 0$). Thus $C^*_{\omega_a}(f)$ is odd. Recall that

$$C^*_{\omega_a}(f) = \frac{1}{1-\bar{\omega}_a z} f(\frac{\omega_a - z}{1-\bar{\omega}_a z}) - \frac{\omega_a}{1-\bar{\omega}_a z} \frac{f(\frac{\omega_a - z}{1-\bar{\omega}_a z}) - f(0)}{\frac{\omega_a - z}{1-\bar{\omega}_a z}}$$

$$= \frac{1}{1-\bar{\omega}_a z} \left(f(\frac{\omega_a - z}{1-\bar{\omega}_a z}) - \omega_a \frac{f(\frac{\omega_a - z}{1-\bar{\omega}_a z})}{\frac{\omega_a - z}{1-\bar{\omega}_a z}} \right).$$

Since f is even with $f(0) = 0$, put $f(z) = \sum_{n=1}^\infty \alpha_n z^{2n}$. Then, after routine computations we get

$$C^*_{\omega_a}(f) = z \frac{|\omega_a|^2 - 1}{(1-\bar{\omega}_a z)^2} \sum_{n=1}^\infty \alpha_n (\frac{\omega_a - z}{1-\bar{\omega}_a z})^{2n-1}.$$

The fact that $C^*_{\omega_a}(f)$ is odd, implies that

$$A(z) = \frac{1}{(1-\bar{\omega}_a z)^2} \sum_{n=1}^\infty \alpha_n (\frac{\omega_a - z}{1-\bar{\omega}_a z})^{2n-1}$$

is even. Note that therefore

$$C_{\omega_a}(A) = \frac{(1-\bar{\omega}_a z)^2}{(1-|\omega_a|^2)^2} \sum_{n=1}^\infty \alpha_n z^{2n-1} \in N(C_a - 1).$$

Let us abreviate $\alpha(z) = \sum_{n=1}^\infty \alpha_n z^{2n-1}$, which is an odd function. Thus

$$(1-\bar{\omega}_a z)^2 \alpha \in N(C_a - 1).$$

Note that $(1-\bar{\omega}_a z)^2$ is outer. Therefore, if $\alpha = pg$ is the inner/outer factorization of α, then

$$(1-\bar{\omega}_a z)^2 \alpha = p\left((1-\bar{\omega}_a z)^2 g\right)$$

is also an inner/outer factorization. Then, by Remark 5.3, we have $p \in N(C_a - 1)$. By a similar argument, since α is odd it follows that p is either odd or even. Note that p even would imply g odd, and thus vanishing at $z = 0$, which cannot happen. Thus $p \in N(C_a - 1) \cap \mathcal{O} = \{0\}$, which is the first assertion of this theorem. Clearly this implies that $f = 0$.

Assertion 2.: the proof of the second assertion is similar. Let us sketch it underlining the differences. The left hand equality: suppose that f is odd and $f \perp N(C_a + 1)$. Then $f = C_{\omega_a}\iota = \iota(\varphi_{\omega_a})$ for some odd function ι. Then $f^2 = \iota^2(\varphi_{\omega_a}) \in N(C_a - 1)$. Then, by the first part of Theorem 5.4, we have that f^2 is constant, then f is constant, and the fact that $f = \iota(\varphi_{\omega_a})$ with ι odd implies that $f = 0$.

The right hand equality of the second assertion, if $f \in N(C_0 + 1) \cap N(C_a + 1)^{\perp}$, then f is odd, $f(z) = \sum_{k\geq 0} \beta_k z^{2k+1}$ and $C^*_{\omega_a} f$ is even. Similarly as above,

$$C^*_{\omega_a} f(z) = z\frac{|\omega_a|^2 - 1}{(1 - \bar{\omega}_a z)^2} \sum_{k\geq 0} \beta_k \left(\frac{\omega_a - z}{1 - \bar{\omega}_a z}\right)^{2k},$$

and thus $B(z) = \frac{1}{(1-\bar{\omega}_a z)^2} \sum_{k\geq 0} \beta_k \left(\frac{\omega_a - z}{1-\bar{\omega}_a z}\right)^{2k}$ is odd. Therefore, if $\beta(z) := \sum_{k\geq 0} \beta_k z^{2k}$, we have

$$C_{\omega_a}(B) = \frac{(1 - \bar{\omega}_a z)^2}{(1 - \bar{\omega}_a z)^2}\beta \in N(C_a + 1), \ i.e., \ (1 - \bar{\omega}_a z)^2\beta \in N(C_a + 1).$$

If $\beta = qh$ is the inner/outer factorization, then q and h are even, and

$$(1 - \bar{\omega}_a z)^2\beta = q\left((1 - \bar{\omega}_a z)^2 h\right)$$

is the inner/outer factorization of an element in $N(C_a + 1)$. Then, again by Remark 5.3, $q \in N(C_a + 1)$. Then q^2 is even and lies in $N(C_a - 1)$, and therefore is constant, by the first part of Theorem 5.4. Thus q is constant in $N(C_a + 1)$, which implies that $q = 0$, and then $f = 0$.

Assertion 3.: For the left hand equality: $f \in N(C_0 - 1)^{\perp} \cap N(C_a - 1)^{\perp}$ is odd, $f(z) = \sum_{n\geq 0} \beta_n z^{2n+1}$, and similarly as above,

$$C^*_{\omega_a} f(z) = \frac{z(\bar{\omega}_a - 1)}{(1 - \bar{\omega}_a z)^2} \sum_{n\geq 0} \beta_n \left(\frac{\omega_a - z}{1 - \bar{\omega}_a z}\right)^{2n} \quad \text{is odd,}$$

so that $D(z) = \frac{1}{(1-\bar{\omega}_a z)^2} \sum_{n\geq 0} \beta_n \left(\frac{\omega_a - z}{1-\bar{\omega}_a z}\right)^{2n}$ is even, and

$$C_{\omega_a} D = \frac{(1 - \bar{\omega}_a z)^2}{(1 - |\omega^2_|)^2} \sum_{n\geq 0} \beta_n z^{2n} \in N(C_a - 1).$$

Denote $\delta(z) = \sum_{n\geq 0} \beta_n z^{2n}$, so that $(1 - \bar{\omega}_a z)^2\delta \in N(C_a - 1)$.

Note that $f(z) = z\delta(z)$ Then we have

$$(1 - \bar{\omega}_a z)^2\delta = (1 + (\bar{\omega}_a z)^2)\delta - 2\bar{\omega}_a f$$

is an orthogonal sum: the left hand term is even and the right hand term is odd. One the other hand, rewriting this equality, we have

$$(1 + (\bar{\omega}_a z)^2)\delta = (1 - \bar{\omega}_a z)^2\delta + 2\bar{\omega}_a f$$

is also and orthogonal sum: the left hand term belongs to $N(C_a - 1)$ and the right hand term is orthogonal to $N(C_a - 1)$. Then we have

$$\|(1-\bar{\omega}_a z)^2\delta\|^2 = \|(1+\bar{\omega}_a z)^2)\delta\|^2 + \|2\bar{\omega}_a f\|^2$$

and

$$\|(1+(\bar{\omega}_a z)^2)\delta\|^2 = \|(1-\bar{\omega}_a z)^2\delta\|^2 + \|2\bar{\omega}_a f\|^2.$$

These imply that $f = 0$.

The right hand equality: let $f \in N(C_a - 1)^\perp$ be even, and suppose first that $f(0) = 0$. Then $f(z) = \sum_{n\geq 1}\alpha_n z^{2n}$. We proceed similarly as in the third assertion, we sketch the proof. We know that

$$C^*_{\omega_a}(f)(z) = \frac{z(\bar{\omega}_a^2 - 1)}{(1-\bar{\omega}_a z)^2}\sum_{n\geq 1}\alpha_n(\frac{\omega_a - z}{1-\bar{\omega}_a z})^{2n-1}$$

is even, so that $E(z) = \frac{1}{(1-\bar{\omega}_a z)^2}\sum_{n\geq 1}\alpha_n(\frac{\omega_a - z}{1-\bar{\omega}_a z})^{2n-1}$ is odd. Then

$$h(z) := C_{\omega_a}(E)(z) = \frac{(1-\bar{\omega}_a z)^2}{1-|\omega_a|^2}\sum_{n\geq 1}\alpha_n z^{2n-1} \in N(C_a + 1).$$

Note that $\sum_{n\geq 1}\alpha_n z^{2n-1} = \frac{f(z)}{z}$. Then we have on one hand that

$$(1-|\omega_a|^2)h(z) = (1+(\bar{\omega}_a z)^2)\sum_{n\geq 1}\alpha_n z^{2n-1} + 2\bar{\omega}_a f(z)$$

is an orthogonal sum, the left hand summand is odd and the right hand summand is even. Thus

$$\|(1-|\omega_a|^2)h\|^2 = \|(1+(\bar{\omega}_a z)^2)\sum_{n\geq 1}\alpha_n z^{2n-1}\|^2 + \|2\bar{\omega}_a f\|^2.$$

On the other hand. the above also means that

$$(1+(\bar{\omega}_a z)^2)\sum_{n\geq 1}\alpha_n z^{2n-1} = (1-|\omega_a|^2)h(z) + 2\bar{\omega}_a f(z)$$

is also an orthogonal sum, the left hanf summand belongs to $N(C_a + 1)$ and the right hand summand belongs to $N(C_a + 1)^\perp$. Then

$$\|(1+(\bar{\omega}_a z)^2)\sum_{n\geq 1}\alpha_n z^{2n-1}\|^2 = \|(1-|\omega_a|^2)h\|^2 + \|2\bar{\omega}_a f\|^2.$$

These two norm identities imply that $f = 0$. Suppose now that $f(0) \neq 0$, by considering a multiple of f, we may assume $f(0) = 1$, i.e., $f(z) = 1 + \sum_{n\geq 1}\alpha_n z^{2b}$. Then

$$g(z) := C^*_{\omega_a}f(z) = \frac{1}{1-\bar{\omega}_a z} + (\bar{\omega}_a - 1)\frac{z}{1-\bar{\omega}_a z}\sum_{n\geq 1}\alpha_n\left(\frac{\omega_a - z}{1-\bar{\omega}_a z}\right)^{2n-1},$$

which is also even. Then $g'(z)$ is odd and $g'(0) = 0$. Note that

$$g'(z) = \frac{\bar{\omega}_a}{(1-\bar{\omega}_a z)^2} + \frac{\bar{\omega}_a - 1}{(1-\bar{\omega}_a z)^2} \sum_{n\geq 1} \alpha_n \left(\frac{\omega_a - z}{1-\bar{\omega}_a z}\right)^{2n-1} +$$

$$+(\bar{\omega}_a - 1)(|\omega_a|^2 - 1)\frac{z}{(1-\bar{\omega}_a z)^3} \sum_{n\geq 1} \alpha_n \left(\frac{\omega_a - z}{1-\bar{\omega}_a z}\right)^{2n-2},$$

so that

$$0 = g'(0) = \bar{\omega}_a + (\bar{\omega}_a - 1)\sum_{n\geq 1} \alpha_n \omega_a^{2n-1}.$$

Note that $f(\omega_a) = 1 + \sum_{n\geq 1} \alpha_n \omega_a^{2n} = 1 + \omega_a \sum_{n\geq 1} \alpha_n \omega_a^{2n-1}$, i.e.,

$$0 = \bar{\omega}_a + (\bar{\omega}_a - 1)\left(\frac{f(\omega_a) - 1}{\omega_a}\right),$$

or $f(\omega_a) = \dfrac{|\omega_a|^2}{1-\bar{\omega}_a} + 1$. Since f is even, $f(\omega_a) = f(-\omega_a)$, i.e., $\frac{1}{1-\bar{\omega}_a} = \frac{1}{1+\bar{\omega}_a}$, or $\omega_a = 0$ (which cannot happen because $a \neq 0$). It follows that $f \equiv 0$. □

Question 5.8. A natural question is wether these properties above hold for arbitrary $a \neq b \in \mathbb{D}$.

Remark 5.9. A straightforward computation shows that if $a \in \mathbb{D}$, the unique $b \in \mathbb{D}$ such that the fixed point ω_b of φ_b (in $\mathbb{D}$) is a is given by $b = \frac{2a}{1+|a|^2}$. Let us denote this element by Ω_a. One may iterate this computation: denote by $\Omega_a^2 := \Omega_{\Omega_a}$, and in general $\Omega_a^{n+1} := \Omega_{\Omega_a^n}$. Then it is easy to see that

$$\Omega_a^n = a_{2^n - 1},$$

where $a_k \in \mathbb{D}$ are the numbers obtained in Lemma 5.5. Note that all these iterations Ω_a^n are multiples of a, with increasing moduli, and $\Omega_a^n \to \frac{a}{|a|}$ as $n \to \infty$.

Moreover, it is easy to see that the sequence a_n is an interpolating sequence: it consists of multiples of $\frac{1-r^{n+1}}{1+r^{n+1}}$ by the number $\frac{a}{|a|}$ of modulus one, with $r < 1$. Therefore Ω_a^n is an interpolating sequence.

6 Geodesics between Eigenspaces of C_a

Recall from the introduction the condition for the existence of a geodesic of the Grassmann manifold of H^2 that joins two given subspaces $\mathcal{S}$ and $\mathcal{T}$, namely, that

$$\dim(\mathcal{S} \cap \mathcal{T}^{\perp}) = \dim(\mathcal{T} \cap \mathcal{S}^{\perp}).$$

This condition clearly holds for $\mathcal{E} = N(C_0 - 1)$ and $\mathcal{O} = N(C_0 + 1) = \mathcal{E}^\perp$: both intersections are, respectively, $\mathcal{E} \cap \mathcal{O}^\perp = \mathcal{E}$ and $\mathcal{O} \cap \mathcal{E}^\perp = \mathcal{O}$, and have the same (infinite) dimension. Our first observation is that this no longer holds for $N(C_a - 1)$ and $N(C_a + 1)$ when $a \neq 0$:

Proposition 6.1. *If $0 \neq a \in \mathbb{D}$, then there does not exist a geodesic of the Grassmann manifold of H^2 joining $N(C_a - 1)$ and $N(C_a + 1)$.*

Proof. The proof follows by direct computation. First, we claim that

$$N(C_a + 1) \cap N(C_a - 1)^\perp = \{0\}. \tag{21}$$

Note that $f \in N(C_a - 1)^\perp$ if and only if $\langle f, g\rangle = 0$ for all $g \in N(C_a - 1) = C_{\omega_a}(\mathcal{E})$, i.e.,

$$0 = \langle C_{\omega_a}^* f, g\rangle,$$

for all $g \in \mathcal{E}$. This is equivalent to $C_{\omega_a}^* f \in \mathcal{O}$, or also that $f \in C_{\omega_a}^*(\mathcal{O})$.

Using the operator C_{ω_a}, our claim (21) is equivalent to

$$\{0\} = C_{\omega_a}(N(C_a + 1)) \cap C_{\omega_a} C_{\omega_a}^*(\mathcal{O}) = \mathcal{O} \cap C_{\omega_a} C_{\omega_a}^*(\mathcal{O}),$$

where the last equality follows from the fact $C_{\omega_a}(N(C_a + 1)) = \mathcal{O}$ observed before. Let $f \in \mathcal{O}$. Then (since $f(0) = 0$)

$$g(z) = C_{\omega_a} C_{\omega_a}^* f(z) = \frac{1 - \bar{\omega}_a z}{1 - |\omega_a|^2} \left(f(z) - \omega_a \frac{f(z)}{z} \right)$$

$$= \frac{1}{1 - |\omega_a|^2} \left(f(z)(1 + |\omega_a|^2) - \left(\omega_a \frac{f(z)}{z} + \bar{\omega}_a z f(z) \right) \right).$$

Then, since g and the first summand are odd, and the second summand is even, the second summand is zero, which implies that $f \equiv 0$.

On the other hand, a similar computation shows that

$$\dim \left(N(C_a - 1) \cap N(C_a + 1)^\perp \right) = 1,$$

which would conclude the proof. Indeed, by a similar argument as above, it suffices to show that

$$\dim(\mathcal{E} \cap C_{\omega_a} C_{\omega_a}^*(\mathcal{E})) = 1.$$

Let g, f be even functions such that

$$g(z) = C_{\omega_a} C_{\omega_a}^* f(z) = \frac{1 - \bar{\omega}_a z}{1 - |\omega_a|^2} \left(f(z) - \omega_a \frac{f(z) - f(0)}{z} \right)$$

$$= \frac{1}{1 - |\omega_a|^2} \left(\left(f(z) + |\omega_a|^2 (f(z) - f(0)) \right) - \left(\bar{\omega}_a f(z) z + \omega_a \frac{f(z) - f(0)}{z} \right) \right).$$

It follows that

$$\bar{\omega}_a f(z)z + \omega_a \frac{f(z)-f(0)}{z} \equiv 0,$$

i.e., $f(z) = \dfrac{c}{\omega_a + \bar{\omega}_a z^2}$. This implies that

$$\mathcal{E} \cap C_{\omega_a} C^*_{\omega_a}(\mathcal{E}) = \langle \frac{1}{\omega_a + \bar{\omega}_a z^2} \rangle.$$

□

Note though that the orthogonal projections onto $N(C_a - 1)$ and $N(C_a + 1)$ are unitarilly equivalent: both subspaces are infinite dimensional and infinite co-dimensional.

Also on the negative side, the subspaces $\mathcal{O}$ and $N(C_a - 1)$, for $a \neq 0$, cannot be joined by a geodesic:

Corollary 6.2. *There exist no geodesics of the Grassmann manifold of H^2 joining $N(C_0 + 1)$ and $N(C_a + 1)$, for $a \neq 0$.*

Proof. Note that, by Theorem 5.4, part 1, for $b = 0$:

$$N(C_0 + 1)^\perp \cap N(C_a - 1) = N(C_0 - 1) \cap N(C_a - 1) = \mathbb{C}1;$$

whereas by Theorem 5.7, Assertion 3, left hand identity, we have that

$$N(C_0 + 1) \cap N(C_a - 1)^\perp = N(C_0 - 1)^\perp \cap N(C_a - 1)^\perp = \{0\}.$$

□

On the affirmative side, a direct consequence of the results in the previous section is the existence of unique normalized geodesics of the Grassmann manifold joining $\mathcal{E} = N(C_0 - 1)$ with $N(C_a - 1)$, $\mathcal{O} = N(C_0 + 1)$ with $N(C_a + 1)$, and $\mathcal{E}$ with $N(C_a + 1)$:

Corollary 6.3. *Let $a \in \mathbb{D}$, $a \neq 0$.*

1. *There exists a unique (geodesic) curve $\delta^-_{0,a}(t) = e^{tZ^-_{0,a}}\mathcal{E}$ of the Grassmann manifold of H^2, with $(Z^-_{0,a})^* = -Z^-_{0,a}$, $Z^-_{0,a}\mathcal{E} \subset \mathcal{O}$ and $\|Z^-_{0,a}\| \leq \pi/2$, such that*

$$e^{Z^-_{0,a}}\mathcal{E} = N(C_a - 1).$$

2. *There exists a unique (geodesic) curve $\delta^+_{0,a}(t) = e^{tZ^+_{0,a}}\mathcal{E}$ of the Grassmann manifold of H^2, with $(Z^+_{0,a})^* = -Z^+_{0,a}$, $Z^+_{0,a}\mathcal{E} \subset \mathcal{O}$ and $\|Z^+_{0,a}\| \leq \pi/2$, such that*

$$e^{Z^+_{0,a}}\mathcal{O} = N(C_a + 1).$$

3. *There exists a unique (geodesic) curve* $\delta_{0,a}^{+,-}(t) = e^{tZ_{0,a}^{+,-}}\mathcal{E}$ *of the Grassmann manifold of* H^2, *with* $(Z_{0,a}^{+,-})^* = -Z_{0,a}^{+,-}$, $Z_{0,a}^{+,-}\mathcal{O} \subset \mathcal{E}$ *and* $\|Z_{0,a}^{+,-}\| \le \pi/2$, *such that*

$$e^{Z_{0,a}^{+,-}}\mathcal{O} = N(C_a - 1).$$

Proof. 1. Follows from assertion 1 in Theorem 5.7.

2. Follows from assertion 2 in Theorem 5.7.

3.
$$N(C_0 - 1) \cap N(C_a + 1)^{\perp} = \{0\},$$
is the right hand side of assertion 2 in Theorem 5.7.
$$N(C_0 - 1)^{\perp} \cap N(C_a + 1) = N(C_0 + 1) \cap N(C_a + 1) = \{0\},$$
is part 2. of Theorem 5.4 for $b = 0$.

□

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Esteban Andruchow
Instituto de Ciencias, Universidad Nacional de Gral. Sarmiento,
J.M. Gutierrez 1150, (1613) Los Polvorines, Argentina
and Instituto Argentino de Matemática, 'Alberto P. Calderón', CONICET,
Saavedra 15, 3er. piso, (1083) Buenos Aires, Argentina.
e-mail: eandruch@ungs.edu.ar

Gustavo Corach
Instituto Argentino de Matemática, 'Alberto P. Calderón', CONICET,
Saavedra 15, 3er. piso, (1083) Buenos Aires, Argentina,
e-mail: gcorach@gmail.com

Lázaro Recht
Instituto Argentino de Matemática, 'Alberto P. Calderón', CONICET,
Saavedra 15, 3er. piso, (1083) Buenos Aires, Argentina,
e-mail: lrecht@gmail.com