

Homogeneous spaces in Hartree-Fock-Bogoliubov theory

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Abstract

We study the action of Bogoliubov transformations on admissible generalized one-particle density matrices arising in Hartree-Fock-Bogoliubov theory. We show that the orbits of this action are reductive homogeneous spaces, and we give several equivalences that characterize when they are embedded submanifolds of natural ambient spaces. We use Lie theoretic arguments to prove that these orbits admit an invariant symplectic form. If, in addition, the operators in the orbits have finite spectrum, or infinite spectrum and trivial kernel, then we obtain that the orbits are actually Kähler homogeneous spaces.

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1 Introduction

Hartree-Fock-Bogoliubov (HFB) theory, also known as generalized Hartree-Fock theory, is a relevant tool for understanding fermionic many-body quantum systems. It is a generalization

of the traditional Hartree-Fock theory for systems in which the particle number is not assumed to be conserved. Although it has been studied earlier in the physics literature [9, 16, 44], pioneering work on rigorous mathematical aspects of HFB theory was done by Bach, Lieb and Solovej [7]. In the present paper we investigate the main objects in HFB theory from the point of view of infinite-dimensional geometry by using Lie groups modeled on Banach spaces and its homogeneous spaces.

We now briefly explain the motivation and framework related to HFB theory (see Appendix A.1 for precise definitions and details). Let \mathfrak{H} be the one-particle Hilbert space, and let $\mathfrak{F} = \mathfrak{F}[\mathfrak{H}]$ be its associated fermionic Fock space. Let \mathcal{Z} be the set of all normal states on \mathfrak{F} . Consider a Hamiltonian \mathbb{H} bounded from below on \mathfrak{F} describing the dynamics of a fermionic system. It is of physical importance to compute the following

$$E_{gs} := \inf\{\omega(\mathbb{H}) : \omega \in \mathcal{Z}\}.$$

This is known as the total ground state energy in the grand canonical ensemble. Approximation methods become relevant for such task, which turns out to be impossible in most cases. In HFB theory the total ground state energy is approximated by taking as trial states the *quasi-free states with finite particle number*, which is a class of states that obeys Wick's theorem. Let us denote by $\mathcal{Z}_{\text{qf}} \subseteq \mathcal{Z}$ the set of all quasi-free states with finite particle number on \mathfrak{F} . The HFB energy is then defined by

$$E_{HFB} := \inf\{\omega(\mathbb{H}) : \omega \in \mathcal{Z}_{\text{qf}}\}.$$

Clearly, we have the upper bound $E_{gs} \leq E_{HFB}$. Now a key point, which is convenient for the calculus of variations, is that one can rewrite the HFB in terms of an infimum over a convex set of operators acting on $\mathfrak{H} \oplus \mathfrak{H}$. Denote by $B_1(\mathfrak{H})$ and $B_2(\mathfrak{H})$ the trace-class and Hilbert-Schmidt operators on \mathfrak{H} , respectively. The set \mathcal{Z}_{qf} is indeed in a bijective correspondence with the set of all *admissible generalized one-particle density matrices (g1-pdm)*

$$\mathcal{D} := \left\{ \begin{pmatrix} \gamma & \alpha \\ \alpha^* & \mathbf{1} - \bar{\gamma} \end{pmatrix} \in B(\mathfrak{H} \oplus \mathfrak{H}) : 0 \leq \begin{pmatrix} \gamma & \alpha \\ \alpha^* & \mathbf{1} - \bar{\gamma} \end{pmatrix} \leq \mathbf{1}_{\mathfrak{H} \oplus \mathfrak{H}}, \gamma = \gamma^* \in B_1(\mathfrak{H}), \alpha^T = -\alpha \right\},$$

where we write $\bar{\gamma} = I_0 \alpha I_0$, $\bar{\alpha} = I_0 \alpha I_0$, $\alpha^T = \bar{\alpha}^*$, and I_0 is a fixed conjugate-linear isometry on \mathfrak{H} . Thus, each $\Gamma = \Gamma[\gamma, \alpha] \in \mathcal{D}$, is described by an operator $\gamma \in B_1(\mathfrak{H})$, $0 \leq \gamma \leq \mathbf{1}$, called the *one-particle density matrix*, and another operator α , called the *pairing matrix*. We observe that one can check that $\alpha \in B_2(\mathfrak{H})$ by elementary computations. Suppose that $\omega \mapsto \Gamma_\omega$ denotes the correspondence between \mathcal{Z}_{qf} and \mathcal{D} , and define the HFB functional by $\mathcal{E}(\Gamma_\omega) := \omega(\mathbb{H})$, then the HFB energy can be rewritten in terms of admissible g1-pdms as

$$E_{HFB} = \inf\{\mathcal{E}(\Gamma) : \Gamma \in \mathcal{D}\}. \quad (1)$$

Let $U(\mathfrak{H} \oplus \mathfrak{H})$ be the unitary group acting on $\mathfrak{H} \oplus \mathfrak{H}$. An useful fact in HFB theory is that the so-called *Bogoliubov transformations satisfying the Shale-Stinespring condition* given by

$$U_{\text{Bog}} := \left\{ \begin{pmatrix} u & v \\ \bar{v} & \bar{u} \end{pmatrix} \in U(\mathfrak{H} \oplus \mathfrak{H}) : v \in B_2(\mathfrak{H}) \right\}$$

defines an action by conjugation

$$U_{\text{Bog}} \times \mathcal{D} \rightarrow \mathcal{D}, \quad U \cdot \Gamma = U \Gamma U^*, \quad U \in U_{\text{Bog}}, \Gamma \in \mathcal{D}.$$

The bijection between quasi-free states and g1-pdms turns out to be an equivariant map, where in the set of quasi-free states one has the action of the subgroup of the unitary group on \mathfrak{F} known as the unitary implementers.

Appart from the previously mentioned work [7], other mathematical results in HFB theory include the following ones. In treating the existence of minimizers in (1) a serious difficulty one has to face is the lack of weak lower semicontinuity of the HFB functional. Remarkably, the existence of minimizers was established by Lenzmann and Lewin [29] for pseudorelativistic fermions interacting with Newtonian gravitational forces. On the other hand, the occurrence of pairing is a phenomena to be understood within HFB theory. This amounts to know when there exist minimizers of the form $\Gamma = \Gamma[\gamma, \alpha]$, $\alpha \neq 0$. Bach, Fröhlich and Jonsson [8] developed a simplification of HFB theory under certain abstract assumptions, to find that the pairing matrix can be expressed in terms of the one-particle matrix. In [6] a generalized Lieb's variational principle was proved, which roughly stated, means that in (1) one can only consider the infimum over those admissible g1-pdms that are projections ($\Gamma^2 = \Gamma$). Finally, we refer to [30] for a numerical approach to the pairing and other problems in HFB theory, and the recent survey [5] for the aspects mentioned in this paragraph and others in relation to HFB theory.

Throughout this paper, \mathfrak{H} is assumed to be an infinite-dimensional Hilbert space. We study the geometric structure of orbits given by the conjugacy action of Bogoliubov transformations on admissible g1-pdms: for $\Gamma \in \mathcal{D}$,

$$\mathcal{O}(\Gamma) := \{UTU^* : U \in \mathbf{U}_{\text{Bog}}\}.$$

We will see that these orbits are concrete examples of infinite-dimensional homogeneous spaces related to quantum mechanics. Geometric structures such as symplectic forms, complex or Kähler homogeneous structures that are pervasive in classical mechanics will be constructed for orbits of g1-pdms by using Lie algebraic arguments. Such kind of arguments are based on general results in the existing literature of infinite-dimensional homogeneous spaces (see for instance [12, 34, 35, 43]). Let us point out that the geometry of homogeneous spaces in the traditional Hartree-Fock was considered in [21]; meanwhile in the physics literature, the above orbits of g1-pdms were investigated in [39] under the assumption that they can be modeled as finite-dimensional manifolds.

In Section 2 we study the reductive homogeneous space structure of orbits of admissible g1-pdms and quasi-free states with finite particle number. We first recall some useful results of the Lie group structure of the Bogoliubov transformations that are scattered in the literature on both Lie groups and HFB theory. In fact, \mathbf{U}_{Bog} belongs to the class of restricted Lie groups studied by Neeb [33], and its Lie algebra is identified with

$$\mathfrak{u}_{\text{Bog}} := \left\{ \begin{pmatrix} x_1 & x_2 \\ \bar{x}_2 & \bar{x}_1 \end{pmatrix} \in \mathbf{B}(\mathfrak{H} \oplus \mathfrak{H}) : x_1 = -x_1^*, x_2 = -x_2^T \in \mathbf{B}_2(\mathfrak{H}) \right\},$$

which has the following restricted norm

$$\left\| \begin{pmatrix} x_1 & x_2 \\ \bar{x}_2 & \bar{x}_1 \end{pmatrix} \right\|_{\text{res}} := 2 \max\{\|x_1\|, \|x_2\|_2\}.$$

We show that the orbits $\mathcal{O}(\Gamma) \simeq \mathbf{U}_{\text{Bog}}/\mathbf{U}_{\text{Bog}}^\Gamma$ can be endowed with the structure of reductive homogeneous space, where $\mathbf{U}_{\text{Bog}}^\Gamma$ is the isotropy group at $\Gamma \in \mathcal{D}$. We also prove that $\mathcal{O}(\Gamma)$ is

connected if and only if $\frac{1}{2}$ is not an eigenvalue of Γ . Otherwise, $\mathcal{O}(\Gamma)$ has two connected components (see Theorem 2.9). To construct the infinite-dimensional homogeneous space structure one has to find an accessible expression of the isotropy group U_{Bog}^Γ to show that its Lie algebra is a complemented subspace of $\mathfrak{u}_{\text{Bog}}$. This can be done thanks to a well-known diagonalization result for g1-pdms by Bogoliubov transformations [7], which implies that each orbit has a diagonal g1-pdm. As a direct consequence of the mentioned correspondence between g1-pdms and quasi-free states, we obtain that orbits of quasi-free states with finite particle number are reductive homogeneous spaces. We observe that the projection P_- on $\mathfrak{H} \oplus \mathfrak{H}$ defined by $P_-(f, g) = (0, g)$ is a g1-pdm, and the orbit $\mathcal{O}(P_-)$ is actually the isotropic restricted Grassmannian related to loop groups (see [38]).

In Section 3 we give several equivalent characterizations to guarantee that orbits of g1-pdms are embedded submanifolds of natural affine spaces. Besides the extrinsic geometric structure of the orbits as homogeneous spaces, which are in this way equipped with the quotient topology, it is desirable to give a more intrinsic type of manifold structure on them. The following inclusion is not difficult to check

$$\mathcal{O}(\Gamma) \subseteq i\mathfrak{u}_{\text{Bog}} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} := i\mathfrak{u}_{\text{Bog}} + P_-.$$

This gives a relative topology on g1-pdms inherited from $i\mathfrak{u}_{\text{Bog}} + P_-$, whose restricted metric is defined by $d(\Gamma_0, \Gamma_1) = \|\Gamma_0 - \Gamma_1\|_{\text{res}}$. We prove that the following conditions are equivalent: tangent spaces of $\mathcal{O}(\Gamma)$ are closed in $i\mathfrak{u}_{\text{Bog}}$, the quotient topology and the relative topology coincide in $\mathcal{O}(\Gamma)$, and Γ has finite spectrum. Furthermore, these are also equivalent to being $\mathcal{O}(\Gamma)$ an embedded submanifold of $i\mathfrak{u}_{\text{Bog}} + P_-$ (Theorem 3.9). In the process, we discuss derivations induced by g1-pdms defined as follows: for each $\Gamma \in \mathcal{D}$, set $\delta_\Gamma : \mathfrak{u}_{\text{Bog}} \rightarrow \mathfrak{u}_{\text{Bog}}$, $\delta_\Gamma(X) = [i\Gamma, X]$, for all $X \in \mathfrak{u}_{\text{Bog}}$. These, in turn, are useful in the construction of continuous local cross sections for the action that allow us to compare the two mentioned topologies. This circle of ideas has recently been studied for unitary orbits in operator ideals by Beltiță and Larotonda (see [13] and the references therein). Also our results on the topology of g1-pdms might be considered as a natural extension of similar ones related to one-particle density matrices (see [17, 26]). In contrast to the aforementioned works that deal with unitary invariant metrics, notice that the action of Bogoliubov transformations is not invariant for the previous restricted metric.

In Section 4 we construct an invariant symplectic form on the orbits. We introduce for each $\Gamma \in \mathcal{D}$ the continuous 2-cocycle $s_\Gamma : \mathfrak{u}_{\text{Bog}} \times \mathfrak{u}_{\text{Bog}} \rightarrow \mathbb{R}$ defined by

$$s_\Gamma(X, Y) := \text{Tr}(X[i\Gamma, Y]).$$

Then a standard construction can be applied to obtain a weakly symplectic homogeneous space $(\mathcal{O}(\Gamma), \omega)$, where $\omega = \Sigma(s_\Gamma)$ is the symplectic form induced by s_Γ . In the case in which Γ has finite spectrum, we show that ω is a strong symplectic form. Furthermore, we prove in Theorem 4.12 that $(\mathcal{O}(\Gamma), \omega)$ is a Kähler homogeneous space in the following two cases: when the spectrum of Γ is finite, or when the spectrum of Γ is infinite and $\ker(\Gamma) = \{0\}$. This is shown by using the notion of Kähler polarizations. In this regard, we refer to [12, 34] for general results in the infinite-dimensional setting, and to [10, 14, 34] for interesting examples.

We include an appendix to make the exposition more self-contained. It contains notation and necessary results on HFB theory, and it treats the needed geometric structures in the setting of Banach manifolds.

2 Reductive homogeneous spaces

2.1 Lie group structure of Bogoliubov transformations

We assume that \mathfrak{H} is an infinite-dimensional separable complex Hilbert space, and we consider the group of Bogoliubov transformations satisfying the Shale-Stinespring condition, namely

$$U_{\text{Bog}} := \left\{ \begin{pmatrix} u & v \\ \bar{v} & \bar{u} \end{pmatrix} \in U(\mathfrak{H} \oplus \mathfrak{H}) : v \in B_2(\mathfrak{H}) \right\}.$$

The conjugate-linear isometry I_0 that defines $\bar{x} = I_0 x I_0$ for every $x \in B(\mathfrak{H})$ is associated to a fixed orthonormal basis $\{\varphi_k\}_{k \geq 1}$ of \mathfrak{H} . This means that I_0 is defined by $I_0(\sum_{k \geq 1} \mu_k \varphi_k) = \sum_{k \geq 1} \bar{\mu}_k \varphi_k$, for every sequence $\{\mu_k\}_{k \geq 1} \in \ell^2(\mathbb{C})$.

We begin by recalling several topological and geometric properties of the group U_{Bog} . Many of them are indeed well-known in the Lie groups literature; our presentation intend to relate them with the Hartree-Fock-Bogoliubov framework in the most convenient way to our purposes. Among other aspects we observe the relation of Bogoliubov transformations with an orthogonal group that appears in the Clifford algebra formulation. Also it turns out that U_{Bog} fits into the class of the so-called *restricted Lie groups*. Our main reference is the work of Neeb ([33]), where the reader can find a detailed account about these groups and its Lie algebras.

Remark 2.1. We recall a Banach algebra and two Lie groups related to U_{Bog} .

i) The restricted algebra. Let P_{\pm} be the orthogonal projections on $\mathfrak{H} \oplus \mathfrak{H}$ defined by $P_+(f, g) = (f, 0)$ and $P_-(f, g) = (0, g)$, for all $f, g \in \mathfrak{H}$. We set

$$D := P_+ - P_- = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}.$$

Consider the following *restricted algebra associated with D and the Hilbert-Schmidt operators*:

$$\begin{aligned} B_2(\mathfrak{H} \oplus \mathfrak{H}, D) &:= \{X \in B(\mathfrak{H} \oplus \mathfrak{H}) : [X, D] \in B_2(\mathfrak{H} \oplus \mathfrak{H})\} \\ &= \left\{ \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \in B(\mathfrak{H} \oplus \mathfrak{H}) : x_{12}, x_{21} \in B_2(\mathfrak{H}) \right\}. \end{aligned}$$

This is a complex Banach $*$ -algebra equipped with the norm

$$\|X\|_{\text{res}} := 2 \max\{\|x_{11}\|, \|x_{22}\|, \|x_{12}\|_2, \|x_{21}\|_2\}, \quad (2)$$

where $\|\cdot\|$ and $\|\cdot\|_2$ are the operator norm and the Hilbert-Schmidt norm, respectively. We refer to this norm as the *restricted norm*. The factor 2 is added to get the submultiplicative property of this norm, which can be checked by elementary estimates with the operator and Hilbert-Schmidt norms. Notice that $U_{\text{Bog}} \subseteq B_2(\mathfrak{H} \oplus \mathfrak{H}, D)$. We remark that multiplication by unitaries in U_{Bog} is not isometric for the restricted norm, i.e. in general $\|UX\|_{\text{res}} \neq \|X\|_{\text{res}}$ and $\|XU\|_{\text{res}} \neq \|X\|_{\text{res}}$, for $U \in U_{\text{Bog}}$ and $X \in B_2(\mathfrak{H} \oplus \mathfrak{H}, D)$.

ii) A restricted general linear group. Recalling that I_0 is the conjugation associated to the fixed orthonormal basis $\{\varphi_k\}_{k \geq 1}$, we put

$$I := \begin{pmatrix} 0 & I_0 \\ I_0 & 0 \end{pmatrix}. \quad (3)$$

Clearly, I is a conjugation acting on $\mathfrak{H} \oplus \mathfrak{H}$. Also note that $I^* = I$. As I is conjugate-linear this means that $\langle (f_1, f_2), I(g_1, g_2) \rangle = \langle (g_1, g_2), I(f_1, f_2) \rangle$, for all $(f_1, f_2), (g_1, g_2) \in \mathfrak{H} \oplus \mathfrak{H}$. Let $\text{GL}(\mathfrak{H} \oplus \mathfrak{H})$ denotes the group of linear invertible operators on $\mathfrak{H} \oplus \mathfrak{H}$. As a particular case of [33, Def. III.3] obtained by taking the aforementioned conjugation I and self-adjoint operator D , we consider the following *restricted general linear group*

$$\begin{aligned} \text{GL}_2(\mathfrak{H} \oplus \mathfrak{H}, I, D) &:= \{G \in \text{GL}(\mathfrak{H} \oplus \mathfrak{H}) : G^{-1} = IG^*I, [G, D] \in \text{B}_2(\mathfrak{H} \oplus \mathfrak{H})\} \\ &= \left\{ \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \in \text{GL}(\mathfrak{H} \oplus \mathfrak{H}) : \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}^{-1} = \begin{pmatrix} g_{22}^T & g_{12}^T \\ g_{21}^T & g_{11}^T \end{pmatrix}, g_{12}, g_{21} \in \text{B}_2(\mathfrak{H}) \right\} \end{aligned}$$

This group carries a Lie group structure endowed with the restricted norm. As a direct consequence of the matrix description of $\text{GL}_2(\mathfrak{H} \oplus \mathfrak{H}, I, D)$, notice that

$$\begin{aligned} \text{U}_{\text{Bog}} &= \{U \in \text{U}(\mathfrak{H} \oplus \mathfrak{H}) : U = IUI, [U, D] \in \text{B}_2(\mathfrak{H} \oplus \mathfrak{H})\} \\ &= \text{U}(\mathfrak{H} \oplus \mathfrak{H}) \cap \text{GL}_2(\mathfrak{H} \oplus \mathfrak{H}, I, D). \end{aligned} \tag{4}$$

The Lie algebra of $\text{GL}_2(\mathfrak{H} \oplus \mathfrak{H}, I, D)$ has the usual commutator of operators as Lie bracket, and it is given by

$$\begin{aligned} \mathfrak{gl}_2(\mathfrak{H} \oplus \mathfrak{H}, I, D) &:= \{X \in \text{B}(\mathfrak{H} \oplus \mathfrak{H}) : X = -IX^*I, [X, D] \in \text{B}_2(\mathfrak{H} \oplus \mathfrak{H})\} \\ &= \left\{ \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & -x_{11}^T \end{pmatrix} \in \text{B}(\mathfrak{H} \oplus \mathfrak{H}) : x_{12} = -x_{12}^T \in \text{B}_2(\mathfrak{H}), x_{21} = -x_{21}^T \in \text{B}_2(\mathfrak{H}) \right\}. \end{aligned} \tag{5}$$

Finally, we note that $\mathfrak{gl}_2(\mathfrak{H} \oplus \mathfrak{H}, I, D)$ is an involutive Lie algebra. This means that is a complex Lie algebra endowed with an involution $*$ such that $(X^*)^* = X$ and $[X, Y]^* = [Y^*, X^*]$. The involution is given by the usual operator adjoint. In particular, the involution determines a real form of \mathfrak{g} , which is the real subalgebra defined by

$$\mathfrak{gl}_2(\mathfrak{H} \oplus \mathfrak{H}, I, D)_{\mathbb{R}} := \{X \in \mathfrak{gl}_2(\mathfrak{H} \oplus \mathfrak{H}, I, D) : X^* = -X\}.$$

The Lie algebra $\mathfrak{gl}_2(\mathfrak{H} \oplus \mathfrak{H}, I, D)$ will be used in Section 4 for the construction of Kähler structures on orbits of gl-pdms.

iii) The restricted orthogonal group. We write $\mathfrak{H}^{\mathbb{R}}$ for the underlying real Hilbert space, and let $J_0 : \mathfrak{H}^{\mathbb{R}} \rightarrow \mathfrak{H}^{\mathbb{R}}$, $J_0 f = if$, be its complex structure. Let $\text{GL}(\mathfrak{H}^{\mathbb{R}})$ be the group of real linear invertible operators on $\mathfrak{H}^{\mathbb{R}}$. The *restricted orthogonal group* is defined by

$$\text{O}_{\text{res}}(\mathfrak{H}^{\mathbb{R}}) := \{O \in \text{GL}(\mathfrak{H}^{\mathbb{R}}) : O^T = O^{-1}, [O, J_0] \in \text{B}_2(\mathfrak{H}^{\mathbb{R}})\}.$$

Here we write O^T for the transpose of O relative to the inner product $\Re \langle \cdot, \cdot \rangle$ of $\mathfrak{H}^{\mathbb{R}}$. Several properties of this group were studied in [20, 33, 36, 38, 40]. In particular, $\text{O}_{\text{res}}(\mathfrak{H}^{\mathbb{R}})$ is a Lie group having two connected components. Its topology is defined by the norm $\|O\|_{\text{res}} := \max\{\|O\|, \|[J_0, O]\|_2\}$, $O \in \text{O}_{\text{res}}(\mathfrak{H}^{\mathbb{R}})$, and its Lie algebra is given by

$$\mathfrak{o}_{\text{res}}(\mathfrak{H}^{\mathbb{R}}) := \{A \in \text{B}(\mathfrak{H}^{\mathbb{R}}) : A^T = -A, [A, J_0] \in \text{B}_2(\mathfrak{H}^{\mathbb{R}})\}.$$

Given an operator $A \in \text{B}(\mathfrak{H}^{\mathbb{R}})$, we may write $A = A_c + A_a$, where $A_a := \frac{1}{2}(A + J_0 A J_0)$ and $A_c := \frac{1}{2}(A - J_0 A J_0)$ are called the *antilinear part* (or *conjugate-linear part*) and *linear part* of A , respectively. Observe that the maps $A \mapsto A_a$ and $A \mapsto A_c$ are projections, $A_a J_0 = -J_0 A_a$ and $A_c J_0 = J_0 A_c$. For $A \in \text{B}(\mathfrak{H}^{\mathbb{R}})$ such that $A^T = -A$, note that $A \in \mathfrak{o}_{\text{res}}(\mathfrak{H}^{\mathbb{R}})$ if and only $A_a \in \text{B}_2(\mathfrak{H}^{\mathbb{R}})$.

Let $\mathfrak{u}(\mathfrak{H} \oplus \mathfrak{H})$ denote the Lie algebra of $U(\mathfrak{H} \oplus \mathfrak{H})$ consisting of all skew-adjoint operators. In the next proposition we collect several facts on the group U_{Bog} , including its relation with the above restricted general linear and orthogonal groups. We abbreviate $G := GL_2(\mathfrak{H} \oplus \mathfrak{H}, I, D)$ and $\mathfrak{g} := \mathfrak{gl}_2(\mathfrak{H} \oplus \mathfrak{H}, I, D)$.

Proposition 2.2. U_{Bog} is a real Lie subgroup of G , whose Lie algebra is given by

$$\begin{aligned} \mathfrak{u}_{\text{Bog}} &:= \{ X \in \mathfrak{u}(\mathfrak{H} \oplus \mathfrak{H}) : X = IXI, [X, D] \in B_2(\mathfrak{H} \oplus \mathfrak{H}) \} \\ &= \left\{ \begin{pmatrix} x_1 & x_2 \\ \bar{x}_2 & \bar{x}_1 \end{pmatrix} \in B(\mathfrak{H} \oplus \mathfrak{H}) : x_1 = -x_1^*, x_2 = -x_2^T \in B_2(\mathfrak{H}) \right\} = \mathfrak{g}_{\mathbb{R}}. \end{aligned}$$

In particular, the topology of U_{Bog} is defined by the norm

$$\left\| \begin{pmatrix} u & v \\ \bar{v} & \bar{u} \end{pmatrix} \right\|_{\text{res}} = 2 \max\{\|u\|, \|v\|_2\}. \quad (6)$$

Furthermore, the following assertions hold:

i) If $U = \begin{pmatrix} u & v \\ \bar{v} & \bar{u} \end{pmatrix} \in U_{\text{Bog}}$, then u and \bar{u} are Fredholm operators of index zero.

ii) The map $\Xi : O_{\text{res}}(\mathfrak{H}^{\mathbb{R}}) \rightarrow U_{\text{Bog}}$ defined by

$$\Xi(O) = \begin{pmatrix} u & v \\ \bar{v} & \bar{u} \end{pmatrix} \text{ if and only if } O = u + vI_0, u = O_c, vI_0 = O_a, \quad (7)$$

is an isomorphism of Lie groups.

iii) Let $i\mathfrak{g}_{\mathbb{R}} := \{X \in \mathfrak{g} : X = X^*\}$ be the self-adjoint part of \mathfrak{g} . The polar decomposition given by

$$U_{\text{Bog}} \times i\mathfrak{g}_{\mathbb{R}} \rightarrow G, \quad (U, X) \mapsto Ue^X,$$

is a diffeomorphism and the inclusion $U_{\text{Bog}} \hookrightarrow G$ is a homotopy equivalence.

iv) $\pi_0(U_{\text{Bog}}) = \mathbb{Z}_2$. The higher homotopy groups of U_{Bog} are 8-periodic, and consequently, they are determined by $\pi_i(U_{\text{Bog}}^0) = \mathbf{0}$, $i = 1, 3, 4, 5$, $\pi_i(U_{\text{Bog}}^0) = \mathbb{Z}$, $i = 2, 6$, and $\pi_i(U_{\text{Bog}}^0) = \mathbb{Z}_2$, $i = 7, 8$.

v) There is a \mathbb{Z}_2 -valued index map defined by

$$\text{ind}_{U_{\text{Bog}}} : U_{\text{Bog}} \rightarrow \mathbb{Z}_2, \quad \text{ind}_{U_{\text{Bog}}} \left(\begin{pmatrix} u & v \\ \bar{v} & \bar{u} \end{pmatrix} \right) = \dim \ker(u) \pmod{2}.$$

This is a continuous morphism of groups, which parametrizes the two connected components of U_{Bog} .

Proof. First, it is easy to see that U_{Bog} is closed in G and $\mathfrak{u}_{\text{Bog}} = \{X \in \mathfrak{g} : e^{tX} \in U_{\text{Bog}}, \forall t \in \mathbb{R}\}$. Also note that $\mathfrak{u}_{\text{Bog}} = \mathfrak{g}_{\mathbb{R}} := \{X \in \mathfrak{g} : X^* = -X\}$ is the real form of \mathfrak{g} , which has a closed supplement in \mathfrak{g} given by $i\mathfrak{g}_{\mathbb{R}} := \{X \in \mathfrak{g} : X^* = X\}$. According to Theorem A.10, it remains to be shown that there are open sets $0 \in \mathcal{W} \subseteq \mathfrak{g}$, $\mathbf{1} = \mathbf{1}_{\mathfrak{H} \oplus \mathfrak{H}} \in \mathcal{V} \subseteq G$, such that the exponential map $\exp : \mathcal{W} \rightarrow \mathcal{V}$ is a diffeomorphism satisfying $\exp(\mathcal{W} \cap \mathfrak{u}_{\text{Bog}}) = \mathcal{V} \cap U_{\text{Bog}}$. The non trivial inclusion

is ' \supseteq '. Pick $U = e^X \in \mathcal{U}_{\text{Bog}}$, for some $X \in \mathcal{W}$. Eventually shrinking \mathcal{V} so that $\|U - 1\|_{\text{res}} < 1$, and then using analytic functional calculus in the Banach algebra $B_2(\mathfrak{H} \oplus \mathfrak{H}, D)$, one gets that $X = \log(U) = \sum_{n \geq 1} (-1)^{n+1} \frac{(U-1)^n}{n} \in B_2(\mathfrak{H} \oplus \mathfrak{H}, D)$. By this expression in terms of a series, it follows that $X^* = -X$ and $X = IXI$. Therefore, $X \in \mathfrak{u}_{\text{Bog}}$. Hence \mathcal{U}_{Bog} is a Lie subgroup of G , whose Lie algebra is given by $\mathfrak{u}_{\text{Bog}}$.

i) Using the expressions in (25), and noting that $v \in B_2(\mathfrak{H})$ and thus a compact operator, it follows that u is invertible in the Calkin algebra. Hence u and \bar{u} are Fredholm operators. We write $\text{ind}(u) = \dim \ker(u) - \dim \ker(u^*)$ for its Fredholm index. But U is unitary, so that it is a Fredholm operator of index zero on $\mathfrak{H} \oplus \mathfrak{H}$. It is a well-known property of the index of block operator matrices that one can calculate $0 = \text{ind}(U) = \text{ind}(u) + \text{ind}(\bar{u})$. Since $\text{ind}(u) = \text{ind}(\bar{u})$, it follows that $\text{ind}(u) = \text{ind}(\bar{u}) = 0$.

ii) We follow the construction in [33, Section IV.2] (see also [37]). We begin by taking the complexification $\mathfrak{H}_{\mathbb{C}} := (\mathfrak{H}^{\mathbb{R}})_{\mathbb{C}}$, and denote by J the extension of J_0 to $\mathfrak{H}_{\mathbb{C}}$. One possible way to introduce the complexification is to put $\mathfrak{H}_{\mathbb{C}} = \mathfrak{H}^{\mathbb{R}} \times \mathfrak{H}^{\mathbb{R}}$ as a set equipped with the usual sum, multiplication by complex scalars and inner product. The extension of an arbitrary linear operator A on $\mathfrak{H}^{\mathbb{R}}$ to $\mathfrak{H}_{\mathbb{C}}$ is given by $A_{\mathbb{C}}(f, g) = (Af, Ag)$, for $f, g \in \mathfrak{H}^{\mathbb{R}}$. Notice that we can decompose the space as $\mathfrak{H}_{\mathbb{C}} = \mathfrak{H}_{\mathbb{C}}^+ \oplus \mathfrak{H}_{\mathbb{C}}^-$, where $\mathfrak{H}_{\mathbb{C}}^{\pm}$ are the $\pm i$ -eigenspaces of J . Each vector in $\mathfrak{H}_{\mathbb{C}}^{\pm}$ can be written as $(f, \mp if)$. It can be checked that the operator $U : \mathfrak{H} \rightarrow \mathfrak{H}_{\mathbb{C}}^+$, $Uf = \frac{1}{\sqrt{2}}(f, -if)$ is a complex linear surjective isometry. Next we define the conjugation $C : \mathfrak{H}_{\mathbb{C}} \rightarrow \mathfrak{H}_{\mathbb{C}}$, $C((f, g)) = (f, -g)$. Clearly, we have that $\mathfrak{H}^{\mathbb{R}} \cong \mathfrak{H}^{\mathbb{R}} \times \{0\} = \{(f, g) \in \mathfrak{H}_{\mathbb{C}} : C((f, g)) = (f, g)\}$. Also note that $C\mathfrak{H}_{\mathbb{C}}^{\pm} = \mathfrak{H}_{\mathbb{C}}^{\mp}$. There is an isometric isomorphism between $\mathfrak{H} \oplus \mathfrak{H}$ and $\mathfrak{H}_{\mathbb{C}} = \mathfrak{H}_{\mathbb{C}}^+ \oplus \mathfrak{H}_{\mathbb{C}}^-$ given by

$$T : \mathfrak{H} \oplus \mathfrak{H} \rightarrow \mathfrak{H}_{\mathbb{C}}, \quad T((f, g)) = Uf + CU I_0 g = \frac{1}{\sqrt{2}}(f + \bar{g}, i(\bar{g} - f)).$$

By using this isomorphism we can identify operators on $\mathfrak{H} \oplus \mathfrak{H}$ with operators on $\mathfrak{H}_{\mathbb{C}}$, i.e. $B(\mathfrak{H} \oplus \mathfrak{H}) \rightarrow B(\mathfrak{H}_{\mathbb{C}})$, $X \mapsto \text{Ad}_T X = T X T^{-1}$. Let $\mathcal{U}(\mathfrak{H}_{\mathbb{C}})$ be the unitary group of $\mathfrak{H}_{\mathbb{C}}$. The extension of orthogonal operators to $\mathfrak{H}_{\mathbb{C}}$ gives the following isomorphism between the Lie groups

$$\mathcal{O}_{\text{res}}(\mathfrak{H}^{\mathbb{R}}) \rightarrow \{U \in \mathcal{U}(\mathfrak{H}_{\mathbb{C}}) : UC = CU, [U, J] \in B_2(\mathfrak{H}_{\mathbb{C}})\}, \quad O \mapsto O_{\mathbb{C}}.$$

Therefore, there is a Lie group isomorphism given by

$$\Xi : \mathcal{O}_{\text{res}}(\mathfrak{H}^{\mathbb{R}}) \rightarrow \mathcal{U}_{\text{Bog}}, \quad \Xi(O) = T^{-1} O_{\mathbb{C}} T. \quad (8)$$

We only observe that the map Ξ actually takes values in \mathcal{U}_{Bog} , which can be deduced from the characterization given in (4), and noting that $iD = \text{Ad}_{T^{-1}} J$ and $I = \text{Ad}_{T^{-1}} C$. All the properties of Lie group isomorphism are straightforward to check. Finally, we remark that this isomorphism can be written in terms of the block matrix operators, and the linear and antilinear parts as in Eq. (7).

iii) These are proved in [33, Prop. III.2] and [33, Prop. III.8].

iv) This is [33, Prop. III.14] combined with the homotopy equivalence of item iii).

v) For instance, we refer to [4] for the construction of this \mathbb{Z}_2 -valued index map. Alternatively, an index was given in [20] for a general restricted orthogonal group on a real Hilbert space with a complex structure. For our particular real Hilbert space $\mathfrak{H}^{\mathbb{R}}$ with complex structure J_0 the mentioned index is given by $\text{ind}_{\mathcal{O}_{\text{res}}} : \mathcal{O}_{\text{res}}(\mathfrak{H}^{\mathbb{R}}) \rightarrow \mathbb{Z}_2$, $\text{ind}_{\mathcal{O}_{\text{res}}}(O) = \dim_{\mathbb{C}} \ker(O - J_0 O J_0) \pmod{2}$, where the subscript \mathbb{C} means that the complex dimension must be computed. One can verify that $\text{ind}_{\mathcal{U}_{\text{Bog}}}(U) = \text{ind}_{\mathcal{O}_{\text{res}}}(\Xi^{-1}(U))$. \square

2.2 A reductive structure on orbits of g1-pdms and quasi-free states

In this subsection, we study the orbits of the conjugacy action of U_{Bog} on admissible g1-pdms as reductive homogeneous spaces of U_{Bog} , or equivalently, orbits of quasi-free states with finite particle number.

Remark 2.3. We first point out some useful spectral properties of g1-pdms.

i) According to Theorem A.8, any $\Gamma \in \mathcal{D}$ can be diagonalized

$$W\Gamma W^* = \begin{pmatrix} \Lambda & 0 \\ 0 & \mathbf{1} - \Lambda \end{pmatrix},$$

for some unitary $W \in U_{\text{Bog}}$. In what follows, it will be convenient to express Λ using the spectral theorem for trace-class operators. Indeed, there is a family of orthogonal projections $\{p_i\}_{i=0}^r$ ($0 \leq r \leq \infty$) satisfying $\sum_{i=0}^r p_i = \mathbf{1}$ (strong operator topology convergence if $r = \infty$), $p_i p_j = \delta_{ij} p_i$, $\dim(\text{ran}(p_i)) := m_i < \infty$ for all $i \geq 1$ and

$$\Lambda = \sum_{i=1}^r \lambda_i p_i.$$

Here the convergence is understood in the $\|\cdot\|_1$ -norm if $r = \infty$; and the eigenvalues satisfy $\lambda_i \in (0, \frac{1}{2}]$, $i \geq 1$, $\lambda_i \neq \lambda_j$ if $i \neq j$ and $\sum_{i=1}^r m_i \lambda_i < \infty$. Observe that one can also assume $p_i = \bar{p}_i = p_i^T$ because Λ is diagonal with respect to the fixed orthonormal basis $\{\varphi_k\}_{k \geq 1}$. Finally, we put $\lambda_0 = 0$, so that p_0 is the projection onto $\ker(\Lambda)$, which satisfies $\dim(\text{ran}(p_0)) := m_0 \in [0, \infty]$.

ii) Notice that the spectrum of Γ is then given by

$$\sigma(\Gamma) = \{0, 1\} \cup \{\lambda_i\}_{i=1}^r \cup \{1 - \lambda_i\}_{i=1}^r.$$

Since Γ is a self-adjoint operator, it is well known that $\sigma(\Gamma) = \sigma_p(\Gamma) \cup \sigma_{\text{ess}}(\Gamma)$, where $\sigma_p(\Gamma)$ and $\sigma_{\text{ess}}(\Gamma)$ indicate the point spectrum (consisting of all eigenvalues of finite multiplicity) and the essential spectrum of Γ , respectively. Clearly, we have $\sigma_p(\Gamma) = \{\lambda_i\}_{i=1}^r \cup \{1 - \lambda_i\}_{i=1}^r$ and $\sigma_{\text{ess}}(\Gamma) = \{0, 1\}$.

Remark 2.4. Notice that U_{Bog} acts on \mathcal{D} by unitary conjugation:

$$U \cdot \Gamma = U\Gamma U^*, \quad U \in U_{\text{Bog}}, \quad \Gamma \in \mathcal{D}.$$

It is well-known in the Hartree-Fock-Bogoliubov literature that this is actually an action (see, e.g., [7]). We add the matrix form of $U \cdot \Gamma$ to briefly show this fact here. If $\Gamma = \Gamma[\gamma, \alpha]$,

$U = \begin{pmatrix} u & v \\ \bar{v} & \bar{u} \end{pmatrix}$, then

$$U \cdot \Gamma = \begin{pmatrix} u\gamma u^* + v\alpha^* u^* + u\alpha v^* + v(\mathbf{1} - \bar{\gamma})v^* & u\gamma \bar{v}^* + v\alpha^* \bar{v}^* + u\alpha \bar{u}^* + v(\mathbf{1} - \bar{\gamma})\bar{u}^* \\ \bar{v}\gamma u^* + \bar{u}\alpha^* u^* + \bar{v}\alpha v^* + \bar{u}(\mathbf{1} - \bar{\gamma})v^* & \bar{v}\gamma \bar{v}^* + \bar{u}\alpha^* \bar{v}^* + \bar{v}\alpha \bar{u}^* + \bar{u}(\mathbf{1} - \bar{\gamma})\bar{u}^* \end{pmatrix}. \quad (9)$$

From this expression, we deduce that the entry $(U \cdot \Gamma)_{11} \in \underline{B_1(\mathfrak{H})}$ because $\alpha, v \in B_2(\mathfrak{H})$ (see Remark A.6 ii)). Also it easily follows that $(U \cdot \Gamma)_{22} = \mathbf{1} - \overline{(U \cdot \Gamma)_{11}}$ using that $\alpha^T = -\alpha$, or equivalently, $\alpha^* = -\bar{\alpha}$. The remaining conditions to ensure that $U \cdot \Gamma \in \mathcal{D}$ are trivial.

Given $\Gamma \in \mathcal{D}$, we consider its orbit

$$\mathcal{O}(\Gamma) = \{U\Gamma U^* : U \in U_{\text{Bog}}\}.$$

In order to prove that these orbits are smooth homogeneous spaces of U_{Bog} , we have to consider the isotropy group of U_{Bog} at Γ , that is,

$$U_{\text{Bog}}^\Gamma := \{U \in U_{\text{Bog}} : U\Gamma U^* = \Gamma\},$$

and prove the following:

Lemma 2.5. U_{Bog}^Γ is a Lie subgroup of U_{Bog} .

Proof. First, it is clear U_{Bog}^Γ is a closed subgroup of U_{Bog} in topology defined by the norm in (6). Set

$$\mathfrak{u}_{\text{Bog}}^\Gamma := \{X \in \mathfrak{u}_{\text{Bog}} : e^{tX} \in U_{\text{Bog}}^\Gamma, \forall t \in \mathbb{R}\},$$

which is then a closed Lie subalgebra of $\mathfrak{u}_{\text{Bog}}$ by Theorem A.10. By the same result we have to check that there are open sets $0 \in \mathcal{W} \subseteq \mathfrak{u}_{\text{Bog}}$, $1 \in \mathcal{V} \subseteq U_{\text{Bog}}$, such that the exponential map $\exp : \mathcal{W} \rightarrow \mathcal{V}$ is a diffeomorphism satisfying $\exp(\mathcal{W} \cap \mathfrak{u}_{\text{Bog}}^\Gamma) = \mathcal{V} \cap U_{\text{Bog}}^\Gamma$, and the subspace $\mathfrak{u}_{\text{Bog}}^\Gamma$ has a closed supplement in $\mathfrak{u}_{\text{Bog}}$.

It will be useful to observe that

$$\mathfrak{u}_{\text{Bog}}^\Gamma = \{X \in \mathfrak{u}_{\text{Bog}} : X\Gamma = \Gamma X\}. \quad (10)$$

Indeed, one inclusion can be shown by taking the derivative of the curve $\gamma(t) = e^{tX}$ at $t = 0$; meanwhile the other inclusion follows from the fact that $X\Gamma = \Gamma X$ implies that $(tX)^n \Gamma = \Gamma(tX)^n$, $n \geq 0$, and consequently, $e^{tX}\Gamma = \Gamma e^{tX}$. In order to check $\exp(\mathcal{W} \cap \mathfrak{u}_{\text{Bog}}^\Gamma) = \mathcal{V} \cap U_{\text{Bog}}^\Gamma$, we take $U = e^X \in U_{\text{Bog}}^\Gamma$, where $X \in \mathcal{W}$. Using analytic functional calculus in the Banach algebra $B_2(\mathfrak{H} \oplus \mathfrak{H}, D)$, we get that $X = \log(U) = \sum_{n \geq 1} (-1)^{n+1} \frac{(U - \mathbf{1})^n}{n}$, whenever \mathcal{V} is a sufficiently small identity neighborhood. Since $U\Gamma = \Gamma U$ yields $(U - \mathbf{1})^n \Gamma = \Gamma(U - \mathbf{1})^n$, $n \geq 1$, we obtain that $\log(U)\Gamma = \Gamma \log(U)$, which proves the desired inclusion. The other inclusion is trivial.

Now we show that $\mathfrak{u}_{\text{Bog}}^\Gamma$ admits a closed supplement in $\mathfrak{u}_{\text{Bog}}$. Note that $WU_{\text{Bog}}^\Gamma W^* = U_{\text{Bog}}^{W\Gamma W^*}$ for every $W \in U_{\text{Bog}}$. Then it follows that U_{Bog}^Γ is a Lie subgroup if and only if $U_{\text{Bog}}^{W\Gamma W^*}$ is. By Theorem A.8 we may therefore assume that Γ has the form $\Gamma = \Gamma[\Lambda, 0]$, where Λ is a trace-class diagonal operator with respect to the fixed orthonormal basis $\{\varphi_k\}_{k \geq 1}$. We now use the notation and properties given in Remark 2.3, where Λ is expressed as $\Lambda = \sum_{i=1}^r \lambda_i p_i$ for a decomposition of the identity $\{p_i\}_{i=0}^r$ ($0 \leq r \leq \infty$). Recall that Γ has spectrum given by $\sigma(\Gamma) = \{0, 1\} \cup \{\lambda_i\}_{i=1}^r \cup \{1 - \lambda_i\}_{i=1}^r$, where $\lambda_i \in (0, \frac{1}{2}]$ are distinct eigenvalues of finite multiplicity.

Then for any operator $X = \begin{pmatrix} x_1 & x_2 \\ \bar{x}_2 & \bar{x}_1 \end{pmatrix} \in \mathfrak{u}_{\text{Bog}}$, the condition $X\Gamma = \Gamma X$ is equivalent to have

$$x_1 \Lambda = \Lambda x_1, \quad x_2 (1 - \Lambda) = \Lambda x_2. \quad (11)$$

From $x_1 \Lambda = \Lambda x_1$, it follows that $\lambda_j p_j x_1 p_j = \lambda_i p_i x_1 p_j$ for $i, j \geq 0$. Thus, $(\lambda_i - \lambda_j) p_i x_1 p_j = 0$, which yields $p_i x_1 p_j = 0$, whenever $i \neq j$, $i, j \geq 0$. Hence x_1 must have the block diagonal form $x_1 = \sum_{i=0}^r p_i x_1 p_i$.

On the other hand, $x_2(\mathbf{1} - \Lambda) = \Lambda x_2$ implies that $(1 - \lambda_j - \lambda_i)p_i x_2 p_j = 0$, $i, j \geq 0$. But $1 - \lambda_j - \lambda_i = 0$ only if $\lambda_i = \lambda_j = \frac{1}{2}$. Thus, we have two cases. In the first case, $x_2 = 0$ when $\lambda = \frac{1}{2}$ is not an eigenvalue of Γ . In the second case, we may w.l.o.g. assume that $\lambda_1 = \frac{1}{2}$ is an eigenvalue of Γ and p_1 is the finite-rank projection onto the corresponding eigenspace. Therefore $p_i x_2 p_j = 0$ for all $i, j \geq 0$, except when $i = j = 1$, which is equivalent to say that $x_2 = p_1 x_2 p_1$. Hence when $\frac{1}{2} \notin \sigma(\Gamma)$ the Lie algebra of the isotropy group can be written as

$$\mathfrak{u}_{\text{Bog}}^\Gamma = \left\{ \begin{pmatrix} x_1 & 0 \\ 0 & \bar{x}_1 \end{pmatrix} \in \mathfrak{u}_{\text{Bog}} : x_1 = \sum_{i=0}^r p_i x_1 p_i \right\},$$

meanwhile when $\frac{1}{2} \in \sigma(\Gamma)$ is

$$\mathfrak{u}_{\text{Bog}}^\Gamma = \left\{ \begin{pmatrix} x_1 & x_2 \\ \bar{x}_2 & \bar{x}_1 \end{pmatrix} \in \mathfrak{u}_{\text{Bog}} : x_1 = \sum_{i=0}^r p_i x_1 p_i, x_2 = p_1 x_2 p_1 \right\}.$$

Then a continuous projection $\mathcal{E}_\Gamma : \mathfrak{u}_{\text{Bog}} \rightarrow \mathfrak{u}_{\text{Bog}}$ with range $\text{ran}(\mathcal{E}_\Gamma) = \mathfrak{u}_{\text{Bog}}^\Gamma$ is given by

$$\mathcal{E}_\Gamma \left(\begin{pmatrix} x_1 & x_2 \\ \bar{x}_2 & \bar{x}_1 \end{pmatrix} \right) = \begin{cases} \begin{pmatrix} \sum_{i=0}^r p_i x_1 p_i & 0 \\ 0 & \sum_{i=0}^r p_i \bar{x}_1 p_i \end{pmatrix} & \text{if } \frac{1}{2} \notin \sigma(\Gamma), \\ \begin{pmatrix} \sum_{i=0}^r p_i x_1 p_i & p_1 x_2 p_1 \\ p_1 \bar{x}_2 p_1 & \sum_{i=0}^r p_i \bar{x}_1 p_i \end{pmatrix} & \text{if } \frac{1}{2} \in \sigma(\Gamma). \end{cases}$$

The convergence of the series is in the strong operator topology when $r = \infty$. This proves that $\mathfrak{u}_{\text{Bog}}^\Gamma$ has the closed supplement $\mathfrak{m}_\Gamma := \ker(\mathcal{E}_\Gamma)$ in both cases. \square

Remark 2.6. Repeating the same computations at the group level that we have done at the Lie algebras level in the previous proof, we find that when $\frac{1}{2} \notin \sigma(\Gamma)$ the isotropy group is given by

$$U_{\text{Bog}}^\Gamma = \left\{ \begin{pmatrix} u & 0 \\ 0 & \bar{u} \end{pmatrix} \in U_{\text{Bog}} : u = \sum_{i=0}^r p_i u p_i \right\}. \quad (12)$$

In the case where $\frac{1}{2} \in \sigma(\Gamma)$, the isotropy group is given by

$$U_{\text{Bog}}^\Gamma = \left\{ \begin{pmatrix} u & v \\ \bar{v} & \bar{u} \end{pmatrix} \in U_{\text{Bog}} : u = \sum_{i=0}^r p_i u p_i, v = p_1 v p_1 \right\}. \quad (13)$$

It will be useful to consider the projections \mathcal{E}_Γ defined in the proof of Lemma 2.5. The following definition is motivated by the conditional expectations associated to normal diagonalizable operators given in [13].

Definition 2.7. Let $\Gamma = \Gamma[\Lambda, 0]$ be a diagonal g1-pdm which defines a family of projections $\{p_i\}_{i=0}^r$ ($0 \leq r \leq \infty$) satisfying the properties of Remark 2.3. The *conditional expectation associated to Γ* is the map $\mathcal{E}_\Gamma : \mathfrak{u}_{\text{Bog}} \rightarrow \mathfrak{u}_{\text{Bog}}$ defined by

$$\mathcal{E}_\Gamma \left(\begin{pmatrix} x_1 & x_2 \\ \bar{x}_2 & \bar{x}_1 \end{pmatrix} \right) = \begin{cases} \begin{pmatrix} \sum_{i=0}^r p_i x_1 p_i & 0 \\ 0 & \sum_{i=0}^r p_i \bar{x}_1 p_i \end{pmatrix} & \text{if } \frac{1}{2} \notin \sigma(\Gamma), \\ \begin{pmatrix} \sum_{i=0}^r p_i x_1 p_i & p_1 x_2 p_1 \\ p_1 \bar{x}_2 p_1 & \sum_{i=0}^r p_i \bar{x}_1 p_i \end{pmatrix} & \text{if } \frac{1}{2} \in \sigma(\Gamma). \end{cases} \quad (14)$$

The convergence is in the strong operator topology when $r = \infty$.

Remark 2.8. It is straightforward to check that the conditional expectations in Eq. (14) satisfy properties similar to that of conditional expectations in operator algebras: $\mathcal{E}_\Gamma^2 = \mathcal{E}_\Gamma$, $\text{ran}(\mathcal{E}_\Gamma) = \mathfrak{u}_{\text{Bog}}^\Gamma$, $\|\mathcal{E}_\Gamma\| = \sup_{\|X\|_{\text{res}}=1} \|\mathcal{E}_\Gamma(X)\|_{\text{res}} = 1$, and $\mathcal{E}_\Gamma(UZU^*) = U\mathcal{E}_\Gamma(Z)U^*$, whenever $U \in \mathfrak{U}_{\text{Bog}}^\Gamma$ and $Z \in \mathfrak{u}_{\text{Bog}}$.

Theorem 2.9. *Let $\Gamma \in \mathcal{D}$. Then the following assertions hold.*

- i) *The orbit $\mathcal{O}(\Gamma) \cong \mathfrak{U}_{\text{Bog}}/\mathfrak{U}_{\text{Bog}}^\Gamma$ is a reductive homogeneous space of $\mathfrak{U}_{\text{Bog}}$. Its tangent space at Γ_1 can be identified with the real Banach space*

$$T_{\Gamma_1}\mathcal{O}(\Gamma) = \{[X, \Gamma_1] : X \in \mathfrak{u}_{\text{Bog}}\},$$

equipped with the norm $\|[X, \Gamma_1]\|_{\Gamma_1} := \inf\{\|X + Y\|_{\text{res}} : Y\Gamma_1 = \Gamma_1 Y\}$.

- ii) *Let us denote by*

$$\mathfrak{U}_{\text{Bog}}^+ := \{U \in \mathfrak{U}_{\text{Bog}} : \text{ind}_{\mathfrak{U}_{\text{Bog}}}(U) = 0\}, \quad \mathfrak{U}_{\text{Bog}}^- := \{U \in \mathfrak{U}_{\text{Bog}} : \text{ind}_{\mathfrak{U}_{\text{Bog}}}(U) = 1\}.$$

If $\frac{1}{2} \notin \sigma(\Gamma)$, then $\mathcal{O}(\Gamma) = \mathfrak{U}_{\text{Bog}}^+ \cdot \Gamma \cup \mathfrak{U}_{\text{Bog}}^- \cdot \Gamma$, is the union of two connected components.

If $\frac{1}{2} \in \sigma(\Gamma)$, then $\mathcal{O}(\Gamma) = \mathfrak{U}_{\text{Bog}}^+ \cdot \Gamma = \mathfrak{U}_{\text{Bog}}^- \cdot \Gamma$ is connected.

Proof. i) The map $\mathcal{O}(\Gamma) \rightarrow \mathfrak{U}_{\text{Bog}}/\mathfrak{U}_{\text{Bog}}^\Gamma$, $WTW^* \mapsto W\mathfrak{U}_{\text{Bog}}^\Gamma$, is a bijection. Thus, we endow $\mathcal{O}(\Gamma)$ with the manifold structure making such map into a diffeomorphism. The manifold structure of $\mathfrak{U}_{\text{Bog}}/\mathfrak{U}_{\text{Bog}}^\Gamma$ follows by using Theorem A.10 and Lemma 2.5. Indeed, by these results $\mathfrak{U}_{\text{Bog}}/\mathfrak{U}_{\text{Bog}}^\Gamma$ becomes a smooth homogeneous space of $\mathfrak{U}_{\text{Bog}}$.

The assertion about tangent spaces now follows by using that the map $\pi : \mathfrak{U}_{\text{Bog}} \rightarrow \mathcal{O}(\Gamma)$, $\pi(U) = UTU^*$, is a submersion. Indeed, its tangent map at $U \in \mathfrak{U}_{\text{Bog}}$, $U \cdot \Gamma = \Gamma_1$, satisfies $T_{\Gamma_1}\mathcal{O}(\Gamma) = \text{ran}(T_U\pi) \cong \mathfrak{u}_{\text{Bog}}/\mathfrak{u}_{\text{Bog}}^{\Gamma_1} \cong \{[X, \Gamma_1] : X \in \mathfrak{u}_{\text{Bog}}\}$, where $[X, \Gamma_1] = X\Gamma_1 - \Gamma_1 X$ and $\ker(T_U\pi) = \mathfrak{u}_{\text{Bog}}^{\Gamma_1}$ is the Lie algebra of the isotropy group at Γ_1 . The norm given in the statement is an expression for the quotient norm in $\mathfrak{u}_{\text{Bog}}/\mathfrak{u}_{\text{Bog}}^{\Gamma_1}$.

To prove that $\mathcal{O}(\Gamma)$ is a reductive homogeneous space we may assume that Γ is diagonal, take \mathcal{E}_Γ its conditional expectation and $\mathfrak{m}_\Gamma = \ker(\mathcal{E}_\Gamma)$. From the proof of Lemma 2.5, we know that $\mathfrak{u}_{\text{Bog}}^\Gamma \oplus \mathfrak{m}_\Gamma = \mathfrak{u}_{\text{Bog}}$. Since $\mathcal{E}_\Gamma(UXU^*) = U\mathcal{E}_\Gamma(X)U^*$, for $U \in \mathfrak{U}_{\text{Bog}}^\Gamma$ and $X \in \mathfrak{u}_{\text{Bog}}$, it follows that $\text{Ad}_U(\mathfrak{m}_\Gamma) = \mathfrak{m}_\Gamma$. Then, for $\Gamma_1 = UTU^*$, set $\mathfrak{m}_{\Gamma_1} := U\mathfrak{m}_\Gamma U^*$. This is well defined since $UTU^* = WTW^*$ implies $W^*U \in \mathfrak{U}_{\text{Bog}}^\Gamma$. Thus, $\mathcal{E}_\Gamma = \text{Ad}_{W^*U} \circ \mathcal{E}_\Gamma \circ \text{Ad}_{U^*W}$, which yields $U\mathfrak{m}_\Gamma U^* = W\mathfrak{m}_\Gamma W^*$. By similar properties the distribution $\{\mathfrak{m}_{\Gamma_1}\}_{\Gamma_1 \in \mathcal{O}(\Gamma)}$ clearly satisfies $\mathfrak{u}_{\text{Bog}} = \mathfrak{m}_{\Gamma_1} \oplus \mathfrak{u}_{\text{Bog}}^{\Gamma_1}$ and $\text{Ad}_U(\mathfrak{m}_{\Gamma_1}) = \mathfrak{m}_{\Gamma_1}$, for $U \in \mathfrak{U}_{\text{Bog}}^{\Gamma_1}$. Fix $\Gamma_1 \in \mathcal{O}(\Gamma)$. For any $U \in \mathfrak{U}_{\text{Bog}}$ such that $\Gamma_1 = UTU^*$, the projection $\mathcal{E}_{\Gamma_1} := U\mathcal{E}_\Gamma(U^* \cdot U)U^*$ is the unique projection satisfying $\text{ran}(\mathcal{E}_{\Gamma_1}) = \mathfrak{u}_{\text{Bog}}^{\Gamma_1}$ and $\ker(\mathcal{E}_{\Gamma_1}) = \mathfrak{m}_{\Gamma_1}$. On the other hand, the map $\pi_{\Gamma_1} : \mathfrak{U}_{\text{Bog}} \rightarrow \mathcal{O}(\Gamma)$, $\pi_{\Gamma_1}(U) = U\Gamma_1 U^*$, is a submersion. This is equivalent to the existence of a smooth local cross section at every point (see [43, Cor. 8.3]). In particular, there are an open set $\mathcal{V}_{\Gamma_1} \subseteq \mathcal{O}(\Gamma)$, $\Gamma_1 \in \mathcal{V}_{\Gamma_1}$ and a smooth map $s_{\Gamma_1} : \mathcal{V}_{\Gamma_1} \rightarrow \mathfrak{U}_{\text{Bog}}$ such that $\pi_{\Gamma_1} \circ s_{\Gamma_1} = \text{id}|_{\mathcal{V}_{\Gamma_1}}$. Consider the smooth map $F : \mathfrak{U}_{\text{Bog}} \rightarrow \mathfrak{B}(\mathfrak{u}_{\text{Bog}})$, $F(U) = \mathbf{1}_{\mathfrak{B}(\mathfrak{u}_{\text{Bog}})} - U\mathcal{E}_\Gamma(U^* \cdot U)U^*$ and the map $f : \mathcal{O}(\Gamma_1) \rightarrow \mathfrak{B}(\mathfrak{u}_{\text{Bog}})$, $f(\Gamma') = \mathbf{1}_{\mathfrak{B}(\mathfrak{u}_{\text{Bog}})} - \mathcal{E}_{\Gamma'}$. Therefore, $f = F \circ s_{\Gamma_1}$ locally at each Γ_1 . Since Γ_1 is arbitrary, we get that f is a smooth map, and consequently, the distribution $\{\mathfrak{m}_{\Gamma_1}\}_{\Gamma_1 \in \mathcal{O}(\Gamma)}$ is smooth. Hence $\mathcal{O}(\Gamma)$ is a reductive homogeneous space.

- ii) According to Proposition 2.2, the \mathbb{Z}_2 -valued index on $\mathfrak{U}_{\text{Bog}}$ parametrizes the two connected components of this group $\mathfrak{U}_{\text{Bog}}^+$ and $\mathfrak{U}_{\text{Bog}}^-$. Therefore, $\mathfrak{U}_{\text{Bog}}^+ \cdot \Gamma = \{UTU^* : U \in \mathfrak{U}_{\text{Bog}}^+\}$ is a

connected set because the map $\pi_\Gamma : \mathcal{U}_{\text{Bog}} \rightarrow \mathcal{O}(\Gamma)$, $\pi_\Gamma(U) = U\Gamma U^*$ is continuous. Let us show that $\mathcal{U}_{\text{Bog}}^+ \cdot \Gamma$ is a connected component of $\mathcal{O}(\Gamma)$. Recall that path and connected components coincide in manifolds. So take a continuous path $\delta : [0, 1] \rightarrow \mathcal{O}(\Gamma)$, $\delta(0) = \Gamma$ and $\delta(1) = \Gamma_1$. We have to prove that $\Gamma_1 \in \mathcal{U}_{\text{Bog}}^+ \cdot \Gamma$.

By item i) the maps $\pi_{\Gamma'} : \mathcal{U}_{\text{Bog}} \rightarrow \mathcal{O}(\Gamma)$, $\pi_{\Gamma'}(U) = U\Gamma'U^*$, are submersions. As we have observed above, there are an open set $\mathcal{V}_{\Gamma'} \subseteq \mathcal{O}(\Gamma)$, $\Gamma' \in \mathcal{V}_{\Gamma'}$ and a smooth map $s_{\Gamma'} : \mathcal{V}_{\Gamma'} \rightarrow \mathcal{U}_{\text{Bog}}$ such that $\pi_{\Gamma'} \circ s_{\Gamma'} = \text{id}|_{\mathcal{V}_{\Gamma'}}$. By a standard compactness argument, there are a partition $0 = t_0 < t_1 < \dots < t_n = 1$, open connected sets $\mathcal{V}_i := \mathcal{V}_{\delta(t_i)}$, and smooth sections $s_i := s_{\delta(t_i)}$ of $\pi_{\delta(t_i)}$, $i = 0, \dots, n$, such that $\delta([0, 1]) \subseteq \bigcup_{i=0}^n \mathcal{V}_i$. Set $\delta(t_i) := \Gamma^{(i)}$, $i = 0, \dots, n$, where $\Gamma = \Gamma^{(0)}$ and $\Gamma_1 = \Gamma^{(n)}$. For $i = 0$, we can have $\text{ind}_{\mathcal{U}_{\text{Bog}}}(s_0(\Gamma)) = 0$ or $\text{ind}_{\mathcal{U}_{\text{Bog}}}(s_0(\Gamma)) = 1$. In the first case, we write $\tilde{s}_0 = s_0$. In the second case, we modify the section by defining $\tilde{s}_0(\Gamma') = s_0(\Gamma')s_0(\Gamma)$. Now this is a smooth section which satisfies $\text{ind}_{\mathcal{U}_{\text{Bog}}}(\tilde{s}_0(\Gamma)) = 0$, and using that \mathcal{V}_0 is connected, it also holds that $\text{ind}_{\mathcal{U}_{\text{Bog}}}(\tilde{s}_0(\Gamma^{(1)})) = 0$. Notice that $\tilde{s}_0(\Gamma^{(1)})\Gamma\tilde{s}_0(\Gamma^{(1)})^* = \Gamma^{(1)}$. We may modify the other sections s_i for $i = 1, \dots, n-1$ to obtain that $\text{ind}_{\mathcal{U}_{\text{Bog}}}(\tilde{s}_i(\Gamma^{(i+1)})) = 0$ and

$$\tilde{s}_{n-1}(\Gamma^{(n)}) \dots \tilde{s}_1(\Gamma^{(2)})\tilde{s}_0(\Gamma^{(1)})\Gamma\tilde{s}_0(\Gamma^{(1)})^*\tilde{s}_1(\Gamma^{(2)})^* \dots \tilde{s}_{n-1}(\Gamma^{(n)})^* = \Gamma^{(n)} = \Gamma_1.$$

This means that $\Gamma_1 \in \mathcal{U}_{\text{Bog}}^+ \cdot \Gamma$. Thus, $\mathcal{U}_{\text{Bog}}^+ \cdot \Gamma$ is a connected component.

Clearly, $\mathcal{O}(\Gamma) = \mathcal{U}_{\text{Bog}}^+ \cdot \Gamma \cup \mathcal{U}_{\text{Bog}}^- \cdot \Gamma$. Fix $R \in \mathcal{U}_{\text{Bog}}^-$. Note that the map $\mathcal{U}_{\text{Bog}}^+ \cdot \Gamma \rightarrow \mathcal{U}_{\text{Bog}}^- \cdot \Gamma$, $UTU^* \rightarrow RUTU^*R^*$, is continuous. Therefore, $\mathcal{U}_{\text{Bog}}^- \cdot \Gamma$ is also connected. To show that it is a connected component, it suffices to prove that $\mathcal{U}_{\text{Bog}}^+ \cdot \Gamma \cap \mathcal{U}_{\text{Bog}}^- \cdot \Gamma = \emptyset$. Assume that $UTU^* = V\Gamma V^*$ for $U \in \mathcal{U}_{\text{Bog}}^+$ and $V \in \mathcal{U}_{\text{Bog}}^-$. Then we have $\text{ind}_{\mathcal{U}_{\text{Bog}}}(V^*U) = 1$ and $V^*U \in \mathcal{U}_{\text{Bog}}^+$. Note that $W\mathcal{U}_{\text{Bog}}^\pm W^* = \mathcal{U}_{\text{Bog}}^\pm$ for every $W \in \mathcal{U}_{\text{Bog}}$. Thus, we may further assume that Γ has a diagonal form. When $\frac{1}{2} \notin \sigma(\Gamma)$ the isotropy group is described in (12). Thus we get that $P_+(V^*U)|_{\mathfrak{H} \oplus \{0\}}$ is a block unitary operator. In particular, $\text{ind}_{\mathcal{U}_{\text{Bog}}}(V^*U) = 0$, a contradiction. This proves that $\mathcal{U}_{\text{Bog}}^- \cdot \Gamma$ is also a connected component.

Finally, we suppose that $\frac{1}{2} \in \sigma(\Gamma)$. Again we can conjugate Γ by a Bogoliubov transformation, so that Γ is assumed to be diagonal. Now we use the expression of the isotropy group in (13). Take a vector $\varphi_1 \in \text{ran}(p_1) = \ker(\Lambda - \frac{1}{2}\mathbf{1})$ ($\lambda_1 = \frac{1}{2}$). Define a Bogoliubov transformation S by $S((\varphi_1, 0)) = (0, \varphi_1)$, $S((0, \varphi_1)) = (\varphi_1, 0)$, and that leaves fixed all the other vectors of the basis $\{(\varphi_i, 0)\}_{i \geq 1} \cup \{(0, \varphi_j)\}_{j \geq 1}$. Then $S\Gamma S^* = \Gamma$. But $\mathcal{U}_{\text{Bog}}^+ S = \mathcal{U}_{\text{Bog}}^-$ because $\text{ind}_{\mathcal{U}_{\text{Bog}}}(S) = 1$, so that $\mathcal{U}_{\text{Bog}}^+ \cdot \Gamma = \mathcal{U}_{\text{Bog}}^+ S \cdot \Gamma = \mathcal{U}_{\text{Bog}}^- \cdot \Gamma = \mathcal{O}(\Gamma)$. \square

Corollary 2.10. \mathcal{D} is a smooth manifold.

Proof. Notice that \mathcal{D} is the disjoint union of orbits, $\mathcal{D} = \bigsqcup_{[\Gamma] \in \mathcal{D}/\approx} \mathcal{O}(\Gamma)$, where we write \approx for the equivalence relation induced by the orbits. \square

Remark 2.11. We apply the previous results on gl-pdms to analyze the set \mathcal{Z}_{qf} of all quasi-free states. According to Theorem A.1, for any $U \in \mathcal{U}_{\text{Bog}}$, there exists a unitary implementer $\mathbb{U} = \mathbb{U}_U : \mathfrak{F} \rightarrow \mathfrak{F}$. The implementer is unique up to a constant in \mathbb{T} . Thus, given $\omega \in \mathcal{Z}_{\text{qf}}$, the state defined by $\omega_U(A) = \omega(\mathbb{U}A\mathbb{U}^*)$, $A \in \mathcal{A}$, is independent of the unitary implementer. Also as we have stated in Remark A.6 i), it can be checked that $\omega_U \in \mathcal{Z}_{\text{qf}}$. Furthermore, the map

$$\mathcal{U}_{\text{Bog}} \times \mathcal{Z}_{\text{qf}} \rightarrow \mathcal{Z}_{\text{qf}}, \quad (U, \omega) \mapsto U \cdot \omega = \omega_U$$

turns out to be an action of \mathcal{U}_{Bog} on \mathcal{Z}_{qf} . This follows immediately by using (28) to show that $\mathbb{U}_{UV} = \lambda \mathbb{U}_U \mathbb{U}_V$, for $U, V \in \mathcal{U}_{\text{Bog}}$ and some $\lambda \in \mathbb{T}$.

Corollary 2.12. *The following assertions hold:*

i) *For each $\omega \in \mathcal{Z}_{\text{qf}}$, the orbit $\mathcal{O}(\omega) := \{\omega_U : U \in \text{U}_{\text{Bog}}\}$ is a reductive homogeneous space of U_{Bog} . Consequently, \mathcal{Z}_{qf} is a smooth manifold.*

ii) *The map $\Phi : \mathcal{Z}_{\text{qf}} \rightarrow \mathcal{D}$, $\Phi(\omega) = \Gamma_\omega$ is a smooth diffeomorphism.*

Proof. i) Using that the map Φ satisfies $\Gamma_{\omega_U} = \Phi(\omega_U) = U^* \Gamma_\omega U$, it follows that the isotropy group at ω is given by

$$\{U \in \text{U}_{\text{Bog}} : \omega_U = \omega\} = \{U \in \text{U}_{\text{Bog}} : U \Gamma_\omega = \Gamma_\omega U\} = \text{U}_{\text{Bog}}^{\Gamma_\omega}.$$

Hence the isotropy group is a Lie subgroup of U_{Bog} by Lemma 2.5, and following the same proof as in Theorem 2.9, the orbit $\mathcal{O}(\omega) \cong \text{U}_{\text{Bog}} / \text{U}_{\text{Bog}}^{\Gamma_\omega}$ is a reductive homogeneous space of U_{Bog} . The assertion about the manifold structure of \mathcal{Z}_{qf} now follows by expressing this set as the union of disjoint orbits.

ii) The manifold structures of both \mathcal{Z}_{qf} and \mathcal{D} are obtained as the disjoint union of orbits. Therefore we may consider the restriction $\Phi : \mathcal{O}(\omega) \rightarrow \mathcal{O}(\Gamma_\omega)$ to prove our statement. The result is a consequence of being $\mathcal{O}(\omega)$ and $\mathcal{O}(\Gamma_\omega)$ both diffeomorphic to $\text{U}_{\text{Bog}} / \text{U}_{\text{Bog}}^{\Gamma_\omega}$. Indeed, a chart at ω is given by $\phi^{-1} = \pi_\omega \circ \exp$, where $\pi_\omega : \text{U}_{\text{Bog}} \rightarrow \mathcal{O}(\omega)$, $\pi_\omega(U) = \omega_U$, and \exp is the exponential map of U_{Bog} (see [12, Lemma 4.21]). Moreover, ϕ is defined in an open set $\mathcal{W} \subseteq \mathcal{O}(\omega)$, $\omega \in \mathcal{W}$, and $\phi(\mathcal{W})$ is an open set in $\mathfrak{m}_{\Gamma_\omega} \cong \mathfrak{u}_{\text{Bog}} / \mathfrak{u}_{\text{Bog}}^{\Gamma_\omega}$, where $\mathfrak{m}_{\Gamma_\omega}$ is the closed subspace from the reductive structure. Similarly, a chart at Γ_ω is of the form $\psi^{-1} = \pi_{\Gamma_\omega} \circ \exp$, and it is a homeomorphism between open sets in $\mathcal{O}(\Gamma_\omega)$ and $\mathfrak{m}_{\Gamma_\omega}$. Thus, the map Φ can be expressed locally as $(\psi \circ \Phi \circ \phi^{-1})(X) = (\psi \circ \Phi)(\omega_{e^X}) = \psi(e^{-X} \Gamma_\omega e^X) = -X$. Hence Φ is a diffeomorphism. \square

Remark 2.13. We end this section with some remarks on a special type of orbits. It is well known that using the bijection between gl-pdms and quasi-free states the projection $P_- = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1} \end{pmatrix}$ corresponds to the state ω_- defined by $\omega_-(A) = \langle \Omega, A \Omega \rangle$, $A \in \text{B}(\mathfrak{F})$, where Ω is the vacuum vector. This is the extension of the Fock state defined in the CAR algebra (see, e.g., [4]).

i) The orbit

$$\mathcal{O}(P_-) = \left\{ U \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1} \end{pmatrix} U^* : U \in \text{U}_{\text{Bog}} \right\}$$

is indeed isomorphic to an *isotropic restricted Grassmannian*, which is a reductive homogeneous space of $\text{O}_{\text{res}}(\mathfrak{H}^{\mathbb{R}})$ and Kähler manifold that naturally shows up in the literature of loop groups (see [38, Sec. 12.4]). Recalling the notation in the proof of Lemma 2.2 ii), we say that a (closed) subspace W of $\mathfrak{H}_{\mathbb{C}}$ is maximal isotropic if $C(W) = W^\perp$, and we write P_W for the orthogonal projection onto W . Then, the isotropic restricted Grassmannian associated to $(\mathfrak{H}^{\mathbb{R}}, \mathfrak{H}_{\mathbb{C}}^-)$ is given by

$$\mathcal{I}_{\text{res}} := \{W : W \text{ is maximal isotropic, } P_W - P_{\mathfrak{H}_{\mathbb{C}}^-} \in \text{B}_2(\mathfrak{H}_{\mathbb{C}})\}.$$

The map $\mathcal{O}(P_-) \rightarrow \mathcal{I}_{\text{res}}$, $UP_-U^* \mapsto \text{Ad}_T(U)(\mathfrak{H}_{\mathbb{C}}^-)$, gives the identification between these homogeneous spaces.

ii) The reductive structure induces a linear connection on every reductive homogeneous space (see [27], or in the infinite-dimensional case [32]). For the orbit $\mathcal{O}(P_-)$ the geodesics of this

connection can be explicitly computed. For instance, the geodesic starting at P_- with velocity vector $[X, P_-]$, for $X = \begin{pmatrix} 0 & y \\ \bar{y} & 0 \end{pmatrix} \in \mathfrak{m}_{P_-}$, $y = -y^T \in B_2(\mathfrak{H})$, is given by

$$\delta(t) = e^{tX} P_- e^{-tX} = \begin{pmatrix} \cos(|ty^*|) & t \operatorname{sinc}(|ty|) \\ ty^* \operatorname{sinc}(|ty^*|) & \cos(|ty|) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1} \end{pmatrix} \begin{pmatrix} \cos(|ty^*|) & t \operatorname{sinc}(|ty^*|)y \\ t \operatorname{sinc}(|ty|)y^* & \cos(|ty|) \end{pmatrix}.$$

3 Embedded submanifolds

3.1 Derivations induced by g1-pdms

We consider the following real Banach space:

$$i\mathfrak{u}_{\text{Bog}} = \left\{ \begin{pmatrix} x_1 & x_2 \\ -\bar{x}_2 & -\bar{x}_1 \end{pmatrix} \in B(\mathfrak{H} \oplus \mathfrak{H}) : x_1 = x_1^*, x_2 = -x_2^T \in B_2(\mathfrak{H}) \right\} \subseteq B_2(\mathfrak{H} \oplus \mathfrak{H}, D)$$

equipped with the restricted norm $\|X\|_{\text{res}} = 2 \max\{\|x_1\|, \|x_2\|_2\}$. Noticing that each $\Gamma = \Gamma[\gamma, \alpha] \in \mathcal{D}$ may be decomposed as

$$\Gamma = \begin{pmatrix} \gamma & \alpha \\ \alpha^* & \mathbf{1} - \bar{\gamma} \end{pmatrix} = \begin{pmatrix} \gamma & \alpha \\ -\bar{\alpha} & -\bar{\gamma} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1} \end{pmatrix} := \Gamma_0 + P_-, \quad \Gamma_0 \in i\mathfrak{u}_{\text{Bog}},$$

and since $\alpha \in B_2(\mathfrak{H})$ by Remark A.6, we have the inclusion $\mathcal{O}(\Gamma) \subseteq i\mathfrak{u}_{\text{Bog}} + P_-$. It is then natural to ask for conditions on Γ that characterize the homogeneous space $\mathcal{O}(\Gamma)$ as an embedded submanifold of the affine space $i\mathfrak{u}_{\text{Bog}} + P_-$. We start with a necessary condition to be a submanifold according to Proposition A.9, which consists in determining when tangent spaces of the orbits are closed in the tangent space of the ambient manifold. Since $\mathcal{O}(\Gamma) \subseteq i\mathfrak{u}_{\text{Bog}} + P_-$, the tangent space at Γ satisfies the inclusion:

$$T_\Gamma \mathcal{O}(\Gamma) = \{X\Gamma - \Gamma X : X \in \mathfrak{u}_{\text{Bog}}\} \subseteq i\mathfrak{u}_{\text{Bog}}.$$

This lead us to introduce the following derivation.

Definition 3.1. Let $\Gamma \in \mathcal{D}$. The *derivation induced by Γ* is given by

$$\delta_\Gamma : \mathfrak{u}_{\text{Bog}} \rightarrow \mathfrak{u}_{\text{Bog}}, \quad \delta_\Gamma(X) = [i\Gamma, X].$$

Observe $\Gamma = \Gamma_0 + P_-$, $\Gamma_0 \in i\mathfrak{u}_{\text{Bog}}$ and $X \in \mathfrak{u}_{\text{Bog}}$, then $[i\Gamma, X] = [i\Gamma_0, X] + [iP_-, X] \in \mathfrak{u}_{\text{Bog}}$. Hence the map δ_Γ actually takes values on $\mathfrak{u}_{\text{Bog}}$. It follows easily that δ_Γ is continuous with $\|\delta_\Gamma\| \leq 2\|\Gamma\|_{\text{res}}$, and it is a derivation of the Lie algebra $\mathfrak{u}_{\text{Bog}}$. Also notice $i\operatorname{ran}(\delta_\Gamma) = T_\Gamma \mathcal{O}(\Gamma)$. Thus, the tangent space $T_\Gamma \mathcal{O}(\Gamma)$ is closed in $i\mathfrak{u}_{\text{Bog}}$ if and only if the derivation δ_Γ has closed range.

We will use the following characterization of closed range linear maps in Banach spaces.

Lemma 3.2. Let E, F be Banach spaces and $T : E \rightarrow F$ a bounded linear map. Assume that there exists a closed subspace \mathcal{M} of E such that $E = \ker(T) \oplus \mathcal{M}$. Let \mathcal{E} be the continuous projection with $\operatorname{ran}(\mathcal{E}) = \ker(T)$ and $\ker(\mathcal{E}) = \mathcal{M}$. Then, the range of T is closed if and only if there exists $c > 0$ such that $\|Te\| \geq c\|e - \mathcal{E}e\|$, for all $e \in E$.

Proof. If the range of T is closed, then $T_0 := T|_{\mathcal{M}} : \mathcal{M} \rightarrow \text{ran}(T)$ is a bijective linear map, whose inverse is continuous by the open mapping theorem. Therefore, we have $\|Te\| = \|T_0(e - \mathcal{E}e)\| \geq \|T_0\|^{-1}\|e - \mathcal{E}e\|$, for all $e \in E$. Conversely, take a sequence $\{Te_n\}_{n \geq 1}$ in $\text{ran}(T)$ such that $Te_n \rightarrow f$. Then, $\|Te_n - Te_m\| \geq \|T((e_n - e_m) - \mathcal{E}(e_n - e_m))\| \geq c\|e_n - \mathcal{E}e_n - (e_m - \mathcal{E}e_m)\|$, which implies that $\{e_n - \mathcal{E}e_n\}_{n \geq 1}$ converges to some $e_0 \in \mathcal{M}$. Since $Te_n = T(e_n - \mathcal{E}e_n)$, we obtain $f = Te_0$. Hence $\text{ran}(T)$ is closed. \square

Proposition 3.3. *Let $\Gamma \in \mathcal{D}$. Then tangent spaces of $\mathcal{O}(\Gamma)$ are closed in $\mathfrak{u}_{\text{Bog}}$ if and only if $\sigma(\Gamma)$ is finite. If any of these conditions hold, then $\mathfrak{u}_{\text{Bog}} = i\mathfrak{u}_{\text{Bog}}^\Gamma \oplus T_\Gamma \mathcal{O}(\Gamma)$.*

Proof. All tangent spaces are closed if and only the tangent space at some Γ is closed. Hence we can assume that $\Gamma = \Gamma[\Lambda, 0] \in \mathcal{D}$ is diagonal, and study when the range of the corresponding derivation δ_Γ is closed.

Now suppose that $\sigma(\Gamma)$ is finite. According to Remark 2.3, this means that $\sigma(\Gamma) = \sigma(\Lambda) \cup \sigma(\mathbf{1} - \Lambda) = \{\lambda_k : k = 0, \dots, r\} \cup \{1 - \lambda_k : k = 0, \dots, r\}$, for some $r < \infty$ ($\lambda_0 = 0$). Using the projections $\{p_i\}_{i=0}^r$ in the mentioned remark: for $X = \begin{pmatrix} x_1 & x_2 \\ \bar{x}_2 & \bar{x}_1 \end{pmatrix} \in \mathfrak{u}_{\text{Bog}}$, notice that

$$[i\Lambda, x_1] = \sum_{i,j=0}^r i(\lambda_i - \lambda_j)p_i x_1 p_j, \quad i\Lambda x_2 - x_2 i(\mathbf{1} - \Lambda) = \sum_{i,j=0}^r i(\lambda_i + \lambda_j - 1)p_i x_2 p_j.$$

Observe that $p_i[\Lambda, x_1]p_i = 0$, for all $i = 0, \dots, r$, and when $\frac{1}{2} \in \sigma(\Gamma)$, $p_1(\Lambda x_2 - x_2(\mathbf{1} - \Lambda))p_1 = 0$. If $\frac{1}{2} \notin \sigma(\Gamma)$, then the coefficients of the preceding sums satisfy $\lambda_i - \lambda_j \neq 0$, $i, j = 0, \dots, r$, $i \neq j$, and $\lambda_i + \lambda_j - 1 \neq 0$, $i, j = 0, \dots, r$. Thus, the range of δ_Γ can be expressed as

$$\text{ran}(\delta_\Gamma) = \left\{ \begin{pmatrix} \sum_{\substack{i,j=0 \\ i \neq j}}^r p_j x_1 p_i & x_2 \\ \bar{x}_2 & \sum_{\substack{i,j=0 \\ i \neq j}}^r p_j \bar{x}_1 p_i \end{pmatrix} : x_1 = -x_1^* \in \text{B}(\mathfrak{H}), x_2 = -x_2^T \in \text{B}_2(\mathfrak{H}) \right\}. \quad (15)$$

In the case where $\frac{1}{2} \in \sigma(\Gamma)$, the above expression should be modified:

$$\text{ran}(\delta_\Gamma) = \left\{ \begin{pmatrix} \sum_{\substack{i,j=0 \\ i \neq j}}^r p_j x_1 p_i & \sum_{\substack{i,j=0 \\ (i,j) \neq (1,1)}}^r p_j x_2 p_i \\ \sum_{\substack{i,j=0 \\ (i,j) \neq (1,1)}}^r p_j \bar{x}_2 p_i & \sum_{\substack{i,j=0 \\ i \neq j}}^r p_j \bar{x}_1 p_i \end{pmatrix} : x_1 = -x_1^* \in \text{B}(\mathfrak{H}), x_2 = -x_2^T \in \text{B}_2(\mathfrak{H}) \right\}. \quad (16)$$

In both cases, $\text{ran}(\delta_\Gamma)$ is clearly a closed subspace of $\mathfrak{u}_{\text{Bog}}$.

For the converse we use the characterization in Lemma 3.2. If we assume that $\text{ran}(\delta_\Gamma)$ is closed in $\mathfrak{u}_{\text{Bog}}$, then there is a constant $c > 0$ satisfying

$$\|\delta_\Gamma(X)\|_{\text{res}} \geq c \|X - \mathcal{E}_\Gamma(X)\|_{\text{res}}, \quad X \in \mathfrak{u}_{\text{Bog}}. \quad (17)$$

Here \mathcal{E}_Γ is the conditional expectation with $\text{ran}(\mathcal{E}_\Gamma) = \ker(\delta_\Gamma) = \mathfrak{u}_{\text{Bog}}^\Gamma$ and $\ker(\mathcal{E}_\Gamma) = \mathfrak{m}_\Gamma$ is the supplement defined in the proof of Lemma 2.5. Let $\text{B}(\mathfrak{H})_{\text{sk}}$ denote the real space of bounded

linear skew-adjoint operators on \mathfrak{H} . Consider the derivation $\delta_\Lambda^{\text{sk}} := \delta_\Lambda|_{\text{B}(\mathfrak{H})_{\text{sk}}} : \text{B}(\mathfrak{H})_{\text{sk}} \rightarrow \text{B}(\mathfrak{H})_{\text{sk}}$, $\delta_\Lambda^{\text{sk}}(x) = [i\Lambda, x]$. Then take a operators $X = \text{diag}(x, \bar{x})$, $x^* = -x$, in (17) to find that $\|\delta_\Lambda^{\text{sk}}(x)\| \geq c\|x - \mathcal{E}_\Lambda(x)\|$, for all $x = -x^*$, where \mathcal{E}_Λ is the continuous projection such that $\text{ran}(\mathcal{E}_\Lambda) = \ker(\delta_\Lambda^{\text{sk}})$ and $\ker(\mathcal{E}_\Lambda) = (\mathfrak{m}_\Lambda)_{\text{sk}} := \{x \in \text{B}(\mathfrak{H})_{\text{sk}} : x = x - \sum_{i=0}^r p_i x_1 p_i\}$, which is a closed supplement of $\ker(\delta_\Lambda^{\text{sk}})$. This shows that $\delta_\Lambda^{\text{sk}}$ has closed range, which implies that $\delta_\Lambda : \text{B}(\mathfrak{H}) \rightarrow \text{B}(\mathfrak{H})$, $\delta_\Lambda(x) = [i\Lambda, x]$ also has closed range since $\Lambda^* = \Lambda$. This is equivalent to have that $\sigma(\Lambda)$ is finite by [2, Thm. 3.3], which means that $\sigma(\Gamma)$ is finite.

Finally, the expressions in (15) and (16) imply that, under the assumption that $\sigma(\Gamma)$ is finite, we have $\text{ran}(\delta_\Gamma) = \ker(\mathcal{E}_\Gamma)$. Hence $\mathfrak{u}_{\text{Bog}} = \text{ran}(\mathcal{E}_\Gamma) \oplus \ker(\mathcal{E}_\Gamma) = \mathfrak{u}_{\text{Bog}}^\Gamma \oplus \text{ran}(\delta_\Gamma)$, o equivalently, $i\mathfrak{u}_{\text{Bog}} = i\mathfrak{u}_{\text{Bog}}^\Gamma \oplus T_\Gamma \mathcal{O}(\Gamma)$. \square

Remark 3.4. Furthermore, we will show later that $\text{Tr}(X\delta_\Gamma(Y)) = 0$, for all $Y \in \mathfrak{u}_{\text{Bog}}$, if and only if $X \in \mathfrak{u}_{\text{Bog}}^\Gamma$ (see Lemma 4.5).

3.2 Embedded submanifod structure of orbits of g1-pdms

Now we consider another necessary condition to be an embedded submanifold. Since we know that $\mathcal{O}(\Gamma) \subseteq i\mathfrak{u}_{\text{Bog}} + P_-$, the orbits can be endowed with the relative topology inherited from the affine space $i\mathfrak{u}_{\text{Bog}} + P_-$. This topology is defined by the metric $d(\Gamma_0, \Gamma_1) = \|\Gamma_0 - \Gamma_1\|_{\text{res}}$, for $\Gamma_0, \Gamma_1 \in \mathcal{O}(\Gamma)$. On the other hand, $\mathcal{O}(\Gamma) \cong \text{U}_{\text{Bog}}/\text{U}_{\text{Bog}}^\Gamma$ is a homogeneous space of U_{Bog} , so that the orbits can be endowed with the quotient topology. Let us write τ_r and τ_q for the relative and quotient topology, respectively. Both topologies must coincide when the orbits are embedded submanifolds of $i\mathfrak{u}_{\text{Bog}} + P_-$. We will see that the coincidence of these topologies is again related to the finite spectrum condition.

Recall that the map $\pi_\Gamma : \text{U}_{\text{Bog}} \rightarrow \mathcal{O}(\Gamma)$, $\pi_\Gamma(U) = UTU^*$ is a submersion when $\mathcal{O}(\Gamma)$ is considered as a homogeneous space, which in particular implies that π_Γ has continuous local cross sections. From this fact, it follows that τ_q is stronger than τ_r . Then, both topologies coincide if and only if the map $\pi_\Gamma : \text{U}_{\text{Bog}} \rightarrow \mathcal{O}(\Gamma)$, $\pi_\Gamma(U) = UTU^*$ admits continuous local cross sections when $\mathcal{O}(\Gamma)$ is considered with the topology τ_r . To construct such a local cross section we essentially follow the argument in [1, Thm. 4.4], but it needs some additional work because the restricted norm is not isometric by multiplication with Bogoliubov transformations.

Remark 3.5. We first need to extend the derivation δ_Γ and the conditional expectation \mathcal{E}_Γ .

i) Given $\Gamma \in \mathcal{D}$, set

$$\tilde{\delta}_\Gamma : \text{B}_2(\mathfrak{H} \oplus \mathfrak{H}, D) \rightarrow \text{B}_2(\mathfrak{H} \oplus \mathfrak{H}, D), \quad \tilde{\delta}_\Gamma(X) = [i\Gamma, X].$$

It is straightforward to check that $\tilde{\delta}_\Gamma$ is a continuous derivation of the Lie algebra $\text{B}_2(\mathfrak{H} \oplus \mathfrak{H}, D)$. For $\Gamma = \Gamma[\Lambda, 0] \in \mathcal{D}$ a diagonal g1-pdm and using the same notation of Definition 2.7 for the spectral decomposition of Λ ($1 \leq r \leq \infty$), we define the linear projection

$$\tilde{\mathcal{E}}_\Gamma : \text{B}_2(\mathfrak{H} \oplus \mathfrak{H}, D) \rightarrow \text{B}_2(\mathfrak{H} \oplus \mathfrak{H}, D),$$

$$\tilde{\mathcal{E}}_\Gamma \left(\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \right) = \begin{cases} \begin{pmatrix} \sum_{i=0}^r p_i x_{11} p_i & 0 \\ 0 & \sum_{i=0}^r p_i x_{22} p_i \end{pmatrix} & \text{if } \frac{1}{2} \notin \sigma(\Gamma), \\ \begin{pmatrix} \sum_{i=0}^r p_i x_{11} p_i & p_1 x_{12} p_1 \\ p_1 x_{21} p_1 & \sum_{i=0}^r p_i x_{22} p_i \end{pmatrix} & \text{if } \frac{1}{2} \in \sigma(\Gamma). \end{cases}$$

We fix the following notation $\tilde{\mathfrak{m}}_\Gamma = \ker(\tilde{\mathcal{E}}_\Gamma)$.

iii) We denote by δ_Γ^0 and \mathcal{E}_Γ^0 for the extensions in the obvious way of δ_Γ and \mathcal{E}_Γ to the whole $B(\mathfrak{H} \oplus \mathfrak{H})$. We also put $\mathfrak{m}_\Gamma^0 = \ker(\delta_\Gamma^0)$.

One can show by similar arguments to that of Proposition 3.3 that the above derivations $\tilde{\delta}_\Gamma$ and δ_Γ^0 also have closed range if and only if $\sigma(\Gamma)$ is finite. Rather than proving this fact, we focus on estimate the constant given by the characterization in Lemma 3.2 in terms of the eigenvalues of Γ . This will be helpful later for the construction of continuous local cross sections.

Lemma 3.6. *Let $\Gamma = \Gamma[\Lambda, 0] \in \mathcal{D}$ be a diagonal $g1$ -pdm with finite spectrum. Then the following assertions hold:*

i) *For every $X \in B_2(\mathfrak{H} \oplus \mathfrak{H}, D)$, we have $\|\tilde{\delta}_\Gamma(X)\|_{\text{res}} \geq \tilde{c}_\Gamma \|X - \tilde{\mathcal{E}}_\Gamma(X)\|_{\text{res}}$, where the constant \tilde{c}_Γ can be taken as*

$$\tilde{c}_\Gamma = \begin{cases} \min\{(\sum_{i \neq j} |\lambda_i - \lambda_j|^{-1})^{-1}, (\sum_{i,j=0}^r |\lambda_i + \lambda_j - 1|^{-1})^{-1}\} & \text{if } \frac{1}{2} \notin \sigma(\Gamma), \\ \min\{(\sum_{i \neq j} |\lambda_i - \lambda_j|^{-1})^{-1}, (\sum_{i,j=0, (i,j) \neq (1,1)}^r |\lambda_i + \lambda_j - 1|^{-1})^{-1}\} & \text{if } \frac{1}{2} \in \sigma(\Gamma). \end{cases}$$

ii) *For every $X \in B(\mathfrak{H} \oplus \mathfrak{H})$, we have $\|\delta_\Gamma^0(X)\| \geq \tilde{c}_\Gamma^0 \|X - \tilde{\mathcal{E}}_\Gamma^0(X)\|$, where*

$$c_\Gamma^0 = \begin{cases} \frac{1}{2} \left(\sum_{i \neq j} |\lambda_i - \lambda_j|^{-1} + \sum_{i,j=0}^r |\lambda_i + \lambda_j - 1|^{-1} \right)^{-1} & \text{if } \frac{1}{2} \notin \sigma(\Gamma), \\ \frac{1}{2} \left(\sum_{i \neq j} |\lambda_i - \lambda_j|^{-1} + \sum_{i,j=0, (i,j) \neq (1,1)}^r |\lambda_i + \lambda_j - 1|^{-1} \right)^{-1} & \text{if } \frac{1}{2} \in \sigma(\Gamma). \end{cases}$$

Proof. i) For $X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \in B_2(\mathfrak{H} \oplus \mathfrak{H}, D)$, the block operator entries of $\tilde{\delta}_\Gamma(X) = [i\Gamma, X]$ are given by

$$\begin{aligned} [i\Lambda, x_{11}] &= \sum_{i,j=0}^r i(\lambda_i - \lambda_j) p_i x_{11} p_j, \quad i\Lambda x_{12} - x_{12} i(\mathbf{1} - \Lambda) = \sum_{i,j=0}^r i(\lambda_i + \lambda_j - 1) p_i x_{12} p_j \\ i(\mathbf{1} - \Lambda) x_{21} - x_{21} i\Lambda &= \sum_{i,j=0}^r i(1 - \lambda_i - \lambda_j) p_j x_{21} p_i, \quad [i(\mathbf{1} - \Lambda), x_{22}] = \sum_{i,j=0}^r i(\lambda_j - \lambda_i) p_i x_{22} p_j. \end{aligned}$$

Motivated by these formulas, for each $x \in B(\mathfrak{H})$, when $\frac{1}{2} \notin \sigma(\Gamma)$, set

$$A_x = - \sum_{\substack{i,j=0 \\ i \neq j}}^r i(\lambda_i - \lambda_j)^{-1} p_i x p_j, \quad B_x = - \sum_{i,j=0}^r i(\lambda_i + \lambda_j - 1)^{-1} p_i x p_j.$$

In the case where $\frac{1}{2} \in \sigma(\Gamma)$, the definition of A_x does not change, and for B_x we now set

$$B_x = - \sum_{\substack{i,j=0 \\ (i,j) \neq (1,1)}}^r i(\lambda_i + \lambda_j - 1)^{-1} p_i x p_j.$$

In both cases define $\beta_\Gamma : B_2(\mathfrak{H} \oplus \mathfrak{H}, D) \rightarrow B_2(\mathfrak{H} \oplus \mathfrak{H}, D)$,

$$\beta_\Gamma \left(\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \right) = \begin{pmatrix} A_{x_{11}} & B_{x_{12}} \\ -B_{x_{21}} & -A_{x_{22}} \end{pmatrix}.$$

This is a bounded linear operator acting on $B_2(\mathfrak{H} \oplus \mathfrak{H}, D)$ satisfying $\beta_\Gamma \circ \tilde{\delta}_\Gamma = \tilde{\delta}_\Gamma \circ \beta_\Gamma = \mathbf{1} - \tilde{\mathcal{E}}_\Gamma$ (here $\mathbf{1} = \mathbf{1}_{B_2(\mathfrak{H} \oplus \mathfrak{H}, D)}$). Therefore, $\|X - \tilde{\mathcal{E}}_\Gamma(X)\|_{\text{res}} = \|(\beta_\Gamma \circ \tilde{\delta}_\Gamma)(X)\|_{\text{res}} \leq \|\beta_\Gamma\| \|\tilde{\delta}_\Gamma(X)\|_{\text{res}}$, which yields $\|\tilde{\delta}_\Gamma(X)\|_{\text{res}} \geq \|\beta_\Gamma\|^{-1} \|X - \tilde{\mathcal{E}}_\Gamma(X)\|_{\text{res}}$ for all X . If $\frac{1}{2} \notin \sigma(\Gamma)$, then one can see $\|\beta_\Gamma\| \leq \max\{\sum_{i \neq j} |\lambda_i - \lambda_j|^{-1}, \sum_{i,j=0}^r |\lambda_i + \lambda_j - 1|^{-1}\}$. This implies that we can take $\tilde{c}_\Gamma = \min\{(\sum_{i \neq j} |\lambda_i - \lambda_j|^{-1})^{-1}, (\sum_{i,j=0}^r |\lambda_i + \lambda_j - 1|^{-1})^{-1}\}$. For the case where $\frac{1}{2} \in \sigma(\Gamma)$ we only need to write $\sum_{i,j=0, (i,j) \neq (1,1)}^r |\lambda_i + \lambda_j - 1|^{-1}$ instead of $\sum_{i,j=0}^r |\lambda_i + \lambda_j - 1|^{-1}$.

ii) Similar arguments apply for the extensions δ_Γ^0 and \mathcal{E}_Γ^0 when one takes the operator norm $\|\cdot\|$ instead of the restricted norm $\|\cdot\|_{\text{res}}$. \square

In general, the restricted norm of a Bogoliubov transformation can have arbitrary large restricted norm. In the following lemma, we control the restricted norm of Bogoliubov transformations acting on g1-pdms that are close enough in the relative topology.

Lemma 3.7. *Suppose that $\Gamma = \Gamma[\Lambda, 0] \in \mathcal{D}$ is a diagonal g1-pdm with finite spectrum. Then there exists a constant $K > 0$, depending only on Γ , such that $\|U\|_{\text{res}} \leq K$, for all $U \in \mathcal{U}_{\text{Bog}}$ satisfying $\|U\Gamma U^* - \Gamma\|_{\text{res}} \leq \frac{c_\Gamma^0}{3}$.*

Proof. Using that Γ is diagonal and expression in (9), we note

$$U = \begin{pmatrix} u & v \\ \bar{v} & \bar{u} \end{pmatrix}, \quad U\Gamma U^* = \begin{pmatrix} * & u\Lambda\bar{v}^* + v(\mathbf{1} - \Lambda)\bar{u}^* \\ * & * \end{pmatrix}.$$

Recall that $\|U\|_{\text{res}} = 2 \max\{\|u\|, \|v\|_2\}$. The operator U is unitary, which implies that $\|u\| \leq 1$ and $\|v\| \leq 1$. Thus, our task is to estimate $\|v\|_2$. Moreover, notice that $\Gamma \in \mathcal{D}$ with finite spectrum says that Λ is a finite-rank operator. Using the notation in Remark 2.3, we have $\Lambda = \sum_{i=0}^r \lambda_i p_i$, where $\sum_{i=0}^r p_i = \mathbf{1}$ and p_0 is the unique projection with infinite rank. Since $v = p_0 v p_0 + (\mathbf{1} - p_0) v p_0 + v(\mathbf{1} - p_0)$ and $\mathbf{1} - p_0$ has finite rank, we only have to estimate $\|p_0 v p_0\|_2$. By assumption $\|U\Gamma U^* - \Gamma\|_{\text{res}} \leq \frac{c_\Gamma^0}{3}$, and therefore, $\|u\Lambda\bar{v}^* + v(\mathbf{1} - \Lambda)\bar{u}^*\|_2 \leq \frac{c_\Gamma^0}{6}$. Since $\Lambda \in B_2(\mathfrak{H})$, we obtain

$$\begin{aligned} \|p_0 v p_0 \bar{u}^*\|_2 &\leq \frac{c_\Gamma^0}{6} + \|u\Lambda\bar{v}^*\|_2 + \|v\Lambda\bar{u}^*\|_2 + \|(\mathbf{1} - p_0)v p_0 + v(\mathbf{1} - p_0)\bar{u}^*\|_2 \\ &\leq \frac{c_\Gamma^0}{6} + \|u\| \|\bar{v}^*\| \|\Lambda\|_2 + \|v\| \|\bar{u}^*\| \|\Lambda\|_2 + \|(\mathbf{1} - p_0)\|_2 \|v p_0\| + \|v\| \|(\mathbf{1} - p_0)\|_2 \|\bar{u}^*\| \\ &\leq \frac{c_\Gamma^0}{6} + 2\|\Lambda\|_2 + 2\|\mathbf{1} - p_0\|_2 := C_1. \end{aligned}$$

Next consider

$$\begin{aligned} \|p_0 v p_0 \bar{u}^*\|_2^2 &= \|\bar{u} p_0 v^* p_0\|_2^2 = \text{Tr}(p_0 v p_0 \bar{u}^* \bar{u} p_0 v^* p_0) \\ &= \text{Tr}(p_0 v p_0 (\mathbf{1} - \bar{v}^* \bar{v}) p_0 v^* p_0) \geq (1 - \|p_0 \bar{v}^* \bar{v} p_0\|) \|p_0 v p_0\|_2^2 \end{aligned}$$

which gives

$$(1 - \|p_0 \bar{v}^* \bar{v} p_0\|) \|p_0 v p_0\|_2^2 \leq \|p_0 v p_0 \bar{u}^*\|_2^2 \leq C_1^2.$$

We will see below that $\|p_0 \bar{v}^* \bar{v} p_0\| = \|p_0 v^* v p_0\| \leq C < 1$ for some $C > 0$. This would imply the desired estimate:

$$\|p_0 v p_0\|_2 \leq (1 - C^2)^{-1/2} C_1. \quad (18)$$

To get a finer estimate the operator norm of v , we first give a bound for the operator norm in terms of the restricted norm: for $X \in \mathcal{B}_2(\mathfrak{H} \oplus \mathfrak{H}, D)$,

$$\begin{aligned} \|X\| &\leq \|P_+XP_+\| + \|P_+XP_-\| + \|P_-XP_+\| + \|P_-XP_-\| \\ &\leq \|P_+XP_+\| + \|P_+XP_-\|_2 + \|P_-XP_+\|_2 + \|P_-XP_-\| \leq 2\|X\|_{\text{res}}. \end{aligned}$$

In the case where $\frac{1}{2} \notin \sigma(\Gamma)$, then by the previous estimate and Lemma 3.6 *ii*) we obtain

$$\begin{aligned} \|vp_0\| &\leq \|v\| = \|P_+(U - \mathcal{E}_\Gamma^0(U))P_-\| \leq \|U - \mathcal{E}_\Gamma^0(U)\| \leq (c_\Gamma^0)^{-1}\|\delta_\Gamma^0(U)\| \\ &= (c_\Gamma^0)^{-1}\|UTU^* - \Gamma\| \leq 2(c_\Gamma^0)^{-1}\|UTU^* - \Gamma\|_{\text{res}} \leq \frac{2}{3}. \end{aligned}$$

In the case where $\frac{1}{2} \in \sigma(\Gamma)$ we have $P_+(U - \mathcal{E}_\Gamma^0(U))P_- = v - p_1vp_1$, and in the similar fashion as above we get $\|vp_0\| = \|(v - p_1vp_1)p_0\| \leq \|v - p_1vp_1\| = \|P_+(U - \mathcal{E}_\Gamma^0(U))P_-\| \leq \frac{2}{3}$. In both cases, $\|p_0v^*vp_0\| = \|vp_0\|^2 < \frac{4}{9}$, so by Eq. (18) we conclude $\|p_0vp_0\|_2 \leq \frac{9}{\sqrt{65}}C_1$. Hence we conclude $\|v\|_2 = \|p_0vp_0 + (\mathbf{1} - p_0)vp_0 + v(\mathbf{1} - p_0)\|_2 \leq \frac{9}{\sqrt{65}}C_1 + 2\|\mathbf{1} - p_0\|_2 := K$, where this constant only depends on Γ . \square

We are now able to prove our main result on the two topologies of orbits of g1-pdms.

Proposition 3.8. *Let $\Gamma \in \mathcal{D}$. Then $\tau_r = \tau_q$ if and only if $\sigma(\Gamma)$ is finite.*

Proof. Again we may assume that $\Gamma = \Gamma[\Lambda, 0]$ is a diagonal g1-pdm. Suppose that $\tau_r = \tau_q$. If $\sigma(\Gamma)$ is not finite, then the derivation $\delta_\Gamma : \mathfrak{u}_{\text{Bog}} \rightarrow \mathfrak{u}_{\text{Bog}}$, $\delta_\Gamma(X) = [i\Gamma, X]$ has non-closed range by Proposition 3.3. This is equivalent to the existence of a sequence $\{X_n\}_{n \geq 1}$ in \mathfrak{m}_Γ such that $\|X_n\|_{\text{res}} = 1$ and $\|\delta_\Gamma(X_n)\|_{\text{res}} \leq 1/n$. Recalling the well-known formula $e^Z \Gamma e^{-Z} = \text{Ad}_{e^Z}(\Gamma) = e^{\delta_Z}(\Gamma) = (\mathbf{1}_{\mathcal{B}(\mathfrak{H} \oplus \mathfrak{H})} + \delta_Z + \delta_Z^2/2 + \delta_Z^3/3! + \dots)(\Gamma)$, where $\delta_Z(X) = [iZ, X]$, and since $\|\delta_{X_n}\| \leq 2$, we find that

$$\begin{aligned} \|e^{X_n} \Gamma e^{-X_n} - \Gamma\|_{\text{res}} &= \|e^{\delta_{X_n}}(\Gamma) - \Gamma\|_{\text{res}} \leq \sum_{k=1}^{\infty} \frac{\|\delta_{X_n}^k(\Gamma)\|_{\text{res}}}{k!} \\ &\leq \sum_{k=1}^{\infty} \frac{\|\delta_{X_n}\|^{k-1} \|\delta_\Gamma(X_n)\|_{\text{res}}}{k!} \leq \frac{1}{n} \sum_{k=1}^{\infty} \frac{2^{k-1}}{k!} \leq \frac{e^2}{n}. \end{aligned}$$

That is, the sequence $\{e^{X_n} \Gamma e^{-X_n}\}_{n \geq 1}$ converges to Γ in the topology τ_r , and consequently, it also converges in the topology τ_q . The maps $\phi : \mathcal{V}_\Gamma \rightarrow \mathcal{W}_\Gamma$, $\phi(Z) = e^Z \Gamma e^{-Z}$, where $0 \in \mathcal{V}_\Gamma \subset \mathfrak{m}_\Gamma$ and $\Gamma \in \mathcal{W}_\Gamma \subseteq \mathcal{O}(\Gamma)$ are open sets, become homeomorphisms when these sets are sufficiently small and $\mathcal{O}(\Gamma)$ is endowed with the topology τ_q . Indeed, these maps are the inverse of the charts used to provide the manifold structure of $\mathcal{O}(\Gamma)$ as homogeneous space of \mathcal{U}_{Bog} (see [12, Lemma 4.21]). In particular, this yields that $\|X_n\|_{\text{res}} \rightarrow 0$, a contradiction. Hence $\sigma(\Gamma)$ must be finite.

Let us assume, conversely, that $\sigma(\Gamma)$ is finite. Suppose that $\Gamma = \Gamma[\Lambda, 0]$ is a diagonal g1-pdm. In order to show that both topologies coincide, we will prove that $\pi_\Gamma : \mathcal{U}_{\text{Bog}} \rightarrow \mathcal{O}(\Gamma)$, $\pi_\Gamma(U) = UTU^*$, admits continuous local cross sections when $\mathcal{O}(\Gamma)$ is considered with the topology τ_r . For G an invertible operator, let $\Omega(G) = G|G|^{-1}$ be the unitary part in the polar decomposition. Consider c_Γ^0 , \tilde{c}_Γ and K the positive constants defined in Lemmas 3.6 and 3.7. Then we set $r_\Gamma := \frac{1}{2} \min \left\{ \frac{c_\Gamma^0}{3}, K^{-2}\tilde{c}_\Gamma \right\}$. The section $s = s_\Gamma$ at Γ is then defined by

$$s : \{UTU^* \in \mathcal{O}(\Gamma) : \|UTU^* - \Gamma\|_{\text{res}} < r_\Gamma\} \rightarrow \mathcal{U}_{\text{Bog}}, \quad s(UTU^*) = U\Omega(\tilde{\mathcal{E}}_\Gamma(U^*)).$$

Clearly, we have $U\Omega(\tilde{\mathcal{E}}_\Gamma(U^*)) \in B_2(\mathfrak{H} \oplus \mathfrak{H}, D)$, for $U \in U_{\text{Bog}}$. We claim that if $\|UTU^* - \Gamma\|_{\text{res}} < r_\Gamma$, then $\tilde{\mathcal{E}}_\Gamma(U^*)$ is invertible in the algebra $B_2(\mathfrak{H} \oplus \mathfrak{H}, D)$. To see this we recall that by our choice of r_Γ , Lemma 3.7 implies $\|U\|_{\text{res}} \leq K$, or equivalently, $\|U^*\|_{\text{res}} \leq K$. Therefore, for $\mathbf{1} = \mathbf{1}_{\mathfrak{H} \oplus \mathfrak{H}}$,

$$\begin{aligned} \|\mathbf{1} - U\tilde{\mathcal{E}}_\Gamma(U^*)\|_{\text{res}} &\leq \|U\|_{\text{res}}\|U^* - \tilde{\mathcal{E}}_\Gamma(U^*)\|_{\text{res}} \leq K\tilde{c}_\Gamma^{-1}\|\tilde{\delta}_\Gamma(U^*)\|_{\text{res}} \\ &\leq K\tilde{c}_\Gamma^{-1}\|U^*\|_{\text{res}}\|UTU^* - \Gamma\|_{\text{res}} \leq K^2\tilde{c}_\Gamma^{-1}\|UTU^* - \Gamma\|_{\text{res}} < 1. \end{aligned}$$

Hence $U\tilde{\mathcal{E}}_\Gamma(U^*)$ invertible in $B_2(\mathfrak{H} \oplus \mathfrak{H}, D)$, and then $\tilde{\mathcal{E}}_\Gamma(U^*)$ is also invertible. This proves our claim.

Now we check that s is well defined. Suppose that $\Gamma_1 = UTU^* = V\Gamma V^*$ for $U, V \in U_{\text{Bog}}$. Thus, $U^*V\Gamma = \Gamma U^*V$, which gives

$$\begin{aligned} s(UTU^*) &= U\Omega(\tilde{\mathcal{E}}_\Gamma(U^*)) = U\Omega(\tilde{\mathcal{E}}_\Gamma(U^*VV^*)) = U\Omega(U^*V\tilde{\mathcal{E}}_\Gamma(V^*)) \\ &= UU^*V\Omega(\tilde{\mathcal{E}}_\Gamma(V^*)) = V\Omega(\tilde{\mathcal{E}}_\Gamma(V^*)) = s(V\Gamma V^*). \end{aligned}$$

Here we have used that $\tilde{\mathcal{E}}_\Gamma(ZX) = Z\tilde{\mathcal{E}}_\Gamma(X)$, whenever $X, Z \in B_2(\mathfrak{H} \oplus \mathfrak{H}, D)$ and $Z\Gamma = \Gamma Z$.

Now note that s actually takes values on U_{Bog} . From the definition of $\tilde{\mathcal{E}}_\Gamma$, $\tilde{\mathcal{E}}_\Gamma(U^*)I = I\tilde{\mathcal{E}}_\Gamma(U^*)$ because $U \in U_{\text{Bog}}$, which yields $\Omega(\tilde{\mathcal{E}}_\Gamma(U^*))I = I\Omega(\tilde{\mathcal{E}}_\Gamma(U^*))$. It is not difficult to check that $\text{ran}(\tilde{\mathcal{E}}_\Gamma)$ is a closed $*$ -subalgebra of $B_2(\mathfrak{H} \oplus \mathfrak{H}, D)$, so that $\Omega(\tilde{\mathcal{E}}_\Gamma(U^*)) \in B_2(\mathfrak{H} \oplus \mathfrak{H}, D)$ by using the analytic functional calculus in the Banach algebra $B_2(\mathfrak{H} \oplus \mathfrak{H})$. From these facts, we conclude that $\Omega(\tilde{\mathcal{E}}_\Gamma(U^*)) \in U_{\text{Bog}}$, and consequently, $s(UTU^*) \in U_{\text{Bog}}$. On the other hand, s is indeed a section. To see this, take $\Gamma_1 = UTU^*$. Since $\tilde{\mathcal{E}}_\Gamma(U^*)\Gamma = \Gamma\tilde{\mathcal{E}}_\Gamma(U^*)$, we find that $\Omega(\tilde{\mathcal{E}}_\Gamma(U^*))\Gamma = \Gamma\Omega(\tilde{\mathcal{E}}_\Gamma(U^*))$. Hence $s(\Gamma_1)\Gamma s(\Gamma_1)^* = U\Omega(\tilde{\mathcal{E}}_\Gamma(U^*))\Gamma\Omega(\tilde{\mathcal{E}}_\Gamma(U^*))^*U^* = UTU^* = \Gamma_1$.

We now prove that s is continuous. It suffices to prove the continuity at Γ . Pick a sequence $\{U_n\Gamma U_n^*\}_{n \geq 1}$ such that $\|U_n\Gamma U_n^* - \Gamma\|_{\text{res}} \rightarrow 0$. By the estimate we have seen above, it follows that $\|\mathbf{1} - U_n\tilde{\mathcal{E}}_\Gamma(U_n^*)\|_{\text{res}} \leq K^2\tilde{c}_\Gamma^{-1}\|U_n\Gamma U_n^* - \Gamma\|_{\text{res}} \rightarrow 0$. Using that $\Omega(G) = G|G|^{-1}$ is continuous as a map in the invertible elements of the Banach $*$ -algebra $B_2(\mathfrak{H} \oplus \mathfrak{H})$ by the continuity of the analytic functional calculus, we get $s(U_n\Gamma U_n^*) = U_n\Omega(\tilde{\mathcal{E}}_\Gamma(U_n^*)) = \Omega(U_n\tilde{\mathcal{E}}_\Gamma(U_n^*)) \rightarrow \Omega(\mathbf{1}) = \mathbf{1} = s(\Gamma)$. Finally, we note that section can be translated to other points by putting $s_{\Gamma_1} = \text{Ad}_U \circ s_\Gamma \circ \text{Ad}_{U^*}$ if $\Gamma_1 = UTU^*$. \square

As a consequence of the previous results we now obtain the following characterization.

Theorem 3.9. *Let $\Gamma \in \mathcal{D}$. The following conditions are equivalent:*

- i) *The spectrum of Γ is finite.*
- ii) *Tangent spaces of $\mathcal{O}(\Gamma)$ are closed in $i\mathfrak{u}_{\text{Bog}}$.*
- iii) *The topology inherited from $i\mathfrak{u}_{\text{Bog}} + P_-$ and the quotient topology coincide in $\mathcal{O}(\Gamma)$.*
- iv) *$\mathcal{O}(\Gamma)$ is an embedded submanifold of the affine space $i\mathfrak{u}_{\text{Bog}} + P_-$.*

Proof. The equivalences between i), ii) and iii) are proved in Propositions 3.3 and 3.8. According to Proposition A.9, these items are equivalent to item iv), since Proposition 3.3 also gives that tangent spaces are complemented subspaces in $i\mathfrak{u}_{\text{Bog}}$. \square

4 Kähler homogeneous spaces

4.1 An invariant symplectic form on orbits of \mathfrak{gl} -pdms

In this section, we will construct an invariant symplectic form on orbits of \mathfrak{gl} -pdms.

Remark 4.1. We will frequently use without mention the following well-known fact for computations involving the trace. If x is a compact operator and y a bounded operator such that xy and yx are trace-class operators, then $\text{Tr}(xy) = \text{Tr}(yx)$ (see [25, Chap.III Thm. 8.2]).

Remark 4.2. *i)* Consider the complex Lie algebra $\mathfrak{g} = \mathfrak{gl}_2(\mathfrak{H} \oplus \mathfrak{H}, I, D)$ defined in Remark 2.1. This corresponds to the case $k_\infty = 1$ in [33, Prop. I.11] (see also [33, Ex. I.9.b]), which says that $H_c^2(\mathfrak{g}, \mathbb{C}) \cong \mathbb{C}$. As we have already observed in the previous section, $\mathfrak{g}_{\mathbb{R}} = \{X \in \mathfrak{g} : X^* = -X\} = \mathfrak{u}_{\text{Bog}}$. Then same arguments to that of [15, Prop. 2.4] give that $H_c^2(\mathfrak{u}_{\text{Bog}}, \mathbb{R}) \cong \mathbb{R}$. Further, a generator of $H_c^2(\mathfrak{u}_{\text{Bog}}, \mathbb{R})$ is given by

$$s_+(X, Y) := \text{Tr}(X[iP_+, Y]), \quad X, Y \in \mathfrak{u}_{\text{Bog}}.$$

This can also be expressed as

$$s_+ \left(\begin{pmatrix} x_1 & x_2 \\ \bar{x}_2 & \bar{x}_1 \end{pmatrix}, \begin{pmatrix} y_1 & y_2 \\ \bar{y}_2 & \bar{y}_1 \end{pmatrix} \right) = 2\Im \text{Tr}(x_2 \bar{y}_2).$$

ii) It is interesting to point out that a multiple of the above generator already appeared in the literature as a 2-cocycle of $\mathfrak{o}_{\text{res}}(\mathfrak{H}^{\mathbb{R}})$. Following the notation in Remark 2.1 *ii)*, recall that for $A \in \mathfrak{B}(\mathfrak{H}^{\mathbb{R}})$, we write A_a for its antilinear part. It was proved by Vershik [40] that the function

$$\alpha : \mathfrak{o}_{\text{res}}(\mathfrak{H}^{\mathbb{R}}) \times \mathfrak{o}_{\text{res}}(\mathfrak{H}^{\mathbb{R}}) \rightarrow \mathbb{R}, \quad \alpha(A, B) = \text{Tr}([A_a, B_a]J_0) = 2 \text{Tr}(A_a B_a J_0),$$

is a continuous real 2-cocycle of $\mathfrak{o}_{\text{res}}(\mathfrak{H}^{\mathbb{R}})$. The trace is taken using the inner product $\Re \langle \cdot, \cdot \rangle$ of $\mathfrak{H}^{\mathbb{R}}$; A_a, B_a belong to $\mathfrak{B}_2(\mathfrak{H}^{\mathbb{R}})$ if $A, B \in \mathfrak{o}_{\text{res}}(\mathfrak{H}^{\mathbb{R}})$, which leads to the different expressions above for α . The differential map at the identity $\mathbf{1} = \mathbf{1}_{\mathfrak{H} \oplus \mathfrak{H}}$ of the Lie group isomorphism in (8) is given by $(d\Xi)_1 : \mathfrak{o}_{\text{res}}(\mathfrak{H}^{\mathbb{R}}) \rightarrow \mathfrak{u}_{\text{Bog}}$, $(d\Xi)_1(A) = T^{-1}A_{\mathbb{C}}T$, which is a Lie algebra isomorphism. Therefore there is a continuous real 2-cocycle $\Xi_*\alpha : \mathfrak{u}_{\text{Bog}} \times \mathfrak{u}_{\text{Bog}} \rightarrow \mathbb{R}$ defined by

$$(\Xi_*\alpha)(X, Y) := \alpha((d\Xi)_1^{-1}(X), (d\Xi)_1^{-1}(Y)).$$

Using analogous relations for the algebras level to the ones stated in (7) for the group level, it is easy to see that

$$(\Xi_*\alpha) \left(\begin{pmatrix} x_1 & x_2 \\ \bar{x}_2 & \bar{x}_1 \end{pmatrix}, \begin{pmatrix} y_1 & y_2 \\ \bar{y}_2 & \bar{y}_1 \end{pmatrix} \right) = -4\Im \text{Tr}(x_2 \bar{y}_2).$$

Hence $\Xi_*\alpha = -2s_+$.

Remark 4.3. *i)* We will need another complex Banach *-algebra:

$$\mathfrak{B}_{1,2}(\mathfrak{H} \oplus \mathfrak{H}, D) := \left\{ \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \in \mathfrak{B}(\mathfrak{H} \oplus \mathfrak{H}) : x_{11}, x_{22} \in \mathfrak{B}_1(\mathfrak{H}), x_{12}, x_{21} \in \mathfrak{B}_2(\mathfrak{H}) \right\},$$

which is endowed with the norm

$$\|X\|_{1,2} := 2 \max\{\|x_{11}\|_1, \|x_{22}\|_1, \|x_{12}\|_2, \|x_{21}\|_2\}.$$

This norm satisfies $\|XY\|_{1,2} \leq \|X\|_{1,2}\|Y\|_{1,2}$. The involution $*$ is the usual adjoint of operators. We remark that this type of block decompositions in terms of trace-class and Hilbert-Schmidt operators appeared in the *anomaly-free orthogonal and symplectic groups* [22], and in the study of infinite dimensional Poisson geometry related to the restricted Grassmannian [15, 42].

ii) We introduce a real subspace of $B_{1,2}(\mathfrak{H} \oplus \mathfrak{H}, D)$. Set

$$\begin{aligned} (\mathfrak{u}_{\text{Bog}})_* &:= \mathfrak{u}_{\text{Bog}} \cap B_{1,2}(\mathfrak{H} \oplus \mathfrak{H}, D) \\ &= \left\{ \begin{pmatrix} x_1 & x_2 \\ \bar{x}_2 & \bar{x}_1 \end{pmatrix} : x_1^* = -x_1 \in B_1(\mathfrak{H}), x_2^T = -x_2 \in B_2(\mathfrak{H}) \right\}. \end{aligned}$$

The notation is justified by the following fact: the pairing

$$\langle \cdot, \cdot \rangle : (\mathfrak{u}_{\text{Bog}})_* \times \mathfrak{u}_{\text{Bog}} \rightarrow \mathbb{R}, \quad \langle X, Y \rangle := \Re \text{Tr}(XY)$$

induces a topological isomorphism $((\mathfrak{u}_{\text{Bog}})_*)^* \cong \mathfrak{u}_{\text{Bog}}$. This is indeed an immediate consequence of [15, Prop. 2.1], where it was proved that

$$\mathfrak{u}_{\text{res}} := \left\{ \begin{pmatrix} x_{11} & x_{12} \\ -x_{12}^* & x_{22} \end{pmatrix} \in B(\mathfrak{H} \oplus \mathfrak{H}) : x_{11} = -x_{11}^*, x_{22} = -x_{22}^*, x_{12} \in B_2(\mathfrak{H}) \right\}$$

and

$$(\mathfrak{u}_{\text{res}})_* := \left\{ \begin{pmatrix} x_{11} & x_{12} \\ -x_{12}^* & x_{22} \end{pmatrix} \in B(\mathfrak{H} \oplus \mathfrak{H}) : x_{11} = -x_{11}^* \in B_1(\mathfrak{H}), x_{22} = -x_{22}^* \in B_1(\mathfrak{H}), x_{12} \in B_2(\mathfrak{H}) \right\}$$

satisfy $((\mathfrak{u}_{\text{res}})_*)^* \cong \mathfrak{u}_{\text{res}}$, where the duality pairing is given by $\langle X, Y \rangle = \text{Tr}(XY) = \Re \text{Tr}(XY)$. In order to prove the topological isomorphism $((\mathfrak{u}_{\text{Bog}})_*)^* \cong \mathfrak{u}_{\text{Bog}}$, given $f \in ((\mathfrak{u}_{\text{Bog}})_*)^*$, one extends this functional by the Hahn-Banach theorem to a functional in $((\mathfrak{u}_{\text{res}})_*)^*$. Therefore there exists $Y \in \mathfrak{u}_{\text{res}}$ such that $f(X) = \Re \text{Tr}(XY)$, for all $X \in (\mathfrak{u}_{\text{Bog}})_*$. We only have to show that the element Y can be changed for another element belonging to $\mathfrak{u}_{\text{Bog}}$. To this end note that for any $X \in (\mathfrak{u}_{\text{Bog}})_*$, one has that $f(X) = f(IXI) = \Re \text{Tr}((IXI)Y) = \Re \text{Tr}(X(IYI))$. Thus, $f(X) = f(\frac{X+IXI}{2}) = \Re \text{Tr}(X(\frac{Y+IYI}{2}))$, where $\frac{Y+IYI}{2} \in \mathfrak{u}_{\text{Bog}}$.

We now show that g1-pdms associated to quasi-free states give continuous real 2-cocycles of $\mathfrak{u}_{\text{Bog}}$.

Lemma 4.4. *For $\Gamma \in \mathcal{D}$, let $s_\Gamma : \mathfrak{u}_{\text{Bog}} \times \mathfrak{u}_{\text{Bog}} \rightarrow \mathbb{R}$ be defined by*

$$s_\Gamma(X, Y) := \text{Tr}(X[i\Gamma, Y]) = \text{Tr}(X\delta_\Gamma(Y)).$$

Then s_Γ is a continuous real 2-cocycle of $\mathfrak{u}_{\text{Bog}}$, which is cohomologous to $-s_+$.

Proof. We first prove that $s_\Gamma(X, Y) < \infty$. If $X = \begin{pmatrix} x_1 & x_2 \\ \bar{x}_2 & \bar{x}_1 \end{pmatrix} \in \mathfrak{u}_{\text{Bog}}$, $Y = \begin{pmatrix} y_1 & y_2 \\ \bar{y}_2 & \bar{y}_1 \end{pmatrix} \in \mathfrak{u}_{\text{Bog}}$, $\Gamma = \Gamma[\gamma, \alpha]$, then

$$[\Gamma, Y] = \begin{pmatrix} [\gamma, y_1] + \alpha \bar{y}_2 - y_2 \alpha^* & \gamma y_2 - y_2(\mathbf{1} - \bar{\gamma}) + \alpha \bar{y}_1 - y_1 \alpha \\ \alpha^* y_1 - \bar{y}_1 \alpha^* + (\mathbf{1} - \bar{\gamma}) \bar{y}_2 - \bar{y}_2 \gamma & \alpha^* y_2 - \bar{y}_2 \alpha + [\mathbf{1} - \bar{\gamma}, \bar{y}_1] \end{pmatrix} := \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & -\bar{z}_1 \end{pmatrix},$$

where $z_1 \in B_1(\mathfrak{H})$, $z_2 \in B_2(\mathfrak{H})$. In fact, the form of the right-hand matrix is derived by using that $\alpha^* = -\bar{\alpha}$. Then, it follows that

$$\begin{aligned} s_\Gamma(X, Y) &= i(\text{Tr}(x_1 z_1) - \text{Tr}(\bar{x}_1 \bar{z}_1)) + i(\text{Tr}(\bar{x}_2 z_2) - \text{Tr}(x_2 \bar{z}_2)) \\ &= -2\Im(\text{Tr}(x_1 z_1) + \text{Tr}(\bar{x}_2 z_2)) < \infty \end{aligned}$$

because $x_2, z_2 \in B_2(\mathfrak{H})$, $z_1 \in B_1(\mathfrak{H})$ and $x_1 \in B(\mathfrak{H})$, which implies $\bar{x}_2 z_2, x_1 z_1 \in B_1(\mathfrak{H})$. The above expression also shows that $s_\Gamma(X, Y) \in \mathbb{R}$. Let us show that s_Γ is a continuous 2-cocycle. For we write

$$\Gamma = \begin{pmatrix} \gamma & \alpha \\ -\bar{\alpha} & -\bar{\gamma} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1} \end{pmatrix} =: \Gamma_0 + P_-,$$

and

$$s_\Gamma(X, Y) = \text{Tr}(X[i\Gamma_0, Y]) + \text{Tr}(X[iP_-, Y]) =: s_{\Gamma_0}(X, Y) + s_-(X, Y).$$

We estimate the first term

$$\begin{aligned} |s_{\Gamma_0}(X, Y)| &= \left| \text{Tr} \left(\begin{pmatrix} x_1 & x_2 \\ \bar{x}_2 & \bar{x}_1 \end{pmatrix} \begin{pmatrix} [\gamma, y_1] + \alpha \bar{y}_2 - y_2 \alpha^* & \gamma y_2 + y_2 \bar{\gamma} + \alpha \bar{y}_1 - y_1 \alpha \\ \alpha^* y_1 - \bar{y}_1 \alpha^* - \bar{\gamma} \bar{y}_2 - \bar{y}_2 \gamma & \alpha^* y_2 - \bar{y}_2 \alpha - [\bar{\gamma}, \bar{y}_1] \end{pmatrix} \right) \right| \\ &\leq 4(\|\gamma\|_1 + \|\alpha\|_2) \|X\|_{\text{res}} \|Y\|_{\text{res}} \\ &\leq 4\|\Gamma_0\|_{1,2} \|X\|_{\text{res}} \|Y\|_{\text{res}}. \end{aligned} \tag{19}$$

The second term can be estimated as follows

$$|s_-(X, Y)| = 2 |\Im \text{Tr}(\bar{x}_2 y_2)| \leq 2 \|x_2\|_2 \|y_2\|_2 \leq \|X\|_{\text{res}} \|Y\|_{\text{res}}.$$

From these estimates we obtain that s_Γ is continuous.

We claim that s_{Γ_0} is a coboundary. Notice that the space of finite-rank operators is dense in the Banach algebra $B_{1,2}(\mathfrak{H} \oplus \mathfrak{H}, D)$ given in Remark 4.3. For Γ_{00} a finite-rank operator, it clearly holds that $\text{Tr}((X\Gamma_{00})Y) = \text{Tr}(Y(X\Gamma_{00}))$, which implies that $s_{\Gamma_{00}}(X, Y) = -\text{Tr}([X, Y]i\Gamma_{00}) := f_{\Gamma_{00}}([X, Y])$. Similar to the above estimates, one can see that $f_{\Gamma_0}(Z) = -\text{Tr}(Zi\Gamma_0)$, where $\Gamma_0 \in B_{1,2}(\mathfrak{H} \oplus \mathfrak{H}, D)$, satisfies $|f_{\Gamma_0}(Z)| \leq \|\Gamma_0\|_{1,2} \|Z\|_{\text{res}}$. Combining this with (19), we can take the limit of finite-rank operators in the norm $\|\cdot\|_{1,2}$ to get that $s_{\Gamma_0}(X, Y) = f_{\Gamma_0}([X, Y])$, which proves our claim.

A simple computation shows that $s_- = -s_+$. Since s_{Γ_0} is a coboundary and s_+ is a 2-cocycle, it follows that s_Γ is a 2-cocycle. This also shows that s_Γ and $-s_+$ are cohomologous. \square

Lemma 4.5. $\mathfrak{u}_{\text{Bog}}^\Gamma = \{X \in \mathfrak{u}_{\text{Bog}} : s_\Gamma(X, Y) = 0, \forall Y \in \mathfrak{u}_{\text{Bog}}\}$.

Proof. Straightforward computations show the following:

$$s_\Gamma(\text{Ad}_V(X), \text{Ad}_V(Y)) = s_{\text{Ad}_{V^*(\Gamma)}}(X, Y), \quad V \in \mathfrak{U}_{\text{Bog}}, X, Y \in \mathfrak{u}_{\text{Bog}}.$$

So we may assume that Γ is a diagonal operator. Thus, to prove one inclusion, we take

$$X = \begin{pmatrix} x_1 & x_2 \\ \bar{x}_2 & \bar{x}_1 \end{pmatrix} \in \mathfrak{u}_{\text{Bog}}^\Gamma, \quad Y = \begin{pmatrix} y_1 & y_2 \\ \bar{y}_2 & \bar{y}_1 \end{pmatrix} \in \mathfrak{u}_{\text{Bog}}, \quad \Gamma = \begin{pmatrix} \Lambda & 0 \\ 0 & \mathbf{1} - \Lambda \end{pmatrix}.$$

We are going to show that $s_\Gamma(X, Y) = 0$. We denote by $\{\cdot, \cdot\}$ the anticommutator of operators. Then,

$$\begin{aligned} s_\Gamma(X, Y) &= i \operatorname{Tr} \left(\begin{pmatrix} x_1 & x_2 \\ \bar{x}_2 & \bar{x}_1 \end{pmatrix} \begin{pmatrix} [\Lambda, y_1] & \{\Lambda, y_2\} - y_2 \\ \bar{y}_2 - \{\Lambda, \bar{y}_2\} & [\bar{y}_1, \Lambda] \end{pmatrix} \right) \\ &= -2\Im (\operatorname{Tr}(x_1[\Lambda, y_1]) + \operatorname{Tr}(x_2(\bar{y}_2 - \{\Lambda, \bar{y}_2\}))). \end{aligned}$$

Now we use the characterization of the Lie algebra $\mathfrak{u}_{\text{Bog}}^\Gamma$ proved in (11). Since $\Lambda \in B_1(\mathfrak{H})$ and $x_1\Lambda = \Lambda x_1$, it follows that the first term vanishes. The second term also vanishes using that $x_2(\mathbf{1} - \Lambda) = \Lambda x_2$. This proves $s_\Gamma(X, Y) = 0$.

In order to show the reversed inclusion, we take an operator $X \in \mathfrak{u}_{\text{Bog}}$ such that $s_\Gamma(X, Y) = 0$ for all $Y \in \mathfrak{u}_{\text{Bog}}$. As usual, we write

$$X = \begin{pmatrix} x_1 & x_2 \\ \bar{x}_2 & \bar{x}_1 \end{pmatrix}, \quad Y = \begin{pmatrix} y_1 & y_2 \\ \bar{y}_2 & \bar{y}_1 \end{pmatrix},$$

with $x_1^* = -x_1$, $-x_2^T = x_2 \in B_2(\mathfrak{H})$, $y_1^* = -y_1$ and $-y_2^T = y_2 \in B_2(\mathfrak{H})$. By the same calculation as above, we have

$$0 = s_\Gamma(X, Y) = -2\Im (\operatorname{Tr}(x_1[\Lambda, y_1]) + \operatorname{Tr}(x_2(\bar{y}_2 - \{\Lambda, \bar{y}_2\}))). \quad (20)$$

First, we take $y_2 = 0$, which gives $0 = \Im \operatorname{Tr}(x_1[\Lambda, y_1]) = \Im \operatorname{Tr}([x_1, \Lambda]y_1)$ for every $y_1^* = -y_1$. Put $z := [x_1, \Lambda]$, which satisfies $z = z^*$. Take $y_1 = iz$ to get $0 = \Im \operatorname{Tr}(ziz) = \|z\|_2^2$, and consequently, $z = 0$. That is, $x_1\Lambda = \Lambda x_1$.

Now we have that (20) turns into $0 = \Im \operatorname{Tr}(x_2(\bar{y}_2 - \{\Lambda, \bar{y}_2\})) = \Im \operatorname{Tr}((x_2(\mathbf{1} - \Lambda) - \Lambda x_2)\bar{y}_2)$, for all $-y_2^T = y_2 \in B_2(\mathfrak{H})$. Note that $z := x_2(\mathbf{1} - \Lambda) - \Lambda x_2$ satisfies $-z^T = z \in B_2(\mathfrak{H})$. Then, take $y_2 = iz$ to find that $0 = \Im \operatorname{Tr}(z(-i\bar{z})) = \Im i \operatorname{Tr}(z z^*) = \|z\|_2^2$. Thus, $\Lambda x_2 = x_2(\mathbf{1} - \Lambda)$. Hence $X \in \mathfrak{u}_{\text{Bog}}^\Gamma$. \square

Now we write the symplectic form of Theorem A.11 in our context. We take $M = \mathcal{O}(\Gamma)$, $G = \mathfrak{U}_{\text{Bog}}$, $K = \mathfrak{U}_{\text{Bog}}^\Gamma$ and $s = s_\Gamma$ in the mentioned theorem. Recall that we may identify $T_\Gamma \mathcal{O}(\Gamma) = \{[X, \Gamma] : X \in \mathfrak{u}_{\text{Bog}}\}$. So at the point Γ , set

$$\omega_\Gamma([X, \Gamma], [Y, \Gamma]) := s_\Gamma(X, Y), \quad X, Y \in \mathfrak{u}_{\text{Bog}}.$$

Then, for $U \in \mathfrak{U}_{\text{Bog}}$,

$$\omega_{U\Gamma U^*}([X, U\Gamma U^*], [Y, U\Gamma U^*]) := \omega_\Gamma([U^* X U, \Gamma], [U^* Y U, \Gamma]), \quad X, Y \in \mathfrak{u}_{\text{Bog}}.$$

Thus, $\omega = \Sigma(s_\Gamma)$ is defined by $\omega := \{\omega_{U\Gamma U^*}\}_{U \in \mathfrak{U}_{\text{Bog}}}$.

Theorem 4.6. *Let $\Gamma \in \mathcal{D}$. Then $(\mathcal{O}(\Gamma), \Sigma(s_\Gamma))$ is a weakly symplectic homogeneous space. Furthermore, if the spectrum of Γ is finite, then $(\mathcal{O}(\Gamma), \Sigma(s_\Gamma))$ is a strongly symplectic homogeneous space.*

Proof. All the necessary conditions to apply Theorem A.11 have been proved in Theorem 2.9, and Lemmas 4.4 and 4.5. Hence $(\mathcal{O}(\Gamma), \Sigma(s_\Gamma))$ becomes a weakly symplectic homogeneous space.

In order to show the second assertion, notice that by the invariance of the symplectic form it is enough to prove that for $X \in \mathfrak{u}_{\text{Bog}}$ the injective map

$$T_\Gamma \mathcal{O}(\Gamma) \rightarrow T_\Gamma^* \mathcal{O}(\Gamma), \quad [X, \Gamma] \mapsto \operatorname{Tr}(X[i\Gamma, \cdot]),$$

is also surjective. In view of (15) and (16) we note that $\text{ran}(\delta_\Gamma) = T_\Gamma \mathcal{O}(\Gamma) \subseteq i(\mathfrak{u}_{\text{Bog}})_* \subseteq B_{1,2}(\mathfrak{H} \oplus \mathfrak{H}, D)$ if Γ has finite spectrum. Since $\|x\| \leq \|x\|_1$ for any operator x , we derive the following relations between norms on the tangent space: for $X \in \mathfrak{u}_{\text{Bog}}$,

$$\|[X, \Gamma]\|_\Gamma = \inf\{\|X + Y\|_{\text{res}} : Y\Gamma = \Gamma Y\} \leq \|[X, \Gamma]\|_{\text{res}} \leq \|[X, \Gamma]\|_{1,2}.$$

Now take $f \in T_\Gamma^* \mathcal{O}(\Gamma)$. From the previous relations between the norms, it follows that f must be also continuous with respect to the norm $\|\cdot\|_{1,2}$. By the Hahn-Banach theorem, there exists a functional $\tilde{f} \in (i(\mathfrak{u}_{\text{Bog}})_*)^*$ such that $\tilde{f}|_{T_\Gamma \mathcal{O}(\Gamma)} = f$. According to Remark 4.3 ii), we have $(i(\mathfrak{u}_{\text{Bog}})_*)^* \cong i\mathfrak{u}_{\text{Bog}}$, so that there is an operator $Z_0 \in i\mathfrak{u}_{\text{Bog}}$ such that $\tilde{f} = \text{Tr}(Z_0 \cdot)$. Therefore, we get $f([Y, \Gamma]) = \tilde{f}([Y, \Gamma]) = \text{Tr}(Z_0[Y, \Gamma]) = \text{Tr}(iZ_0[i\Gamma, Y])$, where $iZ_0 \in \mathfrak{u}_{\text{Bog}}$. \square

4.2 Orbits of g1-pdms as Kähler homogeneous spaces

We now show that orbits of g1-pdms are Kähler homogenous spaces of U_{Bog} using a characterization of these spaces in Lie algebraic terms. Thus we will construct Kähler polarizations. We consider the finite and infinite spectrum cases separately.

Remark 4.7. i) Notice that $(\mathfrak{u}_{\text{Bog}})_\mathbb{C} = \mathfrak{gl}_2(\mathfrak{H} \oplus \mathfrak{H}, I, D) =: \mathfrak{g}$ (see (5)), meanwhile $X \mapsto \bar{X} = -X^*$ is the involution on \mathfrak{g} whose set of fixed points is $\mathfrak{u}_{\text{Bog}}$.

Remark 4.8. In the following lemma we assume that Γ is expressed as

$$\Gamma = \begin{pmatrix} \Lambda & 0 \\ 0 & \mathbf{1} - \Lambda \end{pmatrix}, \quad \Lambda = \sum_{i=1}^r \lambda_i p_i, \quad (21)$$

where $\{p_i\}_{i=1}^{r+1}$ ($r < \infty$) is a decomposition of the identity with the properties stated in Remark 2.3 for $i = 1, \dots, r$, but we modify the notation $p_{r+1} := p_0$ and $\lambda_{r+1} := \lambda_0 = 0$. In particular, recall that each p_i is the projection onto the finite-dimensional subspace $\ker(\Lambda - \lambda_i \mathbf{1})$ for $i = 1, \dots, r$, and p_{r+1} is the projection onto the infinite-dimensional subspace $\ker(\Lambda)$. Further, we suppose that the eigenvalues satisfy $\lambda_i > \lambda_j$ if $i < j$, for all $i, j = 1, \dots, r+1$.

Lemma 4.9 (Finite spectrum). *Suppose that $\Gamma = \Gamma[\Lambda, 0] \in \mathcal{D}$ has finite spectrum and Γ can be written as in (21). Then the following hold:*

i) *If $\frac{1}{2} \notin \sigma(\Gamma)$, then*

$$\mathfrak{p} := \left\{ \begin{pmatrix} x & 0 \\ y & -x^T \end{pmatrix} \in \mathfrak{g} : p_i x p_j = 0 \text{ if } i = 1, \dots, r+1, j < i \right\}$$

is a strongly Kähler polarization of $\mathfrak{u}_{\text{Bog}}$ in s_Γ .

ii) *If $\frac{1}{2} \in \sigma(\Gamma)$, then*

$$\mathfrak{p} := \left\{ \begin{pmatrix} x & z \\ y & -x^T \end{pmatrix} \in \mathfrak{g} : p_i x p_j = 0 \text{ if } i = 1, \dots, r+1, j < i, z = p_1 z p_1 \right\}$$

is a strongly Kähler polarization of $\mathfrak{u}_{\text{Bog}}$ in s_Γ .

Proof. *i)* We say that an operator x is upper triangular with respect to the decomposition of the identity $\{p_i\}_{i=1}^{r+1}$ if $p_i x p_j = 0$ if $i = 1, \dots, r+1, j < i$. Observe that \mathfrak{p} is a closed subalgebra of \mathfrak{g} since it is clearly a complex subspace, and

$$\left[\begin{pmatrix} x_1 & 0 \\ y_1 & -x_1^T \end{pmatrix}, \begin{pmatrix} x_2 & 0 \\ y_2 & -x_2^T \end{pmatrix} \right] = \begin{pmatrix} [x_1, x_2] & 0 \\ w & -[x_1, x_2]^T \end{pmatrix},$$

where $[x_1, x_2]$ is upper triangular with respect to the decomposition of the identity $\{p_i\}_{i=1}^{r+1}$ if x_1, x_2 are, and $w = y_1 x_2 - x_1^T y_2 - y_2 x_1 + x_2^T y_1$ satisfies $w^T = -w \in \mathcal{B}_2(\mathfrak{H})$ if $y_i^T = -y_i \in \mathcal{B}_2(\mathfrak{H})$, $i = 1, 2$. Also it is clear that \mathfrak{p} is closed in \mathfrak{g} , and \mathfrak{p} admits the closed complement

$$\mathfrak{n} := \left\{ \begin{pmatrix} x & z \\ 0 & -x^T \end{pmatrix} \in \mathfrak{g} : p_i x p_j = 0 \text{ if } i = 1, \dots, r+1, j \geq i \right\}.$$

Recall the description of $\mathfrak{u}_{\text{Bog}}^\Gamma$ in the proof of Lemma 2.5, which after taking into account the notation change $p_0 := p_{r+1}$, leads to

$$(\mathfrak{u}_{\text{Bog}}^\Gamma)_{\mathbb{C}} = \left\{ \begin{pmatrix} x & 0 \\ 0 & -x^T \end{pmatrix} \in \mathfrak{g} : x = \sum_{i=1}^{r+1} p_i x p_i \right\}.$$

Straightforward computations show that $\mathfrak{p} + \bar{\mathfrak{p}} = \mathfrak{g}$, $\mathfrak{p} \cap \bar{\mathfrak{p}} = (\mathfrak{u}_{\text{Bog}}^\Gamma)_{\mathbb{C}}$ and $\text{Ad}_U \mathfrak{p} \subseteq \mathfrak{p}$ for all $U \in \mathfrak{U}_{\text{Bog}}^\Gamma$ (see (12)).

Now take

$$X_i = \begin{pmatrix} x_i & 0 \\ y_i & -x_i^T \end{pmatrix} \in \mathfrak{p}, \quad i = 1, 2.$$

Recall that $\text{Tr}(x) = \text{Tr}(x^T)$ for any $x \in \mathcal{B}_1(\mathfrak{H})$. By this fact we obtain

$$\begin{aligned} s_\Gamma(X_1, X_2) &= i \text{Tr}(X_1[\Gamma, X_2]) = i \text{Tr} \left(\begin{pmatrix} x_1[\Lambda, x_2] & 0 \\ * & -x_1^T[\Lambda, x_2^T] \end{pmatrix} \right) \\ &= i \text{Tr}(x_1[\Lambda, x_2] - x_1^T[\Lambda, x_2^T]) \\ &= i(\text{Tr}(x_1 \Lambda x_2 - x_2 \Lambda x_1)) = i \sum_{i=1}^r \lambda_i (\text{Tr}(p_i x_1 p_i x_2 p_i) - \text{Tr}(p_i x_2 p_i x_1 p_i)) = 0. \end{aligned}$$

Hence $s_\Gamma(\mathfrak{p} \times \mathfrak{p}) = 0$.

Let us show the last condition:

$$-i s_\Gamma(X, \bar{X}) > 0, \quad X = \begin{pmatrix} x & 0 \\ y & -x^T \end{pmatrix} \in \mathfrak{p} \setminus (\mathfrak{u}_{\text{Bog}}^\Gamma)_{\mathbb{C}}.$$

To this end, recall that $\Lambda = \sum_{i=1}^{r+1} \lambda_i p_i = \sum_{i=1}^r \lambda_i p_i$, and notice

$$\begin{aligned} -i s_\Gamma(X, \bar{X}) &= -\text{Tr}(X[\Gamma, X^*]) = -\text{Tr} \left(\begin{pmatrix} x & 0 \\ y & -x^T \end{pmatrix} \begin{pmatrix} [\Lambda, x^*] & \Lambda y^* - y^*(\mathbf{1} - \Lambda) \\ 0 & [\Lambda, \bar{x}] \end{pmatrix} \right) \\ &= -\text{Tr} \left(\begin{pmatrix} x[\Lambda, x^*] & * \\ * & y \Lambda y^* - y y^*(\mathbf{1} - \Lambda) - x^T[\Lambda, \bar{x}] \end{pmatrix} \right) \\ &= -2 \text{Tr}(x[\Lambda, x^*]) + \text{Tr}(y y^* - (y^* y + y y^*) \Lambda) := A + B. \end{aligned}$$

In the last line we have used that $\text{Tr}(x[\Lambda, x^*]) = -\text{Tr}(x^T[\Lambda, \bar{x}])$. Now observe x is upper triangular with respect to $\{p_i\}_{i=1}^{r+1}$ if and only if $x = \sum_{i=1}^{r+1} \sum_{j \geq i} p_i x p_j$. Then, we have

$$\begin{aligned} A &= -2 \text{Tr}(x[\Lambda, x^*]) = 2 \text{Tr}((xx^* - x^*x)\Lambda) \\ &= \text{Tr} \left(\sum_{i=1}^r \sum_{j \geq i} \lambda_i p_i x p_j x^* p_i \right) - \text{Tr} \left(\sum_{i=1}^{r+1} \sum_{j \geq i} \lambda_j p_j x^* p_i x p_j \right) \\ &= 2 \sum_{i=1}^r \sum_{j > i} (\lambda_i - \lambda_j) \|p_i x p_j\|_2^2. \end{aligned}$$

Since $\lambda_1 > \dots > \lambda_r > \lambda_{r+1} = 0$, it follows from this expression that $A > 0$ when $p_i x p_j \neq 0$ for some $i = 1, \dots, r, j = i+1, \dots, r+1$.

On the other hand, the second term can be estimated as follows

$$\begin{aligned} B &= \text{Tr}(yy^* - (y^*y + yy^*)\Lambda) = \|y\|_2^2 - \sum_{i=1}^r (\lambda_i \text{Tr}(y^* y p_i) + \lambda_i \text{Tr}(y y^* p_i)) \\ &\geq \|y\|_2^2 - \lambda_1 \left(\sum_{i=1}^r \text{Tr}(p_i y^* y p_i) + \text{Tr}(p_i y y^* p_i) \right) \\ &\geq (1 - 2\lambda_1) \|y\|_2^2. \end{aligned}$$

The last inequality is a consequence of the fact $\|\sum_{i=1}^r p_i z p_i\|_1 \leq \|z\|_1$, for all $z \in B_1(\mathfrak{H})$ (see e.g. [25, Chap. III Thm. 4.2]) and $\|\sum_{i=1}^r p_i z p_i\|_1 = \text{Tr}(\sum_{i=1}^r p_i z p_i)$ for $z \geq 0$. Notice that $B > 0$ if $y \neq 0$ because $\lambda_1 = \max_{i=1, \dots, r} \lambda_i < \frac{1}{2}$. We conclude that $-is(X, \bar{X}) > 0$ whenever $X \in \mathfrak{p} \setminus (\mathfrak{u}_{\text{Bog}}^\Gamma)_{\mathbb{C}}$, which occurs when $p_i x p_j \neq 0$ for some $i = 1, \dots, r, j = i+1, \dots, r+1$, or when $y \neq 0$. Hence \mathfrak{p} is a polarization of $\mathfrak{u}_{\text{Bog}}$ in \mathfrak{s}_Γ , which is strong by Theorem 4.6.

ii) The definition of \mathfrak{p} is changed for this case where $\frac{1}{2} \in \sigma(\Gamma)$. In order to see that \mathfrak{p} is a subalgebra of \mathfrak{g} , we consider the four entries $[X_1, X_2]_{ij}$ of the following operator:

$$[X_1, X_2] = \begin{pmatrix} x_1 & z_1 \\ y_1 & -x_1^T \end{pmatrix} \begin{pmatrix} x_2 & z_2 \\ y_2 & -x_2^T \end{pmatrix} - \begin{pmatrix} x_2 & z_2 \\ y_2 & -x_2^T \end{pmatrix} \begin{pmatrix} x_1 & z_1 \\ y_1 & -x_1^T \end{pmatrix}.$$

First, we see that

$$[X_1, X_2]_{11} = [x_1, x_2] + p_1(z_1 p_1 y_2 - z_2 p_1 y_1),$$

where both terms are clearly upper triangular with respect to the decomposition $\{p_i\}_{i=1}^{r+1}$. Next note

$$\begin{aligned} [X_1, X_2]_{12} &= x_1 z_2 - z_1 x_2^T - (x_2 z_1 - z_2 x_1^T) \\ &= p_1(x_1 p_1 z_2 + x_2 p_1 z_1 - z_1 p_1 x_2^T - z_2 p_1 x_1^T) p_1 := p_1 w p_1, \end{aligned}$$

where $w^T = -w$. The next term is given by

$$[X_1, X_2]_{21} = y_1 x_2 - x_1^T y_2 - (y_2 x_1 - x_2^T y_1).$$

Here we observe that $[X_1, X_2]_{21} \in B_2(\mathfrak{H})$ because $y_i \in B_2(\mathfrak{H})$, and $[X_1, X_2]_{21}^T = -[X_1, X_2]_{21}$. The last term satisfies

$$[X_1, X_2]_{22} = [x_1^T, x_2^T] + (y_1 p_1 z_2 - y_2 p_1 z_1) p_1 = -[X_1, X_2]_{11}^T.$$

This completes the proof that \mathfrak{p} is a subalgebra of \mathfrak{g} . Clearly, a closed supplement of \mathfrak{p} in \mathfrak{g} is given by

$$\mathfrak{n} := \left\{ \begin{pmatrix} x & z \\ 0 & -x^T \end{pmatrix} \in \mathfrak{g} : p_i x p_j = 0 \text{ if } i = 1, \dots, r+1, j \geq i, \text{ and } p_1 z p_1 = 0 \right\}.$$

In this case, the complexification of $\mathfrak{u}_{\text{Bog}}^\Gamma$ turns out to be

$$(\mathfrak{u}_{\text{Bog}}^\Gamma)_\mathbb{C} = \left\{ \begin{pmatrix} x & z \\ y & -x^T \end{pmatrix} \in \mathfrak{g} : x = \sum_{i=1}^{r+1} p_i x p_i, z = p_1 z p_1, z = -z^T, y = p_1 y p_1, y = -y^T \right\}. \quad (22)$$

Again the reader can check the three conditions $\mathfrak{p} + \bar{\mathfrak{p}} = \mathfrak{g}$, $\mathfrak{p} \cap \bar{\mathfrak{p}} = (\mathfrak{u}_{\text{Bog}}^\Gamma)_\mathbb{C}$ and $\text{Ad}_U \mathfrak{p} \subseteq \mathfrak{p}$ for all $U \in \text{U}_{\text{Bog}}^\Gamma$ (see (13)). Now we show that $s_\Gamma(X_1, X_2) = 0$, for all $X_1, X_2 \in \mathfrak{p}$. For notice

$$\begin{aligned} s_\Gamma(X_1, X_2) &= \text{Tr} \left(\begin{pmatrix} x_1 & z_1 \\ y_1 & -x_1^T \end{pmatrix} \left[i \begin{pmatrix} \Lambda & 0 \\ 0 & \mathbf{1} - \Lambda \end{pmatrix}, \begin{pmatrix} x_2 & z_2 \\ y_2 & -x_2^T \end{pmatrix} \right] \right) \\ &= i \text{Tr}(x_1[\Lambda, x_2] - x_1^T[\Lambda, x_2^T]) + i \text{Tr}(z_1((\mathbf{1} - \Lambda)y_2 - y_2\Lambda)) + i \text{Tr}(y_1(\Lambda z_2 - z_2(\mathbf{1} - \Lambda))) \\ &= 0 \end{aligned}$$

This follows because the first term vanishes as in the previous case. For the second term observe that $z_1 = p_1 z_1 p_1$, where p_1 is the projection onto $\ker(\Lambda - \frac{1}{2})$. Therefore,

$$\text{Tr}(z_1((\mathbf{1} - \Lambda)y_2 - y_2\Lambda)) = \text{Tr}(z_1 p_1((\mathbf{1} - \Lambda)y_2 - y_2\Lambda) p_1) = 0.$$

For the third term, note $\Lambda z_2 - z_2(\mathbf{1} - \Lambda) = \frac{1}{2}(p_1 z_2 p_1 - p_1 z_2 p_1) = 0$.

In order to prove the last condition of a Kähler polarization, we compute for $X \in \mathfrak{p} \setminus (\mathfrak{u}_{\text{Bog}}^\Gamma)_\mathbb{C}$

$$\begin{aligned} -is_\Gamma(X, \bar{X}) &= -\text{Tr}(X[\Gamma, X^*]) = -\text{Tr} \left(\begin{pmatrix} x & z \\ y & -x^T \end{pmatrix} \left[\begin{pmatrix} \Lambda & 0 \\ 0 & \mathbf{1} - \Lambda \end{pmatrix}, \begin{pmatrix} x^* & y^* \\ z^* & -\bar{x} \end{pmatrix} \right] \right) \\ &= -2 \text{Tr}(x[\Lambda, x^*]) + \text{Tr}(yy^* - (y^*y + yy^*)\Lambda) - \text{Tr}(z((\mathbf{1} - \Lambda)z^* - z^*\Lambda)) \\ &:= A + B + C. \end{aligned}$$

Again since $z = p_1 z p_1$, it follows that $C = 0$. Using the description of $(\mathfrak{u}_{\text{Bog}}^\Gamma)_\mathbb{C}$ in (22), we have that $X \in \mathfrak{p} \setminus (\mathfrak{u}_{\text{Bog}}^\Gamma)_\mathbb{C}$ if and only if $p_i x p_j \neq 0$ for some $i = 1, \dots, r, j > i$, or when $y \neq 0, y \neq p_1 y p_1$. The term A can be estimated as in the first case, but for the other term note $\|y\|_2^2 = \|p_1 y\|_2^2 + \|(\mathbf{1} - p_1)y\|_2^2$, which yields

$$\begin{aligned} B &= \text{Tr}(yy^* - (y^*y + yy^*)\Lambda) = \|(\mathbf{1} - p_1)y\|_2^2 - \sum_{i=2}^r (\lambda_i \text{Tr}(y^* y p_i) + \lambda_i \text{Tr}(y y^* p_i)) \\ &\geq \|(\mathbf{1} - p_1)y\|_2^2 - \lambda_2 \left(\sum_{i=2}^r \text{Tr}(p_i (y(\mathbf{1} - p_1))^* y (\mathbf{1} - p_1) p_i) + \text{Tr}(p_i (\mathbf{1} - p_1) y ((\mathbf{1} - p_1) y)^* p_i) \right) \\ &\geq (1 - 2\lambda_2) \|(\mathbf{1} - p_1)y\|_2^2, \end{aligned}$$

where $\lambda_2 = \max_{i=2, \dots, r} \lambda_i < \frac{1}{2}$ and $(\mathbf{1} - p_1)y \neq 0$, for $y \neq p_1 y p_1$. Hence $-is_\Gamma(X, \bar{X}) > 0$ for $X \in \mathfrak{p} \setminus (\mathfrak{u}_{\text{Bog}}^\Gamma)_\mathbb{C}$. \square

Now we consider the infinite spectrum case.

Remark 4.10. For following lemma we suppose that Γ has infinite spectrum and $\ker(\Gamma) = \{0\}$. Thus, it can be written as

$$\Gamma = \begin{pmatrix} \Lambda & 0 \\ 0 & \mathbf{1} - \Lambda \end{pmatrix}, \quad \Lambda = \sum_{i=1}^{\infty} \lambda_i p_i, \quad (23)$$

where $\lambda_i \neq 0$, $i \geq 1$, and $\{p_i\}_{i=1}^{\infty}$ is a decomposition of the identity with the properties stated in Remark 2.3. Thus, each p_i is the projection onto the finite-dimensional subspace $\ker(\Lambda - \lambda_i \mathbf{1})$ for $i \geq 1$. We again suppose that the eigenvalues satisfy $\lambda_i > \lambda_j$ if $i < j$, for all $i, j \geq 1$.

Lemma 4.11 (Infinite spectrum). *Suppose that $\Gamma = \Gamma[\Lambda, 0] \in \mathcal{D}$ has infinite spectrum, $\ker(\Gamma) = \{0\}$ and Γ can be expressed as in (23). Then the following hold:*

i) If $\frac{1}{2} \notin \sigma(\Gamma)$, then

$$\mathfrak{p} := \left\{ \begin{pmatrix} x & 0 \\ y & -x^T \end{pmatrix} \in \mathfrak{g} : p_i x p_j = 0 \text{ if } i \geq 1, j < i \right\}$$

is a weakly Kähler polarization of $\mathfrak{u}_{\text{Bog}}$ in s_{Γ} .

ii) If $\frac{1}{2} \in \sigma(\Gamma)$, then

$$\mathfrak{p} := \left\{ \begin{pmatrix} x & z \\ y & -x^T \end{pmatrix} \in \mathfrak{g} : p_i x p_j = 0 \text{ if } i \geq 1, j < i, z = p_1 z p_1 \right\}$$

is a weakly Kähler polarization of $\mathfrak{u}_{\text{Bog}}$ in s_{Γ} .

Proof. i) The proof follows the same lines as the finite-dimensional case with the following additional remarks. For a decomposition of the identity $\{p_i\}_{i \geq 1}$ and $x \in B(\mathfrak{H})$, it is well-known that $x = \sum_{i,j \geq 1} p_i x p_j$, where the convergence of this series is in the WOT (weak operator topology). Furthermore, $\|\sum_{i=1}^N \sum_{j=1}^M p_i x p_j\| \leq \|x\|$ for all $M, N \geq 1$. Notice that x is then upper triangular with respect to $\{p_i\}_{i \geq 1}$ if and only if $x = \sum_{i=1}^{\infty} \sum_{j \geq i} p_i x p_j$. From these facts we may write $x = \sum_{i=1}^{\infty} \sum_{j \geq i} p_i x p_j + \sum_{i=1}^{\infty} \sum_{j < i} p_i x p_j$, where both series converge in the WOT. Thus, the following subspace

$$\mathfrak{n} := \left\{ \begin{pmatrix} x & z \\ 0 & -x^T \end{pmatrix} \in \mathfrak{g} : p_i x p_j = 0 \text{ if } i \geq 1, j \geq i \right\}$$

becomes a closed supplement for \mathfrak{p} in \mathfrak{g} . Using that the adjoint and multiplication by a fixed operator are both continuous maps in the WOT, all but the last of the conditions of a Kähler polarization are straightforward to check.

The last condition reads $-is_{\Gamma}(X, \overline{X}) = -2 \text{Tr}(x[\Lambda, x^*]) + \text{Tr}(yy^* - (y^*y + yy^*)\Lambda) > 0$ for all $X \in \mathfrak{p} \setminus (\mathfrak{u}_{\text{Bog}}^{\Gamma})_{\mathbb{C}}$. Following the finite spectrum case treated in Lemma 4.9, in order to deal with the first term of $-is_{\Gamma}(X, \overline{X})$ we need to compute $\text{Tr}((xx^* - x^*x)\Lambda)$, where $x = \sum_{i=1}^{\infty} \sum_{j \geq i} p_i x p_j$ is now a WOT-convergent series whose partial sums are bounded in norm. To handle $\text{Tr}(xx^*\Lambda)$

we apply twice the fact that functionals of the form $f(y) = \text{Tr}(yz)$, $z \in B_1(\mathfrak{H})$, $y \in B(\mathfrak{H})$, are WOT continuous on the unit ball ([23, Thm. 5.3]). That is, we have

$$\begin{aligned} \text{Tr}(xx^*\Lambda) &= \sum_{i=1}^{\infty} \sum_{j \geq i} \text{Tr}(p_i x p_j x^* \Lambda) \\ &= \sum_{i=1}^{\infty} \sum_{j \geq i} \text{Tr} \left(p_i x p_j \left(\sum_{k=1}^{\infty} \sum_{l \geq k} p_l x^* p_k \right) \left(\sum_{s=1}^{\infty} \lambda_s p_s \right) \right) \\ &= \sum_{i=1}^{\infty} \sum_{j \geq i} \lambda_i \text{Tr}(p_i x p_j x^* p_i). \end{aligned}$$

The term $\text{Tr}(x^*x\Lambda)$ can be handled analogously. Hence,

$$-2 \text{Tr}(x[\Lambda, x^*]) = 2 \sum_{i=1}^{\infty} \sum_{j > i} (\lambda_i - \lambda_j) \|p_i x p_j\|_2^2.$$

Since $\ker(\Gamma) = \{0\}$ and $\lambda_1 > \lambda_2 > \dots$, we conclude that $\lambda_i - \lambda_j > 0$ for all $j > i$. The second term of $-is_{\Gamma}(X, \bar{X})$ can be treated as in the finite spectrum case. Finally, we observe that Theorem 4.6 implies now that \mathfrak{p} is weakly polarization because the spectrum of Γ is infinite.

ii) Similar remarks apply to this case. \square

The previous results now imply the following.

Theorem 4.12. *Let $\Gamma \in \mathcal{D}$. Then the following assertions hold:*

- i) *If Γ has finite spectrum, then $(\mathcal{O}(\Gamma), \Sigma(s_{\Gamma}))$ is a strongly Kähler homogeneous space.*
- ii) *If Γ has infinite spectrum and $\ker(\Gamma) = \{0\}$, then $(\mathcal{O}(\Gamma), \Sigma(s_{\Gamma}))$ is a weakly Kähler homogeneous space.*

Proof. According to Theorem A.8 there exists a diagonal g1-pdm $\Gamma = \Gamma[\Lambda, 0]$ satisfying the conditions of Lemmas 4.9 or 4.11. Then the assertion of each item i) or ii) follows by Theorem A.13. \square

Remark 4.13. Each orbit $\mathcal{O}(\Gamma)$ can be endowed with a complex manifold structure such that the maps $\alpha_U : \mathcal{O}(\Gamma) \rightarrow \mathcal{O}(\Gamma)$, $\alpha_U(\Gamma_1) = U\Gamma_1 U^*$, $\Gamma_1 \in \mathcal{O}(\Gamma)$, are holomorphic for every $U \in U_{\text{Bog}}$. Indeed, this follows by noting that the requirements of Remark A.12 have been checked in Lemmas 4.9 and 4.11 without any assumption on the kernel of Γ .

A Appendix

A.1 Hartree-Fock-Bogoliubov theory

In this appendix we provide the necessary background on Hartree-Fock-Bogoliubov (HFB) theory. Most of the material is presented following [6, 7, 8], where the reader can find the proofs omitted here. We prefer the basis-dependent approach considered in these references, which has

the benefit of dealing with linear operators. A basis independent approach, where one deals with conjugate-linear operators, is also used in the literature (see, e.g., [41]).

Fock space formalism. The one-particle Hilbert space \mathfrak{H} is a complex separable Hilbert space endowed with an inner product $\langle \cdot, \cdot \rangle$, which is linear in the second variable and conjugate-linear in the first. The fermionic Fock space \mathfrak{F} over \mathfrak{H} is defined as the orthogonal sum

$$\mathfrak{F} := \mathfrak{F}[\mathfrak{H}] = \bigoplus_{N=0}^{\infty} \mathfrak{F}^{(N)},$$

where $\mathfrak{F}^{(N)} = \bigwedge_{n=1}^N \mathfrak{H}$ is the N -fold skew-symmetric tensor product of \mathfrak{H} for $N \geq 1$; and $\mathfrak{F}^{(0)} = \mathbb{C}\Omega$, where $\Omega = (1, 0, 0, \dots) \in \mathfrak{F}$ is the vacuum vector. Thus, each vector in $\mathfrak{F}^{(N)}$ is a possibly infinite combination of simple vectors of the form

$$f_1 \wedge \dots \wedge f_N := \frac{1}{\sqrt{N!}} \sum_{\pi \in S_N} \text{sign}(\pi) f_{\pi(1)} \otimes f_{\pi(2)} \otimes \dots \otimes f_{\pi(N)}, \quad f_i \in \mathfrak{H},$$

where S_N denotes the symmetric group and $\text{sign}(\pi)$ the signature of the permutation π . Notice that the inner product between simple vectors on each $\mathfrak{F}^{(N)}$ is given by

$$\langle f_1 \wedge \dots \wedge f_N, g_1 \wedge \dots \wedge g_N \rangle_{\mathfrak{F}^{(N)}} = \det(\langle f_i, g_j \rangle)_{1 \leq i, j \leq N}, \quad f_i, g_i \in \mathfrak{H}.$$

The Fock space \mathfrak{F} becomes a Hilbert space equipped with the inner product defined by

$$\langle \Psi_1, \Psi_2 \rangle_{\mathfrak{F}} := \sum_{N=0}^{\infty} \langle f_1^{(N)}, f_2^{(N)} \rangle_{\mathfrak{F}^{(N)}}, \quad \Psi_i = (f_i^{(0)}, f_i^{(1)}, \dots), \quad i = 1, 2.$$

We now introduce some useful linear operators on the Fock space. Associated to each vector $f \in \mathfrak{H}$, there are creation operators $c^*(f)$ and annihilation operators $c(f)$ acting on \mathfrak{F} . The creation operator is defined on simple vectors by

$$c^*(f)\Omega = f; \quad c^*(f)(f_1 \wedge \dots \wedge f_N) := f \wedge f_1 \wedge \dots \wedge f_N.$$

Then, $c^*(f)$ extends to \mathfrak{F} by linearity. The annihilation operator $c(f)$ is defined as the adjoint $c(f) := (c^*(f))^*$. It is noteworthy that $c^*(f)$ and $c(f)$ are linear and conjugate-linear in f , respectively. Further, we note $c(f)\Omega = 0$. Using that $f_1 \wedge \dots \wedge f_N = c^*(f_1)c^*(f_2)\dots c^*(f_N)\Omega$, it follows that any vector of \mathfrak{F} can be expressed as a limit of polynomials in creation operators acting on the vacuum vector. Creation and annihilation operators satisfy the *canonical anticommutation relations (CARs)*:

$$\begin{aligned} \{c(f), c^*(g)\} &= \langle f, g \rangle \mathbf{1}_{\mathfrak{F}}; \\ \{c^*(f), c^*(g)\} &= \{c(f), c(g)\} = 0. \end{aligned}$$

for every $f, g \in \mathfrak{H}$. Here $\{A, B\} := AB + BA$ is the anticommutator of A and B . Let $B(\mathfrak{F})$ denote the algebra of bounded linear operators on \mathfrak{F} . In particular, the CARs imply that $c(f), c^*(f) \in B(\mathfrak{F})$. Moreover, $c(f)$ and $c^*(f)$ are a partial isometries on \mathfrak{F} , and the map c isometric, i.e. $\|c(f)\| = \|f\|$ and $\|c^*(f)\| = \|f\|$. The unital C^* -algebra generated by $\{c(f) : f \in \mathfrak{H}\}$ in $B(\mathfrak{F})$ is a concrete construction of the *CAR algebra* $\mathcal{A} = \mathcal{A}(\mathfrak{H})$ of \mathfrak{H} .

Another important operator on the Fock space is the *particle number operator* $\widehat{\mathbb{N}}$, which is defined by

$$\widehat{\mathbb{N}} := \bigoplus_{N \geq 0} N \mathbf{1}_{\mathfrak{F}^{(N)}}.$$

This is an unbounded operator, its maximal domain is given by those vectors $\Psi = (f^{(0)}, f^{(1)}, \dots)$ such that $\sum_{N=1}^{\infty} N^2 \|f^{(N)}\|_{\mathfrak{F}^{(N)}}^2 < \infty$.

Bogoliubov transformations. Throughout, we fix an orthonormal basis $\{\varphi_k\}_{k \geq 1}$ of \mathfrak{H} . Associated to this basis, we define a conjugation on \mathfrak{H} by

$$I_0 f := \bar{f} := \sum_{k \geq 1} \bar{\mu}_k \varphi_k \quad (24)$$

if $f = \sum_{k \geq 1} \mu_k \varphi_k$. By a conjugation we mean that I_0 is a conjugate-linear isometry acting on \mathfrak{H} such that $I_0^2 = \mathbf{1}$. Then, for any operator x , we define the operators

$$\bar{x} := I_0 x I_0, \quad \text{and} \quad x^T := (\bar{x})^* = \overline{x^*}.$$

Let $B(\mathfrak{H})$ be the algebra of bounded operators on \mathfrak{H} . Consider two operators $u, v \in B(\mathfrak{H})$ satisfying the following relations

$$u u^* + v v^* = \mathbf{1} = u^* u + v^T \bar{v}; \quad (25)$$

$$u^* v + v^T \bar{u} = 0 = u v^T + v u^T. \quad (26)$$

Notice that equations (25) and (26) are equivalent to say that the following operator

$$U = \begin{pmatrix} u & v \\ \bar{v} & \bar{u} \end{pmatrix} \quad (27)$$

is unitary on $\mathfrak{H} \oplus \mathfrak{H}$. Unitary operators on $\mathfrak{H} \oplus \mathfrak{H}$ having this form are known as *Bogoliubov transformations*.

For each Bogoliubov transformation U as in (27), the operators $d^*(f) := c^*(uf) + c(v\bar{f})$, $d(f) := (d^*(f))^*$, also fulfill the CARs. Then there is a unique $*$ -automorphism β_U of the CAR algebra such that $\beta_U(c(f)) = d(f)$, for all $f \in \mathfrak{H}$ ([19, Thm. 5.2.5]). It is of interest to know whether there exists a unitary operator that implements the automorphism β_U on the Fock space. There is a well-known characterization (see [3]):

Theorem A.1. *Let U be a Bogoliubov transformation as in (27). Then there exists a unitary operator $\mathbb{U} := \mathbb{U}_U : \mathfrak{F} \rightarrow \mathfrak{F}$ such that*

$$\mathbb{U} c^*(f) \mathbb{U}^* = c^*(uf) + c(v\bar{f}), \quad (28)$$

for all $f \in \mathfrak{H}$, if and only if $\text{Tr}(v^* v) < \infty$.

Any such a unitary operator \mathbb{U} is called a *unitary implementer of U* , or simply an *implementer of U* ; it is uniquely determined up to a scalar factor of modulus one. The condition of the operator v being a Hilbert-Schmidt operator is usually referred to as the *Shale-Stinespring condition*. The group of all unitary implementers is also known as the *current group*, and it is a topological group in the strong operator topology (see [4]).

Quasi-free states. A state ω on the algebra $B(\mathfrak{F})$ is a complex linear map such that $\omega(\mathbf{1}_{\mathfrak{F}}) = 1$ and $\omega(A^*A) \geq 0$, for all $A \in B(\mathfrak{F})$. In particular, this implies that ω is continuous. A state is normal if there exists a density matrix ρ (i.e. ρ is a trace-class operator on \mathfrak{F} , $\rho \geq 0$ and $\text{Tr}(\rho) = 1$) such that $\omega(A) = \text{Tr}_{\mathfrak{F}}(\rho A)$, for all $A \in B(\mathfrak{F})$. Important examples of normal states for us are the pure states, which have the form $\omega(A) = \langle \Psi, A\Psi \rangle_{\mathfrak{F}}$, for all $A \in B(\mathfrak{F})$, and some unit vector $\Psi \in \mathfrak{F}$.

We are interested in the following special class of states. A normal state ω is said to be *quasi-free* if it satisfies Wick's theorem, i.e. given vectors $h_1, \dots, h_{2n} \in \mathfrak{H}$ and $2n$ creation and annihilation operators $e_1, \dots, e_{2n} \in \{c^*(h_1), c(h_1), \dots, c^*(h_{2n}), c(h_{2n})\}$, we have $\omega(e_1 \dots e_{2n-1}) = 0$ and

$$\omega(e_1 \dots e_{2n}) = \sum_{\pi \in P_{2n}} \text{sign}(\pi) \omega(e_{\pi(1)} e_{\pi(2)}) \dots \omega(e_{\pi(2n-1)} e_{\pi(2n)}),$$

where P_{2n} denotes the set of all permutations $\pi \in S_{2n}$ such that $\pi(1) < \pi(3) < \dots < \pi(2n-1)$ and $\pi(2j-1) < \pi(2j)$, for all $1 \leq j \leq n$. For instance, when $n = 2$, this means that $\omega(e_1 e_2 e_3 e_4) = \omega(e_1 e_4) \omega(e_2 e_3) - \omega(e_1 e_3) \omega(e_2 e_4) + \omega(e_1 e_2) \omega(e_3 e_4)$. Additionally, we will often restrict to the following class of states of physical interest. A state ω is said to have *finite particle number* if

$$\omega(\widehat{\mathbb{N}}) := \sum_{N=1}^{\infty} N \omega(\Pi^{(N)}) < \infty. \quad (29)$$

where $\Pi^{(N)}$ is the orthogonal projection onto $\mathfrak{F}^{(N)}$.

Example A.2. In the case in which ω is a pure state associated to a vector $\Psi \in \mathfrak{F}^{(N)}$ for some $N \geq 0$, then ω is a quasi-free state if and only if $\Psi = f_1 \wedge \dots \wedge f_N$, for some orthonormal vectors f_1, \dots, f_N (take $\Psi = \Omega$ when $N = 0$). Vectors having this form are known as *Slater determinants*. Clearly, the corresponding pure state has finite particle number.

Notation A.3. We will use the following notation:

$$\begin{aligned} \mathcal{Z} &:= \{\omega : \omega \text{ is a normal state}\}; \\ \mathcal{Z}_{\text{qf}} &:= \{\omega \in \mathcal{Z} : \omega \text{ is a quasi-free state with finite particle number}\}. \end{aligned}$$

Generalized one-particle density matrices. Each state ω gives rise to a *generalized one-particle density matrix (g1-pdm)* Γ_{ω} , which is defined as the self-adjoint operator on $\mathfrak{H} \oplus \mathfrak{H}$ satisfying

$$\left\langle \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \Gamma_{\omega} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \right\rangle_{\mathfrak{H} \oplus \mathfrak{H}} = \omega([c^*(g_1) + c(\bar{g}_2)][c(f_1) + c^*(\bar{f}_2)]), \quad f_i, g_i \in \mathfrak{H}.$$

Since the conjugation I_0 might change (or not) when the basis $\{\varphi_k\}_{k \geq 1}$ changes, the definition of Γ_{ω} depends on the fixed basis. A straightforward computation using the CARs shows that

$$0 \leq \Gamma_{\omega} \leq \mathbf{1}_{\mathfrak{H} \oplus \mathfrak{H}}. \quad (30)$$

It is convenient to express Γ_{ω} as a 2×2 matrix of operators on $\mathfrak{H} \oplus \mathfrak{H}$,

$$\Gamma_{\omega} = \begin{pmatrix} \gamma_{\omega} & \alpha_{\omega} \\ \alpha_{\omega}^* & \mathbf{1} - \bar{\gamma}_{\omega} \end{pmatrix} =: \Gamma_{\omega}[\gamma_{\omega}, \alpha_{\omega}]. \quad (31)$$

Recalling that $\{\varphi_k\}_{k \geq 1}$ is our fixed orthonormal basis of \mathfrak{H} , we abbreviate $c_k^* := c^*(\varphi_k)$ and $c_k := c(\varphi_k)$. The operator blocks of Γ_ω can be expressed as

$$\langle \varphi_m, \gamma_\omega \varphi_k \rangle = \omega(c_k^* c_m), \quad \langle \varphi_m, \alpha_\omega^* \varphi_k \rangle = \omega(c_k^* c_m^*). \quad (32)$$

Notice that

$$\gamma_\omega^* = \gamma_\omega, \quad \alpha_\omega^T = -\alpha_\omega. \quad (33)$$

The operator γ_ω is called *one-particle density matrix (1-pdm)* of the state ω . In the special case of the pure state $\omega_\Psi(A) = \langle \Psi, A\Psi \rangle_{\mathfrak{F}}$, for some unit vector $\Psi \in \mathfrak{F}^{(N)}$, the 1-pdm γ_{ω_Ψ} satisfies $0 \leq \gamma_{\omega_\Psi} \leq \mathbf{1}$ and it has trace equal to N . Conversely, Lieb [31] proved that for each self-adjoint operator γ on \mathfrak{H} such that $0 \leq \gamma \leq \mathbf{1}$ and $\text{Tr}(\gamma) = N$, there exists a unit vector $\Psi \in \mathfrak{F}^{(N)}$ such that $\gamma = \gamma_{\omega_\Psi}$. The 1-pdm is related to the particle number operator. Noticing that $\widehat{\mathbf{N}} = \sum_{k \geq 1} c_k^* c_k$ as a quadratic form, then

$$\omega(\widehat{\mathbf{N}}) = \sum_{k \geq 1} \omega(c_k^* c_k) = \text{Tr}(\gamma_\omega). \quad (34)$$

It follows that γ_ω is a trace-class operator if and only if Γ_ω is the g1-pdm of a state ω with finite particle number. The operator α_ω is called the *pairing matrix*. When $\omega(A) = \langle \Psi, A\Psi \rangle$, for a vector $\Psi = (f^{(0)}, f^{(1)}, \dots) \in \mathfrak{F}$, is a (pure) quasi-free state with finite particle number, then $\omega(\widehat{\mathbf{N}}^2) < \infty$, and its variance is expressed as

$$\omega(\widehat{\mathbf{N}}^2) - \omega(\widehat{\mathbf{N}})^2 = \sum_{n \geq 0} (n - N)^2 \|f^{(N)}\|_{\mathfrak{F}^{(N)}}^2 = 2 \text{Tr}(\alpha_\omega^* \alpha_\omega).$$

Thus, the Hilbert-Schmidt norm of α_ω measures the spreading of Ψ among the subspaces $\mathfrak{F}^{(N)}$.

One can define generalized one-particle density matrices without reference to a state, only taking into account the properties that these operators have when they are associated to a state. That is, a g1-pdm can be defined as an operator of the form

$$\Gamma = \Gamma[\gamma, \alpha] = \begin{pmatrix} \gamma & \alpha \\ \alpha^* & \mathbf{1} - \bar{\gamma} \end{pmatrix} \in \text{B}(\mathfrak{H} \oplus \mathfrak{H})$$

satisfying the obvious corresponding conditions (30) and (33). It is then natural to address the question of which g1-pdms are associated to a state. Using the relation in (34), one can prove the following sufficient condition ([7, Thm. 2.3]).

Theorem A.4. *Let $\Gamma = \Gamma[\gamma, \alpha]$ be a g1-pdm such that $\text{Tr}(\gamma) < \infty$. Then there exists a unique quasi-free state ω with finite particle number such that $\Gamma = \Gamma_\omega$.*

Any g1-pdm $\Gamma = \Gamma[\gamma, \alpha]$ such that $\text{Tr}(\gamma) < \infty$ is called an *admissible generalized one-particle density matrix*.

Notation A.5. Let $\text{B}_1(\mathfrak{H})$ and $\text{B}_2(\mathfrak{H})$ be the ideals of trace-class and Hilbert-Schmidt operators on \mathfrak{H} , respectively. The set of all admissible g1-pdms is denoted by

$$\mathcal{D} := \left\{ \begin{pmatrix} \gamma & \alpha \\ \alpha^* & \mathbf{1} - \bar{\gamma} \end{pmatrix} \in \text{B}(\mathfrak{H} \oplus \mathfrak{H}) : 0 \leq \begin{pmatrix} \gamma & \alpha \\ \alpha^* & \mathbf{1} - \bar{\gamma} \end{pmatrix} \leq \mathbf{1}_{\mathfrak{H} \oplus \mathfrak{H}}, \gamma = \gamma^* \in \text{B}_1(\mathfrak{H}), \alpha^T = -\alpha \right\}. \quad (35)$$

Remark A.6. i) Note that, by the previous result, there is a bijection between quasi-free states and the (convex) set \mathcal{D} , namely $\Phi : \mathcal{Z}_{\text{qf}} \rightarrow \mathcal{D}$, $\Phi(\omega) = \Gamma_\omega$. Let $U : \mathfrak{H} \oplus \mathfrak{H} \rightarrow \mathfrak{H} \oplus \mathfrak{H}$ be a Bogoliubov transformation for which there exists a unitary implementer $\mathbb{U}_U : \mathfrak{F} \rightarrow \mathfrak{F}$. For each state $\omega \in \mathcal{Z}_{\text{qf}}$, define $\omega_U(A) := \omega(\mathbb{U}_U A \mathbb{U}_U^*)$ for any $A \in \mathcal{A}$. Using (28), it easily follows that $\omega_U \in \mathcal{Z}_{\text{qf}}$. Furthermore, it can be shown that $\Phi(\omega_U) = U^* \Gamma_\omega U$.

ii) We claim that $\Gamma = \Gamma[\gamma, \alpha] \in \mathcal{D}$ implies $\alpha \in \text{B}_2(\mathfrak{H})$. To see this, notice that $0 \leq \Gamma \leq \mathbf{1}_{\mathfrak{H} \oplus \mathfrak{H}}$ is equivalent to

$$0 \leq \Gamma - \Gamma^2 = \begin{pmatrix} \gamma - \gamma^2 - \alpha\alpha^* & (\mathbf{1} - \gamma)\alpha + \alpha(\mathbf{1} - \bar{\gamma}) \\ \alpha^*(\mathbf{1} - \gamma) + (\mathbf{1} - \bar{\gamma})\alpha^* & \bar{\gamma} - \bar{\gamma}^2 - \alpha^*\alpha \end{pmatrix}.$$

In particular, we get that $\gamma \geq \gamma^2 + \alpha\alpha^*$, from which our claim follows by using $\gamma \in \text{B}_1(\mathfrak{H})$.

We end this collection of results on g1-pdms with the well-known relation between pure states and projections, and a remarkable diagonalization result (see [7, Thm. 2.6] and the proof of [7, Thm. 2.3]).

Theorem A.7. *A quasi-free state ω with finite particle number is a pure state if and only if its g1-pdm $\Gamma_\omega \in \mathcal{D}$ is a projection on $\mathfrak{H} \oplus \mathfrak{H}$ (i.e. $\Gamma_\omega^2 = \Gamma_\omega$).*

Theorem A.8. *Let $\Gamma \in \mathcal{D}$. Then there exists a Bogoliubov transformation W satisfying the Shale-Stinespring condition such that*

$$W\Gamma W^* = \begin{pmatrix} \Lambda & 0 \\ 0 & \mathbf{1} - \Lambda \end{pmatrix}, \quad (36)$$

where Λ is a diagonal operator with respect to the fixed orthonormal basis $\{\varphi_k\}_{k \geq 1}$ satisfying $\text{Tr}(\Lambda) < \infty$ and $0 \leq \Lambda \leq \frac{1}{2}\mathbf{1}$.

Hartree-Fock-Bogoliubov theory. The Hamiltonian \mathbb{H} is a self-adjoint operator on the Fock space, which we additionally assume to be bounded from below. This Hamiltonian generates the dynamics of a many-particle fermion system, that is, particles satisfying the Pauli exclusion principle. It is of relevance to compute the total ground state energy in the grand canonical ensemble, which is given by the Rayleigh-Ritz principle:

$$E_{gs} := \inf\{\omega(\mathbb{H}) : \omega \in \mathcal{Z}\}.$$

Notice that the expression $\omega(\mathbb{H}) \in [0, \infty]$ is well defined because \mathbb{H} is bounded from below (see [7, Section 2.c]). This is indeed an extension of the original formulation of the Rayleigh-Ritz principle that only uses pure states for the computation of above infimum. Determining the ground state energy E_{gs} is an almost impossible task (except in a few well-known examples). One of the most widely used approximation methods is the *(traditional) Hartree-Fock method* in quantum chemistry to calculate the energy of atoms and molecules. This method replaces the set of all states for the set of all pure states given by Slater determinants:

$$E_{HF} := \inf\{\langle \Psi, \mathbb{H}\Psi \rangle_{\mathfrak{F}} : N \in \mathbb{N}, \Psi = f_1 \wedge \dots \wedge f_N, \langle f_i, f_j \rangle = \delta_{ij}\}.$$

In this case, one assumes that the Hamiltonian \mathbb{H} conserves the particle number (i.e. $[\mathbb{H}, \widehat{\mathbb{N}}] = 0$). Notice the upper bound $E_{gs} \leq E_{HF}$.

In this work we are interested in the Hartree-Fock-Bogoliubov method, which generalizes the Hartree-Fock method and it relies on the machinery described above between Bogoliubov transformations, quasi-free states and generalized one-particle density matrices. In this method the Hamiltonian is not assumed to conserve the particle number. The resulting Hartree-Fock-Bogoliubov energy is obtained by restricting the infimum to the set of all quasi-free states:

$$\begin{aligned} E_{HFB} &:= \inf\{\omega(\mathbb{H}) : \omega \in \mathcal{Z}_{\text{qf}}\} \\ &= \inf\{\mathcal{E}(\Gamma) : \Gamma \in \mathcal{D}\}. \end{aligned}$$

In the second line, we have reformulated the infimum with the aid of the Hartree-Fock-Bogoliubov functional defined by $\mathcal{E}(\Gamma_\omega) := \omega(\mathbb{H})$, where Γ_ω is the gl-pdm of the quasi-free state ω . Observe again that $E_{gs} \leq E_{HFB}$. We are not dealing with domain requirements. The quasi-free states ω for which the kinetic energy is finite (i.e. $\omega(\mathbb{H}) < \infty$) were studied for particular classes of Hamiltonians (see [8, 29]).

Let us mention examples of Hamiltonians that are usually considered. Let h be a self-adjoint operator on \mathfrak{H} , the one-body operator, and let W be a self-adjoint operator acting on $\mathfrak{H} \otimes \mathfrak{H}$, the two-body operator. We write h_j for the operator $\mathbf{1} \otimes \dots \otimes \mathbf{1} \otimes h \otimes \mathbf{1} \otimes \dots \otimes \mathbf{1}$, where h is placed in the j th factor of the N -th fold tensor product. Similarly, W_{ij} acts only the i th and j th factors. Then one considers a Hamiltonian \mathbb{H} on the Fock space formally defined by

$$\mathbb{H} = \bigoplus_{N \geq 0} \left(\sum_{i=1}^N h_i + \sum_{1 \leq i, j \leq N} W_{ij} \right).$$

The fermions to consider can be electrons, and h the Hamiltonian from molecular physics given by taking $h = -\Delta - \sum_{k=1}^K \frac{Z_k}{|x-r_k|} - \mu$ acting on $\mathfrak{H} = L^2(\mathbb{R}^3, \mathbb{C}^2)$, and $W = \frac{1}{|x-y|}$ interpreted as a multiplication operator on $\mathfrak{H} \otimes \mathfrak{H} \cong L^2(\mathbb{R}^6, \mathbb{C}^2)$. Here $r_k \in \mathbb{R}^3$, $r = 1, \dots, K$, represents the positions of the K nuclei, $Z_i > 0$ are their charges, and $\mu < 0$ is a sufficiently small chemical potential. Another example is the pseudo-relativistic Hamiltonian for describing fermions obtained by taking $h = \sqrt{-\Delta + m^2} - m$, $m > 0$ is the mass, and W the same as before (see [29]). As a third example we refer to the Hubbard model treated in [7], where the Hilbert space \mathfrak{H} is now finite dimensional. In all cases, \mathbb{H} fulfills the condition of being bounded from below.

A.2 Banach manifolds basics

We recall definitions and elementary properties of several geometric structures in the setting of manifolds modeled on Banach spaces. Basic definitions of notions such as Banach manifold, smooth or C^∞ maps between manifolds, Banach-Lie group, exponential map, etc. can be found in [28]. The special setting of analytic Banach manifolds is treated in [43]. For several geometric structures in Banach manifolds and their interplay with operator theory we refer to [12]. We write manifold for refer to a real or complex smooth manifold modeled on a Banach space. Also we omit the terminology Banach manifold, Banach-Lie group, etc., which are replaced by manifold, Lie group, etc. The tangent space of a manifold M at a point p (resp. cotangent space) is denoted by $T_p M$ (resp. $T_p^* M$). When $f : M \rightarrow N$ is a smooth map between two manifolds, we write $T_p f : T_p M \rightarrow T_{f(p)} N$ for the tangent of f at p . Given Lie groups G, K, \dots , we use the notation $\mathfrak{g}, \mathfrak{k}, \dots$ for its Lie algebras. The exponential map of Lie group G is denoted by \exp_G .

Homogeneous spaces and submanifolds. Let M be a manifold. A Lie group G acts smoothly on M if there is a smooth map $G \times M \rightarrow M$, $(g, p) \mapsto g \cdot p$, such that $(gh) \cdot p = g \cdot (h \cdot p)$, and $1 \cdot p = p$ for every $g, h \in G$ and $p \in M$. Notice that in this case the map $\pi_p : G \rightarrow M$, $\pi_p(g) = g \cdot p$ is also smooth. The orbit of a point $p \in M$ is defined by $\mathcal{O}(p) := \{g \cdot p : g \in G\}$. The action is said to be *transitive* if $\mathcal{O}(p) = M$ for some $p \in M$. The *isotropy group* at $p \in M$ is $G_p := \{g \in G : g \cdot p = p\}$. Clearly, there is a bijection $\mathcal{O}(p) \rightarrow G/G_p$, $g \cdot p \mapsto gG_p$. A smooth map $f : M \rightarrow N$ between two manifolds is called a *submersion* at $p \in M$ if $T_p f : T_p M \rightarrow T_{f(p)} N$ is surjective and $\ker(T_p f)$ is a complemented subspace in $T_p M$. In the case that f is a submersion at every point of M , it is said that f is a *submersion*.

Let G be a Lie group acting smoothly and transitively on a manifold M . If there is some point $p \in M$ such that the map $\pi_p : G \rightarrow M$, $\pi_p(g) = g \cdot p$ is a submersion at $1 \in G$, then M is called a *smooth homogeneous space* of G . We observe that the existence of such a point $p \in M$ in the above definition is actually equivalent to the fact that every point in M have that property.

Let M be manifold and N a subset of M . N is an *embedded submanifold* of M if for each point $p \in N$ there exists a Banach space E and a chart (\mathcal{W}, ϕ) at p , $\phi : \mathcal{W} \subseteq E \rightarrow M$, $\phi(0) = p$, such that $\phi(\mathcal{W} \cap (F \oplus \{0\})) = \phi(\mathcal{W}) \cap N$, where F is a closed and complemented subspace in E . A smooth map $f : M \rightarrow N$ between two manifolds is called an *immersion* at $p \in M$ if $T_p f : T_p M \rightarrow T_{f(p)} N$ is injective and $\text{ran}(T_p f)$ is a complemented closed subspace in $T_p M$. In the case that f is an immersion at every point of M , it is said that f is an *immersion*. The following criterion is useful.

Proposition A.9. *Let M, N be two manifolds, $N \subseteq M$. Then N is an embedded submanifold of M if and only if the topology of N coincides with the topology inherited from M and the inclusion map $N \hookrightarrow M$ is an immersion.*

The notion of Lie subgroup is related to the construction of homogeneous spaces. This notion is not uniform in the literature, see for instance [34] where various notions are recalled; and also some pathologies are pointed out, which only show up in this context of infinite dimensional Lie groups modeled on Banach spaces. In this work we use the most restrictive notion available following [18]: a subgroup K of a Lie group G is a *Lie subgroup* if K is an embedded submanifold of G . In particular, this allows us to state the next result as follows.

Theorem A.10. *Let G be a Lie group with Lie algebra \mathfrak{g} , K a closed subgroup of G , and set*

$$\mathbf{L}(K) := \{X \in \mathfrak{g} : \exp_G(tX) \in K, \forall t \in \mathbb{R}\}.$$

The following assertions hold:

- i) $\mathbf{L}(K)$ is a closed Lie subalgebra of \mathfrak{g} .
- ii) If there are open neighborhoods $0 \in \mathcal{V} \subseteq \mathfrak{g}$ and $1 \in \mathcal{W} \subseteq G$ such that $\exp_G : \mathcal{V} \rightarrow \mathcal{W}$ is a diffeomorphism and $\exp_G(\mathcal{V} \cap \mathbf{L}(K)) = \mathcal{W} \cap K$, then K carries a Lie group structure endowed with the relative topology inherited from G such that $\mathbf{L}(K)$ is the Lie algebra of K and $\exp_K = \exp_G|_{\mathbf{L}(K)} : \mathbf{L}(K) \rightarrow K$.
- iii) If, in addition, $\mathbf{L}(K)$ is a complemented subspace in \mathfrak{g} , then K is a Lie subgroup of G and the quotient space $M := G/K$ carries the structure of a manifold equipped with the quotient topology such that the natural projection $\pi : G \rightarrow M$, $\pi(g) = gK$, is a submersion.

Furthermore, G acts smoothly on M by the left translation action defined by $G \times M \rightarrow M$, $g \cdot hK = ghK$.

We observe that the manifold structure on G/K is uniquely determined by the condition that the projection π is a submersion. Also the previous result has the following converse. A smooth homogeneous space M of a Lie group G must be diffeomorphic to G/G_p , where G_p is the isotropy group of G at any point $p \in M$, and G_p turns out to be a Lie subgroup of G . Let \mathfrak{g}_p be the Lie algebra of G_p . A *reductive structure* for a smooth homogeneous space M of a Lie group G is a smooth distribution of closed subspaces $\{\mathfrak{m}_p\}_{p \in M}$ of \mathfrak{g} such that $\mathfrak{g}_p \oplus \mathfrak{m}_p = \mathfrak{g}$ and $\text{Ad}_g(\mathfrak{m}_p) = \mathfrak{m}_p$, for every $g \in G_p$ and $p \in M$. A smooth homogeneous space equipped with a reductive structure is called a *reductive homogeneous space*.

Symplectic homogeneous spaces. A *weakly symplectic manifold* is a pair (M, ω) of a manifold M and a non-degenerate closed differential 2-form ω taking values on \mathbb{R} . It is said to be a *strongly symplectic manifold* if for every $p \in M$ the injective map $T_p M \rightarrow T_p^* M$, $v \rightarrow \omega_p(v, \cdot)$, is also surjective. Notice that the notions of weakly and strongly symplectic manifolds clearly coincide for finite dimensional manifolds. In the case in which $M = G/K$ is a smooth homogeneous space of G , consider the smooth map $\alpha_g : M \rightarrow M$ defined by $\alpha_g(hK) = ghK$. A 2-form ω is called *invariant* if $(\alpha_g)^*\omega = \omega$ for all $g \in G$. Here $(\alpha_g)^*\omega$ indicates the pull-back of ω by α_g . A *weakly (resp. strongly) homogeneous space* is a weakly (resp. strongly) symplectic manifold (M, ω) , where $M = G/K$ is a homogeneous space of G and the form ω is invariant.

Although the forthcoming definitions can be given for a Banach space \mathfrak{z} , we only assume that $\mathfrak{z} = \mathbb{R}$ or $\mathfrak{z} = \mathbb{C}$, which are the only cases needed in this work. A *continuous \mathfrak{z} -valued 2-cocycle* of a Lie algebra \mathfrak{g} is a continuous skew-symmetric function $s : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{z}$ such that

$$s(X, [Y, Z]) + s(Z, [X, Y]) + s(Y, [Z, X]) = 0,$$

for all $X, Y, Z \in \mathfrak{g}$. It is called a *coboundary* if there is a continuous linear map $f : \mathfrak{g} \rightarrow \mathfrak{z}$ such that $s(X, Y) = f([X, Y])$. Let $Z_c^2(\mathfrak{g}, \mathfrak{z})$ and $B_c^2(\mathfrak{g}, \mathfrak{z})$ be the space of continuous \mathfrak{z} -valued 2-cocycles and the subspace of coboundaries, respectively. The *second continuous cohomology Lie algebra space* is defined as the following quotient of vector spaces $H_c^2(\mathfrak{g}, \mathfrak{z}) := Z_c^2(\mathfrak{g}, \mathfrak{z})/B_c^2(\mathfrak{g}, \mathfrak{z})$. Two continuous \mathfrak{z} -valued 2-cocycles are said to be *cohomologous* if their classes coincide in $H_c^2(\mathfrak{g}, \mathfrak{z})$.

Let G be a Lie group, K a Lie subgroup of G and $M = G/K$. Let \mathfrak{g} and \mathfrak{k} be the Lie algebras of G and K , and $\pi : G \rightarrow G/K$ the natural projection, $p = \pi(1)$. Suppose that $s \in Z_c^2(\mathfrak{g}, \mathbb{R})$ is such that $\mathfrak{k} = \{X \in \mathfrak{g} : s(X, Y) = 0, \forall Y \in \mathfrak{g}\}$. Since $T_p M \cong \mathfrak{g}/\mathfrak{k}$, and using the relation between \mathfrak{k} and s , the following non-degenerate skew-symmetric bilinear map is well defined:

$$\omega_p(X + \mathfrak{k}, Y + \mathfrak{k}) := s(X, Y), \quad X, Y \in \mathfrak{g}.$$

Then it can be smoothly translated to other points of M by using the above defined maps α_g , namely

$$\omega_{g \cdot p}(v, w) := \omega_p(T_{g \cdot p}(\alpha_{g^{-1}})(v), T_{g \cdot p}(\alpha_{g^{-1}})(w)), \quad v, w \in T_{g \cdot p} M.$$

We write $\omega = \Sigma(s)$ for the differential 2-form defined by $\omega := \{\omega_{g \cdot p}\}_{g \in G}$. For finite-dimensional Lie groups this construction provides an invariant symplectic form on their coadjoint orbits. We now state its generalization to the setting of Banach manifolds.

Theorem A.11. *Let G be a Lie group, K a Lie subgroup of G and $M = G/K$. Assume that $s \in Z_c^2(\mathfrak{g}, \mathbb{R})$ satisfies $\mathfrak{k} = \{X \in \mathfrak{g} : s(X, Y) = 0, \forall Y \in \mathfrak{g}\}$. Then $(M, \Sigma(s))$ is a weakly symplectic homogeneous space.*

Kähler homogenous spaces. Let G be a real Lie group and K a Lie subgroup of G . Then, $M = G/K$ has the structure of real smooth homogenous space. Assume that (M, ω) is a weakly (resp. strongly) symplectic homogeneous space of G satisfying the following conditions: M is actually a complex manifold and the maps $\alpha_g : M \rightarrow M$, $\alpha_g(hK) = ghK$ are holomorphic for all $g \in G$; $\omega_p(v, w) = \omega_p(J_p v, J_p w)$ for all $v, w \in T_p M$ and $p \in M$, where J_p is the complex structure on $T_p M$ giving the multiplication by i ; and $\omega_p(v, J_p v) > 0$ for all $v \in T_p M$, $v \neq 0$, and $p \in M$. In this case, (M, ω) is said to be *weakly (resp. strongly) Kähler homogenous space of G* .

Next we describe how Kähler homogenous space can be constructed in Lie algebraic terms. Given a real Lie group G with Lie algebra \mathfrak{g} we write $\mathfrak{g}_{\mathbb{C}}$ for the complexification of \mathfrak{g} , and we denote by $a \mapsto \bar{a}$ the involution of $\mathfrak{g}_{\mathbb{C}}$ whose set of fixed points is given by \mathfrak{g} .

Remark A.12. The following criterion was proved in [11]. For real smooth homogeneous space $M = G/K$ there is a bijective correspondence between, on one hand, complex manifold structures on M such that the smooth structure underlying the complex structure is just the smooth manifold structure of M and the maps α_g , $g \in G$, are holomorphic, and on the other hand, closed complex subalgebras \mathfrak{p} of $\mathfrak{g}_{\mathbb{C}}$ such that

$$\mathfrak{p} + \bar{\mathfrak{p}} = \mathfrak{g}_{\mathbb{C}}, \quad \mathfrak{p} \cap \bar{\mathfrak{p}} = \mathfrak{k}_{\mathbb{C}}, \quad \text{and} \quad \text{Ad}_h \mathfrak{p} \subseteq \mathfrak{p}, \quad h \in K. \quad (37)$$

Theorem A.13. *Let G be a real Lie group, K a Lie subgroup of G and $M = G/K$ a smooth homogeneous space. Let $s \in Z_c^2(\mathfrak{g}, \mathbb{R})$ satisfying $\mathfrak{k} = \{X \in \mathfrak{g} : s(X, Y) = 0, \forall Y \in \mathfrak{g}\}$. Denote by $\omega = \Sigma(s)$, and consider the \mathbb{C} -linear extension $s : \mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}} \rightarrow \mathbb{C}$. Moreover assume that \mathfrak{p} is a complemented subalgebra of $\mathfrak{g}_{\mathbb{C}}$ with the properties stated in (37), and assume that M has the structure of a complex manifold given by \mathfrak{p} . Then (M, ω) is a weakly Kähler homogenous space if and only if the following hold:*

$$s(\mathfrak{p} \times \mathfrak{p}) = 0, \quad \text{and} \quad -is(a, \bar{a}) > 0, \quad a \in \mathfrak{p} \setminus \mathfrak{k}_{\mathbb{C}}. \quad (38)$$

Furthermore, if the injective map $\mathfrak{g}/\mathfrak{k} \rightarrow (\mathfrak{g}/\mathfrak{k})^* \simeq T_p^* M$, $X + \mathfrak{k} \mapsto \omega_p(X + \mathfrak{k}, \cdot)$, is surjective, then (M, ω) is a strongly Kähler homogenous space.

A complemented subalgebra \mathfrak{p} of $\mathfrak{g}_{\mathbb{C}}$ satisfying the conditions in (37) and (38) is called a *weakly Kähler polarization* of \mathfrak{g} in s . If the last condition in the previous theorem is also satisfied, then it is called a *strongly Kähler polarization* of \mathfrak{g} in s . For a proof of Theorem A.13 we refer to [12].

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