Local minimizers of the distances to the majorization flows

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Abstract

Let $\mathcal{D}(d)$ denote the convex set of density matrices of size d and let $\rho,\,\sigma\in\mathcal{D}(d)$ be such that $\rho\not\prec\sigma$. Consider the majorization flows $\mathcal{L}(\sigma)=\{\mu\in\mathcal{D}(d)\ :\ \mu\prec\sigma\}$ and $\mathcal{U}(\rho)=\{\nu\in\mathcal{D}(d)\ :\ \rho\prec\nu\}$, where $\prec$ stands for the majorization pre-order relation. We endow $\mathcal{L}(\sigma)$ and $\mathcal{U}(\rho)$ with the metric induced by the spectral norm. Let $N(\cdot)$ be a strictly convex unitarily invariant norm and let $\mu_0\in\ \mathcal{L}(\sigma)$ and $\nu_0\in\mathcal{U}(\rho)$ be local minimizers of the distance functions $\Phi_N(\mu)=N(\rho-\mu)$, for $\mu\in\mathcal{L}(\sigma)$ and $\Psi_N(\nu)=N(\sigma-\nu)$, for $\nu\in\mathcal{U}(\rho)$. In this work we show that, for every unitarily invariant norm $\tilde{N}(\cdot)$ we have that

$$\tilde{N}(\rho-\mu_0)\leq\tilde{N}(\rho-\mu)\,,\ \mu\in\mathcal{L}(\sigma)\quad\text{ and }\quad\tilde{N}(\sigma-\nu_0)\leq\tilde{N}(\sigma-\nu)\,,\ \nu\in\mathcal{U}(\rho)\,.$$

That is, μ_0 and ν_0 are global minimizers of the distances to the corresponding majorization flows, with respect to every unitarily invariant norm. We describe the (unique) spectral structure (eigenvalues) of μ_0 and ν_0 in terms of a simple finite step algorithm; we also describe the geometrical structure (eigenvectors) of μ_0 and ν_0 in terms of the geometrical structure of σ and ρ, respectively. We include a discussion of the physical and computational implications of our results. We also compare our results to some recent related results in the context of quantum information theory.

Keywords: majorization flows, state optimization, local minimizer, approximate majorization.

1 Introduction

Majorization has become a fundamental mathematical tool in the quantum realm (see [1, 2, 3] for a detailed account of majorization theory and [4, 5], and the references therein, for applications of majorization in quantum information theory). For instance, Nielsen's theorem [6] showed the role of majorization in the context of transformations of bipartite entangled pure states using local operations and classical communication. It turns out that this problem can be cast in the more general setting of resource theories (see [7]), where the deterministic and exact transformations between resources using free operations are governed by a majorization arrow between probability vectors associated with the resources.

We can model the previous settings as follows: let $\mathcal{D}(d)$ denote the convex set of density matrices of size d and let $\rho,\,\sigma\in\mathcal{D}(d)$ describe the target and initial states of a physical system, respectively. Then, we can transform the system from the state described by σ into the state described by ρ using free operations if the majorization relation $\rho\prec\sigma$ holds. This majorization relation can be

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described using elementary inequalities based on the eigenvalue lists of ρ and σ (see Section 2 for details).

Notice that in the previous model we have considered free operations transforming σ exactly into ρ. However, we are very often more interested in the question of whether a process approximately transforms the physical system from the state σ into the state ρ. For example, if one allows some degree of error, transformations that approximate the target state or that are applied to approximations of the initial state arise. For instance, approximate transformations within entanglement [8, 9], coherence [10, 11, 12] and thermodynamics [13, 14, 15] quantum resource theories have already been studied.

These types of ideas have led to the notion of approximate majorizations with respect to some physical distances, in the sense considered in [13, 16, 17, 18, 19, 20]. Briefly, given a norm $N(\cdot)$ in the algebra of $d \times d$ complex matrices, two density matrices $\rho, \sigma \in \mathcal{D}(d)$ such that $\rho \not\prec \sigma$ and $\varepsilon > 0$ we can ask whether there exists $\tilde{\mu} \in \mathcal{D}(d)$ such that $\tilde{\mu} \prec \sigma$ and $N(\rho - \tilde{\mu}) \leq \varepsilon$; in this case, we can transform the state described by σ into the state described by $\tilde{\mu}$ (which is close to ρ). On the other hand, we can also ask whether there exists $\tilde{\nu} \in \mathcal{D}(d)$ such that $\rho \prec \tilde{\nu}$ and $N(\sigma - \tilde{\nu}) \leq \varepsilon$; in this case, we can transform the state described by $\tilde{\nu}$ (which is close to σ) into the state described by ρ. To determine whether any of these approximate majorizations hold we introduce the sets

$$\mathcal{U}(\rho) = \{\nu \in \mathcal{D}(d) : \rho \prec \nu\} \quad \text{and} \quad \mathcal{L}(\sigma) = \{\mu \in \mathcal{D}(d) : \mu \prec \sigma\}.$$

We call $\mathcal{U}(\rho)$ the *upward majorization flow* at ρ and $\mathcal{L}(\sigma)$ the *downward majorization flow* at σ. Assume that we have computed

$$d_1 = \min\{N(\rho - \mu) \ : \ \mu \in \mathcal{L}(\sigma)\} \quad \text{and} \quad d_2 = \min\{N(\sigma - \nu) \ : \ \nu \in \mathcal{U}(\rho)\},$$

where $d_1 = d_1(N, \rho, \sigma)$ and similarly $d_2 = d_2(N, \rho, \sigma)$. Then, the approximate majorizations above hold if $\varepsilon \geq d_1$ (for the existence of $\tilde{\mu}$) or $\varepsilon \geq d_2$ (for the existence of $\tilde{\nu}$), respectively.

The computation of the distances d_1 and d_2 for the $\|\cdot\|_1$ (trace or nuclear norm) were obtained in [18, 19] and for the $\|\cdot\|_\infty$ (operator norm) in [20]. Indeed, in [18, 19] the authors base their approach on the existence of extremal elements with respect to majorization, in the set $\{\xi \in \mathcal{D}(d) : \|\eta - \xi\|_1 \leq \varepsilon\}$, for $\eta \in \mathcal{D}(d)$ and $\varepsilon > 0$. The approach considered in [20] is similar (see Section 3.2 for details). But, as shown in [20], this approach can not be extended to compute $d_1(\|\cdot\|_p, \rho, \sigma)$ and $d_2(\|\cdot\|_p, \rho, \sigma)$ with respect to the p-norms, for $1 < p < \infty$. On the other hand, the optimal approximations of ρ from $\mu \in \mathcal{L}(\sigma)$ found in [18, 19] and [20] do not coincide; that is, optimal approximations obtained using the techniques in these works depend on the particular choice of the norm being considered. This last fact poses the problem of choosing one of the previous norms, to consider optimal approximations (see the examples and discussion in [20]).

In [21] (see [22] for a modern perspective on this and related results) Li and Tsing obtained a fundamental result related to one of the approximation problems above. Indeed, motivated by a question of Y. Nakamura, it is shown that there exist density matrices $\tilde{\mu} \in \mathcal{L}(\sigma)$ such that *for every unitarily invariant norm* $N(\cdot)$ on $\mathcal{M}_d(\mathbb{C})$,

$$N(\rho - \tilde{\mu}) = \min\{N(\rho - \mu) \ : \ \mu \in \mathcal{L}(\sigma)\}.$$

In particular, $d_1(N, \rho, \sigma) = N(\rho - \tilde{\mu})$, for every unitarily invariant norm $N(\cdot)$. In this case, such a $\tilde{\mu}$ is a universal minimizer of the distance of ρ to $\mathcal{L}(\sigma)$. Moreover, it is also shown that the spectral structure of $\tilde{\mu}$ is unique and can be described in terms of a simple (finite step) algorithm. In particular, $\tilde{\mu}$ is an optimal approximation of ρ from $\mathcal{L}(\sigma)$ with respect to the Schatten p-norms, for $1 < p < \infty$.

The approach developed in [21] raises several questions. On the one hand, it would be interesting to show the existence of universal minimizers $\tilde{\nu}$ of the distance $N(\sigma - \nu)$, for $\nu \in \mathcal{U}(\rho)$, with respect to arbitrary unitarily invariant norms $N(\cdot)$. This can be considered as a dual problem to that considered in [21]. On other hand, motivated by some computational aspects of these problems, it would be interesting to obtain the structure of the *local* minimizers of the functions $N(\rho-\mu)$ for $\mu\in\mathcal{L}(\sigma)$ and $N(\sigma-\nu)$ for $\nu\in\mathcal{U}(\rho)$, where $N(\cdot)$ is an arbitrary *strictly convex* unitarily invariant norm; here we endow $\mathcal{L}(\sigma)$ and $\mathcal{U}(\rho)$ with the metric induced by the spectral norm. Notice that the problem of determining the structure of local minimizers of the functions above plays a key role in the analysis of convergence of iterative methods (such as steepest gradient descent) applied to the objective functions given by the distances to $\mathcal{L}(\sigma)$ and $\mathcal{U}(\rho)$ above.

In this work, we show the existence of universal minimizers $\tilde{\nu}$ of the distance $N(\sigma-\nu)$, for $\nu\in\mathcal{U}(\rho)$. We show that these universal minimizers have unique spectral structure (eigenvalues) that can be computed in terms of a simple algorithm; moreover, we parametrize the set of such universal solutions in terms of the geometric structure (eigenvectors) of ρ. Furthermore, we show that local minimizers of the distance functions to ρ and σ, from $\mathcal{L}(\sigma)$ and $\mathcal{U}(\rho)$ respectively, are also global universal minimizers when $N(\cdot)$ is a strictly convex unitarily invariant norm. Hence, local minimizers of the Schatten p-norms for $1<p<\infty$ are also global universal minimizers of the distance functions above; in particular, our results completely describe these local minimizers. We discuss both physical implications (in the context of quantum information theory) and computational foundations provided by our results. We also apply these results in the context of approximate majorization. Our approach is based on an adaption of the techniques used to tackle some seemingly unrelated problem in the context of finite frame theory and matrix Procrustes problems [23, 24].

The paper is organized as follows. In Section 2 we include the preliminary material on majorization theory and Schur-convex functions, and the main result from [21]. We also include a result about the local minimizers of Schur-convex functions defined on unitary orbits. In Section 3.1 we state our main results; on the one hand, we complement the main result from [21] by obtaining the structure of local minimizers associated with the distance function to ρ from $\mathcal{L}(\sigma)$ induced by a strictly convex unitarily invariant norm. Then we state our main result about the (local) universal minimizers of the distance function to σ from $\mathcal{U}(\rho)$. We point out that the structures of the sets $\mathcal{L}(\sigma)$ and $\mathcal{U}(\rho)$ are quite different (for example, $\mathcal{L}(\sigma)$ is a convex set while $\mathcal{U}(\rho)$ is not); thus, although we follow similar paths in the analysis of both cases, the detailed analysis is different. We also discuss the physical and computational implications of our results. In Section 3.2 we discuss some related results in [18, 19, 20, 21, 22]. In Section 3.3 we apply our results in the context of approximate majorization. In Section 4 we develop detailed proofs of our main results. Finally, in Section 5 we include the proof of some results stated in Section 2.

2 Preliminaries

In this section, we introduce the notation, terminology and different notions that we need in the rest of the paper. In particular, we consider majorization (in $\mathbb{R}^d$ and $\mathcal{H}(d)$), Schur-convex functions and the relations between these notions. We further include a related result from [21] and a technical result (that we prove in Section 5).

Notation and terminology. We let $\mathcal{M}_d(\mathbb{C})$ denote the $*$-algebra of $d\times d$ complex matrices; if $\omega\in\mathcal{M}_d(\mathbb{C})$ then we let ω^* denote the adjoint of ω. We denote by $\mathcal{H}(d)\subset\mathcal{M}_d(\mathbb{C})$ the real vector space of self-adjoint matrices. We let $\mathcal{M}_d(\mathbb{C})^+\subset\mathcal{M}_d(\mathbb{C})$ denote the cone of positive semi-definite matrices and $\mathcal{U}_d\subset\mathcal{M}_d(\mathbb{C})$ the group of unitary matrices. We also consider

$$\mathcal{D}(d)=\{\sigma\in\mathcal{M}_d(\mathbb{C})^+\ :\ \mathrm{tr}(\sigma)=1\}$$

the convex set of density matrices, where $\mathrm{tr}(\eta)\in\mathbb{C}$ denotes the (un-normalized) trace of $\eta\in\mathcal{M}_d(\mathbb{C})$. We let $\mathbb{I}_k=\{1,\ldots,k\}$ for $k\in\mathbb{N}$. Given $\gamma\in\mathcal{H}(d)$ then $\lambda(\gamma)=(\lambda_i(\gamma))_{i\in\mathbb{I}_d}\in\mathbb{R}^d$ denotes the vector of eigenvalues of γ, counting multiplicities and arranged in non-increasing order i.e., $\lambda_1(\gamma)\geq\ldots\geq$

$\lambda_d(\gamma)$. Given $\eta \in \mathcal{M}_d(\mathbb{C})$ we let $s(\eta)$ denote the singular values of η counting multiplicities and arranged in non-increasing order i.e., $s(\eta) = \lambda(|\eta|)$, where $|\eta| = (\eta^*\eta)^{1/2} \in \mathcal{M}_d(\mathbb{C})^+$.

On the other hand, given a real vector $x = (x_i)_{i\in\mathbb{I}_d} \in \mathbb{R}^d$ we let $x^\downarrow = (x_i^\downarrow)_{i\in\mathbb{I}_d}$ denote the vector obtained from x by re-arranging its entries in non-increasing order i.e. $x_1^\downarrow \geq \ldots \geq x_d^\downarrow$. Given a subset $\mathcal{S} \subset \mathbb{R}$ we let $\mathcal{S}^\downarrow = \{x^\downarrow \ : \ x \in \mathcal{S}\}$. Given $k \in \mathbb{N}$ we let $\mathbb{1}_k = (1, \ldots, 1) \in \mathbb{R}^k$. △

Definition 2.1. *Given $x,\, y \in \mathbb{R}^d$ we say that x is majorized by y (or that y majorizes x), denoted $x \prec y$, if*

$$\sum_{i\in\mathbb{I}_k} x_i^\downarrow \leq \sum_{i\in\mathbb{I}_k} y_i^\downarrow \quad \textit{for} \quad k \in \mathbb{I}_{d-1} \quad \textit{and} \quad \operatorname{tr}(x) := \sum_{i\in\mathbb{I}_d} x_i = \sum_{i\in\mathbb{I}_d} y_i = \operatorname{tr}(y)\,.$$

We say that x is strictly majorized by y if $x \prec y$ and $x^\downarrow \neq y^\downarrow$. △

As usual, we extend the notion of majorization to the context of self-adjoint matrices. Indeed, given $\mu, \nu \in \mathcal{H}(d)$ we say that μ is majorized by ν, denoted $\mu \prec \nu$, if $\lambda(\mu) \prec \lambda(\nu) \in \mathbb{R}^d$. Given two density matrices $\rho,\, \sigma \in \mathcal{D}(d)$ we will consider

$$\mathcal{U}(\rho) = \{\nu \in \mathcal{D}(d) : \rho \prec \nu\} \quad \text{and} \quad \mathcal{L}(\sigma) = \{\mu \in \mathcal{D}(d) : \mu \prec \sigma\}\,.$$

We call $\mathcal{U}(\rho)$ the upward majorization flow at ρ and $\mathcal{L}(\sigma)$ the downward majorization flow at σ.

In what follows we consider Schur-convex functions i.e., functions $f : \mathbb{R}^d \to \mathbb{R}$ such that $f(x) \leq f(y)$ whenever $x \prec y$. We will further consider *strictly* Schur-convex functions f i.e., such that $f(x) < f(y)$ whenever x is strictly majorized by y. For example, if $\phi : \mathbb{R} \to \mathbb{R}$ is a (strictly) convex function then $f(x) = \sum_{i\in\mathbb{I}_d} \phi(x_i)$ is a (strictly) Schur-convex function on $\mathbb{R}^d$; on the other hand, if $N(\cdot)$ is a (strictly convex) unitarily invariant norm in $\mathcal{M}_d(\mathbb{C})$ and $g_N(\cdot)$ is its associated gauge symmetric function (so that $N(\eta) = g_N(s(\eta))$ for $\eta \in \mathcal{M}_d(\mathbb{C})$, see [1, 3]) then g_N is a (strictly) Schur-convex function.

Remark 2.2. In what follows we consider the following elementary (but useful) properties of majorization (see [1, 3] for detailed proofs of these facts).

1. If $x(j) \prec y(j) \in \mathbb{R}^{d_j}$ for $j \in \mathbb{I}_k$ then the vectors obtained by juxtaposition satisfy
$$x := (x(1), \ldots, x(k)) \prec y := (y(1), \ldots, y(k)) \in \mathbb{R}^d \quad \text{where} \quad d = \sum_{j\in\mathbb{I}_k} d_j\,.$$
If $x(i)$ is strictly majorized by $y(i)$ for some $i \in \mathbb{I}_k$, then x is strictly majorized by y.

2. If $x \prec y \in \mathbb{R}^d$ and f is a strictly Schur-convex function such that $f(x) = f(y)$ then $x^\downarrow = y^\downarrow$.

3. If $x,\, y \in \mathbb{R}^d$ are such that for every strictly Schur-convex function $f : \mathbb{R}^d \to \mathbb{R}$ we have that $f(x) \leq f(y)$ then $x \prec y$.

4. If $\mu,\, \nu \in \mathcal{H}(d)$ then: $\lambda(\mu) - \lambda(\nu) \prec \lambda(\mu - \nu)$ (Lidskii's inequality) and $\lambda(\mu + \nu) \prec \lambda(\mu) + \lambda(\nu)$ (Weyl's inequality).

Let $\rho, \sigma \in \mathcal{D}(d)$ be such that $\rho \not\prec \sigma$. Given a unitarily invariant norm $N(\cdot)$ in $\mathcal{M}_d(\mathbb{C})$, in what follows we consider the function

$$\Phi_N : \mathcal{L}(\sigma) \to \mathbb{R}_{\geq 0} \quad \text{given by} \quad \Phi_N(\mu) = N(\rho - \mu)\,.$$

We further endow the domain $\mathcal{L}(\sigma)$ of Φ_N with the metric induced by the spectral norm.

The following result was originally obtained in [21] and also included in the more recent work [22]. Notice that it completely solves the problem of the spectral and geometrical structure of *global* minimizers of the distance to the downward majorization flow of a density matrix, with respect to any unitarily invariant norm. In Theorem 3.1 below we further obtain that local minimizers of the distance to the downward majorization flow also share this structure.

Theorem 2.3 ([21, 22])**.** *Consider $\rho, \sigma \in \mathcal{D}(d)$ such that $\rho \not\prec \sigma$. Then, there exists $b^{\rm op} = (b_i^{\rm op})_{i\in\mathbb{I}_d} \in (\mathbb{R}^d)^\downarrow$ such that given $\mu_0 \in \mathcal{L}(\sigma)$, the following statements are equivalent:*

1. *There exists an strictly convex unitarily invariant norm $N(\cdot)$ in $\mathcal{M}_d(\mathbb{C})$ such that μ_0 is a minimizer of Φ_N in $\mathcal{L}(\sigma)$.*
2. *For every unitarily invariant norm $N(\cdot)$ in $\mathcal{M}_d(\mathbb{C})$ μ_0 is a minimizer of Φ_N in $\mathcal{L}(\sigma)$.*
3. *$\lambda(\rho-\mu_0) = \lambda(\rho) - \lambda(\mu_0) = b^{\rm op}$; in this case $\lambda(\rho-\mu_0) \prec \lambda(\rho-\mu)$ for every $\mu \in \mathcal{L}(\sigma)$.*
4. *There exists an onb $\{u_i\}_{i\in\mathbb{I}_d}$ of $\mathbb{C}^d$ such that*
$$\rho = \sum_{i\in\mathbb{I}_d} \lambda_i(\rho)\ u_i\otimes u_i \quad \text{and} \quad \mu_0 = \sum_{i\in\mathbb{I}_d} (\lambda_i(\rho) - b_i^{\rm op})\ u_i \otimes u_i\,.$$

Moreover, $b^{\rm op} = (b_i^{\rm op})_{i\in\mathbb{I}_d} \in (\mathbb{R}^d)^\downarrow$ can be computed in terms of simple algorithm. □

Remark 2.4 (On universal optimal approximations from $\mathcal{L}(\sigma)$)**.** Consider the notation of Theorem 2.3. As a consequence of this result we conclude that for every onb $\{u_i\}_{i\in\mathbb{I}_d}$ of $\mathbb{C}^d$ such that $\rho u_i = \lambda_i(\rho)\,u_i$, for $i \in \mathbb{I}_d$, then

$$\mu_0 = \sum_{i\in\mathbb{I}_d} (\lambda_i(\rho) - b_i^{\rm op})\ u_i\otimes u_i \in \mathcal{L}(\sigma)$$

is such that for *every* unitarily invariant norm $N(\cdot)$ in $\mathcal{M}_d(\mathbb{C})$ we have that $N(\rho-\mu_0) \leq N(\rho-\mu)$, for $\mu \in \mathcal{L}(\sigma)$. In this sense, μ_0 is a universal optimal approximation of $\rho \in \mathcal{D}(d)$ from the downward majorization flow $\mathcal{L}(\sigma)$, for $\sigma \in \mathcal{D}(d)$. The spectral structure of all such μ_0 is uniquely determined and can be computed in terms of the spectral information $\lambda(\rho)$ and $\lambda(\sigma)$ using a simple algorithm (the one that computes $b^{\rm op}$). On the other hand, notice that the geometrical structures of all such μ_0 are determined by the geometrical structure of ρ, i.e. by all possible onb $\{u_i\}_{i\in\mathbb{I}_d}$ as above. In (the generic) case that ρ is multiplicity free (that is, $\lambda_i > \lambda_{i+1}$ for $i \in \mathbb{I}_{d-1}$) then the geometric structure of μ_0 is also uniquely determined. △

In what follows we will need the next result, which can be proved by adapting the techniques of [28]. Given $\sigma \in \mathcal{D}(d)$ we consider its unitary orbit

$$\mathcal{O}(\sigma) = \{\omega^*\sigma\omega\ :\ \omega \in \mathcal{U}_d\}$$

endowed with the metric induced by the spectral norm.

Theorem 2.5. *Let $\rho, \sigma \in \mathcal{D}(d)$ and let $f : \mathbb{R}^d \to \mathbb{R}$ be a strictly Schur-convex function. Let $\Theta_f : \mathcal{O}(\sigma) \to \mathbb{R}$ be given by $\Theta_f(\tilde\sigma) = f(\lambda(\tilde\sigma - \rho))$. If $\sigma_0 \in \mathcal{O}(\sigma)$ is a local minimizer of Θ_f then there exists an onb $\{u_i\}_{i\in\mathbb{I}_d}$ of $\mathbb{C}^d$ such that*

$$\rho = \sum_{i\in\mathbb{I}_d} \lambda_i(\rho)\ u_i\otimes u_i \quad \text{and} \quad \sigma_0 = \sum_{i\in\mathbb{I}_d} \lambda_i(\sigma_0)\ u_i\otimes u_i\,.$$

In particular, $\lambda(\sigma_0 - \rho) = (\lambda(\sigma_0) - \lambda(\rho))^\downarrow$ so σ_0 is a global minimizer of Θ_f on $\mathcal{O}(\sigma)$.

Proof. See Section 5. □

3 Local minimizers of the distance to the majorization flows

In this section we state our main results related to the structure of local minimizers of the distance to the majorization flows, where the distance is induced by an arbitrary strictly convex unitarily invariant norm. We also elaborate on some consequences of our results, involving physical interpretations (in the context of quantum information theory) as well as computational aspects of these problems. We include a comparison of our results and related results from [19, 18, 21, 22, 20]. We then apply our results to obtain the complete characterization of approximate majorization (see [17, 20]) with respect to arbitrary unitarily invariant norms in $\mathcal{M}_d(\mathbb{C})$.

3.1 Main results

Our first main result complements Theorem 2.3. As we will see, it has both computational and physical implications that we consider in Remarks 3.2 and 3.3 below.

Theorem 3.1. *Consider $\rho, \sigma \in \mathcal{D}(d)$ such that $\rho \not\prec \sigma$. If $N(\cdot)$ is a strictly convex unitarily invariant norm in $\mathcal{M}_d(\mathbb{C})$ and $\mu_0 \in \mathcal{L}(\sigma)$, then the following are equivalent:*

1. *μ_0 is a local minimizer of Φ_N.*
2. *μ_0 is a global minimizer of Φ_N i.e. $N(\rho - \mu_0) = \min\{N(\rho - \mu) \ : \ \mu \in \mathcal{L}(\sigma)\}$.*

In this case, $\mu_0 \prec \rho$. Moreover, $\lambda(\rho - \mu_0) = \lambda(\rho) - \lambda(\mu_0) = b^{\rm op}$, where $b^{\rm op} = (b_i^{\rm op})_{i\in\mathbb{I}_d} \in (\mathbb{R}^d)^\downarrow$ is as in Theorem 2.3 and can be computed in terms of a simple algorithm with input $\lambda(\rho)$, $\lambda(\sigma) \in (\mathbb{R}^d)^\downarrow$.

Proof. See Sections 4.1 and 4.2. □

We point out that the algorithm mentioned in Theorem 3.1 differs from the algorithm obtained in [21, 22] (see Section 4.2). On the other hand, both algorithms compute the same vector and seem to have the same order of complexity.

Remark 3.2 (On some computational implications of Theorem 3.1)**.** In case we consider physical systems with an increasing number of particles, it turns out that the size of the density matrices involved in the quantum description of the systems increases exponentially. Thus, arguments based on the exact computation of eigenvectors and eigenvalues become less applicable in these situations. Take for example $\rho, \sigma \in \mathcal{D}(d)$ such that $\rho \not\prec \sigma$. When d becomes large, the complexity of the computation of an onb $\{u_i\}_{i\in\mathbb{I}_d}$ of $\mathbb{C}^d$ such that

$$\rho = \sum_{i\in\mathbb{I}_d} \lambda_i(\rho) \ u_i \otimes u_i$$

makes this task unfeasible. Notice that the construction of a universal $\mu_0 \in \mathcal{L}(\sigma)$ as in Theorem 2.3 depends on the computation of $\{u_i\}_{i\in\mathbb{I}_d}$ as above.

A possible way out of this situation, also based on Theorem 2.3, is to consider a smooth strictly convex unitarily invariant norm, e.g. $\|\cdot\|_2$, and an iterative algorithm that decreases the value of $\Phi_{\|\cdot\|_2}(\mu) = \|\rho - \mu\|_2$, for $\mu \in \mathcal{L}(\sigma)$ starting from some initial value $\tilde{\mu}$. For example, we can consider the steepest descent in the direction of minus the gradient of $\Phi_{\|\cdot\|_2}$ at $\tilde{\mu}$; continuing this process, we generate a sequence $\{\tilde{\mu}_j\}_{j\geq 1}$, where $\tilde{\mu}_1 = \tilde{\mu}$. Indeed, a similar approach (in a broader context) has been considered in [25] (see also [26]); a suitable translation of the iterative method developed in [27] also applies in this context (see [23] for an explicit translation between distance problems and frame completion problems). Notice that local minimizers of $\Phi_{\|\cdot\|_2}$ are attractors of sequences $\{\tilde{\mu}_j\}_{j\geq 1}$ constructed as before. Thus, in this setting, we are led to consider the problem of whether local minimizers of $\Phi_{\|\cdot\|_2}$ are global minimizers. Theorem 3.1 provides a key theoretical support for considering these alternative approaches. △

Remark 3.3 (On some physical implications of Theorem 3.1)**.** Consider the notation of Theorem 3.1. There is an interesting feature of the universal optimal approximations $\mu_0 \in \mathcal{L}(\sigma)$ of ρ from the downward majorization flow $\mathcal{L}(\sigma)$ namely, that $\mu_0 \prec \rho$. Indeed, the fact that $\mu_0 \prec \rho$ implies that μ_0 can be also obtained by *evolving* from ρ (that is, transforming the system using free operations).

The previous remarks have practical implications in the following experimental setting: in case we consider the problem of transforming σ into ρ in terms of free operations, the fact that $\rho \not\prec \sigma$ makes this task not possible. A natural way out in this practical setting would be, for example, to consider

$$\mathcal{C}(\rho, \sigma) = \{\mu \in \mathcal{D}(d) \ : \ \mu \prec \rho \quad \text{and} \quad \mu \prec \sigma\} = \mathcal{L}(\rho) \cap \mathcal{L}(\sigma)\,.$$

Our strategy in this context could be to choose a convenient $\mu \in \mathcal{C}(\rho,\sigma)$ and obtain μ by evolving from ρ. In this case, we would end up with the pair $\mu \prec \sigma$, where we consider μ as a perturbation of ρ. We further consider $N_0(\rho-\mu)$ (for some unitarily invariant norm $N_0(\cdot)$ in $\mathcal{M}_d(\mathbb{C})$) as a measure of the deformation to obtain $\mu \in \mathcal{C}(\rho,\sigma)$ from ρ using free operations in the corresponding physical setting. Then, the previous remarks show that $\mu_0 \in \mathcal{C}(\rho,\sigma)$ is such that it minimizes $N_0(\rho-\mu_0) \leq N_0(\rho-\mu)$, for $\mu \in \mathcal{C}(\rho,\sigma) \subset \mathcal{L}(\sigma)$. △

Given a unitarily invariant norm $N(\cdot)$ in $\mathcal{M}_d(\mathbb{C})$, in what follows we consider the function

$$\Psi_N : \mathcal{U}(\rho) \to \mathbb{R}_{\geq 0} \quad \text{given by} \quad \Psi_N(\nu) = N(\sigma-\nu)\,.$$

We further endow the domain $\mathcal{U}(\rho)$ of Ψ_N with the metric induced by the spectral norm.

Theorem 3.4. *Consider $\rho,\sigma \in \mathcal{D}(d)$ and such that $\rho \not\prec \sigma$. Then, there exists $c^{\rm op} = (c_i^{\rm op})_{i\in\mathbb{I}_d} \in \mathbb{R}^d$ such that given $\nu_0 \in \mathcal{U}(\rho)$, the following statements are equivalent:*

1. *There exists a strictly convex unitarily invariant norm $N(\cdot)$ in $\mathcal{M}_d(\mathbb{C})$ such that ν_0 is a local minimizer of Ψ_N on $\mathcal{U}(\rho)$.*
2. *There exists a strictly convex unitarily invariant norm $N(\cdot)$ in $\mathcal{M}_d(\mathbb{C})$ such that ν_0 is a minimizer of Ψ_N on $\mathcal{U}(\rho)$.*
3. *For every unitarily invariant norm $N(\cdot)$ in $\mathcal{M}_d(\mathbb{C})$, ν_0 is a minimizer of Ψ_N on $\mathcal{U}(\rho)$.*
4. *$\lambda(\sigma-\nu_0) = (\lambda(\sigma)-\lambda(\nu_0))^{\downarrow}$ and $\lambda(\sigma)-\lambda(\nu_0) = c^{\rm op}$; in this case, $\lambda(\sigma-\nu_0) \prec \lambda(\sigma-\nu)$, for every $\nu \in \mathcal{U}(\rho)$.*
5. *There exists an onb $\{u_i\}_{i\in\mathbb{I}_d}$ of $\mathbb{C}^d$ such that*

$$\sigma = \sum_{i\in\mathbb{I}_d} \lambda_i(\sigma)\ u_i \otimes u_i \quad \text{and} \quad \nu_0 = \sum_{i\in\mathbb{I}_d} (\lambda_i(\sigma) - c_i^{\rm op})\ u_i \otimes u_i\,.$$

   Furthermore $\lambda(\nu_0) = \lambda(\sigma) - c^{\rm op} \in (\mathbb{R}^d)^{\downarrow}$.

In this case, $\sigma \prec \nu_0$. Moreover, $c^{\rm op} = (c_i^{\rm op})_{i\in\mathbb{I}_d} \in \mathbb{R}^d$ can be computed in terms of a simple algorithm with input $\lambda(\rho)$, $\lambda(\sigma) \in \mathbb{R}^d$.

Proof. See Sections 4.3 and 4.4. □

Remarks analogous to Remarks 2.4, 3.2 and 3.3 also apply to Theorem 3.4. We point out that Strawn's work on the optimization over finite frame varieties applies in this context after a convenient interpretation of the results in [25]. On the other hand, we remark that there is a difference in the physical implications of Theorem 3.4. Indeed, consider the same experimental setting as in Remark 3.2. Hence, we consider ρ, $\sigma \in \mathcal{D}(d)$ such that $\rho \not\prec \sigma$. We could now consider the following possible way out: let

$$\mathcal{E}(\rho,\sigma) = \{\nu \in \mathcal{D}(d)\ :\ \rho \prec \nu \quad \text{and} \quad \sigma \prec \nu\} = \mathcal{U}(\rho) \cap \mathcal{U}(\sigma)\,.$$

Our strategy is to apply a procedure to obtain $\nu \in \mathcal{E}(\rho,\sigma)$ from σ, but in this case it is possible that $\nu \not\prec \sigma$ (although $\sigma \prec \nu$); that is, in this case we can not transform the system from the state σ into the state ν in terms of free operations. Nevertheless, in case we could transform the system from the state σ into the state ν we would be interested in measuring the deformation needed for this process in terms of $N_1(\sigma-\nu)$ (where $N_1(\cdot)$ is a unitarily invariant norm in $\mathcal{M}_d(\mathbb{C})$). Then, ν_0 as in Theorem 3.4 is such that $\nu_0 \in \mathcal{E}(\rho,\sigma)$ so that $N_1(\sigma-\nu_0)$ is the minimal deformation needed for such transformation.

Example 3.5. Consider $\rho,\sigma\in\mathcal{D}(d)$ such that

$$\lambda(\rho)=(1/2\,,\,1/3\,,\,1/6\,,\,0)\quad\text{ and }\quad\lambda(\sigma)=(4/11\,,\,7/22\,,\,2/11\,,\,3/22)\,.$$

In this case $\rho\not\prec\sigma$. We apply Algorithm 4.8 and compute $b^{\rm op}=(3/22\,,\,1/66\,,\,-1/66\,,\,-3/22)\in(\mathbb{R}^4)^\downarrow$. Hence, $\lambda(\mu_0)=\lambda(\rho)-b^{\rm op}=\lambda(\sigma)$; indeed, by Theorem 3.1 we get that $\lambda(\sigma)=\lambda(\mu_0)\prec\rho$. On the other hand, by Theorem 2.3 we have that for every $1\le p\le\infty$,

$$\text{dist}_{\|\cdot\|_p}(\rho\,,\,\mathcal{L}(\sigma))=\min\{\|\rho-\mu\|_p\ :\ \mu\prec\sigma\}=\|(3/22\,,\,1/66\,,\,-1/66\,,\,-3/22)\|_p\,.$$

In particular, $\text{dist}_{\|\cdot\|_1}(\rho\,,\,\mathcal{L}(\sigma))=10/33$, $\text{dist}_{\|\cdot\|_2}(\rho\,,\,\mathcal{L}(\sigma))\approx 0.156$ and $\text{dist}_{\|\cdot\|_\infty}(\rho\,,\,\mathcal{L}(\sigma))=3/22$. $\triangle$

Example 3.6. Consider $\rho,\sigma\in\mathcal{D}(d)$ such that

$$\lambda(\rho)=(4/11\,,\,3/11\,,\,3/11\,,\,1/11)\quad\text{ and }\quad\lambda(\sigma)=(1/3\,,\,7/24\,,\,7/24\,,\,1/12)\,.$$

In this case $\rho\not\prec\sigma$. We apply Algorithm 4.19 and compute $c^{\rm op}=(-1/33\,,\,1/99\,,\,1/99\,,\,1/99)\in(\mathbb{R}^4)^\uparrow$. Hence,

$$\lambda(\nu_0)=\lambda(\sigma)-c^{\rm op}=(4/11\,,\,7/24-1/99\,,\,7/24-1/99\,,\,1/12-1/99)\,.$$

By Theorem 3.4 we get that $\sigma\prec\nu_0$. On the other hand, for every $1\le p\le\infty$ we get that

$$\text{dist}_{\|\cdot\|_p}(\sigma\,,\,\mathcal{U}(\rho))=\min\{\|\sigma-\nu\|_p\ :\ \rho\prec\nu\}=\|(-1/33\,,\,1/99\,,\,1/99\,,\,1/99)\|_p\,.$$

In particular, $\text{dist}_{\|\cdot\|_1}(\sigma\,,\,\mathcal{U}(\rho))=2/33$, $\text{dist}_{\|\cdot\|_2}(\sigma\,,\,\mathcal{U}(\rho))\approx 0.035$ and $\text{dist}_{\|\cdot\|_\infty}(\sigma\,,\,\mathcal{U}(\rho))=1/33$. $\triangle$

3.2 Relation to previous work

Relation to [21, 22]. We have already mentioned how is that the results of these works are related to our present work (see Sections 1 and 2 and Theorem 2.3). We remark that Theorem 3.1 complements Theorem 2.3, while Theorem 3.4 can be considered as a dual version of the conjunction of Theorems 2.3 and 3.1. We also recall that Algorithm 4.8 mentioned in Theorem 3.1 (described in Section 4.2) differs from that obtained in [21] (see the discussion in [21] about the nature the algorithm obtained there for computing $b^{\rm op}$).

Relation to [18, 19, 20]. Consider $\mathcal{P}(d)$ the convex set of probability vectors in $\mathbb{R}^d_{\ge 0}$. Let $P,Q\in\mathcal{P}(d)^\downarrow$, with their entries arranged in non-increasing order, be such that $Q\not\prec P$. In [18] the authors consider a two-step strategy that allows computing the distances to the majorization flows

$$\mathcal{U}(Q)=\{R\in\mathcal{P}(d)\ :\ Q\prec R\}\quad\text{ and }\quad\mathcal{L}(P)=\{S\in\mathcal{P}(d)\ :\ S\prec P\}\,.$$

in terms of the 1-norm, denoted $\|\cdot\|_1$. Indeed, as a first step, given P, Q as above and $\delta>0$ the authors in [18] show that there exist $\overline{P}^{(\delta)}=\overline{P}^{(\delta)}_{\|\cdot\|_1}$, $\underline{Q}^{(\delta)}=\underline{Q}^{(\delta)}_{\|\cdot\|_1}\in\mathcal{P}(d)^\downarrow$ such that:

1. $\|P-\overline{P}^{(\delta)}\|_1\le\delta$, $\|Q-\underline{Q}^{(\delta)}\|_1\le\delta$;
2. If $R\in\mathcal{P}(d)$ is such that $\|P-R\|_1\le\delta$ then $R\prec\overline{P}^{(\delta)}$; similarly, if $S\in\mathcal{P}(d)$ is such that $\|Q-S\|_1\le\delta$ then $\underline{Q}^{(\delta)}\prec S$.

The vectors $\overline{P}^{(\delta)}$, $\underline{Q}^{(\delta)}\in\mathcal{P}(d)^\downarrow$ can actually be computed according to simple algorithms. As a second step, the authors consider $\delta_1=\delta_{1\,,\,\|\cdot\|_1}$ which is the minimal $\delta\ge 0$ such that $Q\prec\overline{P}^{(\delta)}$; similarly, they consider $\delta_2=\delta_{2\,,\,\|\cdot\|_1}$ which is the minimal $\delta\ge 0$ such that $\underline{Q}^{(\delta)}\prec P$. Then, (see [18, 20]) we get that

$$\min\{\|P-R\|_1\ :\ R\in\mathcal{U}(Q)\}=\|P-\overline{P}^{(\delta_1)}\|_1\quad\text{ and }\quad\min\{\|Q-S\|_1\ :\ S\in\mathcal{L}(P)\}=\|Q-\underline{Q}^{(\delta_2)}\|_1\,.$$

In [19] the authors extend the approach developed in [18] to the context of density matrices $\rho, \sigma \in \mathcal{D}(d)$ such that $\rho \not\prec \sigma$. Indeed, the authors construct $\mu_0 \in \mathcal{L}(\sigma)$ and $\nu_0 \in \mathcal{U}(\rho)$ such that

$$\|\rho - \mu_0\|_1 = \min\{\|\rho - \mu\|_1 \ : \ \mu \in \mathcal{L}(\sigma)\} \quad \text{and} \quad \|\sigma - \nu_0\|_1 = \min\{\|\sigma - \nu\|_1 \ : \ \nu \in \mathcal{U}(\rho)\}\,.$$

In this case the spectra of μ_0 and ν_0 are actually computed using the method developed in [18].

In [20] the authors complement the results in [18] by developing a similar two-step strategy that allows them to compute the distance to the majorization flows in terms of the ∞-norm. It is also observed that the optimal approximations obtained in [18] and [20] for the $\|\cdot\|_1$ and $\|\cdot\|_\infty$ do not coincide. This last fact poses the problem of choosing a convenient norm (between $\|\cdot\|_1$ and $\|\cdot\|_\infty$) with respect to which the previous optimal approximation problems should be considered (see the discussion and examples in [20]). Moreover, this last fact shows that the solutions obtained in terms of the results in [19, 18] are intrinsic to the $\|\cdot\|_1$ norm and differ from the universal solutions in Theorems 2.3, 3.4 and 3.7 above. Notice that these remarks do not contradict our results since $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are not strictly convex.

In [20][Theorem 3.1.] it is shown that these types of (two-step) strategies to solve the optimal approximation problems to the majorization flows can not be carried out for the (strictly convex) p-norms, for $1 < p < \infty$. For example, the case of the Frobenius or 2-norm (which is a significant case, due to its physical interpretation as well as its rich geometric structure) can not be tackled with extensions of the techniques in [18, 20].

As a consequence of Theorems 2.3 and 3.4 we conclude the following result in the context of probability vectors. The proof of this result is straightforward, since it can be reduced to statements about density matrices by considering diagonal matrices with main diagonals given by probability vectors; we leave the details to the reader.

Theorem 3.7. *Let $P, Q \in \mathcal{P}(d)^\downarrow$ be such that $Q \not\prec P$. Then, there exist vectors $b^{\rm op}, c^{\rm op} \in \mathbb{R}^d$, that can be computed in terms of simple algorithms, such that*

1. *$\mathcal{P}(d) \ni S_0 := Q - b^{\rm op} \prec P$ and $Q - S_0 \prec Q - S$, for every $S \in \mathcal{L}(P)$.*
2. *$Q \prec R_0 := P - c^{\rm op} \in \mathcal{P}(d)$ and $P - R_0 \prec P - R$, for every $R \in \mathcal{U}(Q)$.*
3. *If $\|\cdot\|$ is any symmetric gauge function then*
$$\|Q - S_0\| = \min\{\|Q - S\| \ : \ S \in \mathcal{L}(P)\} \quad \text{and} \quad \|P - R_0\| = \min\{\|P - R\| \ : \ R \in \mathcal{U}(Q)\}\,. \quad (1)$$
4. *In this case we also get $S_0 \prec Q$ and $P \prec R_0$.* □

Notice that the previous result provides structural (universal) solutions of the optimal approximation to the majorization flows with respect to a family of norms that include $\|\cdot\|_1$ and $\|\cdot\|_\infty$. Moreover, it turns out that if $\|\cdot\|$ is a strictly convex gauge symmetric function then S_0 and R_0 as in Theorem 3.7 above are uniquely determined by the identities in Eq. (1).

3.3 Application: on approximate majorizations in $\mathcal{D}(d)$

In this section we apply Theorems 2.3 and 3.4 in the context of approximate majorization. We first recall pre- and post- approximate majorization (see [17, 20]) in the context of density matrices and unitarily invariant norms.

Definition 3.8. *Let $\rho, \sigma \in \mathcal{D}(d)$ and let $N(\cdot)$ be a unitarily invariant norm in $\mathcal{M}_d(\mathbb{C})$. Given $\varepsilon > 0$ we say that*

1. *ρ is (N,ε)-post-majorized by σ, denoted $\rho\ _{(N,\varepsilon)}\!\prec \sigma$, whenever there exists $\mu \in \mathcal{L}(\sigma)$ such that $N(\rho-\mu) \leq \varepsilon$;*

2. *ρ is (N,ε)-pre-majorized by σ, denoted $\rho \prec_{(N,\varepsilon)} \sigma$, whenever there exists $\nu \in \mathcal{U}(\rho)$ such that $N(\sigma-\nu) \leq \varepsilon$.* △

In [20] the authors considered pre- and post- approximate majorization $\rho\ _{(\infty,\varepsilon)}\!\prec \sigma$ and $\rho \prec_{(\infty,\varepsilon)} \sigma$, with respect to the spectral norm $\|\cdot\|_\infty$ on $\mathcal{M}_d(\mathbb{C})$ and $\varepsilon > 0$. Moreover, they obtained simple algorithms in terms of $\lambda(\rho)$, $\lambda(\sigma) \in (\mathbb{R}^d)^\downarrow$ that allow checking whether these approximate majorizations hold for any given $\varepsilon > 0$. Similarly, the results from [18] also allow us to obtain simple algorithms that check whether post- and pre-approximate majorization with respect to the nuclear (or trace) norm $\|\cdot\|_1$ holds. We remark that, as stated in [20], the techniques of [18, 20] can not be adapted to deal, for example, with Schatten-p norms $\|\cdot\|_p$, for $1 < p < \infty$.

Theorem 3.9. *Let ρ, $\sigma \in \mathcal{D}(d)$ be such that $\rho \not\prec \sigma$ and let $N(\cdot)$ be a unitarily invariant norm in $\mathcal{M}_d(\mathbb{C})$. Let $b^{\rm op}$, $c^{\rm op} \in \mathbb{R}^d$ be as in Theorems 2.3 and 3.4, constructed with input given by $\lambda(\rho)$, $\lambda(\sigma) \in (\mathbb{R}^d)^\downarrow$. Then, given $\varepsilon > 0$ we have that*

1. *$\rho\ _{(N,\varepsilon)}\!\prec \sigma$ if and only if $\varepsilon \geq N(b^{\rm op})$.*

2. *$\rho \prec_{(N,\varepsilon)} \sigma$ if and only if $\varepsilon \geq N(c^{\rm op})$.*

Proof. The result is an immediate consequence of the properties of the vectors $b^{\rm op}$, $c^{\rm op} \in \mathbb{R}^d$, described in Theorems 2.3 and 3.4. □

We point out that the computations of $b^{\rm op}$, $c^{\rm op} \in \mathbb{R}^d$ (as in Theorems 3.1 and 3.4) can be carried out in terms of simple algorithms whose input is given by $\lambda(\rho)$, $\lambda(\sigma) \in \mathbb{R}^d$. Hence, our approach also allows us to decide in an efficient way whether these approximate majorization relations hold, with respect to any unitarily invariant norm $N(\cdot)$.

4 Proofs of the main results

In this section we present the proofs of our main results. Indeed, in sections 4.1 and 4.2 we develop the proof of Theorem 3.1, while in sections 4.3 and 4.4 we obtain the proof of Theorem 3.4. Briefly, our approach is as follows: we consider a function (i.e. Φ_f or Ψ_f, see below) induced by a fixed strictly Schur-convex function f and a local minimizer of that function (Φ_f or Ψ_f). Then, using geometrical perturbation arguments, we show a series of spectral and geometrical properties (eigenvalues and eigenvectors) of the local minimizer. We end our argument by showing that the spectral structure of the local minimizer is unique, and does not depend on the strictly Schur-convex function f being considered. As a consequence of this approach, we deduce a finite step algorithm that allows us to compute the (unique) spectral structure of these minimizers.

4.1 Proof of Theorem 3.1 - first part

In this section we consider two (fixed) densities $\rho, \sigma \in \mathcal{D}(d)$ such that $\rho \not\prec \sigma$. Given a strictly Schur-convex function f in $\mathbb{R}^d$ we consider $\Phi_f : \mathcal{L}(\sigma) \to \mathbb{R}$ given by

$$\Phi_f(\mu) = f(\lambda(\rho-\mu)) \quad \text{for} \quad \mu \in \mathcal{L}(\sigma)\,.$$

We endow $\mathcal{L}(\sigma)$ with the metric induced by the spectral norm. Since $\mathcal{L}(\sigma)$ is compact and Φ_f is a continuous function then Φ_f attains its minimum value on $\mathcal{L}(\sigma)$. In what follows we consider $\mu_0 \in \mathcal{L}(\sigma)$ that is a *local* minimizer of Φ_f.

Theorem 4.1. *Let $\mu_0 \in \mathcal{L}(\sigma)$ be a local minimizer of Φ_f on $\mathcal{L}(\sigma)$. Then there exists an onb $\{u_i\}_{i\in\mathbb{I}_d}$ of $\mathbb{C}^d$ such that*

$$\rho = \sum_{i\in\mathbb{I}_d} \lambda_i(\rho)\ u_i \otimes u_i \quad \text{and} \quad \mu_0 = \sum_{i\in\mathbb{I}_d} \lambda_i(\mu_0)\ u_i \otimes u_i\,. \tag{2}$$

In particular, $\lambda(\rho-\mu_0) = (\lambda(\rho)-\lambda(\mu_0))^\downarrow$.

Proof. Consider the unitary orbit $\mathcal{O}(\mu_0) = \{\omega^*\,\mu_0\,\omega\ :\ \omega \in \mathcal{U}_d\}$, endowed with the distance induced by the spectral norm. Notice that $\mathcal{O}(\mu_0) \subset \mathcal{L}(\sigma)$ and μ_0 is a local minimizer of Φ_N restricted to $\mathcal{O}(\mu_0)$. Hence, by Theorem 2.5 we get that there exists an onb $\{u_i\}_{i\in\mathbb{I}_d}$ of $\mathbb{C}^d$ with the desired properties. □

Remark 4.2. Let $\mu_0 \in \mathcal{L}(\sigma)$ be a local minimizer of Φ_f on $\mathcal{L}(\sigma)$. By Theorem 4.1 we get that there exists an onb $\{u_i\}_{i\in\mathbb{I}_d}$ of $\mathbb{C}^d$ for which Eq. (2) holds. Hence, we see that

$$\lambda(\rho-\mu_0) = (\lambda(\rho)-\lambda(\mu_0))^\downarrow = b := (b_1\,\mathbb{1}_{t_1},\ldots,b_q\,\mathbb{1}_{t_q})$$

for some $b_1 > \ldots > b_q$ and $t_1+\ldots+t_q = d$, where $\mathbb{1}_t = (1,\ldots,1) \in \mathbb{R}^t$. We further introduce the sets of indices

$$J_k = \{\ell \in \mathbb{I}_d\ :\ \lambda_\ell(\rho)-\lambda_\ell(\mu_0) = b_k\} \quad \text{for} \quad k \in \mathbb{I}_q\,.$$

Notice that $\{J_k\}_{k\in\mathbb{I}_q}$ is a partition of $\mathbb{I}_d$. △

Proposition 4.3. *Let $\mu_0 \in \mathcal{L}(\sigma)$ be a local minimizer of Φ_f on $\mathcal{L}(\sigma)$. With the notation of Remark 4.2 there exist $i_0 = 0 < i_1 < \ldots < i_{q-1} < i_q = d$ such that*

$$J_k = \{\ell \in \mathbb{I}_d\ :\ i_{k-1}+1 \le \ell \le i_k\} \quad \text{for} \quad k \in \mathbb{I}_q\,. \tag{3}$$

Hence, $\lambda(\rho)-\lambda(\mu_0) = (b_1\,\mathbb{1}_{t_1},\ldots,b_q\,\mathbb{1}_{t_q}) = \lambda(\rho-\mu_0) \in (\mathbb{R}^d)^\downarrow$. Moreover,

1. *$0_d \prec (b_1\,\mathbb{1}_{t_1},\ldots,b_q\,\mathbb{1}_{t_q})$;*
2. *$\lambda(\mu_0) \prec \lambda(\mu_0) + (b_1\,\mathbb{1}_{t_1},\ldots,b_q\,\mathbb{1}_{t_q}) = \lambda(\rho)$.*

Proof. In case $q = 1$ we just set $i_1 = d$ and the result is clear. Thus, we assume that $q \ge 2$. We first prove the identity in Eq. (3) for $k = 1$. Hence, let $\ell \in J_1$ and $h \in \mathbb{I}_d \setminus J_1$; assume that $h < \ell$ (and we reach a contradiction).

Assume further that $0 \le \lambda_\ell(\mu_0) < \lambda_h(\mu_0)$. By assumption we have that

$$\lambda_\ell(\rho)-\lambda_\ell(\mu_0) = b_1 \quad \text{and} \quad \lambda_h(\rho)-\lambda_h(\mu_0) = b_p \quad \text{for some} \quad 2 \le p \le q\,. \tag{4}$$

For $\varepsilon > 0$ we let

$$\tilde\mu(\varepsilon) = \sum_{i\in\mathbb{I}_d\setminus\{h\,,\,\ell\}} \lambda_i(\mu_0)\,u_i\otimes u_i + (\lambda_h(\mu_0)-\varepsilon)\,u_h\otimes u_h + (\lambda_\ell(\mu_0)+\varepsilon)\,u_\ell\otimes u_\ell\,,$$

where $\{u_i\}_{i\in\mathbb{I}_d}$ is an onb of $\mathbb{C}^d$ for which Eq. (2) holds. By construction, $\lim_{\varepsilon\to 0^+}\tilde\mu(\varepsilon) = \mu_0$. On the other hand, notice that

$$\lambda(\tilde\mu(\varepsilon)) = ((\lambda_i(\mu_0))_{i\in\mathbb{I}_d\setminus\{h\,,\,\ell\}}\,,\,(\lambda_h(\mu_0)-\varepsilon),(\lambda_\ell(\mu_0)+\varepsilon))^\downarrow\,.$$

Since

$$((\lambda_h(\mu_0)-\varepsilon),(\lambda_\ell(\mu_0)+\varepsilon)) \prec (\lambda_h(\mu_0)\,,\,\lambda_\ell(\mu_0))$$

and $\tilde\mu(\varepsilon) \in \mathcal{M}_d(\mathbb{C})^+$ for sufficiently small $\varepsilon > 0$, we conclude that $\tilde\mu(\varepsilon) \prec \mu_0 \prec \sigma$ i.e. $\tilde\mu(\varepsilon) \in \mathcal{L}(\sigma)$, for sufficiently small $\varepsilon > 0$; furthermore, by construction we get that

$$\lambda(\rho-\tilde\mu(\varepsilon)) = ((\lambda_i(\rho-\mu_0))_{i\in\mathbb{I}_d\setminus\{h\,,\,\ell\}}\,,\,b_1-\varepsilon\,,\,b_p+\varepsilon)^\downarrow\,.$$

Since $(b_1 - \varepsilon\,,\, b_p + \varepsilon) \prec (b_1\,,\, b_p)$ strictly for sufficiently small $\varepsilon > 0$ (recall that $b_1 > b_p$). We now see that $\lambda(\rho - \tilde\mu(\varepsilon)) \prec \lambda(\rho - \mu_0)$ strictly and hence $\Phi_f(\tilde\mu(\varepsilon)) < \Phi_f(\mu_0)$, for sufficiently small $\varepsilon > 0$. This last fact contradicts our assumption that μ_0 is a local minimizer of $\Phi_f(\cdot)$ on $\mathcal{L}(\sigma)$.

Therefore, we conclude that $\lambda_\ell(\mu_0) = \lambda_h(\mu_0)$; by Eq. (4) we get that $\lambda_\ell(\rho) > \lambda_h(\rho)$, which contradicts that $h < \ell$. Hence we now see that $\ell < h$, for every $\ell \in J_1$ and $h \in \mathbb{I}_d \setminus J_1$. This last fact implies that there exists $i_1 = \max J_1$ such that $J_1 = \big\{\ell \in \mathbb{I}_d \ :\ i_0 + 1 \le \ell \le i_1\big\}$.

We can consider $\ell \in J_2$ and $h \in \mathbb{I}_d \setminus (J_1 \cup J_2)$; following an argument analogous to that considered above, we conclude that $\ell < h$. Thus, there exists $i_2 = \max J_2$ such that $J_2 = \big\{\ell \in \mathbb{I}_d \ :\ i_1 + 1 \le \ell \le i_2\big\}$. The first part of the statement follows after applying this argument q times.

By the definition of the partition $\{J_k\}_{k\in\mathbb{I}_q}$ we now see that $\lambda(\rho) - \lambda(\mu_0) = b = \lambda(\rho - \mu_0) \in (\mathbb{R}^d)^\downarrow$, where $b = (b_1\,\mathbb{1}_{t_1}, \ldots, b_q\,\mathbb{1}_{t_q})$. Notice that in particular,

$$\operatorname{tr}(b) = \operatorname{tr}((b_1\,\mathbb{1}_{t_1}, \ldots, b_q\,\mathbb{1}_{t_q})) = \operatorname{tr}(\rho - \mu_0) = 0 \implies 0_d \prec (b_1\,\mathbb{1}_{t_1}, \ldots, b_q\,\mathbb{1}_{t_q})\,.$$

Moreover, we also get that $\lambda(\rho) = \lambda(\mu_0) + b \in (\mathbb{R}^d)^\downarrow$. In particular, if $\ell \in \mathbb{I}_d$ then

$$\sum_{j\in\ell} \lambda_j(\rho) = \sum_{j\in\ell} (\lambda_j(\mu_0) + b_j) = \sum_{j\in\ell} \lambda_j(\mu_0) + \sum_{j\in\ell} b_j \ge \sum_{j\in\ell} \lambda_j(\mu_0)\,,$$

where we used that $\sum_{j\in\ell} b_j \ge 0$ since $0_d \prec b$. These last inequalities show that $\lambda(\mu_0) \prec \lambda(\rho)$. □

Proposition 4.4. *Let $\mu_0 \in \mathcal{L}(\sigma)$ be a local minimizer of Φ_N on $\mathcal{L}(\sigma)$. Consider the notation in Remark 4.2. Let $i_0 = 0 < i_1 < \ldots < i_{q-1} < i_q = d$ be as in Proposition 4.3. Then,*

$$\sum_{j=1}^{i_k} \lambda_j(\mu_0) = \sum_{j=1}^{i_k} \lambda_j(\sigma) \quad \textit{for} \quad k \in \mathbb{I}_q\,.$$

Proof. The result is clearly true when $q = 1$; hence, we assume that $q \ge 2$. We first prove the case when $k = 1$. Indeed, assume that

$$\sum_{j=1}^{i_1} \lambda_j(\mu_0) < \sum_{j=1}^{i_1} \lambda_j(\sigma)$$

where we have used that $\mu_0 \prec \sigma$ so, in case the equality in the statement fails, we should have the strict inequality above. Set

$$\mathcal{M} = \Big\{i_0 + 1 = 1 \le t \le\ i_1 :\ \sum_{j=1}^{t} \lambda_j(\mu_0) = \sum_{j=1}^{t} \lambda_j(\sigma)\Big\}\,.$$

If $\mathcal{M} \ne \emptyset$ we set $r := \max \mathcal{M}$; otherwise we set $r := 0$. Notice that by construction $r < i_1$. We also set

$$h = \min\Big\{i_1 \le t \le d\ :\ \sum_{j=1}^{t} \lambda_j(\mu_0) = \sum_{j=1}^{t} \lambda_j(\sigma)\Big\} > i_1\,.$$

Using that $\mu_0 \prec \sigma$ and the definitions of r and h we conclude that

1. $(\lambda_\ell(\mu_0))_{\ell=1}^r \prec (\lambda_\ell(\sigma))_{\ell=1}^r$;
2. $(\lambda_\ell(\mu_0))_{\ell=r+1}^h \prec (\lambda_\ell(\sigma))_{\ell=r+1}^h$;
3. $(\lambda_\ell(\mu_0))_{\ell=h+1}^d \prec (\lambda_\ell(\sigma))_{\ell=h+1}^d$.

Notice that item 1. only applies when $r \geq 1$; similarly, item 3. only applies when $h \leq d-1$. On the other hand item 2. always applies since $r+1 \leq i_1 < h$. Further, we get that

$$\sum_{\ell=r+1}^{t} \lambda_\ell(\mu_0) < \sum_{\ell=r+1}^{t} \lambda_\ell(\sigma) \quad \text{for} \quad r+1 \leq t \leq h-1\,.$$

This last fact shows that for sufficiently small $\varepsilon > 0$ we get that

$$(\lambda_{r+1}(\mu_0) + \varepsilon\,,\, \lambda_{r+2}(\mu_0)\,,\, \ldots\,,\, \lambda_{h-1}(\mu_0)\,,\, \lambda_h(\mu_0) - \varepsilon) \prec (\lambda_\ell(\sigma))_{\ell=r+1}^{h}\,. \tag{5}$$

Hence, for $\varepsilon > 0$ we define

$$\tilde\mu(\varepsilon) = \sum_{i\in \mathbb{I}_d\setminus\{r\,,\,h\}} \lambda_i(\mu_0)\; u_i \otimes u_i + (\lambda_{r+1}(\mu_0)+\varepsilon)\; u_{r+1}\otimes u_{r+1} + (\lambda_h(\mu_0)-\varepsilon)\; u_h\otimes u_h\,,$$

where $\{u_i\}_{i\in\mathbb{I}_d}$ is an onb of $\mathbb{C}^d$ for which Eq. (2) holds. By construction, $\lim_{\varepsilon\to 0^+}\tilde\mu(\varepsilon) = \mu_0$. Notice that in case $\lambda_h(\mu_0) = 0$ then $\lambda_h(\sigma) = 0$ (since $\mu_0 \in \mathcal{L}(\sigma)$); so in this case we get that

$$\sum_{j=1}^{h-1} \lambda_j(\mu_0) = 1 = \sum_{j=1}^{h-1} \lambda_j(\sigma)$$

that contradicts the definition of h; thus, $\lambda_h(\mu_0) > 0$. Hence, using items 1. and 3., Eq. (5) and that $\lambda_h(\mu_0) > 0$ we get that $\tilde\mu(\varepsilon) \in \mathcal{M}_d(\mathbb{C})^+$ and $\tilde\mu(\varepsilon) \prec \sigma$ (i.e. $\mu(\varepsilon) \in \mathcal{L}(\sigma)$) for sufficiently small $\varepsilon > 0$. Moreover, by construction

$$\lambda(\rho - \tilde\mu(\varepsilon)) = (\,(\lambda_i(\rho-\mu_0))_{i\in\mathbb{I}_d\setminus\{r+1\,,\,h\}}\,,\, b_1 - \varepsilon, b_p + \varepsilon)$$

for some $2 \leq p \leq q$, since $r+1 \leq i_1 < h$ (and the definition of i_1). Since $(b_1 - \varepsilon, b_p + \varepsilon) \prec (b_1\,,\, b_p)$ strictly for sufficiently small $\varepsilon > 0$, we conclude that $\rho - \tilde\mu(\varepsilon) \prec \rho - \mu_0$ strictly for sufficiently small $\varepsilon > 0$. Therefore,

$$\Phi_f(\tilde\mu(\varepsilon)) < \Phi_f(\mu_0) \quad \text{for sufficiently small} \quad \varepsilon > 0\,.$$

This last fact contradicts our assumption that μ_0 is a local minimizer of $\Phi_f(\cdot)$. Hence, we now see that

$$\sum_{j=1}^{i_1} \lambda_j(\mu_0) = \sum_{j=1}^{i_1} \lambda_j(\sigma)\,,$$

and the result is established for $k = 1$. In case

$$\sum_{j=1}^{i_2} \lambda_j(\mu_0) < \sum_{j=1}^{i_2} \lambda_j(\sigma)$$

we argue as above: we consider $r = \max\{i_1 + 1 \leq t \leq i_2 \;:\; \sum_{j=1}^t \lambda_j(\mu_0) = \sum_{j=1}^t \lambda_j(\sigma)\}$ in case the set is not empty or $r = i_1$ otherwise. Similarly, we set $h = \min\{i_2 \leq t \leq d \;:\; \sum_{j=1}^t \lambda_j(\mu_0) = \sum_{j=1}^t \lambda_j(\sigma)\}$ and notice that by construction $r+1 \leq i_2 < h$ and we can partition $\mathbb{I}_d$ in terms of r and h so that the majorization relations in items 1.-3. above hold. Then, we can repeat the rest of the argument and contradict that μ_0 is a local minimizer of $\Phi_f(\cdot)$. The result follows by applying this argument q times. □

Proposition 4.5. *Let $\mu_0 \in \mathcal{L}(\sigma)$ be a local minimizer of Φ_N on $\mathcal{L}(\sigma)$. Consider the notation in Remark 4.2. Let $i_0 = 0 < i_1 < \ldots < i_{q-1} < i_q = d$ be as in Proposition 4.3. Then*

$$b_j = \frac{1}{i_j - i_{j-1}} \sum_{\ell=i_{j-1}+1}^{i_j} \lambda_\ell(\rho) - \lambda_\ell(\sigma) \quad \textit{for} \quad j \in \mathbb{I}_q\,.$$

Moreover, in this case we also get

$$(\lambda_\ell(\rho) - b_j)_{\ell=i_{j-1}+1}^{i_j} \prec (\lambda_\ell(\sigma))_{\ell=i_{j-1}+1}^{i_j} \quad \textit{for} \quad j \in \mathbb{I}_q\,.$$

Proof. As a consequence of Proposition 4.4 we get that

$$\sum_{\ell=i_{j-1}+1}^{i_j} \lambda_\ell(\mu_0) = \sum_{\ell=i_{j-1}+1}^{i_j} \lambda_\ell(\sigma) \quad \text{for} \quad j \in \mathbb{I}_q \,. \tag{6}$$

These last identities together with the majorization relation $\lambda(\mu_0) \prec \lambda(\sigma)$ imply that

$$(\lambda_\ell(\mu_0))_{\ell=i_{j-1}+1}^{i_j} \prec (\lambda_\ell(\sigma))_{\ell=i_{j-1}+1}^{i_j} \,. \tag{7}$$

On the other hand, by construction (see Remark 4.2) we have that, for $j \in \mathbb{I}_q$

$$\lambda_\ell(\rho) - \lambda_\ell(\mu_0) = b_j \quad \text{for} \quad i_{j-1}+1 \le \ell \le i_j \,. \tag{8}$$

Thus, by Eq. (6) we get that

$$b_j = \frac{1}{i_j - i_{j-1}} \sum_{\ell=i_{j-1}+1}^{i_j} \lambda_\ell(\rho) - \lambda_\ell(\mu_0) = \frac{1}{i_j - i_{j-1}} \sum_{\ell=i_{j-1}+1}^{i_j} \lambda_\ell(\rho) - \lambda_\ell(\sigma) \,.$$

Finally, by Eqs. (7) and (8) we see that

$$(\lambda_\ell(\rho) - b_j)_{\ell=i_{j-1}+1}^{i_j} = (\lambda_\ell(\mu_0))_{\ell=i_{j-1}+1}^{i_j} \prec (\lambda_\ell(\sigma))_{\ell=i_{j-1}+1}^{i_j} \,.$$

□

Proposition 4.6. *Let $\sigma,\ \rho \in \mathcal{D}(d)$. Then, there exist unique $q \in \mathbb{I}_d$ and indices $i_0 = 0 < i_1 < \ldots < i_{q-1} < i_q = d$ such that if we let*

$$b_k = \frac{1}{i_k - i_{k-1}} \sum_{\ell=i_{k-1}+1}^{i_k} \lambda_\ell(\rho) - \lambda_\ell(\sigma) \quad \textit{for} \quad k \in \mathbb{I}_q$$

then $b_1 > b_2 > \ldots > b_q$ and

$$(\lambda_\ell(\rho) - b_k)_{\ell=i_{k-1}+1}^{i_k} \prec (\lambda_\ell(\sigma))_{\ell=i_{k-1}+1}^{i_k} \quad \textit{for} \quad k \in \mathbb{I}_q \,.$$

Proof. Proposition 4.5 shows that there are indices and constants as in the statement. Assume that there are some other indices $i'_0 = 0 < i'_1 < \ldots < i'_p = d$, for some $p \in \mathbb{I}_d$, such that if we let

$$b'_k = \frac{1}{i'_k - i'_{k-1}} \sum_{\ell=i'_{k-1}+1}^{i'_k} \lambda_\ell(\rho) - \lambda_\ell(\sigma) \quad \text{for} \quad k \in \mathbb{I}_p$$

then $b'_1 > b'_2 > \ldots > b'_p$ and

$$(\lambda_\ell(\rho) - b'_k)_{\ell=i'_{k-1}+1}^{i'_k} \prec (\lambda_\ell(\sigma))_{\ell=i'_{k-1}+1}^{i'_k} \quad \text{for} \quad k \in \mathbb{I}_p \,.$$

We first assume that $i_1 \neq i'_1$, say $i_1 < i'_1$. Let $2 \le k \le q$ be such that $i_{k-1} < i'_1 \le i_k$ and let $t = i'_1 - i_{k-1} \ge 1$. In this case

$$b'_1 = \frac{1}{i'_1} \sum_{\ell=1}^{i_{k-1}+t} \lambda_\ell(\rho) - \lambda_\ell(\sigma) = \frac{1}{i'_1} \left(\sum_{j=1}^{k-1} (i_j - i_{j-1})\, b_j + \sum_{\ell=i_{k-1}+1}^{i'_1} \lambda_\ell(\rho) - \lambda_\ell(\sigma) \right) \,.$$

Notice that

$$\sum_{\ell=i_{k-1}+1}^{i'_1} \lambda_\ell(\rho) - b_k \le \sum_{\ell=i_{k-1}+1}^{i'_1} \lambda_\ell(\sigma) \implies \sum_{\ell=i_{k-1}+1}^{i'_1} \lambda_\ell(\rho) - \lambda_\ell(\sigma) \le t\, b_k \,.$$

Hence, the previous inequalities imply that

$$b'_1 \le \frac{1}{i'_1} \left(\sum_{j=1}^{k-1} (i_j - i_{j-1})\, b_j + t\, b_k \right) < b_1 \,,$$

where we used that $\sum_{j=1}^{k-1}(i_j - i_{j-1}) + t = i'_1$ and that $b_1 > \ldots > b_k$, with $k \ge 2$ and $t \ge 1$. Since $i_1 < i'_1$ then $(\lambda_\ell(\rho) - b'_1)_{\ell=1}^{i'_1} \prec (\lambda_\ell(\sigma))_{\ell=1}^{i'_1}$ implies that

$$\sum_{\ell=1}^{i_1} \lambda_\ell(\rho) - b_1 < \sum_{\ell=1}^{i_1} \lambda_\ell(\rho) - b'_1 \le \sum_{\ell=1}^{i_1} \lambda_\ell(\sigma) \,,$$

that contradicts the majorization relation $(\lambda_\ell(\rho) - b_k)_{\ell=1}^{i_1} \prec (\lambda_\ell(\sigma))_{\ell=1}^{i_1}$. Hence we see that $i_1 = i'_1$. In case $q = 1$ (so $i_1 = i'_1 = d$ and hence $p = q = 1$) we are done; if $q > 1$ and we assume that $i_2 < i'_2$ we can argue as above and get a contradiction. Thus, the result follows from the application of the previous argument q times. □

Corollary 4.7. *Consider the notation of Proposition 4.5. Then, we have that*

1. *The spectral structure $\lambda(\rho - \mu_0)$ for all local minimizers $\mu_0 \in \mathcal{L}(\sigma)$ of Φ_f is the same. In particular, local minimizers of Φ_f are global minimizers.*

2. *The spectral structure $\lambda(\rho - \mu_0)$ for all local minimizers $\mu_0 \in \mathcal{L}(\sigma)$ of Φ_f does not depend on the particular strictly Schur-convex function $f : \mathbb{R}^d \to \mathbb{R}$. In particular, $\lambda(\rho - \mu_0) \prec \lambda(\rho - \mu)$ for every $\mu \in \mathcal{L}(\sigma)$.*

Proof. Recall that since $\mathcal{L}(\sigma)$ is compact (with respect to the topology induced by the spectral norm) and Φ_f is a continuous function then Φ_f attains its minimum value at some $\tilde{\mu} \in \mathcal{L}(\sigma)$. In particular, $\tilde{\mu}$ is a local minimizer of Φ_f. Let $\mu_0 \in \mathcal{L}(\sigma)$ be any local minimizer of Φ_f. Notice that Propositions 4.5 and 4.6 show that the vectors $\lambda(\rho - \tilde{\mu})$ and $\lambda(\rho - \mu_0)$ coincide in this case. In particular, $\Phi_f(\rho - \tilde{\mu}) = \Phi_f(\rho - \mu_0)$. That is, μ_0 is also a global minimizer of Φ_f. Moreover, notice that Proposition 4.6 shows the spectral structure of $\lambda(\rho - \mu_0)$ is uniquely determined by the spectral structure of ρ and σ (and does not depend on the strictly Schur-convex function $f : \mathbb{R}^d \to \mathbb{R}$). In particular, if $\mu \in \mathcal{L}(\sigma)$ then for every strictly Schur-convex function $g : \mathbb{R}^d \to \mathbb{R}$ we get that $g(\lambda(\rho - \mu_0)) = \Phi_g(\mu_0) \le \Phi_g(\mu) = g(\lambda(\rho - \mu))$; it is well known that this last fact implies that $\lambda(\rho - \mu_0) \prec \lambda(\rho - \mu)$. □

Proof of Theorem 3.1. Let $N(\cdot)$ be a strictly convex unitarily invariant norm and let g_N be its associated strictly Schur-convex gauge symmetric function. Set $f = g_N$ and notice that Φ_N and Φ_f coincide in this case. Hence, the equivalence between items 1. and 2. is a straightforward consequence of Corollary 4.7 above. The majorization relation follows from Proposition 4.3. The remaining part of Theorem 3.1, namely that $b^{\rm op} \in (\mathbb{R}^d)^\downarrow$ can be computed in terms of a simple algorithm with input $\lambda(\sigma)$, $\lambda(\rho) \in \mathbb{R}^d$, will be a consequence of Theorem 4.10 in the next section. □

4.2 A new algorithmic construction of b^{op}

In this section we consider two (fixed) density matrices $\rho, \sigma \in \mathcal{D}(d)$ such that $\rho \not\prec \sigma$. Recall that Theorem 2.3 shows the fundamental role played by the vector $b^{\text{op}} \in (\mathbb{R}^d)^{\downarrow}$. In this section we describe a finite step (and simple) algorithm that computes b^{op} in terms of the spectral structure of ρ and σ. Once this vector is computed then we can describe all $\mu \in \mathcal{L}(\sigma)$ such that $\lambda(\rho - \mu) = b^{\text{op}}$ in terms of all possible spectral representations of ρ.

Algorithm 4.8. Given $\rho, \sigma \in \mathcal{D}(d)$ such that $\rho \not\prec \sigma$, consider the input: $\lambda(\rho)$, $\lambda(\sigma) \in (\mathbb{R}^d)^{\downarrow}$. We first define $i_0 := 0$. Assuming that we have defined $0 \le i_{j-1} < d$ then:

1. For $i_{j-1} + 1 \le k \le d$ we define the auxiliary numbers

$$f_k = \frac{1}{k - i_{j-1}} \sum_{\ell = i_{j-1}+1}^{k} \lambda_\ell(\rho) - \lambda_\ell(\sigma)\,.$$

2. We set

$$i_j = \max\left\{ i_{j-1} + 1 \le k \le d \ : \ (\lambda_\ell(\rho) - f_k)_{\ell = i_{j-1}+1}^{k} \prec (\lambda_\ell(\sigma))_{\ell = i_{j-1}+1}^{k} \right\}\,.$$

After a finite number $q \in \mathbb{I}_d$ of iterations we compute the output: $i_0 = 0 < i_1 < \ldots < i_{q-1} < i_q = d$ and the constants

$$b_j = \frac{1}{i_j - i_{j-1}} \sum_{\ell = i_{j-1}+1}^{i_j} \lambda_\ell(\rho) - \lambda_\ell(\sigma) \quad \text{for} \quad j \in \mathbb{I}_q\,.$$

△

Proposition 4.9. *Consider the output of Algorithm 4.8: $i_0 = 0 < i_1 < \ldots < i_{q-1} < i_q = d$ and the constants $b_1, \ldots, b_q$. Then $b_1 > b_2 > \ldots > b_q$.*

Proof. Let $1 \le j < q$ and assume that $b_j \le b_{j+1}$. Recall that

$$b_j = \frac{1}{i_j - i_{j-1}} \sum_{\ell = i_{j-1}+1}^{i_j} \lambda_\ell(\rho) - \lambda_\ell(\sigma) \quad \text{and} \quad b_{j+1} = \frac{1}{i_{j+1} - i_j} \sum_{\ell = i_j+1}^{i_{j+1}} \lambda_\ell(\rho) - \lambda_\ell(\sigma)\,.$$

We also consider

$$f_{i_{j+1}} = \frac{1}{i_{j+1} - i_{j-1}} \sum_{\ell = i_{j-1}+1}^{i_{j+1}} \lambda_\ell(\rho) - \lambda_\ell(\sigma) = \frac{i_j - i_{j-1}}{i_{j+1} - i_{j-1}}\, b_j + \frac{i_{j+1} - i_j}{i_{j+1} - i_{j-1}}\, b_{j+1}\,.$$

Hence, $b_j \le f_{i_{j+1}} \le b_{j+1}$. In this case, it turns out that

$$(\lambda_\ell(\rho) - f_{i_{j+1}})_{\ell = i_{j-1}+1}^{i_{j+1}} \prec (\lambda_\ell(\sigma))_{\ell = i_{j-1}+1}^{i_{j+1}} \tag{9}$$

To see this, we consider the following cases: if $i_{j-1} + 1 \le k \le i_j$ then, by construction of i_j and b_j (see Algorithm 4.8) we get that

$$\sum_{\ell = i_{j-1}+1}^{k} (\lambda_\ell(\rho) - f_{i_{j+1}}) \le \sum_{\ell = i_{j-1}+1}^{k} (\lambda_\ell(\rho) - b_j) \le \sum_{\ell = i_{j-1}+1}^{k} \lambda_\ell(\sigma)\,.$$

If $i_j + 1 \le k \le i_{j+1}$ then

$$\sum_{\ell=i_{j-1}+1}^{k} (\lambda_\ell(\rho) - f_{i_{j+1}}) = \sum_{\ell=i_{j-1}+1}^{i_j} (\lambda_\ell(\rho) - b_j) + (i_j - i_{j-1})\ (b_j - f_{i_{j+1}}) + \sum_{\ell=i_j+1}^{k} (\lambda_\ell(\rho) - b_{j+1}) + (k - i_j)\ (b_{j+1} - f_{i_{j+1}}) \le \sum_{\ell=i_{j-1}+1}^{k} \lambda_\ell(\sigma)$$

where we used that

$$(\lambda_\ell(\rho) - b_j)_{\ell=i_{j-1}+1}^{i_j} \prec (\lambda_\ell(\sigma))_{\ell=i_{j-1}+1}^{i_j} \quad , \quad (\lambda_\ell(\rho) - b_{j+1})_{\ell=i_j+1}^{i_{j+1}} \prec (\lambda_\ell(\sigma))_{\ell=i_j+1}^{i_{j+1}}$$

and that $b_{j+1} - f_{i_{j+1}} \ge 0$ so that

$$(i_j - i_{j-1})\ (b_j - f_{i_{j+1}}) + (k - i_j)\ (b_{j+1} - f_{i_{j+1}}) \le$$

$$(i_j - i_{j-1})\ (b_j - f_{i_{j+1}}) + (i_{j+1} - i_j)\ (b_{j+1} - f_{i_{j+1}}) = 0\,.$$

These facts prove the majorization relation in Eq. (9); but notice that this majorization relation contradicts the definition of i_j (see Algorithm 4.8). Hence, we conclude that $b_j > b_{j+1}$. □

Theorem 4.10. *Consider the output of Algorithm 4.8: $i_0 = 0 < i_1 < \ldots < i_{q-1} < i_q = d$ and the constants $b_1 > \ldots > b_q$. Set $t_j = i_j - i_{j-1}$, for $j \in \mathbb{I}_q$. If $b^{\rm op} \in (\mathbb{R}^d)^\downarrow$ is as in Theorem 2.3 we have that*

$$b^{\rm op} = (b_1\, \mathbb{1}_{t_1}, \ldots, b_q\, \mathbb{1}_{t_q})\,.$$

Proof. Let us denote $\tilde b = (b_1\, \mathbb{1}_{t_1}, \ldots, b_q\, \mathbb{1}_{t_q}) \in (\mathbb{R}^d)^\downarrow$, where $i_0 = 0 < i_1 < \ldots < i_{q-1} < i_q = d$ and the constants $b_1 > \ldots > b_q$ are the output of Algorithm 4.8 and $t_j = i_j - i_{j-1}$, for $j \in \mathbb{I}_q$. By construction (see Algorithm 4.8) we see that

$$(\lambda_\ell(\rho) - b_k)_{\ell=i_{k-1}+1}^{i_k} \prec (\lambda_\ell(\sigma))_{\ell=i_{k-1}+1}^{i_k} \quad \text{for} \quad k \in \mathbb{I}_q\,.$$

Let N be a strictly convex unitarily invariant norm and let $\mu_0 \in \mathcal{L}(\sigma)$ be a minimizer of Φ_N on $\mathcal{L}(\sigma)$. By Theorem 2.3 we get that $b^{\rm op} = \lambda(\rho - \mu_0) \in (\mathbb{R}^d)^\downarrow$. In this case (see Remark 4.2 and Proposition 4.3) there exist indices $i'_0 = 0 < i'_1 < \ldots < i'_p = d$, for some $p \in \mathbb{I}_d$, such that if we let

$$b'_k = \frac{1}{i'_k - i'_{k-1}} \sum_{\ell=i'_{k-1}+1}^{i'_k} \lambda_\ell(\rho) - \lambda_\ell(\sigma) \quad \text{for} \quad k \in \mathbb{I}_p$$

then $b^{\rm op} = (b'_1\, \mathbb{1}_{t'_1}, \ldots, b'_p\, \mathbb{1}_{t'_p}) \in (\mathbb{R}^d)^\downarrow$, where $t'_j = i'_j - i'_{j-1}$, for $j \in \mathbb{I}_p$. Moreover, in this case $b'_1 > b'_2 > \ldots > b'_p$ and

$$(\lambda_\ell(\rho) - b'_k)_{\ell=i'_{k-1}+1}^{i'_k} \prec (\lambda_\ell(\sigma))_{\ell=i'_{k-1}+1}^{i'_k} \quad \text{for} \quad k \in \mathbb{I}_p\,.$$

By Proposition 4.6 we now see that $q = p$, $i_j = i'_j$, for $j \in \mathbb{I}_q$. In particular, $b_j = b'_j$ and $t_j = t'_j$, for $j \in \mathbb{I}_q$ and hence $\tilde b = b^{\rm op}$. □

4.3 Proof of Theorem 3.4 - first part

In this section we consider two (fixed) densities $\rho, \sigma \in \mathcal{D}(d)$ such that $\rho \not\prec \sigma$. Given a strictly Schur-convex function $f : \mathbb{R}^d \to \mathbb{R}$, let $\Psi_f : \mathcal{U}(\rho) \to \mathbb{R}$ be given by

$$\Psi_f(\nu) = f(\lambda(\sigma - \nu)) \quad \text{for} \quad \nu \in \mathcal{U}(\rho).$$

We endow $\mathcal{U}(\rho)$ with the metric induced by the spectral norm. Since $\mathcal{U}(\rho)$ is compact and Ψ_f is a continuous function then Ψ_f attains its minimum value on $\mathcal{U}(\rho)$. In what follows we consider $\nu_0 \in \mathcal{U}(\rho)$ that is a *local* minimizer of Ψ_f.

Theorem 4.11. *Let $\nu_0 \in \mathcal{U}(\rho)$ be a local minimizer of Ψ_f on $\mathcal{U}(\rho)$. Then there exists an onb $\{v_i\}_{i\in\mathbb{I}_d}$ of $\mathbb{C}^d$ such that*

$$\sigma = \sum_{i\in\mathbb{I}_d} \lambda_i(\sigma)\ v_i \otimes v_i \quad \text{and} \quad \nu_0 = \sum_{i\in\mathbb{I}_d} \lambda_i(\nu_0)\ v_i \otimes v_i \,. \tag{10}$$

In particular, $\lambda(\sigma - \nu_0) = (\lambda(\sigma) - \lambda(\nu_0))^\downarrow$. Moreover, if $i_0 \in \mathbb{I}_d$ is such that $\lambda_{i_0}(\sigma) = 0$ or $\lambda_{i_0}(\rho) = 0$ then $\lambda_{i_0}(\nu_0) = 0$.

Proof. Consider the unitary orbit $\mathcal{O}(\nu_0)$ endowed with the distance induced by the spectral norm. Notice that the unitary orbit $\mathcal{O}(\nu_0) \subset \mathcal{U}(\rho)$ and ν_0 is a local minimizer of Ψ_f restricted to $\mathcal{O}(\nu_0)$. Hence, by Theorem 2.5 we get that there exists an onb $\{v_i\}_{i\in\mathbb{I}_d}$ of $\mathbb{C}^d$ with the desired properties.

Assume that $i_0 \in \mathbb{I}_d$ is such that $\lambda_{i_0}(\sigma) = 0$ and let $2 \le h = \min\{i \in \mathbb{I}_d \ :\ \lambda_i(\sigma) = 0\} \le i_0$. Assume that $\lambda_h(\nu_0) > 0$. Notice that in this case,

$$\sum_{\ell=1}^{h-1} \lambda_\ell(\sigma) - \lambda_\ell(\nu_0) = 1 - \sum_{\ell=1}^{h-1} \lambda_\ell(\nu_0) \ge \lambda_h(\nu_0) > 0\,.$$

Then, there exists $1 \le \ell \le h-1$ such that $\lambda_\ell(\sigma) - \lambda_\ell(\nu_0) > 0$. For $\varepsilon > 0$ we consider

$$\tilde\nu(\varepsilon) = \sum_{i\in\mathbb{I}_d\setminus\{\ell,h\}} \lambda_i(\nu_0)\, v_i \otimes v_i + (\lambda_\ell(\nu_0) + \varepsilon)\, v_\ell \otimes v_\ell + (\lambda_h(\nu_0) - \varepsilon)\, v_h \otimes v_h \,.$$

By construction $\lim_{\varepsilon\to 0^+} \tilde\nu(\varepsilon) = \nu_0$; moreover,

$$\lambda(\tilde\nu(\varepsilon)) = ((\lambda_i(\nu_0))_{i\in\mathbb{I}_d\setminus\{\ell,h\}}\,,\ \lambda_\ell(\nu_0) + \varepsilon\,,\ \lambda_h(\nu_0) - \varepsilon)^\downarrow\,.$$

Notice that for sufficiently small $\varepsilon > 0$ we have that

$$(\lambda_\ell(\nu_0)\,,\ \lambda_h(\nu_0)) \prec (\lambda_\ell(\nu_0) + \varepsilon\,,\ \lambda_h(\nu_0) - \varepsilon) \in \mathbb{R}^2_{\ge 0}\,.$$

Hence, for sufficiently small $\varepsilon > 0$ we get that $\tilde\nu(\varepsilon) \in \mathcal{M}_d(\mathbb{C})^+$ and $\lambda(\nu_0) \prec \lambda(\tilde\nu(\varepsilon))$; in particular, $\tilde\nu(\varepsilon) \in \mathcal{U}(\rho)$. On the other hand,

$$\lambda(\sigma - \tilde\nu(\varepsilon)) = ((\lambda_i(\sigma) - \lambda_i(\nu_0))_{i\in\mathbb{I}_d\setminus\{\ell,h\}}\,,\ \lambda_\ell(\sigma) - \lambda_\ell(\nu_0) - \varepsilon\,,\ \lambda_h(\sigma) - \lambda_h(\nu_0) + \varepsilon)^\downarrow\,. \tag{11}$$

Notice that for sufficiently small $\varepsilon > 0$ we have that

$$(\lambda_\ell(\sigma) - \lambda_\ell(\nu_0) - \varepsilon\,,\ \lambda_h(\sigma) - \lambda_h(\nu_0) + \varepsilon) \prec (\lambda_\ell(\sigma) - \lambda_\ell(\nu_0)\,,\ \lambda_h(\sigma) - \lambda_h(\nu_0))$$

with strict majorization (where we have used that $\lambda_\ell(\sigma) - \lambda_\ell(\nu_0) > 0$ and that $\lambda_h(\sigma) - \lambda_h(\nu_0) = -\lambda_h(\nu_0) < 0$, since $\lambda_h(\sigma) = 0$). This last fact together with Eq. (11) show that $\lambda(\sigma - \tilde\nu(\varepsilon)) \prec \lambda(\sigma - \nu_0)$ strictly, for sufficiently small $\varepsilon > 0$; in turn, this last strict majorization relation implies that $\Psi_f(\tilde\nu(\varepsilon)) < \Psi_f(\nu_0)$, for sufficiently small $\varepsilon > 0$, that contradicts our assumption that ν_0 is a local minimizer of Ψ_f in $\mathcal{U}(\rho)$. Hence, we conclude that $\lambda_h(\nu_0) = 0$ and in particular, $\lambda_{i_0}(\nu_0) = 0$ (since $h \le i_0$). On the other hand, the majorization relation $\rho \prec \nu_0$ implies that if $\lambda_{i_0}(\rho) = 0$ then $\lambda_{i_0}(\nu_0) = 0$. □

Remark 4.12. Let $\rho, \sigma \in \mathcal{D}(d)$ be such that $\rho \not\prec \sigma$. Let $\nu_0 \in \mathcal{U}(\rho)$ be a local minimizer of Ψ_f on $\mathcal{U}(\rho)$. By Theorem 4.11 we get that there exists an onb $\{v_i\}_{i\in\mathbb{I}_d}$ of $\mathbb{C}^d$ for which Eq. (10) holds. In this case we set

$$1 \leq m = m_{\nu_0} := \max\{i \in \mathbb{I}_d \ : \ \lambda_i(\nu_0) > 0\} \leq d\,.$$

By Theorem 4.11 we get that $\lambda_m(\sigma)$, $\lambda_m(\rho) > 0$ and

$$\lambda(\sigma - \nu_0)^\uparrow = (\lambda(\sigma) - \lambda(\nu_0))^\uparrow = ((\lambda_\ell(\sigma) - \lambda_\ell(\nu_0))_{\ell\in\mathbb{I}_m}, (\lambda_\ell(\sigma))_{\ell=m+1}^d)^\uparrow\,.$$

Notice that there is an abuse of notation above, in case $m = d$. In what follows we consider

$$(\lambda_\ell(\sigma) - \lambda_\ell(\nu_0))^\uparrow_{\ell\in\mathbb{I}_m} = (c_1\,\mathbb{1}_{s_1}, \ldots, c_p\,\mathbb{1}_{s_p}) \in (\mathbb{R}^m)^\uparrow$$

for some $c_1 < \ldots < c_p$ and $s_1 + \ldots + s_p = m$. We further introduce the set of indices

$$H_k = \{\ell \in \mathbb{I}_m \ : \ \lambda_\ell(\sigma) - \lambda_\ell(\nu_0) = c_k\} \quad \text{for} \quad k \in \mathbb{I}_p\,.$$

Notice that $\{H_k\}_{k\in\mathbb{I}_p}$ is a partition of $\mathbb{I}_m$. △

Proposition 4.13. *Let $\nu_0 \in \mathcal{U}(\rho)$ be a local minimizer of Ψ_f on $\mathcal{U}(\rho)$. With the notation of Remark 4.12 there exist $h_0 := 0 < h_1 < \ldots < h_{p-1} < h_p = m$ such that*

$$H_k = \{\ell \in \mathbb{I}_m \ : \ h_{k-1} + 1 \leq \ell \leq h_k\} \quad \textit{for} \quad k \in \mathbb{I}_p\,.$$

Hence,

1. $(\lambda_\ell(\sigma) - \lambda_\ell(\nu_0))_{\ell\in\mathbb{I}_m} = (c_1\,\mathbb{1}_{s_1}, \ldots, c_p\,\mathbb{1}_{s_p}) \in (\mathbb{R}^m)^\uparrow$;
2. $0_m \prec_w -(c_1\,\mathbb{1}_{s_1}, \ldots, c_p\,\mathbb{1}_{s_p}) \in (\mathbb{R}^m)^\downarrow$;
3. $\lambda(\sigma) \prec \lambda(\sigma) - (c_1\,\mathbb{1}_{s_1}, \ldots, c_p\,\mathbb{1}_{s_p}, (\lambda_\ell(\sigma))_{\ell=m+1}^d) = \lambda(\nu_0)$.

Proof. In case $p = 1$ we just set $i_1 = m$ and the result follows. Thus, we assume that $p \geq 2$. We first prove the identity above for $k = 1$. Hence, we assume that $\ell \in H_1$ and $h \in \mathbb{I}_m \setminus H_1$; assume further $h < \ell$ so that $0 < \lambda_\ell(\nu_0) \leq \lambda_h(\nu_0)$ (and we reach a contradiction). By assumption we have that

$$\lambda_\ell(\sigma) - \lambda_\ell(\nu_0) = c_1 \quad \text{and} \quad \lambda_h(\sigma) - \lambda_h(\nu_0) = c_k \quad \text{for some} \quad 2 \leq k \leq p\,.$$

For $\varepsilon > 0$ we let

$$\tilde{\nu}(\varepsilon) = \sum_{i\in\mathbb{I}_d\setminus\{h\,,\ell\}} \lambda_i(\nu_0)\, v_i \otimes v_i + (\lambda_h(\nu_0) + \varepsilon)\, v_h \otimes v_h + (\lambda_\ell(\nu_0) - \varepsilon)\, v_\ell \otimes v_\ell\,.$$

By construction, $\lim_{\varepsilon\to 0^+} \tilde{\nu}(\varepsilon) = \nu_0$. On the other hand, notice that

$$\lambda(\tilde{\nu}(\varepsilon)) = ((\lambda_i(\nu_0))_{i\in\mathbb{I}_d\setminus\{h\,,\ell\}}\,, (\lambda_h(\nu_0) + \varepsilon), (\lambda_\ell(\nu_0) - \varepsilon))^\downarrow\,.$$

Notice that for sufficiently small $\varepsilon > 0$ we have that

$$(\lambda_h(\nu_0)\,, \lambda_\ell(\nu_0)) \prec ((\lambda_h(\nu_0) + \varepsilon), (\lambda_\ell(\nu_0) - \varepsilon)) \in \mathbb{R}^2_{\geq 0}\,.$$

Hence, we see that $\tilde{\nu}(\varepsilon) \succ \nu_0 \succ \rho$ and $\tilde{\nu}(\varepsilon) \in \mathcal{M}_d(\mathbb{C})^+$ so that $\tilde{\nu}(\varepsilon) \in \mathcal{U}(\rho)$, for sufficiently small $\varepsilon > 0$. Furthermore,

$$\lambda(\sigma - \tilde{\nu}(\varepsilon)) = ((\lambda_i(\sigma - \nu_0))_{i\in\mathbb{I}_d\setminus\{h\,,\ell\}}\,, c_1 + \varepsilon\,, c_k - \varepsilon)^\downarrow\,.$$

Since $(c_1 + \varepsilon\,, c_k - \varepsilon) \prec (c_1\,, c_k)$ strictly (recall that $c_1 < c_k$) we now see that $\lambda(\sigma - \tilde{\nu}(\varepsilon)) \prec \lambda(\sigma - \nu_0)$ strictly so that $\Psi_f(\tilde{\nu}(\varepsilon)) < \Psi_f(\nu_0)$, for sufficiently small $\varepsilon > 0$. This last fact contradicts our assumption that ν_0 is a local minimizer of $\Psi_f(\cdot)$.

Therefore, we now see that $\ell < h$, for every $\ell \in J_1$ and $h \in \mathbb{I}_m \setminus J_1$. This last claim implies that there exists $h_1 = \max H_1$ such that $H_1 = \{\ell \in \mathbb{I}_m \ : \ h_0 + 1 = 1 \le \ell \le h_1\}$.

We can now consider $\ell \in H_2$ and $h \in \mathbb{I}_m \setminus (J_1 \cup J_2)$; following an argument analogous to that considered above, we conclude that $\ell < h$. Thus, there exists $h_2 = \max H_2$ such that $H_2 = \{\ell \in \mathbb{I}_m \ : \ h_1 + 1 \le \ell \le h_2\}$. The result follows after applying this argument p times.

Notice that item 1. follows from the previous facts. In particular, we get that

$$\lambda(\sigma - \nu_0) = \lambda(\sigma) - \lambda(\nu_0) = (c_1 \mathbb{1}_{s_1}, \ldots, c_p \mathbb{1}_{s_p}, \lambda_{m+1}(\sigma), \ldots, \lambda_d(\sigma)) \,, \tag{12}$$

where $s_i = h_i - h_{i-1}$ for $i \in \mathbb{I}_p$. Since $\lambda(\sigma) \in (\mathbb{R}^d_{\ge 0})^\downarrow$ and $\operatorname{tr}(\sigma - \rho) = 0$ we see that

$$\operatorname{tr}(c_1 \mathbb{1}_{s_1}, \ldots, c_p \mathbb{1}_{s_p}) \le 0 \implies 0_m \prec_w -(c_1 \mathbb{1}_{s_1}, \ldots, c_p \mathbb{1}_{s_p}) \in (\mathbb{R}^m)^\downarrow \,.$$

These last facts show item 2. Item 3 is a straightforward consequence of item 1. (see also Eq. (12)) and item 2. □

Proposition 4.14. *Let $\nu_0 \in \mathcal{U}(\rho)$ be a local minimizer of Ψ_f on $\mathcal{U}(\rho)$. Consider the notation in Remark 4.12. Let $h_0 = 0 < h_1 < \ldots < h_{p-1} < h_p = m$ be as in Proposition 4.13. Then,*

$$\sum_{j=1}^{h_k} \lambda_j(\nu_0) = \sum_{j=1}^{h_k} \lambda_j(\rho) \quad \text{for} \quad k \in \mathbb{I}_{p-1} \quad \text{and} \quad \sum_{j=1}^{h_p} \lambda_j(\nu_0) = \sum_{j=1}^{d} \lambda_j(\rho) = 1 \,.$$

Proof. The result is clearly true when $p = 1$; hence, we assume that $p \ge 2$. We first prove the case when $k = 1$. Indeed, assume that

$$\sum_{j=1}^{h_1} \lambda_j(\rho) < \sum_{j=1}^{h_1} \lambda_j(\nu_0) \,,$$

where we have used that $\rho \prec \nu_0$ so in case the equality in the statement fails, we should have the strict inequality above. Set

$$\mathcal{M} = \{h_0 + 1 = 1 \le t \le \ h_1 : \ \sum_{j=1}^{t} \lambda_j(\nu_0) = \sum_{j=1}^{t} \lambda_j(\rho)\} \,.$$

If $\mathcal{M} \ne \emptyset$ we set $r := \max \mathcal{M}$; otherwise we set $r := 0$; notice that $r < h_1$. We also set

$$s = \min\{h_1 \le t \le d \ : \ \sum_{j=1}^{t} \lambda_j(\nu_0) = \sum_{j=1}^{t} \lambda_j(\rho)\} > h_1 \,.$$

Using that $\rho \prec \nu_0$ and the definitions of r and s we conclude that

1. $(\lambda_\ell(\rho))_{\ell=1}^r \prec (\lambda_\ell(\nu_0))_{\ell=1}^r$;
2. $(\lambda_\ell(\rho))_{\ell=r+1}^s \prec (\lambda_\ell(\nu_0))_{\ell=r+1}^s$;
3. $(\lambda_\ell(\rho))_{\ell=s+1}^d \prec (\lambda_\ell(\nu_0))_{\ell=s+1}^d$.

Notice that item 1. only applies when $r \ge 1$; similarly, item 3. only applies when $s \le d - 1$. On the other hand item 2. always applies since $r + 1 \le h_1 < s$. Further, we get that

$$\sum_{\ell=r+1}^{t} \lambda_\ell(\rho) < \sum_{\ell=r+1}^{t} \lambda_\ell(\nu_0) \quad \text{for} \quad r + 1 \le t \le s - 1 \,.$$

This last fact shows that for sufficiently small $\varepsilon > 0$ we have that

$$(\lambda_\ell(\rho))_{\ell=r+1}^s \prec (\lambda_{r+1}(\nu_0) - \varepsilon\,,\ \lambda_{r+2}(\nu_0)\,,\ \ldots\,,\ \lambda_{s-1}(\nu_0)\,,\ \lambda_s(\nu_0) + \varepsilon) \in \mathbb{R}_{\geq 0}^{s-r}\,, \tag{13}$$

where we have used that $\lambda_{r+1}(\nu_0) > 0$, since $r + 1 \leq h_1 \leq m$. Hence, for $\varepsilon > 0$ we define

$$\tilde{\nu}(\varepsilon) = \sum_{i \in \mathbb{I}_d} \lambda_i(\nu_0)\ v_i \otimes v_i + (\lambda_{r+1}(\nu_0) - \varepsilon)\ v_{r+1} \otimes v_{r+1} + (\lambda_s(\nu_0) + \varepsilon)\ v_s \otimes v_s\,.$$

By construction, $\lim_{\varepsilon \to 0^+} \tilde{\nu}(\varepsilon) = \nu_0$. Also, using items 1. and 3. above together with Eq. (13) we get that $\rho \prec \tilde{\nu}(\varepsilon)$ and $\tilde{\nu}(\varepsilon) \in \mathcal{M}_d(\mathbb{C})^+$ (i.e. $\tilde{\nu}(\varepsilon) \in \mathcal{U}(\rho)$) for sufficiently small $\varepsilon > 0$. Moreover, by construction

$$\lambda(\sigma - \tilde{\nu}(\varepsilon)) = (\,(\lambda_i(\sigma - \nu_0))_{i \in \mathbb{I}_d \setminus \{r+1\,,\,s\}}\,,\ c_1 + \varepsilon, c_n - \varepsilon)$$

for some $2 \leq n \leq p$, since $r + 1 \leq h_1 < s$. Since $(c_1 + \varepsilon, c_n - \varepsilon) \prec (c_1\,,\ c_n)$ strictly (recall that $c_1 < c_n$) we conclude that $\sigma - \tilde{\nu}(\varepsilon) \prec \sigma - \nu_0$ and that

$$\Psi_f(\nu(\varepsilon)) < \Psi_f(\nu_0) \quad \text{ for sufficiently small } \quad \varepsilon > 0\,.$$

This last fact contradicts our assumption that ν_0 is a local minimizer of $\Psi_f(\cdot)$. Hence, we see that

$$\sum_{j=1}^{h_1} \lambda_j(\nu_0) = \sum_{j=1}^{h_1} \lambda_j(\rho)\,,$$

and the result is established for $k = 1$. In case $2 < p$ and we assume that

$$\sum_{j=1}^{h_2} \lambda_j(\nu_0) < \sum_{j=1}^{h_2} \lambda_j(\rho)$$

then we argue as above: we consider $r = \max\{h_1+1 \leq t \leq h_2\ :\ \sum_{j=1}^t \lambda_j(\nu_0) = \sum_{j=1}^t \lambda_j(\rho)\}$ in case the set is not empty or $r = h_1$ otherwise. Similarly, we set $s = \min\{h_2 \leq t \leq d\ :\ \sum_{j=1}^t \lambda_j(\nu_0) = \sum_{j=1}^t \lambda_j(\rho)\}$ and notice that by construction $r + 1 \leq h_2 < s$ and we can partition $\mathbb{I}_d$ in terms of r and s so that the majorization relations in items 1.-3. above hold. Then, we can repeat the rest of the argument and contradict that ν_0 is a local minimizer of $\Psi_f(\cdot)$. The result follows by applying this argument $p - 1$ times. Notice that the last claim in the statement (i.e. when $k = p$) follows from the definition of $h_p = m$. □

Proposition 4.15. *Let $\nu_0 \in \mathcal{U}(\rho)$ be a local minimizer of Ψ_f on $\mathcal{U}(\rho)$. Consider the notation in Remark 4.12. Let $h_0 = 0 < h_1 < \ldots < h_{p-1} < h_p = m$ be as in Proposition 4.13 and assume that $p \geq 2$. Then*

$$c_j = \frac{1}{h_j - h_{j-1}} \sum_{\ell=h_{j-1}+1}^{h_j} (\lambda_\ell(\sigma) - \lambda_\ell(\rho)) \quad \textit{for} \quad j \in \mathbb{I}_{p-1}$$

and

$$c_p = \frac{1}{h_p - h_{p-1}} \sum_{\ell=h_{p-1}+1}^{h_p} (\lambda_\ell(\sigma) - \lambda_\ell(\rho)) - \frac{1}{h_p - h_{p-1}} \sum_{\ell=h_p+1}^{d} \lambda_\ell(\rho)$$

Moreover, if $s_j = h_j - h_{j-1}$ for $j \in \mathbb{I}_p$ then we also get that

$$(\lambda_\ell(\rho))_{\ell=h_{j-1}+1}^{h_j} \prec (\lambda_\ell(\sigma) - c_j)_{\ell=h_{j-1}+1}^{h_j} \in (\mathbb{R}_{>0}^{s_j})^{\downarrow} \quad \textit{for} \quad j \in \mathbb{I}_{p-1}\,.$$

Proof. As a consequence of Proposition 4.14 we get that

$$\sum_{\ell=h_{j-1}+1}^{h_j} \lambda_\ell(\nu_0) = \sum_{\ell=h_{j-1}+1}^{h_j} \lambda_\ell(\rho) \quad \text{for} \quad j \in \mathbb{I}_{p-1}\,. \tag{14}$$

These last identities together with the majorization relation $\lambda(\rho) \prec \lambda(\nu_0)$ imply that

$$(\lambda_\ell(\rho))_{\ell=h_{j-1}+1}^{h_j} \prec (\lambda_\ell(\nu_0))_{\ell=h_{j-1}+1}^{h_j}\,. \tag{15}$$

On the other, by construction (see Remark 4.12) we have that, for $j \in \mathbb{I}_{p-1}$

$$\lambda_\ell(\sigma) - \lambda_\ell(\nu_0) = c_j \quad \text{for} \quad h_{j-1} + 1 \le \ell \le h_j\,. \tag{16}$$

Thus, Eq. (14) implies that

$$c_j = \frac{1}{h_j - h_{j-1}} \sum_{\ell=h_{j-1}+1}^{h_j} \lambda_\ell(\sigma) - \lambda_\ell(\nu_0) = \frac{1}{h_j - h_{j-1}} \sum_{\ell=h_{j-1}+1}^{h_j} \lambda_\ell(\sigma) - \lambda_\ell(\rho)\,.$$

The formula for c_p is obtained analogously. Finally, by Eqs. (15) and (16) we see that

$$(\lambda_\ell(\rho))_{\ell=h_{j-1}+1}^{h_j} \prec (\lambda_\ell(\nu_0))_{\ell=h_{j-1}+1}^{h_j} = (\lambda_\ell(\sigma) - c_j)_{\ell=h_{j-1}+1}^{h_j} \in (\mathbb{R}_{>0}^{s_j})^{\downarrow}\,.$$

□

Proposition 4.16. *Let $\nu_0 \in \mathcal{U}(\rho)$ be a local minimizer of Ψ_f on $\mathcal{U}(\rho)$. Consider the notation in Remark 4.12. Let $h_0 = 0 < h_1 < \ldots < h_{p-1} < h_p = m$ be as in Proposition 4.13. Then*

1. *$c_p < \lambda_m(\sigma)$.*
2. *If we let $n := \max\{\ell \in \mathbb{I}_d \; : \; \lambda_\ell(\rho) > 0\}$ then $n \ge m$.*
3. *If we assume that $n > m$ then $\lambda_{m+1}(\sigma) \le c_p$.*
4. *$c_p = \max\{\lambda_\ell(\sigma) - \lambda_\ell(\nu_0) \; : \; \ell \in \mathbb{I}_n\}$ and $m = h_p = \max\{h_{p-1} + 1 \le \ell \le n \; : \; c_p < \lambda_\ell(\sigma)\}$.*
5. *$(\lambda_\ell(\rho))_{\ell=h_{p-1}+1}^{n} \prec (\,(\lambda_\ell(\sigma) - c_p)^+\,)_{\ell=h_{p-1}+1}^{n}$.*
6. *c_p is the unique solution $x \in \mathbb{R}$ of the equation $\sum_{\ell=h_{p-1}+1}^{n} (\lambda_\ell(\sigma) - x)^+ = \sum_{\ell=h_{p-1}+1}^{n} \lambda_\ell(\rho) > 0$.*

Proof. In order to prove item 1. recall that $c_p = \lambda_m(\sigma) - \lambda_m(\nu_0) < \lambda_m(\sigma)$, since $\lambda_m(\nu_0) > 0$ (by definition of m).

Item 2. is a straightforward consequence of Theorem 4.11.

In order to prove item 3., assume that $n > m$ and that $c_p < \lambda_{m+1}(\sigma)$. Since $n > m$ then $\lambda_{m+1}(\rho) > 0$. In particular,

$$\sum_{\ell=1}^{m} \lambda_\ell(\rho) \le 1 - \lambda_{m+1}(\rho) < 1\,.$$

Let $0 < \varepsilon$ be such that

$$0 < \varepsilon \le \min\Big\{\frac{1}{2}\lambda_m(\nu_0),\ \frac{\lambda_{m+1}(\sigma) - c_p}{2},\ 1 - \sum_{\ell=1}^{m} \lambda_\ell(\rho)\Big\}$$

and consider

$$\tilde{\nu}(\varepsilon) = \sum_{i\in\mathbb{I}_d\setminus\{m\,,m+1\}} \lambda_i(\nu_0)\ v_i \otimes v_i + (\lambda_m(\nu_0) - \varepsilon)\, v_m \otimes v_m + \varepsilon\, v_{m+1} \otimes v_{m+1}\,.$$

Hence, the assumptions about ε give that $\lambda(\tilde{\nu}(\varepsilon)) = (\lambda_1(\nu_0), \ldots, \lambda_{m-1}(\nu_0), \lambda_m(\nu_0) - \varepsilon, \varepsilon, 0, \ldots, 0)$ so that $\tilde{\nu}(\varepsilon_1) \in \mathcal{M}_d(\mathbb{C})^+$. Moreover, it is clear that we have also that $\rho \prec \tilde{\nu}(\varepsilon)$; indeed, it follows from the fact that $\rho \prec \nu_0$ and that $\varepsilon \le 1 - \sum_{\ell=1}^m \lambda_\ell(\rho)$. Hence $\tilde{\nu}(\varepsilon) \in \mathcal{U}(\rho)$.

Furthermore, we have that $(c_p + \varepsilon, \lambda_{m+1}(\sigma) - \varepsilon) \prec (c_p, \lambda_{m+1}(\sigma))$ with strict majorization; thus, the identity

$$\lambda(\sigma - \tilde{\nu}(\varepsilon)) = (\,(\lambda_\ell(\sigma - \nu_0))_{\ell\in\mathbb{I}_d\setminus\{m\,,m+1\}}\,,\ c_p + \varepsilon\,,\ \lambda_{m+1}(\sigma) - \varepsilon)$$

shows that $\sigma - \tilde{\nu}(\varepsilon) \prec \sigma - \nu_0$ strictly. As before, these facts contradict our assumption that ν_0 is a local minimizer of Ψ_f on $\mathcal{U}(\rho)$. Therefore $c_p \ge \lambda_{m+1}$, which establishes item 3.

To prove item 4., notice that if $m = n$ then $(\lambda_\ell(\sigma) - \lambda_\ell(\nu_0))_{\ell\in\mathbb{I}_n} = (c_1\,\mathbb{1}_{s_1}, \ldots, c_p\,\mathbb{1}_{s_p})$, with $c_1 < \ldots < c_p$ and the claim follows from these facts. On the other hand, if $n > m$ then by item 3. we get that $c_p \ge \lambda_{m+1}(\sigma) \ge \ldots \ge \lambda_d(\sigma)$. Then,

$$(\lambda_\ell(\sigma) - \lambda_\ell(\nu_0))_{\ell\in\mathbb{I}_n} = (c_1\,\mathbb{1}_{s_1}, \ldots, c_p\,\mathbb{1}_{s_p}\ ,\ (\lambda_j(\sigma))_{\ell=m+1}^n)$$

so we see that $c_p \ge \lambda_\ell(\sigma) - \lambda_\ell(\nu_0)$ for $\ell \in \mathbb{I}_n$. The second claim follows from a similar argument.

To show item 5. we first notice that

$$(\lambda_\ell(\rho))_{\ell=h_{p-1}+1}^n \prec (\lambda_\ell(\nu_0))_{\ell=h_{p-1}+1}^n$$

since $\rho \prec \nu_0$, $\lambda_{n+1}(\rho) = \lambda_{n+1}(\nu_0) = 0$ and hence $\sum_{\ell=h_{p-1}+1}^n \lambda_\ell(\rho) = \sum_{\ell=h_{p-1}+1}^m \lambda_\ell(\nu_0)$ (by Proposition 4.14). On the other hand, $\lambda_\ell(\nu_0) = \lambda_\ell(\sigma) - c_p = (\lambda_\ell(\sigma) - c_p)^+ > 0$ for $h_{p-1} + 1 \le \ell \le m \le n$, where we have used item 1. above. Moreover, by item 3. we get that in case $n > m$ then $\lambda_\ell(\nu_0) = 0 = (\lambda_\ell(\sigma) - c_p)^+$, for $m + 1 \le \ell \le n$. Thus, $(\lambda_\ell(\nu_0))_{\ell=h_{p-1}+1}^n = (\,(\lambda_\ell(\sigma) - c_p)^+\,)_{\ell=h_{p-1}+1}^n$ and item 5. follows from these remarks.

To show item 6. notice that by item 5. we get the equality of traces

$$0 < \sum_{\ell=h_{p-1}+1}^{n} \lambda_\ell(\rho) = \sum_{\ell=h_{p-1}+1}^{n} (\lambda_\ell(\sigma) - c_p)^+\,,$$

where we have used that $h_{p-1} + 1 \le m \le n$ and hence $\lambda_{h_{p-1}+1}(\rho) > 0$. On the other hand, it is straightforward to see that for every $y > 0$ there exists a unique $x \in \mathbb{R}$ such that

$$\Gamma(x) = \sum_{\ell=h_{p-1}+1}^{n} (\lambda_\ell(\sigma) - x)^+ = y\,.$$

Item 6. follows from these remarks. □

Proposition 4.17. *Let σ, $\rho \in \mathcal{D}(d)$ and let $n = \max\{\ell \in \mathbb{I}_d\ :\ \lambda_\ell(\rho) > 0\}$. Then, there exist a unique $p \in \mathbb{I}_d$ and indices $h_0 = 0 < h_1 < \ldots < h_{p-1} < n$ and constants $c_1 < \ldots < c_p$ such that for $k \in \mathbb{I}_{p-1}$*

$$(\lambda_\ell(\rho))_{\ell=h_{k-1}+1}^{h_k} \prec ((\lambda_\ell(\sigma) - c_k)^+)_{\ell=h_{k-1}+1}^{h_k} \quad \text{and} \quad (\lambda_\ell(\rho))_{\ell=h_{p-1}+1}^{n} \prec (\,(\lambda_\ell(\sigma) - c_p)^+\,)_{\ell=h_{p-1}+1}^n\,.$$

Proof. Propositions 4.15 and 4.16 show that there exist indices and constants as in the statement. Assume that there are other indices $h'_0 = 0 < h'_1 < \ldots < h'_{q-1} < n$ and constants $c'_1 < \ldots < c'_q$ for some $q \in \mathbb{I}_d$, such that for $k \in \mathbb{I}_{q-1}$

$$(\lambda_\ell(\rho))_{\ell=h'_{k-1}+1}^{h'_k} \prec (\,(\lambda_\ell(\sigma) - c'_k)^+)_{\ell=h'_{k-1}+1}^{h'_k} \quad \text{and} \quad (\lambda_\ell(\rho))_{\ell=h'_{q-1}+1}^{n} \prec (\,(\lambda_\ell(\sigma) - c'_q)^+\,)_{\ell=h'_{q-1}+1}^n\,.$$

Notice that the previous majorization relations determine uniquely the constants $c_1, \ldots, c_p$ and $c'_1, \ldots, c'_q$.

Assume that $q = 1$ and that $p \geq 2$. In this case we have that

$$\sum_{\ell=1}^{n} \lambda_\ell(\rho) = \sum_{\ell=1}^{n} (\lambda_\ell(\sigma) - c'_1)^+$$

and

$$\sum_{\ell=1}^{n} \lambda_\ell(\rho) = \sum_{j=1}^{p-1} \sum_{\ell=h_{j-1}+1}^{h_j} (\lambda_\ell(\sigma) - c_j)^+ + \sum_{\ell=h_{p-1}+1}^{n} (\lambda_\ell(\sigma) - c_p)^+ \,,$$

where we have used the majorization relations in the statement of the result in this last claim. Since $p \geq 2$, $c_1 < c_2$ and $\lambda(\sigma) \in (\mathbb{R}^d)^\downarrow$ we conclude that

$$\sum_{\ell=1}^{n} (\lambda_\ell(\sigma) - c_1)^+ > \sum_{j=1}^{p-1} \sum_{\ell=h_{j-1}+1}^{h_j} (\lambda_\ell(\sigma) - c_j)^+ + \sum_{\ell=h_{p-1}+1}^{n} (\lambda_\ell(\sigma) - c_p)^+$$

and then $\sum_{\ell=1}^{n} (\lambda_\ell(\sigma) - c_1)^+ > \sum_{\ell=1}^{n} \lambda_\ell(\rho)$. The previous facts imply that $c_1 < c'_1$. Then, we see that

$$\sum_{\ell=1}^{h_1} \lambda_\ell(\rho) = \sum_{\ell=1}^{h_1} (\lambda_\ell(\sigma) - c_1)^+ > \sum_{\ell=1}^{h_1} (\lambda_\ell(\sigma) - c'_1)^+ \geq \sum_{\ell=1}^{h_1} \lambda_\ell(\rho) \,.$$

The previous contradiction implies that we should have $p = 1$ in this case. Since the roles of p and q are symmetric we conclude that $q = 1$ if and only if $p = 1$. Furthermore, in this case we get that $c_1 = c'_1$.

Assume now that $q \geq 2$ so $p \geq 2$. Moreover, assume that $h_1 \neq h'_1$, say $h_1 < h'_1$. In this case there exists $1 \leq k \leq p-2$ such that $h_k < h'_1 \leq h_{k+1}$ or $h_{p-1} < h'_1 \leq n$.

In case $h_k < h'_1 \leq h_{k+1}$ for some $1 \leq k \leq p-2$ then

$$\sum_{\ell=1}^{h'_1} (\lambda_\ell(\sigma) - c'_1)^+ = \sum_{\ell=1}^{h'_1} \lambda_\ell(\rho) \leq \sum_{j=1}^{k} \sum_{\ell=h_{j-1}+1}^{h_j} (\lambda_\ell(\sigma) - c_j)^+ + \sum_{\ell=h_k+1}^{h'_1} (\lambda_\ell(\sigma) - c_{k+1})^+ < \sum_{\ell=1}^{h'_1} (\lambda_\ell(\sigma) - c_1)^+$$

where we have used that $1 \leq k$ and $c_1 < c_j$ for $2 \leq j \leq k+1$. Thus, we conclude that $c_1 < c'_1$; we now see that

$$\sum_{\ell=1}^{h_1} \lambda_\ell(\rho) = \sum_{\ell=1}^{h_1} (\lambda_\ell(\sigma) - c_1)^+ > \sum_{\ell=1}^{h_1} (\lambda_\ell(\sigma) - c'_1)^+ \geq \sum_{\ell=1}^{h_1} \lambda_\ell(\rho) \,,$$

where in the last inequality we have used that $h_1 < h'_1$. The previous contradiction implies that we should have $h_1 \geq h'_1$ in this case. By the symmetry of the roles of the indices and constants we conclude that $h_1 = h'_1$ and hence $c_1 = c'_1$.

In case $h_{p-1} < h'_1 \leq n$ we can reach a contradiction following an analogous argument. Thus, in this case we also get that $h_1 = h'_1$ and $c_1 = c'_1$.

In case $q = 2$ then we can argue as at the beginning of the proof and conclude that we should have $p = 2$; moreover, the previous facts imply that $c_2 = c'_2$ and we are done.

In case $q > 2$ then we get that $p > 2$; if we assume that $h_2 \neq h'_2$, say $h_2 < h'_2$ we can argue as before and reach a contradiction. Thus, we get that $h_2 = h'_2$ and then that $c_2 = c'_2$. In the general case, we argue as before q times. The result follows from these remarks. $\square$

Corollary 4.18. *Consider the notation of Proposition 4.16. Then, we have that*

1. *The spectral structure of $\lambda(\sigma-\nu_0)$ for local minimizers $\nu_0 \in \mathcal{U}(\rho)$ of Ψ_f is the same. In particular, $\nu_0 \in \mathcal{U}(\rho)$ is a global minimizer of Ψ_f.*

2. *The spectral structure of $\lambda(\sigma-\nu_0)$ for local minimizers $\nu_0 \in \mathcal{U}(\rho)$ of Ψ_f does not depend on the particular strictly Schur-convex function $f:\mathbb{R}^d \to \mathbb{R}$. In particular, $\lambda(\sigma-\nu_0) \prec \lambda(\sigma-\nu)$ for every $\nu \in \mathcal{U}(\rho)$.*

Proof. 1. Recall that since $\mathcal{U}(\sigma)$ is compact (with respect to the topology induced by the spectral norm) and $f:\mathbb{R}^d \to \mathbb{R}$ is a strictly Schur-convex function then Ψ_f attains its minimum value at some $\tilde{\nu} \in \mathcal{U}(\rho)$. Let $\nu_0 \in \mathcal{U}(\rho)$ be a local minimizer of Ψ_f. Then, with the notation of Remark 4.12 (notice that $\tilde{\nu} \in \mathcal{U}(\rho)$ is also a local minimizer of Ψ_f), Propositions 4.15, 4.16 and 4.17 show that

$$\lambda(\sigma)-\lambda(\tilde{\nu}) = (c_1\,\mathbb{1}_{s_1},\ldots,c_p\,\mathbb{1}_{s_p},(\lambda_\ell(\sigma))_{\ell=h_p+1}^d) = \lambda(\sigma)-\lambda(\nu_0)\,,$$

with $h_p=\max\{\ell \in \mathbb{I}_n \ : \ c_p<\lambda_\ell(\sigma)\}$, and $s_j=h_j-h_{j-1}$ for $j \in \mathbb{I}_p$. Indeed, Proposition 4.17 shows that the indices $0=h_0<h_1<\ldots<h_p \le n \le d$ as well as the constants $c_1<\ldots<c_p$ are unique. Moreover, by Theorem 4.11 $\lambda(\sigma-\tilde{\nu})=(\lambda(\sigma)-\lambda(\tilde{\nu}))^\downarrow$ and $\lambda(\sigma-\nu_0)=(\lambda(\sigma)-\lambda(\nu_0))^\downarrow$. These facts show that $\lambda(\sigma-\tilde{\nu})=\lambda(\sigma-\nu_0)$.

2. Notice that Proposition 4.17 also shows the spectral structure of $\lambda(\sigma-\nu_0)$ is uniquely determined by those of ρ and σ (and does not depend on the strictly Schur-convex function $f:\mathbb{R}^d \to \mathbb{R}$). In particular, if $\nu \in \mathcal{U}(\rho)$ then for every strictly Schur-convex function $g:\mathbb{R}^d \to \mathbb{R}$ we get that $g(\lambda(\sigma-\nu_0))=\Psi_g(\nu_0) \le \Psi_g(\nu)=g(\lambda(\sigma-\nu))$. It is well known that this last fact implies that $\lambda(\sigma-\nu_0) \prec \lambda(\sigma-\nu)$ (see [1, 3]). □

Proof of Theorem 3.4. Given a strictly convex unitarily invariant norm $N(\cdot)$ in $\mathcal{M}_d(\mathbb{C})$, consider its associated the strictly Schur-convex gauge symmetric function g_N. Set $f=g_N$ and notice that in this case the functions Ψ_N and Ψ_f coincide. Since $\mathcal{U}(\rho)$ is compact and Ψ_N is a continuous function then Ψ_N attains its minimum value at some $\tilde{\nu} \in \mathcal{U}(\rho)$

By the previous remarks we can apply our previous results in this setting; in particular, using the notation of Remark 4.12 we can define

$$c^{\rm op} := (c_1\,\mathbb{1}_{s_1},\ldots,c_p\,\mathbb{1}_{s_p},\lambda_{h_p+1}(\sigma),\ldots,\lambda_d(\sigma))\,.$$

Arguing as in the proof of Corollary 4.18 we get that $c^{\rm op}=\lambda(\sigma)-\lambda(\tilde{\nu})=\lambda(\sigma-\tilde{\nu})$.

5. $\implies$ 4. It is clear that item 5. implies the first part of item 4. As a consequence of item 2. in Corollary 4.18 we conclude that $c^{\rm op} \prec \lambda(\sigma-\nu)$ for every $\nu \in \mathcal{U}(\rho)$; notice that the second claim in 4. follows from this last fact.

It is clear that 4. $\implies$ 3. $\implies$ 2. $\implies$ 1. If we assume item 1. then arguing as in the proof of Corollary 4.18 we see that $\lambda(\sigma)-\lambda(\nu_0)=c^{\rm op}$ and hence $\lambda(\nu_0)=\lambda(\sigma)-c^{\rm op}$. This last fact together with Theorem 4.11 prove the representation in item 5. with respect to some onb of $\mathbb{C}^d$.

Also notice that as a consequence of item 3. in Proposition 4.13 we get that $\sigma \prec \nu_0$. The remaining part of Theorem 3.4, namely that $c^{\rm op}$ can be computed in terms of a simple algorithm with input $\lambda(\rho),\,\lambda(\sigma) \in \mathbb{R}^d$, will be a consequence of Theorem 4.21 in the next section. □

4.4 Algorithmic construction of $c^{\rm op}$

The first part of Theorem 3.4 shows the fundamental role played by the vector $c^{\rm op}$. Next, we develop a finite step (and simple) algorithm to construct $c^{\rm op}$. Once this vector is computed then we can describe all $\nu \in \mathcal{U}(\rho)$ such that $\lambda(\sigma-\nu)=(c^{\rm op})^\downarrow$ in terms of all possible spectral representations of σ (see Theorem 3.4).

Algorithm 4.19. Given $\rho,\sigma \in \mathcal{D}(d)$ such that $\rho \not\prec \sigma$, consider the input $\lambda(\rho),\,\lambda(\sigma) \in (\mathbb{R}^d)^\downarrow$. Let $n := \max\{\ell \in \mathbb{I}_d \ : \ \lambda_\ell(\rho)>0\}$. We first define $h_0 := 0$. Assuming that we have defined $0 \le h_{j-1} < n$ then:

1. For $h_{j-1}+1 \le k \le n$ we define g_k which is the unique solution $x \in \mathbb{R}$ of the equation

$$\sum_{\ell=h_{j-1}+1}^{k} (\lambda_\ell(\sigma) - x)^+ = \sum_{\ell=h_{j-1}+1}^{k} \lambda_\ell(\rho) > 0\,.$$

2. We set

$$h_j = \max\left\{ h_{j-1}+1 \le k \le n \;:\; (\lambda_\ell(\rho))_{\ell=h_{j-1}+1}^{k} \prec ((\lambda_\ell(\sigma) - g_k)^+)_{\ell=h_{j-1}+1}^{k} \right\}\,.$$

After a finite number $p-1 \in \mathbb{I}_d$ of iterations we obtain the output: $h_0 = 0 < h_1 < \ldots < h_{p-1} < n$ and the constants $c_1, \ldots, c_p$ that are uniquely determined by the equations: for $j \in \mathbb{I}_{p-1}$

$$\sum_{\ell=h_{j-1}+1}^{h_j} (\lambda_\ell(\sigma) - c_j)^+ = \sum_{\ell=h_{j-1}+1}^{h_j} \lambda_\ell(\rho) \quad \text{and} \quad \sum_{\ell=h_{p-1}+1}^{n} (\lambda_\ell(\sigma) - c_p)^+ = \sum_{\ell=h_{p-1}+1}^{n} \lambda_\ell(\rho)\,.$$

Finally we reset the value of $h_p = n$ to $h_p := \max\{h_{p-1}+1 \le \ell \le n \;:\; \lambda_\ell(\sigma) - c_p > 0\}$. △

Proposition 4.20. *Consider the output of Algorithm 4.19: $h_0 = 0 < h_1 < \ldots < h_{p-1} < h_p \le n \le d$ and the constants $c_1, \ldots, c_p$. If $p \ge 2$ then $c_1 < c_2 < \ldots < c_p$.*

Proof. We assume that $p \ge 2$. Let $1 \le j < j+1 \le p-1$ and assume that $c_j \ge c_{j+1}$. Notice that

$$\sum_{\ell=h_{j-1}+1}^{h_{j+1}} (\lambda_\ell(\sigma) - c_j)^+ \le \sum_{\ell=h_{j-1}+1}^{h_j} (\lambda_\ell(\sigma) - c_j)^+ + \sum_{\ell=h_j+1}^{h_{j+1}} (\lambda_\ell(\sigma) - c_{j+1})^+ = \sum_{\ell=h_{j-1}+1}^{h_{j+1}} \lambda_\ell(\rho)\,.$$

Similarly, $\sum_{\ell=h_{j-1}+1}^{h_{j+1}} (\lambda_\ell(\sigma) - c_{j+1})^+ \ge \sum_{\ell=h_{j-1}+1}^{h_{j+1}} \lambda_\ell(\rho)$. Hence, if $\tilde{c}$ is determined by the equation

$$\sum_{\ell=h_{j-1}+1}^{h_{j+1}} (\lambda_\ell(\sigma) - \tilde{c})^+ = \sum_{\ell=h_{j-1}+1}^{h_{j+1}} \lambda_\ell(\rho) \tag{17}$$

then the previous facts show that $c_j \ge \tilde{c} \ge c_{j+1}$. We claim that

$$(\lambda_\ell(\rho))_{\ell=h_{j-1}+1}^{h_{j+1}} \prec ((\lambda_\ell(\sigma) - \tilde{c})^+)_{\ell=h_{j-1}+1}^{h_{j+1}}\,. \tag{18}$$

Indeed, notice that both vectors above have their entries arranged in non-increasing order. If we let $h_{j-1}+1 \le k \le h_j$ then

$$\sum_{\ell=h_{j-1}+1}^{k} (\lambda_\ell(\sigma) - \tilde{c})^+ \ge \sum_{\ell=h_{j-1}+1}^{k} (\lambda_\ell(\sigma) - c_j)^+ \ge \sum_{\ell=h_{j-1}+1}^{k} \lambda_\ell(\rho)\,. \tag{19}$$

If we now consider $h_j + 1 \le k \le h_{j+1}$ then notice that

$$\sum_{\ell=k}^{h_{j+1}} \lambda_\ell(\rho) \ge \sum_{\ell=k}^{h_{j+1}} (\lambda_\ell(\sigma) - c_{j+1})^+ \ge \sum_{\ell=k}^{h_{j+1}} (\lambda_\ell(\sigma) - \tilde{c})^+\,. \tag{20}$$

We remark that Eqs. (17), (19) and (20) imply the majorization relation in Eq. (18). This last fact contradicts the definition of h_j (see Algorithm 4.19). Therefore, we now see that $c_j < c_{j+1}$, for $1 \le j < j+1 \le p-1$.

In case $j = p-1$ then an argument analogous to that above also shows that $c_{p-1} < c_p$.

□

Theorem 4.21. *Consider the output of Algorithm 4.19: $h_0 = 0 < h_1 < \ldots < h_{p-1} < h_p \leq n$ and the constants $c_1 < \ldots < c_p$, where $n = \max\{\ell \in \mathbb{I}_d \ : \ \lambda_\ell(\rho) > 0\}$. Set $s_j = h_j - h_{j-1}$, for $j \in \mathbb{I}_p$. If $c^{\rm op} \in \mathbb{R}^d$ is as in Theorem 3.4 we have that*

$$c^{\rm op} = (c_1 \, \mathbb{1}_{s_1}, \ldots, c_p \, \mathbb{1}_{s_p}, (\lambda_\ell(\sigma))_{\ell=h_p+1}^d) \, .$$

Proof. Let us denote $\tilde c = (c_1 \, \mathbb{1}_{s_1}, \ldots, c_p \, \mathbb{1}_{s_p}, (\lambda_\ell(\sigma))_{\ell=h_p+1}^d) \in \mathbb{R}^d$, where $h_0 = 0 < h_1 < \ldots < h_{p-1} < h_p \leq n$ and the constants $c_1 < \ldots < c_p$ are the output of Algorithm 4.19 and $s_j = h_j - h_{j-1}$, for $j \in \mathbb{I}_p$. By construction (see Algorithm 4.19) we see that for $k \in \mathbb{I}_{p-1}$

$$(\lambda_\ell(\rho))_{\ell=h_{k-1}+1}^{h_k} \prec (\,(\lambda_\ell(\sigma) - c_k)^+)_{\ell=h_{k-1}+1}^{h_k} \ , \ (\lambda_\ell(\rho))_{\ell=h_{p-1}+1}^{n} \prec (\,(\lambda_\ell(\sigma) - c_p)^+)_{\ell=h_{p-1}+1}^{n} \, .$$

Let $f : \mathbb{R}^d \to \mathbb{R}$ be a strictly Schur-convex function and let $\nu_0 \in \mathcal{U}(\rho)$ be a minimizer of Ψ_f on $\mathcal{U}(\rho)$. Then (see the proof of Theorem 3.4) we get that $c^{\rm op} = \lambda(\sigma) - \lambda(\nu_0) \in \mathbb{R}^d$. In this case (see Remark 4.12 and Proposition 4.13) there exist constants $c'_1 < \ldots < c'_q$ and indices $h'_0 = 0 < h'_1 < \ldots < h'_q \leq n$, for some $q \in \mathbb{I}_d$, such that $c^{\rm op} = (c'_1 \, \mathbb{1}_{s'_1}, \ldots, c'_q \, \mathbb{1}_{s'_q}, (\lambda(\sigma))_{\ell=h'_q+1}^d)$, where $s'_k = h'_k - h'_{k-1}$ for $k \in \mathbb{I}_q$. By Propositions 4.15 and 4.16 we see that, for $k \in \mathbb{I}_{q-1}$

$$(\lambda_\ell(\rho))_{\ell=h'_{k-1}+1}^{h'_k} \prec (\,(\lambda_\ell(\sigma) - c'_k)^+)_{\ell=h'_{k-1}+1}^{h'_k} \ , \ (\lambda_\ell(\rho))_{\ell=h'_{q-1}+1}^{n} \prec (\,(\lambda_\ell(\sigma) - c'_q)^+)_{\ell=h'_{q-1}+1}^{n} \, .$$

By Proposition 4.17 we now see that $q = p$, $h_j = h'_j$ and $c_j = c'_j$, for $j \in \mathbb{I}_p$. In particular $s_j = s'_j$, for $j \in \mathbb{I}_p$; hence $\tilde c = c^{\rm op} \in \mathbb{R}^d$. □

5 Appendix

We now recall the statement of a result included in Section 2 and present its proof. We remark that we consider $\mathcal{O}(\sigma)$ endowed with the metric induced by the spectral norm.

Theorem 2.3. *Let $\rho, \sigma \in \mathcal{D}(d)$ and let $f : \mathbb{R}^d \to \mathbb{R}$ be a strictly Schur-convex function. Let $\Theta_f : \mathcal{O}(\sigma) \to \mathbb{R}$ be given by $\Theta_f(\tilde\sigma) = f(\lambda(\tilde\sigma - \rho))$. If $\sigma_0 \in \mathcal{O}(\sigma)$ is a local minimizer of Θ_f then there exists an onb $\{u_i\}_{i\in\mathbb{I}_d}$ of $\mathbb{C}^d$ such that*

$$\rho = \sum_{i\in\mathbb{I}_d} \lambda_i(\rho) \ u_i \otimes u_i \quad \textit{and} \quad \sigma_0 = \sum_{i\in\mathbb{I}_d} \lambda_i(\sigma_0) \ u_i \otimes u_i \, .$$

In particular, $\lambda(\sigma_0 - \rho) = (\lambda(\sigma_0) - \lambda(\rho))^\downarrow$ so σ_0 is a global minimizer of Θ_f on $\mathcal{O}(\sigma)$.

Proof. The proof is essentially the same as the proof of [28, Theorem 3.5] so we sketch it. Let $\sigma_0 \in \mathcal{O}(\sigma)$ be a local minimizer of Θ_f. We let $\Gamma : \mathcal{U}_d \times \mathcal{U}_d \to \mathcal{H}(d)_0 := \{\eta \in \mathcal{H}(d) \ : \ \operatorname{tr}(\eta) = 0\}$ be given by $\Gamma(\omega, \xi) = \omega^* \sigma_0 \omega - \xi^* \rho \, \xi$. We endow $\mathcal{U}_d \times \mathcal{U}_d$ with the (product) metric $d((\omega, \xi), (\tilde\omega, \tilde\xi)) = \max\{\|\omega - \tilde\omega\|, \|\xi - \tilde\xi\|\}$, where $\|\cdot\|$ denotes the spectral norm. We further consider the function $\Delta : \mathcal{U}_d \times \mathcal{U}_d \to \mathbb{R}$ given by $\Delta(\omega, \xi) = f(\lambda(\Gamma(\omega, \xi)))$. Arguing as in the proof of [28, Lemma 3.2.] we conclude that $(I, I) \in \mathcal{U}_d \times \mathcal{U}_d$ is a local minimizer of Δ. Arguing as in the proof of [28, Lemma 3.4.] we now see that $[\sigma_0, \rho] = \sigma_0 \, \rho - \rho \, \sigma_0 = 0$; thus, there exists of an onb $\{u_i\}_{i\in\mathbb{I}_d}$ of $\mathbb{C}^d$ such that

$$\rho = \sum_{i\in\mathbb{I}_d} \lambda_i(\rho) \ u_i \otimes u_i \quad \text{and} \quad \sigma_0 = \sum_{i\in\mathbb{I}_d} a_i \ u_i \otimes u_i \, ,$$

where $(a_i)_{i\in\mathbb{I}_d}^\downarrow = \lambda(\sigma_0)$. Finally, arguing as in the proof of [28, Theorem 3.5.] we get that $\lambda(\sigma_0) = (a_i)_{i\in\mathbb{I}_d}^\downarrow = (a_i)_{i\in\mathbb{I}_d}$. The fact that σ_0 is a global minimizer of Θ_f is a consequence of Lidskii's inequality for self-adjoint matrices (see [1, 3]). □

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