Sphere bundle over the set of inner products in a Hilbert space

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Abstract

Let $(\mathcal{H}, \langle\, , \,\rangle)$ be a complex Hilbert space and $\mathcal{B}(\mathcal{H})$ the space of bounded linear operators in $\mathcal{H}$. Any other equivalent inner product in $\mathcal{H}$ is of the form $\langle f, g\rangle_A = \langle Af, g\rangle$ $(f, g \in \mathcal{H})$ for some positive invertible operator $A \in \mathcal{B}(\mathcal{H})$. In this paper we study the bundle $\mathcal{M}$ which consist of the unit sphere $\{f \in \mathcal{H} : \langle f, f\rangle_A = 1\}$ over each (equivalent) inner product $\langle\, , \,\rangle_A$, which due to the observation above can be defined

$$\mathcal{M} = \{(A, f) \in \mathcal{B}(\mathcal{H}) \times \mathcal{H} : A \text{ is positive and invertible and } \langle Af, f\rangle = 1\}.$$

We prove that $\mathcal{M}$ is a complemented submanifold of the Banach space $\mathcal{B}(\mathcal{H}) \times \mathcal{H}$ and a homogeneous space of the Banach-Lie group $\mathcal{G}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$ of invertible operators. We introduce a reductive structure in $\mathcal{M}$, and study properties of the geodesics of the linear connection induced by this reductive structure. We consider certain submanifolds of $\mathcal{M}$, for instance, the one obtained when the positive elements A describing the inner products lie in a prescribed C*-algebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$.

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1 Introduction

Let $(\mathcal{H}, \langle\, , \,\rangle)$ be a Hilbert space, denote by $\mathcal{G}(\mathcal{H})$ the group of invertible operators in $\mathcal{H}$, and by $\mathcal{G}^+(\mathcal{H})$ the subset of $\mathcal{G}(\mathcal{H})$ of positive operators. The set $\mathcal{G}^+(\mathcal{H})$ parametrizes the inner products in $\mathcal{H}$, which induce norms that are equivalent to the original norm of $\mathcal{H}$: any such inner product is of the form $\langle f, g\,\rangle_A = \langle Af, g\rangle$ $(f, g \in \mathcal{H})$, for a unique operator $A \in \mathcal{G}^+(\mathcal{H})$. In this paper we study the set

$$\mathcal{M} := \{([\ ,\ ], f) : [\ ,\ ] \text{ is an inner product in } \mathcal{H} \text{ equivalent to } \langle\ ,\ \rangle \text{ and } [f, f] = 1\},$$

as a bundle over the set of inner products. That is, over each inner product we put its unit sphere. Due to the remark that $\mathcal{G}^+(\mathcal{H})$ parametrizes this set of inner products, we can (and choose to) describe $\mathcal{M}$ as operators and vectors:

$$\mathcal{M} = \{(A, g) \in \mathcal{B}(\mathcal{H}) \times \mathcal{H} : A \in \mathcal{G}^+(\mathcal{H}) \text{ and } \langle Ag, g\rangle = 1\}.$$

We call *canonical bundle* the map

$$\pi_{\mathcal{M}} : \mathcal{M} \to \mathcal{G}^+(\mathcal{H}), \quad \pi_{\mathcal{M}}(A, g) = A, \tag{1}$$

whose fiber over every $A \in \mathcal{G}^+(\mathcal{H})$ (i.e., every inner product) is the unit sphere of the given inner product. The group $\mathcal{G}(\mathcal{H})$ acts on $\mathcal{M}$: if $G \in \mathcal{G}(\mathcal{H})$ and $(A, g) \in \mathcal{M}$,

$$G \cdot (A, f) = ((G^*)^{-1} A G^{-1}, Gf).$$

Note that it is indeed a well defined left action: $(G^*)^{-1} A G^{-1} \in \mathcal{G}^+(\mathcal{H})$ and

$$\langle Gf, Gf \rangle_{(G^*)^{-1} A G^{-1}} = \langle (G^*)^{-1} A G^{-1} Gf, Gf \rangle = \langle Af, f \rangle = 1.$$

We prove in Section 3 that $\mathcal{M}$ becomes a homogeneous space of $\mathcal{G}(\mathcal{H})$. The purpose of this paper is the geometric study of $\mathcal{M}$ and this action. We shall profit by the thorough study of the space $\mathcal{G}^+(\mathcal{H})$ done by Corach, Porta and Recht in several papers, of which we mainly cite [2]: the space $\mathcal{G}^+(\mathcal{H})$ behaves like a non positively curved metric space, when endowed with a natural Finsler metric. Let us state some of the properties of $\mathcal{G}^+(\mathcal{H})$

Remark 1.1. (See [2])

1. There is a natural left action of the Banach-Lie group $\mathcal{G}(\mathcal{H})$ on $\mathcal{G}^+(\mathcal{H})$:

$$G \cdot A = (G^*)^{-1} A G, \quad G \in \mathcal{G}(\mathcal{H}), A \in \mathcal{G}^+(\mathcal{H}).$$

The action is transitive. If one considers the element $1 \in \mathcal{G}^+(\mathcal{H})$, the subgroup of $\mathcal{G}(\mathcal{H})$ which fixes 1 (usually called the isotropy group at 1) is the unitary group $\mathcal{U}(\mathcal{H})$ of $\mathcal{H}$. The Banach-Lie algebras of the groups $\mathcal{G}(\mathcal{H})$ and $\mathcal{U}(\mathcal{H})$ are, respectively, $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}_{ah}(\mathcal{H})$ (the anti-Herimitian operators in $\mathcal{H}$). The natural choice of a complement for $\mathcal{B}_{ah}(\mathcal{H})$ is $\mathcal{B}_h(\mathcal{H})$ (the Hermitian operators in $\mathcal{H}$). The decomposition

$$\mathcal{B}(\mathcal{H}) = \mathcal{B}_{ah}(\mathcal{H}) \oplus \mathcal{B}_h(\mathcal{H})$$

can be pushed to every other element $A \in \mathcal{G}^+(\mathcal{H})$ with the action of $\mathcal{G}(\mathcal{H})$. The distribution of decompositions of $\mathcal{B}(\mathcal{H})$ so obtained induces a linear connection in $\mathcal{G}^+(\mathcal{H})$. This type of construction is usually called a *reductive structure* in classical differential geometry.

2. The geodesics of this connection can be explicitly computed. Given $A, B \in \mathcal{G}^+(\mathcal{H})$, there is a unique geodesic $\gamma_{A,B}$ with $\gamma_{A,B}(0) = A$ and $\gamma_{A,B}(1) = B$. It is given by

$$\gamma_{A,B}(t) = A^{1/2}(A^{-1/2} B A^{-1/2})^t A^{1/2}. \tag{2}$$

In particular, note that if $A = 1$ then $\gamma_{(1,A)}(t) = B^t$.

3. The action also induces a norm in the tangent spaces (which are all identified with $\mathcal{B}_h(\mathcal{H})$): at 1 one chooses the usual operator norm of $\mathcal{B}(\mathcal{H})$. At a point $A \in \mathcal{G}^+(\mathcal{H})$ one puts

$$|X|_A := \|A^{-1/2} X A^{-1/2}\|, \ X \in \mathcal{B}_h(\mathcal{H}). \tag{3}$$

The action of $\mathcal{G}(\mathcal{H})$ is isometric.

4. The geodesics of the linear connection given in (2) are also geodesics in the metric sense (3): $\gamma_{A,B}$ is the shortest possible curve joining any pair of points in its path. Therefore the geodesic distance $d(A, B)$ between A and B is given by the length of $\gamma_{A,B}$, which can be explicitly computed.

5. The metric space $(\mathcal{G}^+(\mathcal{H}), d)$ behaves like a non-positively curved space. For instance, if γ_1 and γ_2 are geodesics in $\mathcal{G}^+(\mathcal{H})$, then the function

$$f(t) = d(\gamma_1(t), \gamma_2(t)),\ t \in \mathbb{R}$$

is convex.

We shall see that also $\mathcal{M}$ admits a reductive structure, and a metric which is invariant under the group action.

We refer the reader to [1] for the basic facts concerning manifolds, submanifolds and homogeneous spaces in the infinite dimensional setting. In particular, we observe that in this setting one distinguishes the notions of *submanifold* M of a Banach space E, and *complemented submanifold* of E. In the former case, the tangent spaces of M are *closed* subspaces of E, in the latter the tangent spaces are *complemented* subspaces of E.

The contents of the paper are the following. In Section 2 we prove that the action is transitive, and that the maps $m_{(A,g)}$ (for fixed $(A, g) \in \mathcal{M}$)

$$m_{(A,g)} : \mathcal{G}(\mathcal{H}) \to \mathcal{M},\ m_{(A,g)}(G) = G \cdot (A, g) = ((G^*)^{-1}AG^{-1}, Gg) \tag{4}$$

which are induced by the action, have continuous local cross sections. In Section 3 we use this fact to prove that $\mathcal{M}$ is a complemented submanifold of $\mathcal{B}(\mathcal{H}) \times \mathcal{H}$, which is contractible, and that the maps $m_{(A,g)}$ are submersions. In Section 4 we consider subets of $\mathcal{M}$, obtained when we restrict the sets of inner product, or the sets of vectors. In particular, we consider the case when the positive operators parametrizing the inner product lie in a prescribed C*-algebra. In Section 5 we introduce the reductive structure of $\mathcal{M}$, and describe the geodesics of the linear connection. We study properties of these geodesics, with respect to the restricted parts of $\mathcal{M}$ considered in Section 4. In the brief Section 6, we introduce an invariant Finsler metric for $\mathcal{M}$.

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2 Transitivy and local cross sections for the action

If $A \in \mathcal{G}^+(\mathcal{H})$, we shall denote by $\mathbb{S}_A(\mathcal{H}) = \{g \in \mathcal{H} : \langle Ag, g\rangle = 1\}$, the unit sphere of the A-inner product, and by $\mathbb{S}(\mathcal{H}) = \mathbb{S}_1(\mathcal{H})$, the usual sphere. Also put $\|g\|_A = \langle g, g\rangle_A^{1/2} = \|A^{1/2}g\|$. Clearly

$$g \in \mathbb{S}_A(\mathcal{H}) \text{ if and only if } A^{1/2}g \in \mathbb{S}(\mathcal{H}).$$

The following notation will be useful. If $f, g \in \mathcal{H}$ and $A \in \mathcal{G}^+$, denote by $f \otimes_A g$ the rank one operator given by

$$f \otimes_A g(h) = \langle h, g\rangle_A f = \langle Ah, g\rangle f.$$

If $A = 1$, we write $f \otimes g = f \otimes_1 g$. Note that

$$f \otimes_A g = (f \otimes g)A = f \otimes Ag.$$

Denote by $\mathcal{U}(\mathcal{H})$ the unitary group of $\mathcal{H}$, $\mathcal{U}(\mathcal{H}) = \{U \in \mathcal{B}(\mathcal{H}) : UU^* = U^*U = 1\}$. For any given linear operator T, $R(T)$ and $N(T)$ will denote the range and the nullspace, respectively.

Remark 2.1. Given $f, g \in \mathbb{S}(\mathcal{H})$ with $\|f - g\| < \sqrt{2}$, there exists a unitary operator $\Theta_{f,g}$ in $\mathcal{H}$, which is a smooth map in both parameters f, g, such that $\Theta_{f,g} g = f$. Namely, put

$$\mathcal{V} := \{(f, g) \in \mathbb{S}(\mathcal{H}) \times \mathbb{S}(\mathcal{H}) : \|f - g\| < \sqrt{2}\};$$

then there exists a C^∞ map

$$\mathcal{V} \to \mathcal{U}(\mathcal{H}) \ , \quad (f, g) \mapsto \Theta_{f,g}$$

satisfying $\Theta_{f,g} g = f$ for all $(f, g) \in \mathcal{V}$. One explicit way to construct $\Theta_{f,g}$ is the following: if $\|f - g\| < \sqrt{2}$ (or, less restrictive, $\langle f, g\rangle \neq 0$), then

$$\|f \otimes f - g \otimes g\| < 1.$$

Indeed, this can be shown using the Krein-Krasnoselski-Milman [6] formula: if P, Q are orthogonal projections, then

$$\|P - Q\| = \max\{\|PQ - P\|, \|QP - Q\|\}.$$

In our case, $f \otimes f, g \otimes g$ are orthogonal projections, and therefore (using the elementary fact that $\|h \otimes h'\| = \|h\|\|h'\|$), after straightforward computations, we have that

$$\|f \otimes f - g \otimes g\| = \max\{\|(f \otimes f)(g \otimes g) - f \otimes f\|, \|(g \otimes g)(f \otimes f) - g \otimes g\|\} = \sqrt{1 - |\langle f, g\rangle|^2}.$$

It is easy to see that $\|f - g\|^2 < 2$ implies that $\langle f, g\rangle \neq 0$, and thus $\|f \otimes f - g \otimes g\| < 1$. Therefore, there exists a unitary operator U, which depends smoothly on the projections $f \otimes f, g \otimes g$ (and thus, also depends smoothly on f, g) such that

$$U(f \otimes f)U^* = Uf \otimes Uf = g \otimes g.$$

This is a well known fact, see for instance [4], or [3]. In this latter work U is obtained as follows: if $\|P - Q\| < 1$, then $P + Q - 1$ is invertible: indeed, this follows from the elementary equality (see [4])

$$(P - Q)^2 + (P + Q - 1)^2 = 1.$$

Then the unitary part U in the polar decomposition $P + Q - 1 = U|P + Q - 1|$ is a unitary operator (in fact, a symmetry, i.e., a selfadjoint unitary operator) such that $UPU^* = UPU = Q$. Also, it is an explicit formula in terms of P and Q:

$$U = (P + Q - 1)|P + Q - 1|^{-1} = (P + Q - 1)\{(P + Q - 1)^2\}^{-1/2}.$$

Continuing our argument, Uf, g are unit vectors generating the same complex line: $g = \alpha U f$, with $\alpha \in \mathbb{C}$, $|\alpha| = 1$, i.e., $\alpha = \langle g, Uf\rangle$. Then

$$g = \Theta f,$$

for the unitary operator $\Theta = \langle g, Uf\rangle U$.

This mode of finding the unitary Θ linking f and g, uses the geometry of the complex Grassmann manifold of $\mathcal{H}$, the unitary U links the complex lines generated by f and g. It can be shown that the line generated by g lies in the range of the exponential map at f of the Grassmann manifold, if and only if $\langle f, g\rangle \neq 0$.

Proposition 2.2. *The action of $\mathcal{G}(\mathcal{H})$ is transitive on $\mathcal{M}$. If $(A_0, g_0) \in \mathcal{M}$, the map (4)*

$$m_{(A_0,g_0)} : \mathcal{G}(\mathcal{H}) \to \mathcal{M}, \quad m_{(A_0,g_0)}(G) = ((G^*)^{-1}A_0G^{-1}, Gg_0)$$

has continuous local cross sections.

Proof. Fix $g_0 \in \mathbb{S}(\mathcal{H})$, so that $(1, g_0) \in \mathcal{M}$, and let $(A, g) \in \mathcal{M}$. We show that there exists $G \in \mathcal{G}(\mathcal{H})$ such that $G \cdot (1, g_0) = (A, g)$, proving that the action is transitive. The usual left action of the unitary group of $\mathcal{H}$ on the unit sphere of $\mathcal{H}$ is transitive. Then there exists a unitary operator U such that $Ug_0 = A^{1/2}g$. If we consider $G = A^{-1/2}U \in \mathcal{G}(\mathcal{H})$, it's easy to see that

$$Gg_0 = A^{-1/2}Ug_0 = A^{-1/2}A^{1/2}g = g$$

and

$$(G^*)^{-1}G^{-1} = (U^*A^{-1/2})^{-1}(A^{-1/2}U)^{-1} = A^{1/2}UU^*A^{1/2} = A,$$

i.e. $G \cdot (1, g_0) = ((G^*)^{-1}G^{-1}, Gg_0) = (A, g)$. Let us construct a local cross section for $m_{(A_0,g_0)}$ on a neighbourhood of (A_0, g_0). By translation with the left action of $\mathcal{G}(\mathcal{H})$ on itself, one obtains cross sections around other points of $\mathcal{M}$. Consider the following open subset of $\mathcal{M}$:

$$\mathcal{B}_{(A_0,g_0)} = \{(A, g) \in \mathcal{M} : \|A^{1/2}g - A_0^{1/2}g_0\| < \sqrt{2}\}.$$

If $(A, g) \in \mathcal{B}_{(A_0,g_0)}$ then there exists (as mentioned in the above Remark) a continuous and smooth formula $\Theta = \Theta_{A_0^{1/2}g_0, A^{1/2}g}$, which is a unitary operator of the Hilbert space $\mathcal{H}$, such that

$$\Theta A_0^{1/2}g_0 = A^{1/2}g.$$

Put $\gamma_{(A_0,g_0)} : \mathcal{B}_{(A_0,g_0)} \to \mathcal{G}(\mathcal{H})$,

$$\gamma_{(A_0,g_0)}(A, g) = A^{-1/2}\Theta A_0^{1/2} = \gamma.$$

Then

$$(\gamma^*)^{-1}A_0\gamma^{-1} = A^{1/2}(\Theta^*)^{-1}A_0^{-1/2}A_0A_0^{-1/2}\Theta^{-1}A^{1/2} = A$$

and

$$\gamma g_0 = A^{-1/2}\Theta A_0^{1/2}g_0 = A^{-1/2}A^{1/2}g = g,$$

i.e. $\gamma_{(A_0,g_0)}$ is a cross section for $m_{(A_0,g_0)}$.

□

Proposition 2.3. *Suppose that $\mathcal{H}$ is a separable infinite dimensional Hilbert space. Then the space $\mathcal{M}$ has trivial homotopy type.*

Proof. Consider $(1, g_0) \in \mathcal{M}$ and the fibre bundle $m_{(1,g_0)} : \mathcal{G}(\mathcal{H}) \to \mathcal{M}$. By Kuiper's theorem [7], the group $\mathcal{G}(\mathcal{H})$ is contractible. The fibre of $m_{(1,g_0)}$ over $(1, g_0)$ is the subgroup

$$m^{-1}_{(1,g_0)}(1, g_0) = \{U \in \mathcal{G}(\mathcal{H}) : (U^{-1})^*U^{-1} = 1, Ug_0 = g_0\},$$

i.e. $U \in m^{-1}_{(1,g_0)}(1, g_0)$ is a unitary operator with $Ug_0 = g_0$. Therefore its matrix in the decomposition $\mathcal{H} =< g_0 > \oplus < g_0 >^{\perp}$ is

$$U = \begin{pmatrix} 1 & 0 \\ 0 & U' \end{pmatrix},$$

where U' is a unitary operator in $< g_0 >^{\perp}$, which is (separable) infinite dimensional. It follows, again by Kuiper's theorem, that $m^{-1}_{(1,g_0)}(1, g_0)$ is contractible. It follows that all the homotopy groups of $\mathcal{M}$ are trivial (see for instance [13]). □

3 Regular structure

Let us prove that $\mathcal{M}$ is a C^∞ differentiable manifold, and that the map $\pi_{\mathcal{M}}$ (given in 1) is a C^∞ fibre bundle. In order to establish the first assertion, the following lemma will be useful. We state it without proof, a complete proof can be found in [11].

Lemma 3.1. *Let G be a Banach-Lie group acting smoothly on a Banach space X. For a fixed $x_0 \in X$, denote by $m_{x_0} : G \to X$ the smooth map $m_{x_0}(g) = g \cdot x_0$. Suppose that*

1. *m_{x_0} is an open mapping, regarded as a map from G onto the orbit $\{g \cdot x_0 : g \in G\}$ of x_0 (with the relative topology of X).*
2. *The differential $d(m_{x_0})_1 : (TG)_1 \to X$ splits: its nullspace and range are closed complemented subspaces.*

Then the orbit $\{g \cdot x_0 : g \in G\}$ is a smooth complemented submanifold of X, and the map

$$m_{x_0} : G \to \{g \cdot x_0 : g \in G\}$$

is a smooth submersion.

Theorem 3.2. *$\mathcal{M}$ is a C^∞ complemented submanifold of $\mathcal{B}(\mathcal{H}) \times \mathcal{H}$. For any $(A_0, g_0) \in \mathcal{M}$, the map*

$$m_{(A_0,g_0)} : \mathcal{G}(\mathcal{H}) \to \mathcal{M},\ m_{(A_0,g_0)}(G) = G \cdot (A_0, g_0) = ((G^*)^{-1} A_0 G^{-1}, G g_0)$$

is a C^∞-submersion.

Proof. Since $m_{(A_0,g_0)}$ has continuous local cross sections, it is an open mapping.

Put $\delta_{(A_0,g_0)} = d(m_{(A_0,g_0)})_1 : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \times \mathcal{H}$, which after easy computations is shown to be

$$\delta_{(A_0,g_0)}(X) = (-X^* A_0 - A_0 X, X g_0).$$

The nullspace of $\delta_{(A_0,g_0)}$ consists of operators Z which are A_0-anti-Hermitian (anti-Hermitian for the inner product $\langle\ ,\ \rangle_{A_0}$):

$$Z^* A_0 = -A_0 Z,$$

such that $Z g_0 = 0$. Note that this last condition is equivalent to $Z(g_0 \otimes_{A_0} g_0) = 0$. Since g_0 is a unit vector for the A_0-inner product, the operator $g_0 \otimes_{A_0} g_0$ is a (rank one) A_0-orthogonal projection. Therefore the matrix of Z, in the A_0-orthogonal decomposition $\mathcal{H} =< g_0 > \oplus_{A_0} < g_0 >^{\perp}$ is

$$Z = \begin{pmatrix} 0 & 0 \\ 0 & Z_{22} \end{pmatrix},$$

where Z_{22} is A_0-anti-Hermitian. A natural supplement for the nullspace of $\delta_{(A_0,g_0)}$ is therefore the set of matrices (again in terms of the decomposition $\mathcal{H} =< g_0 > \oplus_{A_0} < g_0 >^{\perp}$)

$$\begin{pmatrix} z g_0 & X_{12} \\ X_{21} & Y \end{pmatrix},$$

where $z \in \mathbb{C}$ and Y is A_0-Hermitian. The range of $\delta_{(A_0,g_0)}$ consists of pairs $(-X^* A_0 - A_0 X, X g_0)$, with X varying over $\mathcal{B}(\mathcal{H})$. Note that the left hand part of this pair is selfadjoint (for the usual inner product), and that any selfadjoint operator Y is of this form. Indeed, given $Y = Y^*$, put $X = -\frac{1}{2}A_0^{-1}Y$. Then

$$-X^*A_0 - A_0X = \frac{1}{2}Y + \frac{1}{2}Y = Y.$$

Any operator decomposes $X = X_h + X_{ah}$ in its A_0- Hermitian and anti-Hermitian parts, i.e., $X_h^*A_0 = A_0X_h$ and $X_{ah}^*A_0 = -A_0X_{ah}$. Then $-X^*A_0 - A_0X = -2A_0X_h$. There is a linear isomorphism ℓ of $\mathcal{B}(\mathcal{H}) \times \mathcal{H}$ given by

$$\ell : \mathcal{B}(\mathcal{H}) \times \mathcal{H} \to \mathcal{B}(\mathcal{H}) \times \mathcal{H},\ \ell(X, h) = (-\frac{1}{2}A_0^{-1}X, h),$$

which maps the range of $\delta_{(A_0,g_0)}$ onto the subspace $\{(X_h, Xg_0) : X \in \mathcal{B}(\mathcal{H})\}$.

Note that $\{Yg_0 \in \mathcal{H} : Y \text{ is } A_0 - \text{Hermitian}\} = \{f \in \mathcal{H} : \langle f, g_0\rangle_{A_0} \in \mathbb{R}\}$ and that $\{Zg_0 \in \mathcal{H} : Z \text{ is } A_0 - \text{anti-Hermitian}\} = \{h \in \mathcal{H} : \langle h, g_0\rangle_{A_0} \in i\mathbb{R}\}$. We have the direct sum decomposition

$$\{(X_h, Xg_0) : X \in \mathcal{B}(\mathcal{H})\} = \{(X_h, X_hg_0) : X \in \mathcal{B}(\mathcal{H})\} \oplus \{(0, X_{ah}g_0) : X \in \mathcal{B}(\mathcal{H})\}. \tag{5}$$

The right hand subspace in (5) equals $\{(0, h) : \langle h, g_0\rangle_{A_0} \in i\mathbb{R}\}$. The left hand subspace in (5) is contained in $\{(Y, f) : Y \text{ is } A_0 - \text{Hermitian and } \langle f, g_0\rangle_{A_0} \in \mathbb{R}\}$, and it is complemented there: a complement is

$$\mathbf{S}_{(A_0,g_0)} := \{(Y, f) : Y \text{ is } A_0 - \text{Hermitian },\ \langle f, g_0\rangle_{A_0} \in \mathbb{R} \text{ and } \langle f, Yg_0\rangle_{A_0} = 0\}.$$

Therefore, a complement for $\ell(\delta_{(A_0,g_0)})$ in $\{(Y, h) : Y \text{ is } A_0 - \text{Hermitian}, h \in \mathcal{H}\}$ is

$$\mathbf{S}_{(A_0,g_0)} \oplus \{(0, h) : \langle h, g_0\rangle_{A_0} \in i\mathbb{R}\}.$$

Clearly $\{(Y, h) : Y \text{ is } A_0 - \text{Hermitian },\ h \in \mathcal{H}\}$ is complemented in $\mathcal{B}(\mathcal{H}) \times \mathcal{H}$: a complement is $\{(Z, 0) : Z \text{ is } A_0 - \text{ anti-Hermitian}\}$.

Putting these facts togeher, we have that

$$\mathbf{S}_{(A_0,g_0)} \oplus \{(0, h) : \langle h, g_0\rangle_{A_0} \in i\mathbb{R}\} \oplus \{(Z, 0) : Z \text{ is } A_0 - \text{ anti-Hermitian}\}$$

is a complement for $\ell(\delta_{(A_0,g_0)})$ in $\mathcal{B}(\mathcal{H}) \times \mathcal{H}$. □

Recall from Proposition 2.3, that $\mathcal{M}$ has trivial homotopy groups. Then, since it is a differentiable manifold, it holds that (see [10]):

Corollary 3.3. *Suppose that $\mathcal{H}$ is separable and infinite dimensional. Then $\mathcal{M}$ is contractible.*

Proposition 3.4. *The map (1)*

$$\pi_{\mathcal{M}} : \mathcal{M} \to \mathcal{G}^+(\mathcal{H}),\ \pi_{\mathcal{M}}(A, g) = A$$

is a locally trivial fibre bundle. In fact, the map $\pi_{\mathcal{M}}$ defines a trivial (product) bundle.

Proof. Consider the map

$$\varphi : \mathcal{M} \to \mathcal{G}^+(\mathcal{H}) \times \mathbb{S}(\mathcal{H}),\ \varphi(A, g) = (A, A^{1/2}g).$$

Note that φ is a diffeomorphism (with inverse $\varphi^{-1}(A, h) = (A, A^{-1/2}h)$) which trivializes the map (1) $\pi_{\mathcal{M}}$:

$$\pi_{\mathcal{M}}\varphi^{-1} : \mathcal{G}^+(\mathcal{H}) \times \mathbb{S}(\mathcal{H}) \to \mathcal{G}^+(\mathcal{H}),\ \pi_{\mathcal{M}}\varphi^{-1}(A, h) = A.$$

□

Let us describe the tangent spaces of $\mathcal{M}$

Proposition 3.5. *If $(A_0, g_0) \in \mathcal{M}$, then*

$$(T\mathcal{M})_{(A_0,g_0)} = R(\delta_{(A_0,g_0)}) = \{-X^*A_0 - A_0X, Xg_0) : X \in \mathcal{B}(\mathcal{H})\},$$

which also consists of

$$(T\mathcal{M})_{(A_0,g_0)} = \{(Z,h) \in \mathcal{B}(\mathcal{H}) \times \mathcal{H} : Z^* = Z \text{ and } \langle Zg_0, g_0\rangle + 2\ Re\langle Ah, g_0\rangle = 0\}.$$

Proof. The first assertion follows because $m_{(A_0,g_0)} : \mathcal{G}(\mathcal{H}) \to \mathcal{M}$ is a submersion, and thus $\delta_{(A_0,g_0)} : \mathcal{B}(\mathcal{H}) \to (T\mathcal{M})_{(A_0,g_0)}$ is surjective.

To prove the second assertion, consider $(A(t), g(t))$ a smooth curve in $\mathcal{M}$ with $A(0) = A_0$, $\dot{A}(0) = Z$, $g(0) = g_0$ and $\dot{g}(0) = h$. Since $A^*(t) = A(t)$, $Z^* = Z$. Differentiating $\langle A(t)g(t), g(t)\rangle = 1$ at $t = 0$, we get

$$\langle Zg_0, g_0\rangle + \langle A_0h, g_0\rangle + \langle A_0g_0, h\rangle = 0.$$

To prove the other inclusion, we make use of the following translation maps in $\mathcal{M}$. Fix $G \in \mathcal{G}(\mathcal{H})$, then $\ell_G : \mathcal{M} \to \mathcal{M}$, $\ell_G(A, f) = G \cdot (A, f)$ is a diffeomorphism, explicitly, $\ell_G(A, f) = ((G^{-1})^*AG^{-1}, Gf)$. Note that ℓ_G is the restriction to $\mathcal{M}$ of a linear isomorphism defined in the whole space $\mathcal{B}(\mathcal{H}) \times \mathcal{H}$. Therefore its tangent map coincides with ℓ_G. Using this translation, we may suppose that $A_0 = 1$, and prove the reverse inclusion in this case. Pick a pair $(Z, h) \in \mathcal{B}(\mathcal{H}) \times \mathcal{H}$ with $Z^* = Z$, satisfying

$$\langle Zg_0, g_0\rangle + \langle h, g_0\rangle + \langle g_0, h\rangle = 0.$$

Consider the vector $w = -\frac{i}{2}Zg_0 - ih$. Note that $\langle w, g_0\rangle \in \mathbb{R}$:

$$\langle w, g_0\rangle = -i\left(\frac{1}{2}\langle Zg_0, g_0\rangle + \langle h, g_0\rangle\right) = -\frac{i}{2}\left(\langle h, g_0\rangle - \langle g_0, h\rangle\right) \in \mathbb{R}.$$

Therefore there exists a selfadjoint operator Y such that $Yg_0 = w$ (short proof: since $\langle w, g_0\rangle \in \mathbb{R}$, pick $Y = w \otimes g_0 + g_0 \otimes w - \langle w, g_0\rangle g_0 \otimes g_0$). Consider

$$X = -\frac{1}{2}Z + iY.$$

Then $-X - X^* = Z$ and $Xg_0 = h$. □

4 Subsets of $\mathcal{M}$

Let us consider the following subsets of $\mathcal{M}$:

Definition 4.1. *For $\mathbf{F} \subset \mathcal{H} \setminus \{0\}$ and $\mathcal{C} \subset \mathcal{G}^+(\mathcal{H})$, consider*

$$\mathcal{M}^{\mathbf{F}} := \{(A, f) \in \mathcal{M} : f \in \mathbf{F}\}; \tag{6}$$

$$\mathcal{M}_{\mathcal{C}} := \{(A, f) \in \mathcal{M} : A \in \mathcal{C}\}; \tag{7}$$

and

$$\mathcal{M}^{\mathbf{F}}_{\mathcal{C}} := \{(A, f) \in \mathcal{M} : f \in \mathbf{F} \text{ and } A \in \mathcal{C}\}. \tag{8}$$

Proposition 4.2.

1. *If* $\mathbf{F} \subset \mathcal{H} \setminus \{0\}$ *is a submanifold, then* $\mathcal{M}^{\mathbf{F}}$ *is a submanifold of* $\mathcal{M}$*.*

2. *If* $\mathcal{C} \subset \mathcal{G}^+(\mathcal{H})$ *is a submanifold, then* $\mathcal{M}_{\mathcal{C}}$ *is a submanifold of* $\mathcal{M}$*.*

3. *If* $\mathbf{F} \subset \mathcal{H} \setminus \{0\}$ *and* $\mathcal{C} \subset \mathcal{G}^+$ *are submanifolds, and* $\mathcal{C}$ *satisfies that* $\mathbb{R}_{>0}1 \subset \mathcal{C}$*, then* $\mathcal{M}^{\mathbf{F}}_{\mathcal{C}}$ *is a submanifold of* $\mathcal{M}$*.*

Moreover, if $\mathbf{F} \subset \mathcal{H}\setminus\{0\}$ *and* $\mathcal{C} \subset \mathcal{G}^+(\mathcal{H})$ *are complemented submanifolds, then the corresponding subsets* $\mathcal{M}^{\mathbf{F}}$*,* $\mathcal{M}_{\mathcal{C}}$ *and* $\mathcal{M}^{\mathbf{F}}_{\mathcal{C}}$ *are complemented submanifolds of* $\mathcal{M}$*.*

Proof. To prove the first assertion, note that the map

$$\rho_{\mathcal{M}} : \mathcal{M} \to \mathcal{H} \setminus \{0\} \ , \quad \rho_{\mathcal{M}}(A, f) = f$$

is a C^∞ retraction. A global cross section for this map is $\mathcal{H} \setminus \{0\} \ni h \overset{\mathbf{s}}{\mapsto} (\frac{1}{\|h\|^2}1, h) \in \mathcal{M}$. Clearly this map is C^∞, and therefore $\mathcal{M}^{\mathbf{F}} = \rho_{\mathcal{M}}^{-1}(\mathbf{F})$ is a submanifold of $\mathcal{M}$. If $\mathbf{F}$ is a complemened submanifold of $\mathcal{H} \setminus \{0\}$, then $\mathcal{M}^{\mathbf{F}}$ is a complemented submanifold (see for instance [1]).

Similarly, for the second assertion, recall that $\pi_{\mathcal{M}} : \mathcal{M} \to \mathcal{G}^+(\mathcal{H})$, $\pi_{\mathcal{M}}(A, f) = A$ is a submersion, and thus $\mathcal{M}_{\mathcal{C}} = \pi_{\mathcal{M}}^{-1}(\mathcal{C})$ is a submanifold of $\mathcal{M}$ (again, a complemented submanifold if $\mathcal{C} \subset \mathcal{G}^+(\mathcal{H})$ is a complemented submanifold) .

To prove the third assertion, note that the hypothesis that $\mathbb{R}_{>0}1 \subset \mathcal{C}$, implies that the map $\mathcal{H} \setminus \{0\} \ni h \overset{\mathbf{s}}{\mapsto} (\frac{1}{\|h\|^2}1, h)$ takes values in $\mathcal{C}$, and therefore is a cross section for $\rho_{\mathcal{M}}|_{\mathcal{M}_{\mathcal{C}}} : \mathcal{M}_{\mathcal{C}} \to \mathcal{H} \setminus \{0\}$. Therefore $\rho|_{\mathcal{M}_{\mathcal{C}}}$ is a submersion, and therefore $\mathcal{M}^{\mathbf{F}}_{\mathcal{C}} = \left(\rho_{\mathcal{M}}|_{\mathcal{M}_{\mathcal{C}}}\right)^{-1}(\mathbf{F})$ is a submanifold of $\mathcal{M}_{\mathcal{C}}$, and therefore also of $\mathcal{M}$. If both $\mathcal{C}$ and $\mathbf{F}$ are complemented submanifolds, then $\mathcal{M}^{\mathbf{F}}_{\mathcal{C}}$ is a complemented submanifold of $\mathcal{M}$. □

Remark 4.3. We shall be interested, for instance, in the cases $\mathbf{F} = \{f\}$, $\mathcal{C} = \{A\}$, or $\mathcal{C} = \mathcal{A}^+_\bullet$, the set of positive invertible elements of a C^*-algebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$. Note that $\mathcal{M}_{\{A\}}$ identifies with $\mathbb{S}_A(\mathcal{H})$, the unit sphere of the inner product given by A.

Remark 4.4. Another relevant example occurs when $\mathcal{A}$ is a von Neumann algebra with a faithful normal trace τ. Consider the Hilbert space $\mathcal{H} = L^2(\mathcal{A}, \tau)$, the GNS Hilbert space of τ, $\mathcal{A}$ (represented faithfully and normally) in $\mathcal{B}(\mathcal{H})$ and let $f_0 = 1 \in \mathcal{A}$ regarded as a vector in $\mathcal{H}$. Then $\tau(a) = \langle af_0, f_0\rangle$, for $a \in \mathcal{A}$. Then

$$\mathcal{M}^{\{f_0\}}_{\mathcal{A}^+_\bullet} \simeq \{a \in \mathcal{A}^+_\bullet : \tau(a) = 1\}.$$

This space is studied by L. Recht and A. Varela in [12].

The tangent spaces of $\mathcal{M}^{\{f\}}$ are, clearly,

$$(T\mathcal{M}^{\{f\}})_{(A,f)} = \{(X, 0) \in \mathcal{B}(\mathcal{H}) \times \{0\} : X^* = X \text{ and } \langle Xf, f\rangle = 0\}.$$

5 Reductive structure

In this section we propose a natural reductive structure for $\mathcal{M}$. A *reductive structure* for a homogeneous space M of a Lie group G (see for instance the classic text [5]) means the following: if $p \in M$, $m_p : G \to M$ denotes the map induced by the left action of G, $m_p(f) = f \cdot p$, and $\delta_p = (dm_p)_1 : (TG)_1 \to (TM)_p$ is its differential at 1, a reductive structure for M is a familiy $\{\mathbf{H}_p\}_{p\in M}$ of supplements for the nullspaces $N(\delta_p)$ in $(TG)_1$ (i.e., $(TG)_1 = N(\delta_p) \oplus \mathbf{H}_p$), which satisfy that

1. the distribution $M \ni p \mapsto \mathbf{H}_p$ is smooth;
2. the supplements are invariant under the action of the corresponding isotropy subgroup $K_p = \{k \in G : h \cdot p = p\}$:
$$Ad(k)(\mathbf{H}_p) = \mathbf{H}_p.$$

The assertion that the distribution is smooth, means that if $\mathbf{P}_p$ denotes the projection from $(TG)_1$ onto $\mathbf{H}_p$ with nullspace $N(\delta_p)$, then the map $M \ni p \mapsto \mathbf{P}_p$ is C^∞, regarded as a map into the Banach space of bounded (real) linear operators acting in $(TG)_1$.

In our case the group is $\mathcal{G}(\mathcal{H})$, whose Banach-Lie group $(T\mathcal{G}(\mathcal{H}))_1$ identifies with $\mathcal{B}(\mathcal{H})$. For each element (A, f_0) in the homogeneous space $\mathcal{M}$, the map $m_{(A,f_0)}$ was already introduced, and its differential at the identity is $\delta_{(A,f_0)}(X) = (-X^*A - AX, Xf_0)$. The isotropy subgroup $K_{(A,f_0)}$ at $(A, f_0) \in \mathcal{M}$ is given by

$$K_{(A,f_0)} = \{G \in \mathcal{G}(\mathcal{H}) : (G^*)^{-1}AG^{-1} = A \text{ and } Gf_0 = f_0\},$$

i.e., the invertible elements G which are unitaries for the A-inner product $\langle\ ,\ \rangle_A$ and fix f_0. The nullspace $N(\delta_{(A,f_0)})$ (which is the Banach-Lie algebra of this group) is

$$N(\delta_{(A,f_0)}) = \{Y \in \mathcal{B}(\mathcal{H}) : Y^*A = -AY \text{ and } Yf_0 = 0\}$$

i.e., the operators which are anti-Hermitian for the A-inner product, and anhihilate f_0. Let us write these elements in matrix form, in terms of the A-orthogonal projection $f_0 \otimes_A f_0$: they are of the form

$$N(\delta_{(A,f_0)}) = \Big\{ \begin{pmatrix} 0 & 0 \\ 0 & Y' \end{pmatrix} : Y' \text{ is } A-\text{anti-Hermitian}\Big\}.$$

Note that the $1,1$ entry is scalar. A natural supplement for $N(\delta_{(A,f_0)})$ is the following:

Definition 5.1. *For $(A, f_0) \in \mathcal{M}$, put*

$$\mathbf{H}_{(A,f_0)} = \Big\{ \begin{pmatrix} a_0 & \mathbf{a} \\ \mathbf{b}^* & Z' \end{pmatrix} : Z' \textit{ is } A-\textit{ selfadjoint}, a_0 \in \mathbb{C}, \mathbf{a}, \mathbf{b} \in \mathcal{B}\left(R(1 - f_0 \otimes_A f_0), R(f_0 \otimes f_0)\right)\}. \tag{9}$$

Here $\mathbf{a}, \mathbf{b} :< f_0 >^\perp \to < f_0 >$, are bounded operators, and $\mathbf{b}^*$ is the adjoint of $\mathbf{b}$ (for the A-inner product). The supplement $\mathbf{H}_{(A,f_0)}$ is usually called the horizontal *space at (A, f_0).*

Let us check that the distribution of spaces $\mathbf{H}_{(A,f)}$ ($(A, f) \in \mathcal{M}$) indeed defines a reductive structure in $\mathcal{M}$. Note that

$$\mathbf{H}_{(A,f_0)} = \{Z \in \mathcal{B}(\mathcal{H}) : (1 - f_0 \otimes_A f_0)Z(1 - f_0 \otimes_A f_0) \text{ is } A-\text{ selfadjoint}\}. \tag{10}$$

Proposition 5.2. *The distribution* $\mathbf{H}_{(A,f)}$ $((A,f)\in\mathcal{M})$ *defines a reductive structure in* $\mathcal{M}$.

Proof. The set $\mathbf{H}_{(A,f_0)}$ is clearly a supplement for the space $N(\delta_{(A,f_0)})$ (see the proof of Theorem 3.2).

If $G\in K_{(A,f_0)}$, then G is A-unitary (i.e., $G^*AG=A$ or equivalently $AG=(G^*)^{-1}A$), and $Gf_0=f_0$. Then

$$G^{-1}(f_0\otimes_A f_0)=G^{-1}f_0\otimes_A f_0=f_0\otimes_A f_0,$$

and

$$(f_0\otimes_A f_0)G=\langle AG\,\cdot,f_0\rangle f_0=\langle (G^*)^{-1}A\,\cdot,f_0\rangle f_0=\langle A\,\cdot,G^{-1}f_0\rangle f_0=\langle A\,\cdot,f_0\rangle f_0=f_0\otimes_A f_0.$$

Then, if $Z\in\mathbf{H}_{(A,f_0)}$

$$(1-f_0\otimes_A f_0)(GZG^{-1})(1-f_0\otimes_A f_0)=(1-f_0\otimes_A f_0)Z(1-f_0\otimes_A f_0).$$

In particular, $G\mathbf{H}_{(A,f_0)}G^{-1}=\mathbf{H}_{(A,f_0)}$.

If $X\in\mathcal{B}(\mathcal{H})$ is written in matrix form in terms of the A-orthogonal projection $f_0\otimes_A f_0$, its decomposition in the direct sum $N(\delta_{(A,f_0)})\oplus\mathbf{H}_{(A,f_0)}$ is

$$X=\begin{pmatrix} x_{11} & x_{12}\\ x_{21} & X'\end{pmatrix}=\begin{pmatrix} 0 & 0\\ 0 & Y'\end{pmatrix}+\begin{pmatrix} x_{11} & x_{12}\\ x_{21} & Z'\end{pmatrix},$$

where $X'=Y'+Z'$ is the decomposition of X' in its A-anti-Hermitian and A-selfadjoint parts. Thus

$$Y'=\frac{1}{2i}(X'-A^{-1}(X')^*A)=\frac{1}{2i}(1-f_0\otimes_A f_0)(X-A^{-1}X^*A)(1-f_0\otimes_A f_0).$$

Then

$$\mathbf{P}_{(A,f_0)}(X)=X-\frac{1}{2i}(1-f_0\otimes_A f_0)(X-A^{-1}X^*A)(1-f_0\otimes_A f_0).$$

Recall that $f_0\otimes_A f_0=(f_0\otimes f_0)A$, and therefore the map

$$\mathcal{M}\ni(A,f_0)\mapsto\mathbf{P}_{(A,f_0)}\in\mathcal{B}_{\mathbb{R}}(\mathcal{B}(\mathcal{H}))$$

is C^∞ (here $\mathcal{B}_{\mathbb{R}}(\mathcal{B}(\mathcal{H}))$ denotes the Banach space of *real* bounded linear operators acting in $\mathcal{B}(\mathcal{H})$). □

As in classical differential geometry, a reductive structure in a homogeneous space induces a linear connection in the space ([5]). The features of the linear connection can be computed in terms of the so called 1-*form of the reductive structure*. Namely, the map

$$\delta_{(A,f_0)}|_{\mathbf{H}_{(A,f_0)}}:\mathbf{H}_{(A,f_0)}\to(T\mathcal{M})_{(A,f_0)}$$

is a linear isomorphism. Its inverse

$$\kappa_{(A,f_0)}:(T\mathcal{M})_{(A,f_0)}\to\mathbf{H}_{(A,f_0)} \tag{11}$$

is the 1-form of the reductive structure.

For $(A,f_0)\in\mathcal{M}$, denote by

$$\Pi_{(A,f_0)}:=\kappa_{(A,f_0)}\circ\delta_{(A,f_0)}:\mathcal{B}(\mathcal{H})\to\mathcal{B}(\mathcal{H}). \tag{12}$$

Note that $\Pi_{(A,f_0)}$ is an idempotent map, whose range is the horizontal space $\mathbf{H}_{(A,f_0)}$, and whose nullspace is the vertical space $N(\delta_{(A,f_0)})$. Then (see [8]):

1. If $(A, f_0) \in \mathcal{M}$ and $\mathcal{V} = (-X^*A - AX, h) \in (T\mathcal{M})_{(A,f_0)}$, then the unique geodesic γ of the linear connection of $\mathcal{M}$ with $\gamma(0) = (A, f_0)$ and $\dot\gamma(0) = \mathcal{V}$ is the curve

$$\gamma(t) = e^{t\kappa_{(A,f_0)}(\mathcal{V})} \cdot (A, f_0) = (e^{-t\kappa^*_{(A,f_0)}(\mathcal{V})} A e^{-t\kappa_{(A,f_0)}(\mathcal{V})}, e^{t\kappa_{(A,f_0)}(\mathcal{V})} f_0).$$

2. If $(A, f_0) \in \mathcal{M}$ and $\mathcal{X}, \mathcal{Y} \in (T\mathcal{M})_{(A,f_0)}$, the torsion T of the linear connection is given by

$$\kappa_{(A,f_0)}(T(\mathcal{X}, \mathcal{Y})) = -\Pi_{(A,f_0)}([X, Y]), \tag{13}$$

   where $X = \kappa_{(A,f_0)}(\mathcal{X})$ and $Y = \kappa_{(A,f_0)}(\mathcal{Y})$.

3. If $(A, f_0) \in \mathcal{M}$ and $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \in (T\mathcal{M})_{(A,f_0)}$, and again we use the notation

$$X = \kappa_{(A,f_0)}(\mathcal{X}), Y = \kappa_{(A,f_0)}(\mathcal{Y}) \text{ and } Z = \kappa_{(A,f_0)}(\mathcal{Z}),$$

   then the curvature tensor R is given by

$$\kappa_{(A,f_0)}(R(\mathcal{X}, \mathcal{Y})\mathcal{Z}) = \Big[Z, (1 - \Pi_{(A,f_0)})\,([X, Y])\Big]. \tag{14}$$

5.1 Computation of the 1-form of the connection

In view of its role in the computations of the invariants of the reductive connection, it is useful to obtain an explicit formula for the 1-form κ. First note the following:

Lemma 5.3. *Let $(A, f_0) \in \mathcal{M}$ and $G \in \mathcal{G}(\mathcal{H})$. Then*

$$Z \in \mathbf{H}_{(A,f_0)} \iff G^{-1}ZG \in \mathbf{H}_{G\cdot(A,f_0)}.$$

Proof. We must check that $(1 - Gf_0 \otimes_{(G^*)^{-1}AG^{-1}} Gf_0)GZG^{-1}(1 - Gf_0 \otimes_{(G^*)^{-1}AG^{-1}} Gf_0)$ is $(G^*)^{-1}AG^{-1}$-selfadjoint (recall that Y is B-selfadjoint if $Y^*B = BY$). Note that

$$(1 - Gf_0 \otimes_{(G^*)^{-1}AG^{-1}} Gf_0)GZG^{-1}(1 - Gf_0 \otimes_{(G^*)^{-1}AG^{-1}} Gf_0)$$
$$= (1 - \langle (G^*)^{-1}AG^{-1}\,\cdot\,, Gf_0\rangle Gf_0)GZG^{-1}(1 - \langle (G^*)^{-1}AG^{-1}\,\cdot\,, Gf_0\rangle Gf_0)$$
$$= (1 - \langle AG^{-1}\,\cdot\,, f_0\rangle Gf_0)GZG^{-1}(1 - \langle AG^{-1}\,\cdot\,, f_0\rangle Gf_0)$$
$$= G(1 - \langle A\,\cdot\,, f_0\rangle f_0)G^{-1}GZG^{-1}G(1 - \langle A\,\cdot\,, f_0\rangle f_0)G^{-1} = G(1 - f_0 \otimes_A f_0)Z(1 - f_0 \otimes_A f_0)G^{-1}.$$

Let us check that this operator is $(G^*)^{-1}AG^{-1}$-symmetric:

$$\big(G(1 - f_0 \otimes_A f_0)Z(1 - f_0 \otimes_A f_0)G^{-1}\big)^* (G^*)^{-1}AG^{-1}$$
$$= (G^*)^{-1}\left((1 - f_0 \otimes_A f_0)Z(1 - f_0 \otimes_A f_0)\right)^* AG^{-1}. \tag{15}$$

Since $Z \in \mathbf{H}_{(A,f_0)}$, we have that

$$((1 - f_0 \otimes_A f_0)Z(1 - f_0 \otimes_A f_0))^* A = A(1 - f_0 \otimes_A f_0)Z(1 - f_0 \otimes_A f_0).$$

Thus, (15) equals

$$(G^*)^{-1}A(1 - f_0 \otimes_A f_0)Z(1 - f_0 \otimes_A f_0)G^{-1} = (G^*)^{-1}AG^{-1}G(1 - f_0 \otimes_A f_0)Z(1 - f_0 \otimes_A f_0)G^{-1},$$

as claimed. ☐

Therefore, for $G \in \mathcal{G}(\mathcal{H})$, we have the map

$$Ad_G : \mathbf{H}_{(A,f_0)} \to \mathbf{H}_{G\cdot(A,f_0)}, \;\; Ad_G(Z) = GZG^{-1}. \tag{16}$$

Clearly it is an isomorphism (it is the restriction to $\mathbf{H}_{(A,f_0)}$ of a linear multiplicative global automorphism of $\mathcal{B}(\mathcal{H})$). Its inverse is $Ad_{G^{-1}}$.

Recall that $\ell_G : \mathcal{M} \to \mathcal{M}$,

$$\ell_G(A,f) = G \cdot (A,f) = ((G^*)^{-1}AG^{-1}, Gf)$$

is the restriction to $\mathcal{M}$ of a global linear isomorphism in $\mathcal{B}(\mathcal{H}) \times \mathcal{H}$, which we will also denote ℓ_G. With this slight abuse of notation, it follows that $(d\ell_G)_{(A,f_0)} = \ell_G$ for any $(A,f_0) \in \mathcal{M}$.

Lemma 5.4. *Let $(A,f_0) \in \mathcal{M}$ and $G \in \mathcal{G}(\mathcal{H})$. The following diagram of linear isomorphisms is commutative:*

$$\begin{array}{ccc} \mathbf{H}_{(A,f_0)} & \xrightarrow{\delta_{(A,f_0)}} & (T\mathcal{M})_{(A,f_0)} \\ {\scriptstyle Ad_G}\downarrow & & \downarrow{\scriptstyle \ell_G} \\ \mathbf{H}_{G\cdot(A,f_0)} & \xrightarrow{\delta_{G\cdot(A,f_0)}} & (T\mathcal{M})_{G\cdot(A,f_0)} \end{array}$$

Proof. Pick $Z \in \mathbf{H}_{(A,f_0)}$. Then

$$\ell_G \circ \delta_{(A,f_0)}(Z) = \ell_G(-Z^*A - AZ, Zf_0) = \left((G^*)^{-1}(-Z^*A - AZ)G^{-1}, GZf_0\right).$$

On the other hand

$$\begin{aligned} \delta_{G\cdot(A,f_0)} \circ Ad_G(Z) &= \left(-(GZG^{-1})^*(G^*)^{-1}AG^{-1} - (G^*)^{-1}AG^{-1}(GZG^{-1}), GZG^{-1}Gf_0\right) \\ &= \left((G^*)^{-1}(-Z^*A - AZ)G^{-1}, GZf_0\right). \end{aligned}$$

□

This lemma allows us to compute the 1-form at a specific element of $\mathcal{M}$, say $(1,f_0)$ (for $\|f_0\| = 1$), and translate the formula to other elements via the automorphims Ad_G. Let us abreviate $(f_0 \otimes f_0)^\perp = 1 - f_0 \otimes f_0$.

Lemma 5.5. *Let $(B,h) \in (T\mathcal{M})_{(1,f_0)}$ Then*

$$\kappa_{(1,f_0)}(B,h) = h \otimes f_0 + \Big(f_0 \otimes (Bf_0 - h)\Big)(f_0 \otimes f_0)^\perp - \frac{1}{2}(f_0 \otimes f_0)^\perp B (f_0 \otimes f_0)^\perp.$$

Proof. Given $(B,h) \in (T\mathcal{M})_{(1,f_0)}$, we look for the unique $Z = \kappa_{(1,f_0)}(B,h) \in \mathbf{H}_{(1,f_0)}$ such that $-Z^* - Z = B$, $Zf_0 = h$, and $(f_0 \otimes f_0)^\perp Z (f_0 \otimes f_0)^\perp$ selfadjoint. Let us compute the four matrix entries of Z in terms of the decomposition $\mathcal{H} = \langle f_0 \rangle \oplus \langle f_0 \rangle^\perp$:

$$(f_0 \otimes f_0)Z(f_0 \otimes f_0), (f_0 \otimes f_0)^\perp Z(f_0 \otimes f_0), (f_0 \otimes f_0)Z(f_0 \otimes f_0)^\perp \text{ and } (f_0 \otimes f_0)^\perp Z(f_0 \otimes f_0)^\perp.$$

The first two are determined by the condition $Zf_0 = h$:

$$(f_0 \otimes f_0)Z(f_0 \otimes f_0) = (f_0 \otimes f_0)(Zf_0 \otimes f_0) = (f_0 \otimes f_0)(h \otimes f_0)$$

and

$$(f_0 \otimes f_0)^{\perp} Z (f_0 \otimes f_0) = (f_0 \otimes f_0)^{\perp} (Z f_0 \otimes f_0) = (f_0 \otimes f_0)^{\perp} (h \otimes f_0).$$

Next, since $-Z^* - Z = B$, we have

$$B f_0 = -Z^* f_0 - Z f_0 = -Z^* f_0 - h, \text{ i.e., } Z^* f_0 = B f_0 - h.$$

Thus

$$(f_0 \otimes f_0) Z (f_0 \otimes f_0)^{\perp} = (f_0 \otimes Z^* f_0)(f_0 \otimes f_0)^{\perp} = (f_0 \otimes (B f_0 - h)(f_0 \otimes f_0)^{\perp}.$$

Finally since $(f_0 \otimes f_0)^{\perp} Z (f_0 \otimes f_0)^{\perp}$ should be selfadjoint, condition $-Z^* - Z = B$ implies that

$$(f_0 \otimes f_0)^{\perp} B (f_0 \otimes f_0)^{\perp} = -(f_0 \otimes f_0)^{\perp} Z^* (f_0 \otimes f_0)^{\perp} - (f_0 \otimes f_0)^{\perp} Z (f_0 \otimes f_0)^{\perp} = -2 (f_0 \otimes f_0)^{\perp} Z (f_0 \otimes f_0)^{\perp},$$

So that

$$(f_0 \otimes f_0)^{\perp} Z (f_0 \otimes f_0)^{\perp} = -\frac{1}{2} (f_0 \otimes f_0)^{\perp} B (f_0 \otimes f_0)^{\perp}.$$

Therefore

$$\begin{aligned} Z &= (f_0 \otimes f_0)(h \otimes f_0) + (f_0 \otimes f_0)^{\perp} (h \otimes f_0) + \Big(f_0 \otimes (B f_0 - h) \Big) (f_0 \otimes f_0)^{\perp} - \frac{1}{2} (f_0 \otimes f_0)^{\perp} B (f_0 \otimes f_0)^{\perp} \\ &= (h \otimes f_0) + \Big(f_0 \otimes (B f_0 - h) \Big) (f_0 \otimes f_0)^{\perp} - \frac{1}{2} (f_0 \otimes f_0)^{\perp} B (f_0 \otimes f_0)^{\perp}. \end{aligned}$$

□

Putting these facts together, we get

Proposition 5.6. *For $(A, f_0) \in \mathcal{M}$ and $(B, h) \in (T\mathcal{M})_{(A,f_0)}$, we have*

$$\kappa_{(A,f_0)}(B, h) = h \otimes_A f_0 + \big(f_0 \otimes_A (A^{-1} B f_0 - h) \big) (1 - f_0 \otimes_A f_0) - \frac{1}{2} (1 - f_0 \otimes_A f_0) A^{-1} B (1 - f_0 \otimes_A f_0).$$

Proof. Using the diagram in Lemma 5.4 for the element $(1, A^{1/2} f_0)$ and $G = A^{-1/2}$, since $A^{-1/2} \cdot (1, A^{1/2} f_0) = (A, f_0)$, we get

$$\begin{array}{ccc} \mathbf{H}_{(1,A^{1/2}f_0)} & \xrightarrow{\delta_{(1,A^{1/2}f_0)}} & (T\mathcal{M})_{(A,f_0)} \\ {\scriptstyle Ad_{A^{-1/2}}} \downarrow & & \downarrow {\scriptstyle \ell_{A^{-1/2}}} \\ \mathbf{H}_{(A,f_0)} & \xrightarrow{\delta_{(A,f_0)}} & (T\mathcal{M})_{(A,f_0)} \end{array}$$

Then

$$\kappa_{(A,f_0)} = Ad_{A^{-1/2}} \kappa_{(1,A^{1/2})} \ell_{A^{1/2}},$$

and

$$\begin{aligned} \kappa_{(A,f_0)}(B, h) &= Ad_{A^{-1/2}} \kappa_{(1,A^{1/2}f_0)} (A^{-1/2} B A^{-1/2}, A^{1/2} h) \\ &= Ad_{A^{-1/2}} \Big(A^{1/2} h \otimes A^{1/2} f_0 + \Big(A^{1/2} f_0 \otimes (A^{-1/2} B f_0 - A^{1/2} h) \Big) (A^{1/2} f_0 \otimes A^{1/2} f_0)^{\perp} - \\ &\quad - \frac{1}{2} (A^{1/2} f_0 \otimes A^{1/2} f_0)^{\perp} A^{-1/2} B A^{-1/2} (A^{1/2} f_0 \otimes A^{1/2} f_0)^{\perp} \Big). \end{aligned}$$

After straightforward computations one gets

$$\kappa_{(A,f_0)}(B,h) = h\otimes Af_0 + (f_0\otimes(Bf_0 - Ah))\,(1-f_0\otimes Af_0) - \frac{1}{2}(1-f_0\otimes Af_0)A^{-1}B(1-f_0\otimes Af_0).$$

Note that $Bf_0 - Ah = A(A^{-1}Bf_0 - h)$, and using that $g\otimes Ag' = g\otimes_A g'$, we can write

$$\kappa_{(A,f_0)}(B,h) = h\otimes_A f_0 + \big(f_0\otimes_A(A^{-1}Bf_0 - h)\big)\,(1-f_0\otimes_A f_0) - \frac{1}{2}(1-f_0\otimes_A f_0)A^{-1}B(1-f_0\otimes_A f_0).$$

□

Remark 5.7. Recall the submanifold $\mathcal{M}^{\{f\}}\subset\mathcal{M}$, for a given fixed $0\neq f\in\mathcal{H}$,

$$\mathcal{M}^{\{f\}} = \{(A,f) : A\in\mathcal{G}^+(\mathcal{H}) \text{ such that } \langle Af,f\rangle = 1\}.$$

Therefore, it can be regarded as a submanifold of $\mathcal{G}^+(\mathcal{H})$, namely, the elements in $A\in\mathcal{G}^+(\mathcal{H})$ such that $\omega_f(A) = 1$, where ω_f is the pure state given by f ($\omega_f(X) = \langle Xf,f\rangle$). Clearly, the submanifold $\mathcal{M}^{\{f\}}$ is a convex set.

A natural question is then the following: If $(A,f),(B,f)\in\mathcal{M}^{\{f\}}$ and $\gamma_{A,B}$ is the unique geodesic of $\mathcal{G}^+(\mathcal{H})$ such that $\gamma_{A,B}(0) = A$ and $\gamma_{A,B}(1) = B$ (see (2) in Remark 1.1), then is it true that $(\gamma_{A,B},f)$ lies in $\mathcal{M}^{\{f\}}$? The answer is no, in general. What fails is the condition $\langle\gamma_{A,B}(t)f,f\rangle = 1$. Consider for instance

$$A = 1,\ B = \begin{pmatrix} 2 & 0\\ 0 & 1/2\end{pmatrix},\quad\text{and}\quad f = \begin{pmatrix} 1/\sqrt{3}\\ \sqrt{2}/\sqrt{3}\end{pmatrix}.$$

Then $\langle f,f\rangle = \langle Bf,f\rangle = 1$ but $\langle\gamma_{1,B}(\frac{1}{2})f,f\rangle = \langle B^{1/2}f,f\rangle < 1$.

We have though the following inequality: let $(A,f),(B,f)\in\mathcal{M}^{\{f\}}$ and let $\gamma_{A,B}$ be the geodesic of $\mathcal{G}^+(\mathcal{H})$ with $\gamma_{A,B}(0) = A$ and $\gamma_{A,B}(1) = B$. Then for all $t\in[0,1]$

$$0 < \langle\gamma_{A,B}(t)f,f\rangle \leq 1.$$

Since $\gamma_{A,B}(t)$ is positive and invertible, it is clear that $\langle\gamma_{A,B}(t)f,f\rangle > 0$. Consider the real function

$$g(t) = \langle\gamma_{A,B}(t)f,f\rangle = \langle B^{1/2}\left(B^{-1/2}AB^{-1/2}\right)^t B^{1/2}f,f\rangle = \langle C^t h,h\rangle,$$

where $C = B^{-1/2}AB^{-1/2}$ and $h = B^{1/2}f$. Clearly g is smooth and convex:

$$\ddot{g}(t) = \langle\log(C)^2C^th,h\rangle = \langle C^t\log(C)h,\log(C)h\rangle \geq 0,$$

because C and $\log(C)$ commute. Note that by hypothesis $g(0) = g(1) = 1$. Then $g(t)\leq 1$ for $t\in[0,1]$.

In other words, the geodesic $\gamma_{A,B}$ joining elements in the set $\mathcal{M}^{\{f\}} = \{(A,f)\in\mathcal{M} : \omega_f(A) = 1\}$ remains inside $\{(A,f)\in\mathcal{M} : \omega_f(A)\leq 1\}$.

Proposition 5.8. *Let $(A,f_0)\in\mathcal{M}$, and $\mathcal{V} = (X,0)$ tangent to the submanifold $\mathcal{M}^{\{f_0\}}$ at (A,f_0). Then*

1. *The geodesic γ of $\mathcal{M}$, with $\gamma(0) = (A, f_0)$ and $\dot\gamma(0) = \mathcal{V}$ is of the form*
$$\gamma(t) = (\Gamma(t), f_0),$$
where $\Gamma(t) = e^{-tZ^*}Ae^{-tZ}$, for $Z = \kappa_{(A,f_0)}(\mathcal{V})$. In particular, γ remains inside $\mathcal{M}^{\{f_0\}}$ for all t.

2. *A necessary and sufficient condition for Γ to be a geodesic of $\mathcal{G}^+(\mathcal{H})$ is*
$$Xf_0 = 0.$$

Proof. Recall that $\mathcal{V}$ is of the form $\mathcal{V} = (X, 0)$, with $X^* = X$ and $\langle Xf_0, f_0\rangle = 0$. Let $Z = \kappa_{(A,f_0)}(\mathcal{V})$. Then, in particular, $Zf_0 = 0$. Thus $e^{tZ}f_0 = f_0$, and
$$\Gamma(t) = (e^{-tZ^*}Ae^{-tZ}, e^{tZ}f_0) = (\Gamma(t), f_0) \in \mathcal{M}^{\{f_0\}}, \text{ for all } t.$$

For the second assertion, let us consider first the case when the geodesic γ starts at $A = 1$. Suppose first that $Xf_0 = 0$, and let $Z = \kappa_{(1,f_0)}(\mathcal{V})$, that is $-Z^* - Z = X$, $Zf_0 = 0$ and $(f_0 \otimes f_0)^\perp Z(f_0 \otimes f_0)^\perp$ is selfadjoint. Since $Z^* = -Z - X$ and $Zf_0 = 0$, our assumption $Xf_0 = 0$ implies that $Z^*f_0 = 0$. Then
$$(f_0 \otimes f_0)^\perp Z(f_0 \otimes f_0)^\perp = (f_0 \otimes f_0)^\perp (Z - Zf_0 \otimes f_0) = (f_0 \otimes f_0)^\perp Z = Z - f_0 \otimes Z^*f_0 = Z,$$
i.e., Z is selfadjoint. Then
$$\gamma(t) = e^{tZ} \cdot (1, f_0) = (e^{-tZ^*}e^{-tZ}, e^{tZ}f_0) = (e^{-2tZ}, f_0),$$
where $\Gamma(t) = e^{-2tZ}$ is a geodesic of $\mathcal{G}^+(\mathcal{H})$.

Conversely, suppose that $e^{-tZ^*}e^{-tZ}$ is a geodesic of $\mathcal{G}^+(\mathcal{H})$. Since it starts at 1, with initial velocity $-Z^* - Z$, it must be $e^{-tZ^*}e^{-tZ} = e^{tX}$. Differentiating, one gets $-Z^*e^{-tZ^*}e^{-tZ} - e^{-tZ^*}e^{-tZ}Z = Xe^{tX}$ for all t, or
$$-Z^*e^{tX} - e^{tX}Z = Xe^{tX} \quad \text{for all } t,$$
which implies that $-e^{-tX}Z^*e^{tX} = Z + X$ for all t, i.e., constant. Thus, differentiating this last identity at $t = 0$ we get $XZ^* - Z^*X = 0$, that is, X commutes with Z^*. Since X is selfadjoint, X commutes also with Z. Then, the identity $X = -Z^* - Z$ implies that Z is normal. Since $Zf_0 = 0$, then $ZZ^*f_0 = Z^*Zf_0 = 0$, and then $Z^*f_0 = 0$. Therefore $Xf_0 = -Z^*f_0 - Zf_0 = 0$.

For the general case, recall that γ is a geodesic of $\mathcal{M}$ if and only if $G \cdot \gamma$ is also a geodesic. Then, $A^{1/2} \cdot \gamma = (A^{-1/2}\Gamma A^{-1/2}, A^{1/2}f_0)$ is a geodesic of $\mathcal{M}$ starting at $(1, A^{1/2}f_0)$ with initial velocity $(A^{-1/2}XA^{-1/2}, 0)$. Also, it is clear that $A^{1/2} \cdot \mathcal{M}^{\{f_0\}} = \mathcal{M}^{\{A^{1/2}f_0\}}$. Thus, by the previous case, $A^{1/2} \cdot \delta$ remains inside $A^{1/2} \cdot \mathcal{M}^{\{f_0\}}$ if and only if
$$A^{-1/2}XA^{-1/2}A^{1/2}f_0 = A^{-1/2}Xf_0 = 0,$$
i.e., $Xf_0 = 0$. □

Remark 5.9. Note that the argument in the above proof, yields the fact that the condition $Xf_0 = 0$ is equivalent to $Z = \kappa_{(A,f_0)}(X, 0)$ being A-selfadjoint. Indeed, in the case $A = 1$, $Z^* = Z$ is obtained explicitly as a necessary condition, and the condition Z normal as a sufficient condition. In the general case, using Lemma 5.3, we have that $A^{-1/2}ZA^{1/2} \in \mathbf{H}_{(1,A^{1/2})}$ is selfadjoint:
$$A^{-1/2}ZA^{1/2} = (A^{-1/2}ZA^{1/2})^* = A^{1/2}Z^*A^{-1/2} \iff AZ^*A^{-1} = Z,$$
i.e., Z is A-selfadjoint.

5.2 Subalgebras of $\mathcal{B}(\mathcal{H})$

Let us consider the case when $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ is a unital C*-algebra. The space of $\mathcal{A}_\bullet^+$ of positive and invertible elements in $\mathcal{A}$ is an open subset of $\{x \in \mathcal{A} : x^* = x\}$, which is a closed (non necesarily complemented) real-linear subspace of $\mathcal{B}_h(\mathcal{H})$. Thus $\mathcal{A}_\bullet^+$ is a submanifold of $\mathcal{G}^+(\mathcal{H})$, modelled in the Banach space $\{x \in \mathcal{A} : x^* = x\}$. Therefore, by Proposition 4.2,

$$\mathcal{M}_{\mathcal{A}_\bullet^+} = \{(a, g) \in \mathcal{M} : a \in \mathcal{A}_\bullet^+\}$$

is a submanifold of $\mathcal{M}$. Clearly, the invertible group $G_\mathcal{A}$ of $\mathcal{A}$ acts on this submanifold: if $g \in G_\mathcal{A}$ and $a \in \mathcal{A}_\bullet^+$, $(g^*)^{-1}ag^{-1} \in \mathcal{A}_\bullet^+$.

The algebra $\mathcal{A}$ is said to *act irreducibly in* $\mathcal{H}$ if there are no subspaces $\mathcal{S} \subset \mathcal{H}$ (other than $\mathcal{S} = \{0\}$ or $\mathcal{S} = \mathcal{H}$) such that $\mathcal{A}\mathcal{S} \subset \mathcal{S}$. Recall Kadison's transivity theorem (see for instance Theorem 5.2.2. in [9]): if $\mathcal{A}$ acts irreducibly in $\mathcal{H}$, $f, g \in \mathcal{H}$ and $f \neq 0$, then there exists $a \in \mathcal{A}$ such that $af = g$; if $\|f\| = \|g\|$, then a can be chosen unitary ($a^*a = aa^* = 1$).

Proposition 5.10. *If $\mathcal{A}$ acts irreducibly in $\mathcal{H}$ and $1 \in \mathcal{A}$, then the action of $G_\mathcal{A}$ on $\mathcal{M}_{\mathcal{A}_\bullet^+}$ is transitive.*

Proof. The argument is similar as the case of the whole algebra $\mathcal{B}(\mathcal{H})$. Namely, pick $g_0 \in \mathbb{S}(\mathcal{H})$, so that $(1, g_0) \in \mathcal{M}_{\mathcal{A}_\bullet^+}$. Pick any element $(a, f) \in \mathcal{M}_{\mathcal{A}_\bullet^+}$, so that also $a^{1/2}f \in \mathbb{S}(\mathcal{H})$. Then using Kadison's transitivity theorem, there exists a unitary element $u \in \mathcal{A}$ such that $ug_0 = a^{1/2}f_0$, and then $g = a^{-1/2}u \in G_\mathcal{A}$ satisfies

$$g \cdot (1, g_0) = (a, f)$$

as in Proposition 2.2. $\square$

The hypothesis that $\mathcal{A}$ is irreducible is necessary. Consider for instance $\mathcal{A} = C([0, 1])$ acting in $L^2(0, 1)$, and pick f a continuous function with $\|f\|_2 = 1$. Then the orbit $\{g \cdot (1, f) : g \in G_\mathcal{A}\}$ consists of pairs on which the the second coordinate gf is a continuous function, and thus the action is not transitive (note that $(1, h) \in \mathcal{M}_{\mathcal{A}_\bullet^+}$ for any $h \in L^2(0, 1)$ with $\|h\|_2 = 1$).

Recall the expression of the tangent spaces of $\mathcal{M}$ in Proposition 3.5:

$$(T\mathcal{M})_{(A_0,g_0)} = \{(Z, h) \in \mathcal{B}(\mathcal{H}) \times \mathcal{H} : Z^* = Z \text{ and } \langle Zg_0, g_0\rangle + 2\ Re\langle Ah, g_0\rangle = 0\}.$$

Then clearly

$$(T\mathcal{M}_{\mathcal{A}_\bullet^+})_{(a_0,g_0)} = \{(z, h) \in \mathcal{A} \times \mathcal{H} : z^* = z \text{ and } \langle zg_0, g_0\rangle + 2\ Re\langle a_0h, g_0\rangle = 0\}. \tag{17}$$

Theorem 5.11. *Suppose that the C*-algebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ contains the compact operators. Then:*

1. *For any $(a, f) \in \mathcal{M}_{\mathcal{A}_\bullet^+}$, the map*

$$m_{(a,f)} : \mathcal{A}_\bullet \to \mathcal{O}_{(a,f)} = \{g \cdot (a, f) : g \in G_\mathcal{A}\} \subset \mathcal{M}_{\mathcal{A}_\bullet^+}$$

is a C^∞ submersion. The orbit $\mathcal{O}_{(a,f)}$ is a union of connected components of $\mathcal{M}_{\mathcal{A}_\bullet^+}$.

2. *If $(a, f) \in \mathcal{M}_{\mathcal{A}_\bullet^+}$ and $(x, h) \in (T\mathcal{A}_\bullet^+)_{(a,f)}$, then the unique geodesic δ of $\mathcal{M}$ with $\delta(0) = (a, f)$ and $\dot\delta(0) = (x, h)$, satisfies that $\delta(t) \in \mathcal{M}_{\mathcal{A}_\bullet^+}$ for all $t \in \mathbb{R}$.*

Proof. To prove 1., note that $m_{(a,f)}$ has continuous local cross sections (with values in $G_{\mathcal{A}}$). More specifically, the construction of local cross sections done in Remark 2.1 and Proposition 2.2, takes values in $G_{\mathcal{A}}$, if the data are taken in $\mathcal{M}_{\mathcal{A}^+_\bullet}$. Indeed, if $f, g \in \mathbb{S}_1(\mathcal{H})$, since $\mathcal{A}$ contains the compact operators, $f \otimes f, g \otimes g \in \mathcal{A}$; if in addition $\|f \otimes f - g \otimes g\| < 1$, the unitary operator U such that $Uf = g$ constructed in remark 2.1, belongs to $\mathcal{A}$. Therefore it is also clear that the cross sections constructed in Proposition 2.2 also take values in $\mathcal{A}$.

To prove 2., similarly, if $a, x \in \mathcal{A}$, since $\mathcal{A}$ contains the compact operators, the rank one operators $h \otimes_a f, f \otimes_a (a^{-1}xf - h)$ and $f \otimes_a f$ belong to $\mathcal{A}$. Therefore $\kappa_{(a,f)}(x, h) \in \mathcal{A}$, and the proof follows.

□

Remark 5.12. Note that if $\mathcal{K}(\mathcal{H}) \subset \mathcal{A} \subset \mathcal{B}(\mathcal{H})$, then $\mathcal{A}$ acts irreducibly in $\mathcal{H}$ (it is well known that $\mathcal{K}(\mathcal{H})$ acts ireducibly in $\mathcal{H}$). Examples of unital subalgebras of $\mathcal{B}(\mathcal{H})$ containing the compacts are:

- The unitization $\hat{\mathcal{K}}(\mathcal{H})$ of $\mathcal{K}(\mathcal{H})$,

$$\hat{\mathcal{K}}(\mathcal{H}) = \{\lambda 1 + K : \lambda \in \mathbb{C}, K \in \mathcal{K}(\mathcal{H})\}.$$

- For $\mathcal{H} = \ell^2 = \ell^2(\mathbb{N})$, the C*-algebra $C^*(S)$ generated by the shift operator $S : \ell^2 \to \ell^2$, $S(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$.

6 $\mathcal{G}(\mathcal{H})$-invariant metric for $\mathcal{M}$

Let us introduce the following natural metric in $\mathcal{M}$:

Definition 6.1. *Let $(A, f) \in \mathcal{M}$ and $(Z, h) \in (T\mathcal{M})_{(A,f)}$. Put*

$$|(Z, h)|_{(A,f)} := \{\|A^{-1/2}ZA^{-1/2}\|^2 + \|A^{1/2}h\|^2\}^{1/2}.$$

Remark 6.2. The metric just defined is the natural metric of $\mathcal{G}^+(\mathcal{H})$ (see [2]), times the natural norm of the sphere $\mathbb{S}_A(\mathcal{H})$.

Proposition 6.3. *The metric defined in (6.1) is invariant under the action of $\mathcal{G}(\mathcal{H})$: if $(A, f) \in \mathcal{M}$, $(Z, h) \in (T\mathcal{M})_{(A,f)}$ and $G \in \mathcal{G}(\mathcal{H})$,*

$$|G \cdot (Z, h)|_{g\cdot(A,f)} = |(Z, h)|_{(A,f)}.$$

Proof. We must prove first that

$$\|[(G^*)^{-1}AG^{-1}]^{-1/2}(G^*)^{-1}ZG^{-1}[(G^*)^{-1}AG^{-1}]^{-1/2}\| = \|A^{-1/2}ZA^{-1/2}\|.$$

This fact was shown in [2], and it is one of the main features of the Geometry of $\mathcal{G}^+(\mathcal{H})$. We include the computation. Note that

$$\|A^{-1/2}ZA^{-1/2}\|^2 = \|A^{-1/2}ZA^{-1/2}A^{-1/2}ZA^{-1/2}\| = \|A^{-1/2}(ZA^{-1}Z)A^{-1/2}\|.$$

Since $Z^* = Z$, $ZA^{-1}Z$ is positive, and we can take its square root. Thus the above norm equals

$$= \|A^{-1/2}(ZA^{-1}Z)^{1/2}[A^{-1/2}(ZA^{-1}Z)^{1/2}]^*\| = \|[A^{-1/2}(ZA^{-1}Z)^{1/2}]^*A^{-1/2}(ZA^{-1}Z)^{1/2}\|$$

$$= \|(ZA^{-1}Z)^{1/2}A^{-1}(ZA^{-1}Z)^{1/2}\|.$$

Using this version of the norm, applied to the element $G \cdot Z$ measured at $G \cdot A$ we get, after trivial simplifications

$$\|\{(G^{+})^{-1}ZA^{-1}ZG^{-1}\}^{1/2}GA^{-1}G^{*}\{(G^{+})^{-1}ZA^{-1}ZG^{-1}\}^{1/2}\| = \|BB^{*}\|,$$

for $B = \{(G^{+})^{-1}ZA^{-1}XG^{-1}\}^{1/2}GA^{-1/2}$, and therefore

$$\|BB^{*}\| = \|B^{*}B\| = \|A^{-1/2}G^{*}\{(G^{+})^{-1}ZA^{-1}ZG^{-1}\}^{1/2}\{(G^{+})^{-1}ZA^{-1}ZG^{-1}\}^{1/2}GA^{-1/2}\| =$$

$$\|A^{-1/2}ZA^{-1}ZA^{-1/2}\| = \|A^{-1/2}ZA^{-1/2}\|^{2}.$$

Next note that the square of the norm of Gh given by $(G^{*})^{-1}AG^{-1}$ is

$$\langle (G^{*})^{-1}AG^{-1}Gh, Gh\rangle = \langle (G^{*})^{-1}Ah, Gh\rangle = \langle Ah, h\rangle,$$

which finishes the proof. □

Remark 6.4. Note that with this metric just defined, the map (1)

$$\pi_{\mathcal{M}} : \mathcal{M} \to \mathcal{G}^{+}(\mathcal{H}),\ \pi_{\mathcal{M}}(A, g) = A$$

is contractive, if $\mathcal{G}^{+}(\mathcal{H})$ is considered with its natural metric (see (3) and Remark 1.1), given by

$$|Z|_{A} = \|A^{-1/2}ZA^{-1/2}\|,$$

for $A \in \mathcal{G}^{+}(\mathcal{H}), Z^{*} = Z$; that is

$$d_{\mathcal{G}^{+}(\mathcal{H})}(A, B) \leq d_{\mathcal{M}}((A, g), (B, g)),$$

if $(A, g), (B, h) \in \mathcal{M}$.

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