IDEMPOTENT LINEAR RELATIONS

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Abstract. A linear relation E acting on a Hilbert space is idempotent if $E^2 = E$. A triplet of subspaces is needed to characterize a given idempotent: $(\operatorname{ran} E, \operatorname{ran}(I-E), \operatorname{dom} E)$, or equivalently, $(\ker(I-E), \ker E, \operatorname{mul} E)$. The relations satisfying the inclusions $E^2 \subseteq E$ (sub-idempotent) or $E \subseteq E^2$ (super-idempotent) play an important role. Lastly, the adjoint and the closure of an idempotent linear relation are studied.

1. Introduction

The introduction of linear relations by von Neumann [17] was motivated by the need to define the adjoint of a non-densely defined operator and in considering the inverses of certain operators. Semi-projections, which form the linear relation counterpart of the class of projection operators, appear when solving least-squares problems of linear relations (see [14]). Linear relations provide the appropriate framework when dealing with control problems subject to generalized or nonstandard boundary conditions. In particular, they naturally occur if the normal equations, which are used to characterize solutions of various standard constrained or unconstrained least-squares problems, involve the adjoint of a non-densely defined linear operator (cf. [15]).

A linear operator E is said to be a *projection* if $E^2 = E$, that is, if $\operatorname{dom} E$ (the domain of E) is $E-$invariant and $E^2x = Ex$ for all $x \in \operatorname{dom} E$. For any given projection E, if $\mathcal{M} := \operatorname{ran} E$ (the range of E) and $\mathcal{N} := \ker E$ (the kernel of E) then

$$(1)\ \mathcal{M} \subseteq \operatorname{dom} E, \text{and } (2)\ \mathcal{M} \cap \mathcal{N} = \{0\}.$$

Unbounded (even non closable) projections were first considered by Ôta [18]. He showed that any projection E is fully determined by its range and kernel, and that the projection determined by $(\mathcal{M}, \mathcal{N})$ is closed if and

2020 *Mathematics Subject Classification*: Primary 47A06; Secondary 47A05.

Key words and phrases: multivalued linear operators, linear relations, projections, idempotents.

only if both $\mathcal{M}$ and $\mathcal{N}$ are closed. Further investigations on closed densely defined projections were carried out by Andô [1]. We extended this work to *semiclosed* projections in [3].

Cross and Wilcox [4] and Labrousse [13] studied the linear relations satisfying (1) and $E^2 = E$. Such linear relations are called *semi-projections* [13] or *multivalued linear projections* [4]. As with projections, semi-projections are fully characterized by the range and kernel, and in this case the multivalued part is given by their intersection. So a semi-projection is a projection if and only if (2) holds. Dropping not just (2), but both (1) and (2), the result is an *idempotent* relation; that is, a linear relation E such that $E^2 = E$.

Any idempotent E verifies the twofold inclusion $E^2 \subseteq E \subseteq E^2$. When a relation E only satisfies the left inclusion, it is termed *sub-idempotent*. Similarly, E is *super-idempotent* if instead the other inclusion holds.

The purpose of this paper is to study idempotents, as well as sub- and super-idempotents. Various characterizations of these classes are given, as well as adjoints and closures of relations in these classes. Much was already done for the class of semi-projections by Cross and Wilcox [4] (see also [13]).

Section 2 serves to introduce the notation and to give some preliminary results. In Section 3 we show that for a full description, three subspaces are needed; $(\operatorname{ran} E, \operatorname{ran}(I-E), \operatorname{dom} E)$ for sub-idempotents and $(\ker(I-E), \ker E, \operatorname{mul} E)$ for super-idempotents. Then we turn our attention to the description of E^2 when E is either sub- or super-idempotent, and we establish in either case that E^2 is an idempotent. In Section 4 the results of Section 3 are applied to obtain several characterizations of idempotents. The main results of this section concern the representation of the class of idempotents. These include two in which a triplet of subspaces uniquely determines an idempotent whenever the so-called *idempotency condition* is satisfied. Section 5 looks at the closure and adjoint of a relation E which is one of the three classes. In general these operations do not yield idempotents. Necessary and sufficient conditions are given for E^* and $\overline{E}$ to be idempotent, and we characterize those idempotents that are closed. Throughout, examples are presented illustrating the very rich structure of all these classes.

2. Preliminaries

Throughout, $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{E}$ are complex and separable Hilbert spaces. As usual, the direct sum of two subspaces $\mathcal{M}$ and $\mathcal{N}$ of a Hilbert space $\mathcal{H}$ is indicated by $\mathcal{M} \dotplus \mathcal{N}$. The orthogonal complement of a subspace $\mathcal{M} \subseteq \mathcal{H}$ is written as $\mathcal{M}^\perp$, or $\mathcal{H} \ominus \mathcal{M}$ interchangeably.

We consider the inner product on $\mathcal{H} \times \mathcal{K}$

$$\langle (h,k),(h',k') \rangle = \langle h,h' \rangle + \langle k,k' \rangle , \ (h,k),(h',k') \in \mathcal{H} \times \mathcal{K},$$

with the associated norm $\|(h,k)\|^2 = \|h\|^2 + \|k\|^2$.

For $\mathcal{S}$ and $\mathcal{T}$ closed subspaces of $\mathcal{H}$, Friedrichs [8] defined the cosine of the *angle* between $\mathcal{S}$ and $\mathcal{T}$ as

$$c(\mathcal{S},\mathcal{T}) := \sup \{ | \langle x,y \rangle | : x \in \mathcal{S} \ominus (\mathcal{S} \cap \mathcal{T}), y \in \mathcal{T} \ominus (\mathcal{S} \cap \mathcal{T}), \|x\|, \|y\| \le 1 \} .$$

On the other hand, the *minimal angle* between $\mathcal{S}$ and $\mathcal{T}$ was defined by Dixmier [6] as the one whose cosine is

$$c_0(\mathcal{S},\mathcal{T}) := \sup \{ | \langle x,y \rangle | \ : \ x \in \mathcal{S}, \ y \in \mathcal{T}, \|x\|, \|y\| \le 1 \} .$$

In general, $c(\mathcal{S},\mathcal{T}) \le c_0(\mathcal{S},\mathcal{T})$. However, when $\mathcal{S} \cap \mathcal{T} = \{0\}$ both angles coincide.

Theorem 2.1 ([5, Theorem 13])**.** *Let $\mathcal{S}$, $\mathcal{T}$ be closed subspaces of $\mathcal{H}$. The following are equivalent:*

i) $c(\mathcal{S},\mathcal{T}) < 1$;
ii) $\mathcal{S} + \mathcal{T}$ *is closed;*
iii) $\mathcal{S}^\perp + \mathcal{T}^\perp$ *is closed.*

Lemma 2.2. *Let $\mathcal{S}$, $\mathcal{T}, \mathcal{W}$ be closed subspaces of $\mathcal{H}$ such that $\mathcal{T} \subseteq \mathcal{W}$ and $\mathcal{T} \cap \mathcal{S} = \mathcal{W} \cap \mathcal{S}$. Then*

$$c(\mathcal{T},\mathcal{S}) \le c(\mathcal{W},\mathcal{S}).$$

Proposition 2.3 ([12, Proposition 2.3.3, Corollary 2.3.1])**.** *Let $\mathcal{M}, \mathcal{N}$ be operator ranges such that $\mathcal{M} + \mathcal{N}$ is closed. Then*

1. $\overline{\mathcal{M} \cap \mathcal{N}} = \overline{\mathcal{M}} \cap \overline{\mathcal{N}}$.
2. $(\mathcal{M} \cap \mathcal{N})^\perp = \mathcal{M}^\perp + \mathcal{N}^\perp$.

Linear relations. A linear relation from $\mathcal{H}$ into $\mathcal{K}$ is a linear subspace T of the cartesian product $\mathcal{H} \times \mathcal{K}$. The set of linear relations from $\mathcal{H}$ into $\mathcal{K}$ will be denoted by $\mathrm{lr}(\mathcal{H},\mathcal{K})$, and $\mathrm{lr}(\mathcal{H}) := \mathrm{lr}(\mathcal{H},\mathcal{H})$. The domain, range, kernel or nullspace and multivalued part of $T \in \mathrm{lr}(\mathcal{H},\mathcal{K})$ are denoted by $\mathrm{dom}\, T$, $\mathrm{ran}\, T$, $\ker T$ and $\mathrm{mul}\, T$, respectively. When $\mathrm{mul}\, T = \{0\}$, T is an operator.

Arens stated the next lemma in [2, 2.02]. We write it in a slightly different form as in [13, Proposition 1.21].

Lemma 2.4. *Let $S, T \in \mathrm{lr}(\mathcal{H},\mathcal{K})$. Then $S = T$ if and only if $S \subseteq T$, $\mathrm{dom}\, T \subseteq \mathrm{dom}\, S$ and $\mathrm{mul}\, T \subseteq \mathrm{mul}\, S$.*

Given $T, S \in \operatorname{lr}(\mathcal{H}, \mathcal{K})$, $T \cap \mathcal{S}$ and $T \hat{+} S$ are the usual intersection and sum of T and S as subspaces, respectively. In particular, $\operatorname{mul}(T \cap S) = \operatorname{mul} T \cap \operatorname{mul} S$ and $\ker(T \cap S) = \ker T \cap \ker S$, $\operatorname{dom}(T \hat{+} S) = \operatorname{dom} T + \operatorname{dom} S$ and $\operatorname{ran}(T \hat{+} S) = \operatorname{ran} T + \operatorname{ran} S$.

The sum of two linear relations $T, S \in \operatorname{lr}(\mathcal{H}, \mathcal{K})$ is the linear relation defined by

$$T + S := \{(x, y + z) : (x, y) \in T \text{ and } (x, z) \in S\}.$$

If $T \in \operatorname{lr}(\mathcal{H}, \mathcal{E})$ and $S \in \operatorname{lr}(\mathcal{E}, \mathcal{K})$, the product ST is the linear relation from $\mathcal{H}$ to $\mathcal{K}$ defined by

$$ST := \{(x, y) : (x, z) \in T \text{ and } (z, y) \in S \text{ for some } z \in \mathcal{E}\}.$$

Given a subspace $\mathcal{M}$ of $\mathcal{H}$, $I_{\mathcal{M}} := \{(u, u) : u \in \mathcal{M}\}$. In particular, the identity is $I := I_{\mathcal{H}}$.

Lemma 2.5. *Let* $T \in \operatorname{lr}(\mathcal{H})$. *Then* $(u, v) \in I - T$ *if and only if* $(u, u - v) \in T$. *As a consequence,* $\ker(I - T) \subseteq \operatorname{ran} T \cap \operatorname{dom} T$, $\ker(I - T) \subseteq \ker(I - T^2)$ *and* $\operatorname{ran}(I - T^2) \subseteq \operatorname{ran}(I - T)$.

The inverse of $T \in \operatorname{lr}(\mathcal{H}, \mathcal{K})$ is $T^{-1} = \{(y, x) : (x, y) \in T\}$. The following identities can be easily checked

$$T^{-1}T = I_{\operatorname{dom} T} \hat{+} (\{0\} \times \ker T) \quad \text{and} \quad TT^{-1} = I_{\operatorname{ran} T} \hat{+} (\{0\} \times \operatorname{mul} T), \tag{2.1}$$

[10, Equation 2.4].

The closure $\overline{T}$ of a linear relation T from $\mathcal{H}$ to $\mathcal{K}$ is the closure of the subspace T in $\mathcal{H} \times \mathcal{K}$, when the product is provided with the product topology. The relation T is *closed* when it is closed as a subspace of $\mathcal{H} \times \mathcal{K}$.

The *adjoint* of T is the linear relation from $\mathcal{K}$ to $\mathcal{H}$ defined by

$$T^* := JT^{\perp} = (JT)^{\perp},$$

where $J(x, y) = i(-y, x)$. The adjoint is automatically a closed linear relation and $\overline{T} = T^{**} := (T^*)^*$. It is immediate that $(\overline{T})^* = T^*$. Since

$$T^* = \{(x, y) \in \mathcal{K} \times \mathcal{H} : \langle\, g, x \,\rangle = \langle\, f, y \,\rangle \text{ for all } (f, g) \in T\},$$

we get that $\operatorname{mul} T^* = (\operatorname{dom} T)^{\perp}$ and $\ker T^* = (\operatorname{ran} T)^{\perp}$. Therefore, if T is closed both $\ker T$ and $\operatorname{mul} T$ are closed subspaces.

Theorem 2.6 ([4, Theorem 3.3])**.** *Let* $T \in \operatorname{lr}(\mathcal{H}, \mathcal{K})$ *be closed. Then* $\operatorname{ran} T$ *is closed if and only if* $\operatorname{ran} T^*$ *is closed.*

If $T \in \operatorname{lr}(\mathcal{H}, \mathcal{E})$ and $S \in \operatorname{lr}(\mathcal{E}, \mathcal{K})$ then

$$T^*S^* \subseteq (ST)^*. \tag{2.2}$$

If $T, S \in \operatorname{lr}(\mathcal{H}, \mathcal{K})$ then

$$(T \hat{+} S)^* = T^* \cap S^*, \tag{2.3}$$

and

$$T^* + S^* \subseteq (T + S)^*. \tag{2.4}$$

Lemma 2.7 ([11, Lemma 2.10])**.** *Let $T, S \in \operatorname{lr}(\mathcal{H}, \mathcal{K})$ be closed linear relations. Then $T \hat{+} S$ is closed if and only if $T^* \hat{+} S^*$ is closed.*

The next result follows from [2, 2.02]. See also [16, Theorem 4.2].

Lemma 2.8. *Let $A, B \in \operatorname{lr}(\mathcal{H}, \mathcal{K})$ such that $A \subseteq B^*$. If*

$$\ker A + \operatorname{ran} B = \mathcal{H} \quad \textit{and} \quad \ker B + \operatorname{ran} A = \mathcal{K}, \tag{2.5}$$

then $A = B^*$ and $B = A^*$ and both A and B are closed with closed ranges.

3. Sub- and super-idempotents

A linear relation $E \subseteq \mathcal{H} \times \mathcal{H}$ is called an *idempotent* if $E^2 = E$. If, in addition $\operatorname{ran} E \subseteq \operatorname{dom} E$, we say that E is a *semi-projection.* If E is an idempotent operator then $\operatorname{ran} E \subseteq \operatorname{dom} E$ and we say that E is a *projection.* Denote by $\operatorname{Id}(\mathcal{H})$ and $\operatorname{Sp}(\mathcal{H})$ the set of idempotents and the set of semi-projections, respectively.

Semi-projections are studied in detail in [4] and [13] where, among other results, it is proved that a semi-projection is uniquely determined by its range and kernel. More precisely, if $\mathcal{M}$ and $\mathcal{N}$ are two subspaces of $\mathcal{H}$ then

$$P_{\mathcal{M},\mathcal{N}} := I_{\mathcal{M}} \hat{+} (\mathcal{N} \times \{0\}) \tag{3.1}$$

is the unique semi-projection with $\operatorname{ran} P_{\mathcal{M},\mathcal{N}} = \mathcal{M}$ and $\ker P_{\mathcal{M},\mathcal{N}} = \mathcal{N}$. Furthermore, $\operatorname{dom} P_{\mathcal{M},\mathcal{N}} = \mathcal{M} + \mathcal{N}$ and $\operatorname{mul} P_{\mathcal{M},\mathcal{N}} = \mathcal{M} \cap \mathcal{N}$.

Semi-projections appear, for example, when solving least-squares problems for linear relations (see [14, Proposition 2.2]). More precisely, if $T \in \operatorname{lr}(\mathcal{H}, \mathcal{K})$ then, by (2.1), $T^{-1}T = P_{\operatorname{dom} T, \ker T}$ and $TT^{-1} = P_{\operatorname{ran} T, \operatorname{mul} T}$.

Proposition 3.1 ([4, Proposition 1.1])**.** *Let $E \in \operatorname{lr}(\mathcal{H})$. Then $E \in \operatorname{Sp}(\mathcal{H})$ if and only if $E = P_{\operatorname{ran} E, \ker E}$.*

One of our goals is to get a representation similar to (3.1) for idempotent relations. The range and kernel are not sufficient to fully describe an idempotent unless it is a semi-projection. We will see that a triplet of subspaces is needed to characterize an idempotent relation.

Example 3.2. If $\mathcal{M}, \mathcal{S}, \mathcal{S}'$ are subspaces of $\mathcal{H}$ with $\mathcal{M}\dot{+}\mathcal{S} = \mathcal{M}\dot{+}\mathcal{S}'$ and $\mathcal{S} \neq \mathcal{S}'$, it can be seen that the relations $E = I_{\mathcal{M}}\hat{+}(\{0\} \times \mathcal{S})$ and $E' = I_{\mathcal{M}}\hat{+}(\{0\} \times \mathcal{S}')$ are idempotent with $\operatorname{ran} E = \operatorname{ran} E' = \mathcal{M} \dot{+} \mathcal{S}$ and $\ker E = \ker E' = \{0\}$ although $E \neq E'$ because $\operatorname{mul} E = \mathcal{S}$ and $\operatorname{mul} E' = \mathcal{S}'$.

Given a linear relation E, there are two semi-projections naturally associated with E, namely $P_{\ker(I-E),\ker E}$ and $P_{\operatorname{ran} E,\operatorname{ran}(I-E)}$, as the following lemma shows.

Lemma 3.3. *Let $E \in \operatorname{lr}(\mathcal{H})$. Then*

$$P_{\ker(I-E),\ker E} \hat{+} (\{0\} \times \operatorname{mul} E) \subseteq E \subseteq P_{\operatorname{ran} E,\operatorname{ran}(I-E)} \cap (\operatorname{dom} E \times \mathcal{H}).$$

Proof. To see the first inclusion we only need to check that $I_{\ker(I-E)} \subseteq E$ but if $u \in \ker(I - E)$ then $(u, 0) \in I - E$ whence $(u, u) \in E$. To prove the second inclusion, let $(u, v) \in E$ then $(u, u - v) \in I - E$ so that $(u, v) = (v, v) + (u - v, 0) \in P_{\operatorname{ran} E,\operatorname{ran}(I-E)}$. □

From now on $\mathcal{M}, \mathcal{N}$ and $\mathcal{S}$ are subspaces of $\mathcal{H}$.

In view of the above lemma, we begin by studying the relations

$$R := P_{\mathcal{M},\mathcal{N}} \cap (\mathcal{S} \times \mathcal{H}) \text{ and } T := P_{\mathcal{M},\mathcal{N}} \hat{+} (\{0\} \times \mathcal{S}). \tag{3.2}$$

Lemma 3.4. *Let $T, R \in \operatorname{lr}(\mathcal{H})$ be defined as in (3.2). Then*

1. $\operatorname{dom} R = (\mathcal{M} + \mathcal{N}) \cap \mathcal{S}$, $\operatorname{ran} R = \mathcal{M} \cap (\mathcal{N} + \mathcal{S})$, $\ker R = \mathcal{N} \cap \mathcal{S}$ *and* $\operatorname{mul} R = \mathcal{M} \cap \mathcal{N}$.
2. $\operatorname{dom} T = \mathcal{M} + \mathcal{N}$, $\operatorname{ran} T = \mathcal{M} + \mathcal{S}$, $\ker T = \mathcal{N} + \mathcal{M} \cap \mathcal{S}$ *and* $\operatorname{mul} T = \mathcal{S} + \mathcal{M} \cap \mathcal{N}$.

Proof. Use the definitions of R and T. □

Lemma 3.5. *Let $R, T \in \operatorname{lr}(\mathcal{H})$ be defined as in (3.2). Then*

1. $I - R = P_{\mathcal{N},\mathcal{M}} \cap (\mathcal{S} \times \mathcal{H})$ *and* $R^{-1} = P_{\mathcal{S},\mathcal{N}} \cap (\mathcal{M} \times \mathcal{H})$.
2. $I - T = P_{\mathcal{N},\mathcal{M}} \hat{+} (\{0\} \times \mathcal{S})$ *and* $T^{-1} = P_{\mathcal{M},\mathcal{S}} \hat{+} (\{0\} \times \mathcal{N})$.

Proof. By Lemma 2.5, $(x, y) \in I - R$ if and only if $(x, x - y) \in R$, or equivalently $(x, x - y) \in P_{\mathcal{M},\mathcal{N}} \cap (\mathcal{S} \times \mathcal{H})$. Then $(x, y) \in I - R$ if and only if $x = m + n \in \mathcal{S}$, $m \in \mathcal{M}$, $n \in \mathcal{N}$ and $x - y = m$. Therefore $y = x - m = n$ so that $(x, y) = (m + n, n)$, $m + n \in \mathcal{S}$. Hence $(x, y) \in P_{\mathcal{N},\mathcal{M}} \cap (\mathcal{S} \times \mathcal{H})$. The other inclusion is similar.

To prove the formula for R^{-1}, let $(x, y) \in R$. Then $(x, y) = (m + n, m)$ where $m \in \mathcal{M}, n \in \mathcal{N}$ and $m + n =: s \in \mathcal{S}$. Therefore $(x, y) = (s, s - n)$ with $s - n = m \in \mathcal{M}$ so that $(y, x) = (s - n, s) \in P_{\mathcal{S},\mathcal{N}} \cap (\mathcal{M} \times \mathcal{H})$. Hence $R^{-1} \subseteq P_{\mathcal{S},\mathcal{N}} \cap (\mathcal{M} \times \mathcal{H})$. The reverse inclusion is similar.

The proof of item 2 follows in a similar fashion. □

Lemma 3.6. *Let $R, T \in \mathrm{lr}(\mathcal{H})$ be defined as in (3.2). Then*

1. $R^2 \subseteq R$.
2. $T \subseteq T^2$.

Proof. 1 : By Lemmas 3.4 and 3.5, it easily follows that $\ker(I-R) = \operatorname{ran} R \cap \operatorname{dom} R$. If $(x,y) \in R^2$ then there exists $z \in \mathcal{H}$ such that $(x,z),(z,y) \in R$. So that $z \in \ker(I-R)$, or equivalently $(z,z) \in R$. Hence $(x-z,0) = (x,z)-(z,z) \in R$ and $(0,y-z) = (z,y)-(z,z) \in R$. Therefore $(x,y) = (x-z,0)+(z,z)+(0,y-z) \in R$.

2: By Lemmas 3.4 and 3.5, it easily follows that $\operatorname{ran}(I-T) = \ker T + \operatorname{mul} T$. If $(x,y) \in T$ then $(x,x-y) \in I-T$ so that $x-y \in \ker T + \operatorname{mul} T$. Hence $x-y = n+s$ for some $n \in \ker T$ and $s \in \operatorname{mul} T$. Then $(x,y+s) = (x,y)+(0,s) \in T$ and $(y+s,y) = (x-n,y) = (x,y)-(n,0) \in T$. Therefore $(x,y) \in T^2$. □

Definition 3.7. $E \in \mathrm{lr}(\mathcal{H})$ is called *sub-idempotent* if $E^2 \subseteq E$ and *super-idempotent* if $E \subseteq E^2$.

Lemma 3.8. *If $E \in \mathrm{lr}(\mathcal{H})$ is sub- (super-idempotent) then E^2 is sub- (super-idempotent).*

Proof. Use that if $A, B \in \mathrm{lr}(\mathcal{H})$ and $A \subseteq B$ then $A^2 \subseteq B^2$. □

Lemma 3.9. *Let $E \in \mathrm{lr}(\mathcal{H})$.*

1. *If E is sub-idempotent then* $\ker E^2 = \ker E$, $\ker(I-E^2) = \ker(I-E)$ *and* $\operatorname{mul} E^2 = \operatorname{mul} E$.
2. *If E is super-idempotent then* $\operatorname{ran} E^2 = \operatorname{ran} E$, $\operatorname{ran}(I-E^2) = \operatorname{ran}(I-E)$ *and* $\operatorname{dom} E^2 = \operatorname{dom} E$.

Proof. By Lemma 2.5, $\ker(I-E) \subseteq \ker(I-E^2)$ always holds. If E is sub-idempotent then $I-E^2 \subseteq I-E$ and then $\ker(I-E^2) \subseteq \ker(I-E)$. The other assertions follow similarly. □

Proposition 3.10. *Let $E \in \mathrm{lr}(\mathcal{H})$. Then the following are equivalent:*

i) E *is sub-idempotent;*
ii) $E = P_{\operatorname{ran} E, \operatorname{ran}(I-E)} \cap (\operatorname{dom} E \times \mathcal{H})$;
iii) $\ker(I-E) = \operatorname{ran} E \cap \operatorname{dom} E$;
iv) $P_{\operatorname{ran} E \cap \operatorname{dom} E, \ker E} \subseteq E$;

In this case, $\operatorname{mul} E \cap \operatorname{dom} E = \operatorname{ran} E \cap \ker E$.

Proof. Set $R := P_{\operatorname{ran} E, \operatorname{ran}(I-E)} \cap (\operatorname{dom} E \times \mathcal{H})$. By Lemma 3.3, $E \subseteq R$ and $\operatorname{dom} E = \operatorname{dom} R$.

i) $\Rightarrow$ *ii)*: Suppose that $E^2 \subseteq E$. To see that $E = R$ we apply Lemma 2.4 by showing that $\operatorname{mul} R \subseteq \operatorname{mul} E$. Let $w \in \operatorname{mul} R = \operatorname{ran} E \cap \operatorname{ran}(I - E)$. Then there exist $u, v \in \mathcal{H}$ such that $(u, w) \in E$ and $(v, w) \in I - E$. Then $(v, v - w) \in E$, so that $(u + v, v) \in E$. Hence $(u + v, v - w) \in E^2 \subseteq E$. Therefore $(0, w) = (u + v, v) - (u + v, v - w) \in E$.

ii) $\Rightarrow$ *iii)*: Follows from Lemma 3.4.

iii) $\Rightarrow$ *iv)*: $P_{\operatorname{ran} E \cap \operatorname{dom} E, \ker E} = P_{\ker(I-E), \ker E} \subseteq E$, by Lemma 3.3.

iv) $\Rightarrow$ *i)*: Let $(u, v) \in E^2$. Then there exists w such that (u, w), $(w, v) \in E$. Then $(w, w) \in E$ because $w \in \operatorname{ran} E \cap \operatorname{dom} E$ and $I_{\operatorname{ran} E \cap \operatorname{dom} E} \subseteq E$. Hence $(u, v) = (u - w, 0) + (w, w) + (0, v - w) \in E$.

In this case, $\operatorname{mul} E = \operatorname{ran} E \cap \operatorname{ran}(I - E)$. By Lemma 3.5, $I - E$ is also sub-idempotent; then $\ker E = \operatorname{ran}(I - E) \cap \operatorname{dom} E$. Therefore $\operatorname{ran} E \cap \ker E = \operatorname{ran} E \cap \operatorname{ran}(I - E) \cap \operatorname{dom} E = \operatorname{mul} E \cap \operatorname{dom} E$. $\square$

Corollary 3.11. *The set of sub-idempotents is*

$$\{P_{\mathcal{M},\mathcal{N}} \cap (\mathcal{S} \times \mathcal{H}) : \mathcal{M}, \mathcal{N}, \mathcal{S} \subseteq \mathcal{H} \textit{ subspaces}\}.$$

Proof. By Proposition 3.10, any sub-idempotent belongs to the set. Conversely, if $R := P_{\mathcal{M},\mathcal{N}} \cap (\mathcal{S} \times \mathcal{H})$, by Lemma 3.6, R is sub-idempotent. $\square$

Remark 3.12. Let $E \in \operatorname{lr}(\mathcal{H})$ be sub-idempotent. Then $E = P_{\mathcal{M},\mathcal{N}} \cap (\mathcal{S} \times \mathcal{H})$ if and only if $\operatorname{ran} E = (\mathcal{N} + \mathcal{S}) \cap \mathcal{M}$, $\operatorname{ran}(I - E) = (\mathcal{M} + \mathcal{S}) \cap \mathcal{N}$ and $\operatorname{dom} E = (\mathcal{M} + \mathcal{N}) \cap \mathcal{S}$.

In fact, since E is sub-idempotent, by Proposition 3.10,

$$E = P_{\operatorname{ran} E, \operatorname{ran}(I-E)} \cap (\operatorname{dom} E \times \mathcal{H}).$$

So that, if $\operatorname{ran} E = (\mathcal{N} + \mathcal{S}) \cap \mathcal{M}$, $\operatorname{ran}(I - E) = (\mathcal{M} + \mathcal{S}) \cap \mathcal{N}$ and $\operatorname{dom} E = (\mathcal{M} + \mathcal{N}) \cap \mathcal{S}$, then $E = P_{(\mathcal{N}+\mathcal{S})\cap\mathcal{M}, (\mathcal{M}+\mathcal{S})\cap\mathcal{N}} \hat{+} (((\mathcal{M} + \mathcal{N}) \cap \mathcal{S}) \times \mathcal{H}) = P_{\mathcal{M},\mathcal{N}} \cap (\mathcal{S} \times \mathcal{H})$. The converse follows from Lemma 3.4.

Proposition 3.13. *Let $E \in \operatorname{lr}(\mathcal{H})$. Then the following are equivalent:*

i) *E is super-idempotent;*
ii) $E = P_{\ker(I-E), \ker E} \hat{+} (\{0\} \times \operatorname{mul} E)$;
iii) $\operatorname{ran}(I - E) = \ker E + \operatorname{mul} E$;
iv) $E \subseteq P_{\operatorname{ran} E, \ker E + \operatorname{mul} E}$.

In this case, $\operatorname{dom} E = \operatorname{ran} E \cap \operatorname{dom} E + \ker E$.

Proof. i) $\Rightarrow$ *ii)*: Suppose that $E \subseteq E^2$ and let $(u, v) \in E$, so that there exists w such that $(u, w), (w, v) \in E$. Then $(u - w, 0) \in E$, $(0, v - w) \in E$ and $(w, w) = (u, v) - (u - w, 0) - (0, v - w) \in E$. Therefore $(w, 0) \in I - E$ and $(u, v) = (w, w) + (u - w, v - w) \in P_{\ker(I-E), \ker E} \hat{+} (\{0\} \times \operatorname{mul} E)$. This shows that $E \subseteq P_{\ker(I-E),\ker E} \hat{+} (\{0\} \times \operatorname{mul} E)$. The other inclusion always holds (see Lemma 3.3).

ii) $\Rightarrow$ *iii)*: Follows from Lemma 3.4.

iii) $\Rightarrow$ *iv)*: By Lemma 3.3, $E \subseteq P_{\operatorname{ran} E,\operatorname{ran}(I-E)} = P_{\operatorname{ran} E,\ker E+\operatorname{mul} E}$.

iv) $\Rightarrow$ *i)*: Let $(u, v) \in E$. Then $(u, v) = (x + y, x)$, with $x \in \operatorname{ran} E$, $y \in \ker E + \operatorname{mul} E$. So $y = y_1 + y_2$ with $(y_1, 0), (0, y_2) \in E$. Then $(x + y, x + y_2), (x + y_2, x) \in E$ so that $(x + y, x) = (u, v) \in E^2$.

In this case, $\operatorname{dom} E \subseteq \operatorname{dom}(P_{\operatorname{ran} E,\ker E+\operatorname{mul} E}) = \operatorname{ran} E + \ker E$. So that $\operatorname{dom} E \subseteq \operatorname{ran} E \cap \operatorname{dom} E + \ker E$. The other inclusion always holds. ☐

Corollary 3.14. *The set of super-idempotents is*

$$\{P_{\mathcal{M},\mathcal{N}} \hat{+} (\{0\} \times \mathcal{S}) : \mathcal{M}, \mathcal{N}, \mathcal{S} \subseteq \mathcal{H} \textit{ subspaces}\}.$$

Proof. By Proposition 3.13, any super-idempotent belongs to the set. Conversely, if $T := P_{\mathcal{M},\mathcal{N}} \hat{+} (\{0\}\times\mathcal{S})$, by Lemma 3.6, T is super-idempotent. ☐

Remark 3.15. Let $E \in \operatorname{lr}(\mathcal{H})$ be super-idempotent.

Then $E = P_{\mathcal{M},\mathcal{N}} \hat{+} (\{0\} \times \mathcal{S})$ if and only if $\ker(I - E) = \mathcal{M} + \mathcal{N} \cap \mathcal{S}$, $\ker E = \mathcal{N} + \mathcal{M} \cap \mathcal{S}$ and $\operatorname{mul} E = \mathcal{S} + \mathcal{M} \cap \mathcal{N}$.

In fact, since E is super-idempotent, by Proposition 3.13,

$$E = P_{\ker(I-E),\ker E} \hat{+} (\{0\} \times \operatorname{mul} E).$$

So that, if $\ker(I-E) = \mathcal{M}+\mathcal{N}\cap\mathcal{S}$, $\ker E = \mathcal{N}+\mathcal{M}\cap\mathcal{S}$ and $\operatorname{mul} E = \mathcal{S}+\mathcal{M}\cap \mathcal{N}$, then $E = P_{\mathcal{M}+\mathcal{N}\cap\mathcal{S},\mathcal{N}+\mathcal{M}\cap\mathcal{S}} \hat{+}(\{0\} \times (\mathcal{S} + \mathcal{M} \cap \mathcal{N})) = P_{\mathcal{M},\mathcal{N}} \hat{+} (\{0\} \times \mathcal{S})$. The converse follows from Lemma 3.4.

Corollary 3.16. *Let $E \in \operatorname{lr}(\mathcal{H})$. Then E is sub-idempotent if and only if $I - E$ is sub-idempotent if and only if E^{-1} is sub-idempotent. An analogue result holds if E is super-idempotent.*

Proof. Use Lemma 3.5 and Corollaries 3.11 and 3.14. ☐

Proposition 3.17. *The following statements hold:*

1. *Let E be sub-idempotent. Then $E \in \operatorname{Id}(\mathcal{H})$ if and only if* $\operatorname{dom} E = \operatorname{ran} E \cap \operatorname{dom} E + \ker E$.
2. *Let E be super-idempotent. Then $E \in \operatorname{Id}(\mathcal{H})$ if and only if* $\operatorname{mul} E \cap \operatorname{dom} E = \operatorname{ran} E \cap \ker E$.

Proof. 1: If $E \in \operatorname{Id}(\mathcal{H})$ then, by Proposition 3.13, $\operatorname{dom} E = \operatorname{ran} E \cap \operatorname{dom} E + \ker E$. Conversely, since E is sub-idempotent, by Proposition 3.10,

$$T := P_{\operatorname{ran} E\cap\operatorname{dom} E,\ker E} \hat{+} (\{0\} \times \operatorname{mul} E) \subseteq E$$

and $\operatorname{mul} E \subseteq \operatorname{mul} T$. Since $\operatorname{dom} E = \operatorname{ran} E \cap \operatorname{dom} E + \ker E = \operatorname{dom} T$, by Lemma 2.4, $E = T$. Then, by Proposition 3.13, E is super-idempotent, so that $E \in \operatorname{Id}(\mathcal{H})$.

2: If $E \in \operatorname{Id}(\mathcal{H})$ then, by Proposition 3.10, $\operatorname{mul} E \cap \operatorname{dom} E = \operatorname{ran} E \cap \ker E$. Conversely, since E is super-idempotent, by Proposition 3.13, $E \subseteq P_{\operatorname{ran} E, \operatorname{ran}(I-E)}$. By Lemma 3.3, $E \subseteq P_{\operatorname{ran} E, \operatorname{ran}(I-E)} \cap (\operatorname{dom} E \times \mathcal{H}) := R$ and $\operatorname{dom} R = (\operatorname{ran} E + \operatorname{ran}(I-E)) \cap \operatorname{dom} E = \operatorname{dom} E$. Also, $\operatorname{mul} R = \operatorname{ran} E \cap \operatorname{ran}(I-E) = \operatorname{ran} E \cap (\ker E + \operatorname{mul} E) = \operatorname{ran} E \cap \ker E + \operatorname{mul} E = \operatorname{mul} E$. Then $E = R$, and by Proposition 3.13, E is sub-idempotent. Therefore $E \in \operatorname{Id}(\mathcal{H})$. $\square$

Theorem 3.18. *Let $E \in \operatorname{lr}(\mathcal{H})$. Then*

1. *E is sub-idempotent if and only if*

$$E^2 = P_{\ker(I-E), \ker E} \hat{+} (\{0\} \times \operatorname{mul} E).$$

2. *E is super-idempotent if and only if*

$$E^2 = P_{\operatorname{ran} E, \operatorname{ran}(I-E)} \cap (\operatorname{dom} E \times \mathcal{H}).$$

In either case, $E^2 \in \operatorname{Id}(\mathcal{H})$.

Proof. 1: Set $P := P_{\ker(I-E), \ker E} \hat{+} (\{0\} \times \operatorname{mul} E)$.

If $E^2 = P$ then, since $P \subseteq E$ always holds, $E^2 \subseteq E$ and E is sub-idempotent.

Conversely, if $E^2 \subseteq E$, by Lemma 3.9, $P = P_{\ker(I-E^2), \ker E^2} \hat{+} (\{0\} \times \operatorname{mul} E^2) \subseteq E^2$. Also, $\operatorname{mul} E^2 = \operatorname{mul} E \subseteq \operatorname{mul} P$. It only remains to see that $\operatorname{dom} E^2 \subseteq \operatorname{dom} P$ to apply Lemma 2.4 and get that $E^2 = P$. Let $x \in \operatorname{dom} E^2$; then there exists $w \in \operatorname{ran} E \cap \operatorname{dom} E$ such that $(x, w), (w, y) \in E$ for some $y \in \mathcal{H}$. But, by Proposition 3.10, $\operatorname{ran} E \cap \operatorname{dom} E = \ker(I-E)$; then $w \in \ker(I-E)$ or $(w, w) \in E$. Hence $(x-w, 0) \in E$ and $x = x - w + w \in \ker E + \ker(I-E) = \operatorname{dom} P$.

2: Set $Q := P_{\operatorname{ran} E, \operatorname{ran}(I-E)} \cap (\operatorname{dom} E \times \mathcal{H})$.

If $E^2 = Q$, since $E \subseteq Q$ always holds, $E \subseteq E^2$ and E is super-idempotent.

Conversely, if $E \subseteq E^2$, by Lemma 3.9, $E^2 \subseteq Q = P_{\operatorname{ran} E^2, \operatorname{ran}(I-E^2)} \cap (\operatorname{dom} E^2 \times \mathcal{H})$. Also, $\operatorname{dom} Q = \operatorname{dom} E = \operatorname{dom} E^2$. It only remains to see that $\operatorname{mul} Q \subseteq \operatorname{mul} E^2$ to apply Lemma 2.4 and get that $E^2 = Q$. By Lemma 3.4 and Proposition 3.13, $\operatorname{mul} Q = \operatorname{ran} E \cap \operatorname{ran}(I-E) = \operatorname{ran} E \cap (\ker E + \operatorname{mul} E) = \operatorname{ran} E \cap \ker E + \operatorname{mul} E \subseteq \operatorname{ran} E \cap \ker E + \operatorname{mul} E^2$, where the inclusion holds because $E \subseteq E^2$. To see that $\operatorname{ran} E \cap \ker E \subseteq \operatorname{mul} E^2$, let

$u \in \operatorname{ran} E \cap \ker E$. Thus $(u,0) \in E$ and $(y,u) \in E$ for some $y \in \operatorname{dom} E$. Hence $(y,0) \in E^2$ and $(y,u) \in E^2$. Therefore $(0,u) \in E^2$, i.e. $u \in \operatorname{mul} E^2$.

Finally, suppose that E is sub-idempotent then E^2 is sub-idempotent. Since $E^2 = P$, it is also super-idempotent and then $E^2 \in \operatorname{Id}(\mathcal{H})$. The case when E is super-idempotent is similar. □

Corollary 3.19. *Let $E \in \operatorname{lr}(\mathcal{H})$. Then:*

1. *E is sub-idempotent if and only if $\ker E^2 = \ker E$, $\ker(I - E^2) = \ker(I - E)$, $\operatorname{mul} E^2 = \operatorname{mul} E$ and E^2 is super-idempotent.*
2. *E is super-idempotent if and only if $\operatorname{ran} E^2 = \operatorname{ran} E$, $\operatorname{ran}(I - E^2) = \operatorname{ran}(I - E)$, $\operatorname{dom} E^2 = \operatorname{dom} E$ and E^2 is sub-idempotent.*

Proof. If E is sub-idempotent then, by Lemma 3.9 and Theorem 3.18, the result follows. Conversely, by Proposition 3.13,

$$E^2 = P_{\ker(I-E^2),\ker E^2} \hat{+} (\{0\} \times \operatorname{mul} E^2) = P_{\ker(I-E),\ker E} \hat{+} (\{0\} \times \operatorname{mul} E).$$

Then, by Theorem 3.18, E is sub-idempotent. The case when E is super-idempotent is similar. □

The following example shows that there are linear relations which are sub-idempotent but not super-idempotent and viceversa.

Example 3.20.

1. Take $R := P_{\mathcal{M},\mathcal{N}} \cap (\mathcal{S} \times \mathcal{H})$, with $\mathcal{M} \cap \mathcal{S} + \mathcal{N} \cap \mathcal{S} \subsetneq (\mathcal{M} + \mathcal{N}) \cap \mathcal{S}$. Then R is sub-idempotent but not super-idempotent. In fact, by Lemma 3.4, $\operatorname{dom} R \cap \operatorname{ran} R + \ker R = \mathcal{M} \cap \mathcal{S} + \mathcal{N} \cap \mathcal{S} \subsetneq (\mathcal{M} + \mathcal{N}) \cap \mathcal{S} = \operatorname{dom} R$. Then, by Proposition 3.17, R is not super-idempotent.
2. Take $T := P_{\mathcal{M},\mathcal{N}} \hat{+} (\{0\} \times \mathcal{S})$, with $\mathcal{S} + \mathcal{N} \cap \mathcal{M} \subsetneq \mathcal{S} + \mathcal{N} \cap (\mathcal{M} + \mathcal{S})$. Then T is super-idempotent but not sub-idempotent. Indeed, by Lemma 3.4, $\operatorname{mul} T = \mathcal{S} + \mathcal{N} \cap \mathcal{M} \subsetneq \mathcal{S} + \mathcal{N} \cap (\mathcal{M} + \mathcal{S}) = \operatorname{ran} T \cap \operatorname{ran}(I - T)$. Therefore, by Proposition 3.10, T is not sub-idempotent.

Remark 3.21. If E is a super-idempotent operator on $\mathcal{H}$, since $\operatorname{mul} E = \{0\}$, by Proposition 3.13, $E = P_{\ker(I-E),\ker E}$ and $\ker(I - E) \cap \ker E = \{0\}$. So that E is a projection.

If E is a sub-idempotent operator then $E = P_{\operatorname{ran} E,\operatorname{ran}(I-E)} \cap (\operatorname{dom} E \times \mathcal{H})$ and $\{0\} = \operatorname{mul} E = \operatorname{ran} E \cap \operatorname{ran}(I - E)$. Then E is a restriction of a projection and E is a projection if and only if $\operatorname{dom} E = \operatorname{ran} E + \operatorname{ran}(I - E)$.

4. Idempotent linear relations

We begin this section with a list of corollaries regarding idempotent relations which follow immediately from the results in the previous section and the fact that $E \in \mathrm{Id}(\mathcal{H})$ if and only if E is sub- and super-idempotent.

Corollary 4.1 ([13, Propositions 2.2 and 2.4])**.** *Let $E, F \in \mathrm{lr}(\mathcal{H})$. Then $E \in \mathrm{Id}(\mathcal{H})$ if and only if $I - E \in \mathrm{Id}(\mathcal{H})$ if and only if $E^{-1} \in \mathrm{Id}(\mathcal{H})$.*

Proof. Apply Corollary 3.16. □

Example 4.2. Suppose that E is a projection on $\mathcal{H}$. By Corollary 4.1, E^{-1} is idempotent but since $\operatorname{dom} E^{-1} = \operatorname{ran} E$ and $\operatorname{ran} E^{-1} = \operatorname{dom} E$, then E^{-1} may not even be a semi-projection.

Corollary 4.3. *Let $E \in \mathrm{lr}(\mathcal{H})$. The following are equivalent:*

i) $E \in \mathrm{Id}(\mathcal{H})$*;*
ii) $E = P_{\ker(I-E), \ker E} \hat{+} (\{0\} \times \operatorname{mul} E) = P_{\operatorname{ran} E, \operatorname{ran}(I-E)} \cap (\operatorname{dom} E \times \mathcal{H})$*;*
iii) $E = P_{\operatorname{ran} E, \operatorname{ran}(I-E)} \cap (\operatorname{dom} E \times \mathcal{H})$ *and* $\operatorname{dom} E = \operatorname{ran} E \cap \operatorname{dom} E + \ker E$*;*
iv) $E = P_{\ker(I-E), \ker E} \hat{+} (\{0\} \times \operatorname{mul} E)$ *and* $\operatorname{mul} E \cap \operatorname{dom} E = \operatorname{ran} E \cap \ker E$.

Proof. Apply Propositions 3.10, 3.13 and 3.17. □

Corollary 4.4. *Let $E \in \mathrm{Id}(\mathcal{H})$. Then*

1. $\operatorname{dom} E = \ker(I - E) + \ker E$.
2. $\operatorname{ran} E = \ker(I - E) + \operatorname{mul} E$.
3. $\ker E = \operatorname{ran}(I - E) \cap \operatorname{dom}(E)$.
4. $\operatorname{mul} E = \operatorname{ran} E \cap \operatorname{ran}(I - E)$.

Proof. Use Propositions 3.10 and 3.13. □

Corollary 4.5. *The set of idempotent linear relations can be expressed as*

$$\{P_{\mathcal{M}\cap\mathcal{S}, \mathcal{N}\cap\mathcal{S}} \hat{+} (\{0\} \times (\mathcal{M} \cap \mathcal{N})) : \mathcal{M}, \mathcal{N}, \mathcal{S} \subseteq \mathcal{H} \text{ subspaces}\}.$$

Alternatively,

$$\{P_{\mathcal{M}+\mathcal{S}, \mathcal{N}+\mathcal{S}} \cap ((\mathcal{M} + \mathcal{N}) \times \mathcal{H}) : \mathcal{M}, \mathcal{N}, \mathcal{S} \subseteq \mathcal{H} \text{ subspaces}\}.$$

Proof. If $E \in \mathrm{Id}(\mathcal{H})$ then E is sub-idempotent and, by Corollary 3.11, $E = P_{\mathcal{M}, \mathcal{N}} \cap (\mathcal{S} \times \mathcal{H})$ for some subspaces $\mathcal{M}, \mathcal{N}, \mathcal{S} \subseteq \mathcal{H}$. Then, by Theorem 3.18 and Lemma 3.4, $E = E^2 = P_{\ker(I-E), \ker E} \hat{+} (\{0\} \times \operatorname{mul} E) = P_{\mathcal{M}\cap\mathcal{S}, \mathcal{N}\cap\mathcal{S}} \hat{+} (\{0\} \times (\mathcal{M} \cap \mathcal{N}))$.

Conversely, if $E = P_{\mathcal{M}\cap\mathcal{S},\mathcal{N}\cap\mathcal{S}} \hat{+} (\{0\} \times (\mathcal{M}\cap\mathcal{N}))$ for some subspaces $\mathcal{M},\mathcal{N},\mathcal{S} \subseteq \mathcal{H}$ then, by Corollary 3.14, E is super-idempotent. By Lemma 3.4, $\operatorname{mul} E \cap \operatorname{dom} E = \mathcal{M}\cap\mathcal{N}\cap\mathcal{S} = \operatorname{ran} E \cap \ker E$. Therefore, by Proposition 3.17, $E \in \operatorname{Id}(\mathcal{H})$. The second equality follows in a similar way. □

Proposition 4.6. *Let* $E \in \operatorname{lr}(\mathcal{H})$. *Then* $E \in \operatorname{Id}(\mathcal{H})$ *if and only if* $\operatorname{dom} E \subseteq \operatorname{ran} E + \ker E$ *and* $I_{\operatorname{ran} E\cap\operatorname{dom} E} \subseteq E$.

Proof. If $E \in \operatorname{Id}(\mathcal{H})$ then, by Proposition 3.13, $\operatorname{dom} E \subseteq \operatorname{ran} E + \ker E$ and, by Proposition 3.10, $I_{\operatorname{ran} E\cap\operatorname{dom} E} \subseteq E$.

Conversely, since $I_{\operatorname{ran} E\cap\operatorname{dom} E} \subseteq E$, $P_{\operatorname{ran} E\cap\operatorname{dom} E,\ker E} \subseteq E$. By Proposition 3.10, $E^2 \subseteq E$. If $\operatorname{dom} E \subseteq \operatorname{ran} E+\ker E$ then $\operatorname{dom} E = \operatorname{ran} E\cap\operatorname{dom} E+\ker E$. Therefore, by Proposition 3.17, $E^2 = E$. □

4.1. **The idempotency condition.** This subsection is devoted to get a representation of idempotent relations similar to the representation of semi-projections (3.1).

Proposition 4.7. *There exists* $E \in \operatorname{Id}(\mathcal{H})$ *such that*

$$\mathcal{M} \subseteq \ker(I - E),\ \mathcal{N} \subseteq \ker E \text{ and } \mathcal{S} \subseteq \operatorname{mul} E. \tag{4.1}$$

Moreover, $E_0 := P_{\mathcal{M}+\mathcal{S},\mathcal{N}+\mathcal{S}} \cap ((\mathcal{M}+\mathcal{N}) \times \mathcal{H})$ *is the smallest idempotent satisfying* (4.1).

Proof. By Lemma 3.4, $\mathcal{M} \subseteq (\mathcal{M}+\mathcal{S}) \cap (\mathcal{M}+\mathcal{N}) = \ker(I - E_0)$, $\mathcal{N} \subseteq (\mathcal{N}+\mathcal{S}) \cap (\mathcal{N}+\mathcal{M}) = \ker E_0$ and $\mathcal{S} \subseteq (\mathcal{S}+\mathcal{M}) \cap (\mathcal{S}+\mathcal{N}) = \operatorname{mul} E_0$. By Corollary 3.11, E_0 is sub-idempotent.

It is easy to check that $E_0 = P_{(\mathcal{M}+\mathcal{N})\cap(\mathcal{M}+\mathcal{S}),\mathcal{N}} \hat{+} (\{0\} \times \mathcal{S})$ so that, by Corollary 3.14, E_0 is super-idempotent. Then the idempotent E_0 satisfies (4.1).

Now, if $E \in \operatorname{Id}(\mathcal{H})$ satisfies (4.1) then $(\mathcal{M}+\mathcal{S}) \cap (\mathcal{M}+\mathcal{N}) \subseteq \operatorname{ran} E \cap \operatorname{dom} E$ so that, by Proposition 4.6, $I_{(\mathcal{M}+\mathcal{N})\cap(\mathcal{M}+\mathcal{S})} \subseteq E$. Since $\mathcal{N} \times \mathcal{S} \subseteq \ker E \times \operatorname{mul} E$, we get that $E_0 \subseteq E$. □

Proposition 4.8. *Let* $\mathcal{X},\mathcal{Y},\mathcal{Z}$ *be subspaces of* $\mathcal{H}$. *Then there exists* $F \in \operatorname{Id}(\mathcal{H})$ *such that*

$$\operatorname{ran} F \subseteq \mathcal{X},\ \operatorname{ran}(I - F) \subseteq \mathcal{Y} \text{ and } \operatorname{dom} F \subseteq \mathcal{Z}. \tag{4.2}$$

Moreover, $F_0 := P_{\mathcal{X}\cap\mathcal{Z},\mathcal{Y}\cap\mathcal{Z}} \hat{+} ((\mathcal{X}\cap\mathcal{Y}) \times \mathcal{H})$ *is the largest idempotent satisfying* (4.2).

Proof. By Lemma 3.4, $\operatorname{ran} F_0 = \mathcal{X}\cap\mathcal{Z}\cap(\mathcal{Y}\cap\mathcal{Z}+\mathcal{X}\cap\mathcal{Y}) \subseteq \mathcal{X}$, $\operatorname{ran}(I-F_0) = \mathcal{Y}\cap\mathcal{Z}\cap(\mathcal{X}\cap\mathcal{Z}+\mathcal{X}\cap\mathcal{Y}) \subseteq \mathcal{Y}$ and $\operatorname{dom} F_0 = (\mathcal{X}\cap\mathcal{Z}+\mathcal{Y}\cap\mathcal{Z})\cap\mathcal{X}\cap\mathcal{Y} \subseteq \mathcal{Z}$.

By Corollary 3.14, F_0 is super-idempotent. It is easy to check that $F_0 = P_{\mathcal{X},\mathcal{Y}} \cap ((\mathcal{Z}\cap\mathcal{X}+\mathcal{Z}\cap\mathcal{Y})\times\mathcal{H})$ so that, by Corollary 3.11, F_0 is sub-idempotent. Then the idempotent F_0 satisfies (4.2).

Now, if $F \in \mathrm{Id}(\mathcal{H})$ satisfies (4.2) then, by Corollary 4.4, $\mathrm{ran}\, F \cap \mathrm{dom}\, F \subseteq \mathcal{X}\cap\mathcal{Z}$, $\ker F = \mathrm{ran}(I-F)\cap \mathrm{dom}\, F \subseteq \mathcal{Y}\cap\mathcal{Z}$ and $\mathrm{mul}\, F = \mathrm{ran}\, F \cap \mathrm{ran}(I-F) \subseteq \mathcal{X}\cap\mathcal{Y}$. Then, by Corollary 4.3, $F = P_{\mathrm{ran}\, F\cap \mathrm{dom}\, F, \ker F} \hat{+} (\{0\}\times \mathrm{mul}\, F) \subseteq F_0$. □

In what follows we characterize the triplets for which there is equality in (4.1) or in (4.2).

Proposition 4.9. *The following are equivalent:*

i) *There exists* $E \in \mathrm{Id}(\mathcal{H})$ *such that* $\mathcal{M} = \ker(I-E)$, $\mathcal{N} = \ker E$ *and* $\mathcal{S} = \mathrm{mul}\, E$;

ii) $(\mathcal{M}+\mathcal{N})\cap\mathcal{S} = \mathcal{M}\cap\mathcal{N}$;

iii) $(\mathcal{M}+\mathcal{N})\cap(\mathcal{M}+\mathcal{S}) = \mathcal{M}$ *and* $\mathcal{M}\cap\mathcal{N} = \mathcal{M}\cap\mathcal{S}$.

In this case, $E = P_{\mathcal{M},\mathcal{N}} \hat{+} (\{0\}\times\mathcal{S})$ *is the unique idempotent satisfying item i).*

Proof. *i)* ⇒ *ii)*: Applying Corollary 4.4 to E and $I-E$, it follows that $(\mathcal{M}+\mathcal{N})\cap\mathcal{S} = (\ker(I-E)+\ker E)\cap \mathrm{mul}\, E = \mathrm{dom}\, E\cap \mathrm{mul}\, E = \mathrm{dom}\, E \cap \mathrm{ran}(I-E)\cap \mathrm{ran}\, E = \ker(I-E)\cap\ker E = \mathcal{M}\cap\mathcal{N}$.

ii) ⇒ *iii)*: Suppose that $(\mathcal{M}+\mathcal{N})\cap\mathcal{S} = \mathcal{M}\cap\mathcal{N}$. Then $\mathcal{M}\cap\mathcal{N}\subseteq\mathcal{S}$. So that $\mathcal{M}\cap\mathcal{N} = \mathcal{S}\cap\mathcal{M}\cap\mathcal{N} := \mathcal{W}$. From $\mathcal{M}\cap\mathcal{S}+\mathcal{N}\cap\mathcal{S}\subseteq(\mathcal{M}+\mathcal{N})\cap\mathcal{S} = \mathcal{M}\cap\mathcal{N}$, we get $\mathcal{M}\cap\mathcal{S}\subseteq\mathcal{N}$ and $\mathcal{N}\cap\mathcal{S}\subseteq\mathcal{M}$. Therefore, $\mathcal{M}\cap\mathcal{S} = \mathcal{N}\cap\mathcal{S} = \mathcal{W}$. Let $x\in(\mathcal{M}+\mathcal{N})\cap(\mathcal{M}+\mathcal{S})$, so that $x = m+n = m'+s$ with $m, m'\in\mathcal{M}$, $n\in\mathcal{N}$ and $s\in\mathcal{S}$. Then $m+n-m' = s\in(\mathcal{M}+\mathcal{N})\cap\mathcal{S} = \mathcal{M}\cap\mathcal{N}$ and $x = m'+s\in\mathcal{M}+\mathcal{M}\cap\mathcal{N} = \mathcal{M}$. The other inclusion always holds.

iii) ⇒ *i)*: Define $E := P_{\mathcal{M},\mathcal{N}} \hat{+} (\{0\}\times\mathcal{S})$. Then $\mathrm{dom}\, E = \mathcal{M}+\mathcal{N}$ and $\mathrm{ran}\, E = \mathcal{M}+\mathcal{S}$. So that $\mathrm{ran}\, E\cap\mathrm{dom}\, E = (\mathcal{M}+\mathcal{S})\cap(\mathcal{M}+\mathcal{N}) = \mathcal{M}$. By Lemma 3.4, $\ker E = \mathcal{M}\cap\mathcal{S}+\mathcal{N} = \mathcal{M}\cap\mathcal{N}+\mathcal{N} = \mathcal{N}$ and $\mathrm{mul}\, E = \mathcal{M}\cap\mathcal{N}+\mathcal{S} = \mathcal{M}\cap\mathcal{S}+\mathcal{S} = \mathcal{S}$. Then, by Corollary 4.3, $E\in\mathrm{Id}(\mathcal{H})$. Finally, if E_1 satisfies (i) then, by Proposition 4.7, $E\subseteq E_1$. Since $\mathrm{dom}\, E = \mathcal{M}+\mathcal{N} = \mathrm{dom}\, E_1$ and $\mathrm{mul}\, E = \mathcal{S} = \mathrm{mul}\, E_1$, by Lemma 2.4, $E = E_1$. □

If $(\mathcal{M}+\mathcal{N})\cap\mathcal{S} = \mathcal{M}\cap\mathcal{N}$, it is easy to check that any triplet having $\mathcal{M},\mathcal{N}$ and $\mathcal{S}$ as components satisfies the corresponding equality (see Corollary 4.15).

Proposition 4.10. *Let* $\mathcal{X},\mathcal{Y},\mathcal{Z}$ *be subspaces of* $\mathcal{H}$. *The following are equivalent:*

i) *There exists* $F \in \operatorname{Id}(\mathcal{H})$ *such that* $\mathcal{X} = \operatorname{ran} F$, $\mathcal{Y} = \operatorname{ran}(I - F)$ *and* $\mathcal{Z} = \operatorname{dom} F$;

ii) $\mathcal{X} \cap \mathcal{Y} + \mathcal{Z} = \mathcal{X} + \mathcal{Y}$;

iii) $\mathcal{X} = \mathcal{X} \cap \mathcal{Y} + \mathcal{X} \cap \mathcal{Z}$ *and* $\mathcal{X} + \mathcal{Y} = \mathcal{X} + \mathcal{Z}$.

In this case, $F = P_{\mathcal{X},\mathcal{Y}} \cap (\mathcal{Z} \times \mathcal{H})$ *is the unique idempotent satisfying item i).*

Proof. *i)* $\Rightarrow$ *ii)*: By Corollary 4.4, $\mathcal{X} \cap \mathcal{Y} + \mathcal{Z} = \operatorname{ran} F \cap \operatorname{ran}(I - F) + \operatorname{dom} F = \operatorname{mul} F + \operatorname{dom} F = \operatorname{mul} F + \operatorname{ran} F \cap \operatorname{dom} F + \ker F = \operatorname{ran} F + \ker F = \mathcal{X} + \mathcal{Y}$.

ii) $\Rightarrow$ *iii)*: $\mathcal{X} + \mathcal{Y} = \mathcal{X} \cap \mathcal{Y} + \mathcal{Z} \subseteq \mathcal{X} + \mathcal{Z} \subseteq \mathcal{X} + \mathcal{Y}$ because $\mathcal{Z} \subseteq \mathcal{X} + \mathcal{Y}$. Then $\mathcal{X} + \mathcal{Y} = \mathcal{X} + \mathcal{Z}$. Since $\mathcal{X} \subseteq \mathcal{X} \cap \mathcal{Y} + \mathcal{Z}$ then $\mathcal{X} \subseteq (\mathcal{X} \cap \mathcal{Y} + \mathcal{Z}) \cap \mathcal{X} = \mathcal{X} \cap \mathcal{Y} + \mathcal{X} \cap \mathcal{Z}$. The other inclusion always holds.

iii) $\Rightarrow$ *i)*: Define $F := P_{\mathcal{X},\mathcal{Y}} \cap (\mathcal{Z} \times \mathcal{H})$. Since $\mathcal{Z} \subseteq \mathcal{X} + \mathcal{Y}$, $\mathcal{Y} \subseteq \mathcal{X} + \mathcal{Z}$, and $\mathcal{X} \subseteq \mathcal{Y} + \mathcal{Z}$, it follows that $\operatorname{dom} F = (\mathcal{X} + \mathcal{Y}) \cap \mathcal{Z} = \mathcal{Z}$, $\operatorname{ran} F = \mathcal{X} \cap (\mathcal{Y} + \mathcal{Z}) = \mathcal{X}$ and $\operatorname{ran}(I - F) = \mathcal{Y} \cap (\mathcal{X} + \mathcal{Z}) = \mathcal{Y}$. Also, $\mathcal{Z} \subseteq \mathcal{X} + \mathcal{Y} = \mathcal{X} \cap \mathcal{Y} + \mathcal{X} \cap \mathcal{Z} + \mathcal{Y} = \mathcal{X} \cap \mathcal{Z} + \mathcal{Y}$. Then $\mathcal{Z} = \mathcal{X} \cap \mathcal{Z} + \mathcal{Y} \cap \mathcal{Z}$, so that $\operatorname{dom} F = \operatorname{ran} F \cap \operatorname{dom} F + \ker F$. Therefore, by Corollary 4.3, $F \in \operatorname{Id}(\mathcal{H})$.

Finally, if F_1 satisfies (*i*) then, by Proposition 4.7, $F_1 \subseteq F$. Since $\operatorname{dom} F = \mathcal{Z} = \operatorname{dom} F_1$ and, by Corollary 4.4, $\operatorname{mul} F = \mathcal{X} \cap \mathcal{Y} = \operatorname{mul} F_1$, then $F = F_1$. □

It is easy to see that item *iii)* of Proposition 4.9 is equivalent to

$$\begin{aligned} \mathcal{M} &= (\mathcal{M} + \mathcal{N}) \cap (\mathcal{M} + \mathcal{S}), \\ \mathcal{N} &= (\mathcal{N} + \mathcal{M}) \cap (\mathcal{N} + \mathcal{S}), \\ \mathcal{S} &= (\mathcal{S} + \mathcal{M}) \cap (\mathcal{S} + \mathcal{N}). \end{aligned}$$

In a symmetric fashion item *iii)* of Proposition 4.10 is equivalent to

$$\begin{aligned} \mathcal{X} &= \mathcal{X} \cap \mathcal{Y} + \mathcal{X} \cap \mathcal{Z}, \\ \mathcal{Y} &= \mathcal{Y} \cap \mathcal{X} + \mathcal{Y} \cap \mathcal{Z} \\ \mathcal{Z} &= \mathcal{Z} \cap \mathcal{X} + \mathcal{Z} \cap \mathcal{Y}. \end{aligned}$$

In view of Propositions 4.9 and 4.10, the set of idempotent linear relations can be given in terms of subspaces.

Corollary 4.11. *The set of idempotent linear relations can be expressed as*

$$\{P_{\mathcal{M},\mathcal{N}} \hat{+} (\{0\} \times \mathcal{S}) : \mathcal{M}, \mathcal{N}, \mathcal{S} \subseteq \mathcal{H} \text{ subspaces, } (\mathcal{M} + \mathcal{N}) \cap \mathcal{S} = \mathcal{M} \cap \mathcal{N}\}.$$

Alternatively;

$$\{P_{\mathcal{X},\mathcal{Y}} \cap (\mathcal{Z} \times \mathcal{H}) : \mathcal{X}, \mathcal{Y}, \mathcal{Z} \subseteq \mathcal{H} \text{ subspaces, } \mathcal{X} \cap \mathcal{Y} + \mathcal{Z} = \mathcal{X} + \mathcal{Y}\}.$$

Proposition 4.12. *If*

$$(\mathcal{M}+\mathcal{N})\cap\mathcal{S}=\mathcal{M}\cap\mathcal{N} \tag{4.3}$$

then

$$\mathcal{X}:=\mathcal{M}+\mathcal{S},\ \mathcal{Y}:=\mathcal{N}+\mathcal{S} \text{ and } \mathcal{Z}:=\mathcal{M}+\mathcal{N}$$

satisfy

$$\mathcal{X}\cap\mathcal{Y}+\mathcal{Z}=\mathcal{X}+\mathcal{Y}. \tag{4.4}$$

Conversely, if the subspaces $\mathcal{X},\mathcal{Y},\mathcal{Z}$ *satisfy* (4.4) *then*

$$\mathcal{M}:=\mathcal{X}\cap\mathcal{Z},\ \mathcal{N}:=\mathcal{Y}\cap\mathcal{Z} \text{ and } \mathcal{S}:=\mathcal{X}\cap\mathcal{Y}$$

satisfy (4.3).

Proof. By Proposition 4.9, $\mathcal{X}\cap\mathcal{Y}=(\mathcal{M}+\mathcal{S})\cap(\mathcal{N}+\mathcal{S})=\mathcal{S}$. Therefore $\mathcal{X},\mathcal{Y},\mathcal{Z}$ satisfiy (4.4). Conversely, by Proposition 4.10, $\mathcal{M}+\mathcal{N}=\mathcal{X}\cap\mathcal{Z}+\mathcal{Y}\cap\mathcal{Z}=\mathcal{Z}$. Therefore $\mathcal{M},\mathcal{N},\mathcal{S}$ satisfy (4.3). □

Summarizing, by Propositions 4.9 and 4.10, we get that $E\in\mathrm{Id}(\mathcal{H})$ is characterized by any of the following triplets:

$$\ker(I-E),\ker E \text{ and } \operatorname{mul}E$$

or

$$\operatorname{ran}E,\operatorname{ran}(I-E) \text{ and } \operatorname{dom}E,$$

and Proposition 4.12 shows how to get one triplet from the other. Any of these triplets provides a unique representation of an $E\in\mathrm{Id}(\mathcal{H})$; namely,

$$E=P_{\ker(I-E),\ker E}\ \hat{+}\ (\{0\}\times\operatorname{mul}E)$$

or

$$E=P_{\operatorname{ran}E,\operatorname{ran}(I-E)}\cap(\operatorname{dom}E\times\mathcal{H}).$$

The first representation of $E\in\mathrm{Id}(\mathcal{H})$ resembles the representation (3.1) of semi-projections. So, from now on, we use this representation rather than the second one.

Definition 4.13. The subspaces $\mathcal{M},\mathcal{N},\mathcal{S}$ satisfy the *idempotency condition (IC)* if

$$(\mathcal{M}+\mathcal{N})\cap\mathcal{S}=\mathcal{M}\cap\mathcal{N}.$$

Example 4.14.
1. $\mathcal{M},\mathcal{N},\mathcal{M}\cap\mathcal{N}$ satisfy the IC.
2. $(\mathcal{M}+\mathcal{N})\cap(\mathcal{M}+\mathcal{S})$, $(\mathcal{N}+\mathcal{M})\cap(\mathcal{N}+\mathcal{S})$ and $(\mathcal{S}+\mathcal{M})\cap(\mathcal{S}+\mathcal{N})$ are the "minimal" subspaces of those with the IC containing $\mathcal{M},\mathcal{N}$ and $\mathcal{S}$, respectively (see Proposition 4.7).

3. $\mathcal{N} \cap \mathcal{S}, \mathcal{M} \cap \mathcal{S}$ and $\mathcal{M} \cap \mathcal{N}$ are the "maximal" subspaces of those with the IC contained in $\mathcal{M}, \mathcal{N}$ and $\mathcal{S}$, respectively (see Proposition 4.8).
4. If $(\mathcal{M} \dot{+} \mathcal{N}) \cap \mathcal{S} = \{0\}$ then $\mathcal{M}, \mathcal{N}$ and $\mathcal{S}$ satisfy the IC.
5. If $\overline{\mathcal{M} + \mathcal{N}} \cap \overline{\mathcal{S}} = \overline{\mathcal{M}} \cap \overline{\mathcal{N}}$ then $\overline{\mathcal{M}}, \overline{\mathcal{N}}$ and $\overline{\mathcal{S}}$ satisfy the IC.
6. If $T \in \operatorname{lr}(\mathcal{H})$ then $\operatorname{ran} T \cap \operatorname{dom} T$, $\ker T$ and $\operatorname{mul} T$ satisfy the IC if and only if $\operatorname{mul} T \cap \operatorname{dom} T = \operatorname{ran} T \cap \ker T$.

In the sequel, given $\mathcal{M}, \mathcal{N}$, $\mathcal{S}$ subspaces of $\mathcal{H}$ satisfying the IC, we write

$$P_{\mathcal{M},\mathcal{N},\mathcal{S}} := P_{\mathcal{M},\mathcal{N}} \hat{+} (\{0\} \times \mathcal{S}).$$

In other words, $P_{\mathcal{M},\mathcal{N},\mathcal{S}}$ is the unique idempotent with $\ker(I - P_{\mathcal{M},\mathcal{N},\mathcal{S}}) = \mathcal{M}$, $\ker P_{\mathcal{M},\mathcal{N},\mathcal{S}} = \mathcal{N}$ and $\operatorname{mul} P_{\mathcal{M},\mathcal{N},\mathcal{S}} = \mathcal{S}$. We emphasize here that throughout this paper, the notation $P_{\mathcal{M},\mathcal{N},\mathcal{S}}$ is used for $P_{\mathcal{M},\mathcal{N}} \hat{+} (\{0\} \times \mathcal{S})$ only when $\mathcal{M}, \mathcal{N}$, $\mathcal{S}$ satisfy the IC. By Proposition 4.12,

$$P_{\mathcal{M},\mathcal{N},\mathcal{S}} = P_{\mathcal{M}+\mathcal{S},\mathcal{N}+\mathcal{S}} \cap ((\mathcal{M} + \mathcal{N}) \times \mathcal{H}).$$

In particular, $P_{\mathcal{M},\mathcal{N}} = P_{\mathcal{M},\mathcal{N},\mathcal{M}\cap\mathcal{N}}$.

Corollary 4.15. *Let $\mathcal{M}, \mathcal{N}, \mathcal{S}$ be subspaces of $\mathcal{H}$ satisfying the IC. Then $P_{\mathcal{M},\mathcal{N},\mathcal{S}}^{-1} = P_{\mathcal{M},\mathcal{S},\mathcal{N}}$ and $I - P_{\mathcal{M},\mathcal{N},\mathcal{S}} = P_{\mathcal{N},\mathcal{M},\mathcal{S}}$.*

5. The closure and adjoint

This section is devoted to study the closure and the adjoint of sub-, super- and idempotent relations. The adjoint and the closure of semi-projections are again semi-projections. The following formulae for E^* and $\overline{E}$ where proved in [4] and [13].

Proposition 5.1. *If $E = P_{\mathcal{M},\mathcal{N}}$ then*

$$E^* = P_{\mathcal{N}^\perp,\mathcal{M}^\perp} \text{ and } \overline{E} = P_{\overline{\mathcal{M}},\overline{\mathcal{N}}}.$$

Moreover, E is closed if and only if $\mathcal{M}$ and $\mathcal{N}$ are closed.

Example 5.2. Let $\mathcal{M}$ and $\mathcal{N}$ be subspaces of $\mathcal{H}$ such that $\mathcal{M} \cap \mathcal{N} = \{0\}$ but $\overline{\mathcal{M}} \cap \overline{\mathcal{N}} \neq \{0\}$ and $\mathcal{M}^\perp \cap \mathcal{N}^\perp \neq \{0\}$. Let E be the projection onto $\mathcal{M}$ with kernel $\mathcal{N}$. Then, by Proposition 5.1, E^* and $\overline{E}$ are both semi-projections but they are not projections.

We begin by studying the closure and the adjoint of sub- and super-idempotents.

Lemma 5.3. *Let $\mathcal{M}, \mathcal{N}, \mathcal{S}$ be closed subspaces of $\mathcal{H}$. Then*

1. *$P_{\mathcal{M},\mathcal{N}} \cap (\mathcal{S} \times \mathcal{H})$ is closed.*
2. *$P_{\mathcal{M},\mathcal{N}} \hat{+} (\{0\} \times \mathcal{S})$ is closed if and only if $\mathcal{M}^\perp + \mathcal{N}^\perp + \mathcal{S}^\perp$ is closed.*

Proof. By Proposition 5.1, $P_{\mathcal{M},\mathcal{N}}$ is closed. Since $\mathcal{S} \times \mathcal{H}$ is closed, item 1 follows.

By Lemma 2.7, $P_{\mathcal{M},\mathcal{N}} \hat{+} (\{0\} \times \mathcal{S})$ is closed if and only if

$$P^*_{\mathcal{M},\mathcal{N}} \hat{+} (\{0\} \times \mathcal{S})^* = P_{\mathcal{N}^\perp,\mathcal{M}^\perp} \hat{+} (\mathcal{S}^\perp \times \mathcal{H})$$

is closed. But it is easy to check that

$$P_{\mathcal{N}^\perp,\mathcal{M}^\perp} \hat{+} (\mathcal{S}^\perp \times \mathcal{H}) = (\mathcal{M}^\perp + \mathcal{N}^\perp + \mathcal{S}^\perp) \times \mathcal{H}.$$

The latter is closed if and only if $\mathcal{M}^\perp + \mathcal{N}^\perp + \mathcal{S}^\perp$ is closed. □

Corollary 5.4.

$$\overline{P_{\mathcal{M},\mathcal{N}} \hat{+} (\{0\} \times \mathcal{S})} = P_{\overline{\mathcal{M}},\overline{\mathcal{N}}} \hat{+} (\{0\} \times \overline{\mathcal{S}})$$

if and only if $\mathcal{M}^\perp + \mathcal{N}^\perp + \mathcal{S}^\perp$ is closed.

Lemma 5.5. *Let $\mathcal{M}, \mathcal{N}, \mathcal{S}$ be operator ranges in $\mathcal{H}$ such that $\mathcal{M} + \mathcal{N} + \mathcal{S}$ is closed. Then*

$$\overline{P_{\mathcal{M},\mathcal{N}} \cap (\mathcal{S} \times \mathcal{H})} = P_{\overline{\mathcal{M}},\overline{\mathcal{N}}} \cap (\overline{\mathcal{S}} \times \mathcal{H}).$$

Proof. Consider $T_1 = P_{\mathcal{M},\mathcal{N}}$ and $T_2 = \mathcal{S} \times \mathcal{H}$. Then T_1 and T_2 are operator ranges (see [7]) and $T_1 \hat{+} T_2 = P_{\mathcal{M},\mathcal{N}} \hat{+} (\mathcal{S} \times \mathcal{H}) = (\mathcal{M} + \mathcal{N} + \mathcal{S}) \times \mathcal{H}$ is closed because $\mathcal{M} + \mathcal{N} + \mathcal{S}$ is closed. Then, applying Propositions 2.3 and 5.1, $\overline{P_{\mathcal{M},\mathcal{N}} \cap (\mathcal{S} \times \mathcal{H})} = \overline{P_{\mathcal{M},\mathcal{N}}} \cap \overline{\mathcal{S} \times \mathcal{H}} = P_{\overline{\mathcal{M}},\overline{\mathcal{N}}} \cap (\overline{\mathcal{S}} \times \mathcal{H})$. □

In the next result we characterize the super-idempotents that are closed and, in particular, the closed idempotent relations.

Proposition 5.6. *Let E be super-idempotent and set $\mathcal{M} := \ker(I-E), \mathcal{N} = \ker E$ and $\mathcal{S} = \operatorname{mul} E$. Then the following are equivalent:*

i) E is closed;
ii) $\mathcal{M}, \mathcal{N}, \mathcal{S}$ and $\mathcal{M}^\perp + \mathcal{N}^\perp + \mathcal{S}^\perp$ are closed.

Proof. By Proposition 3.13, $E = P_{\mathcal{M},\mathcal{N}} \hat{+} (\{0\} \times \mathcal{S})$. If E is closed, $\mathcal{N} = \ker E$ and $\mathcal{S} = \operatorname{mul} E$ are closed. Also, since I is bounded, $\overline{I-E} = (I-E)^{**} = I - E^{**} = I - \overline{E} = I - E$. So that $I-E$ is closed and $\mathcal{M} = \ker(I-E)$ is closed. Then, by Lemma 5.3, $\mathcal{M}^\perp + \mathcal{N}^\perp + \mathcal{S}^\perp$ is closed. The converse follows applying Lemma 5.3 once again. □

As a corollary we get a characterization of the closed idempotents.

Theorem 5.7. *Let $E = P_{\mathcal{M},\mathcal{N},\mathcal{S}}$. Then E is closed if and only if $\mathcal{M}, \mathcal{N}, \mathcal{S}$ and $\mathcal{M}^\perp + \mathcal{N}^\perp + \mathcal{S}^\perp$ are closed.*

Our next goal is to study the adjoint and closure of idempotent relations. In general, these operations are not closed on $\mathrm{Id}(\mathcal{H})$ (see Examples 5.18 and 5.19).

Proposition 5.8.

$$(P_{\mathcal{M},\mathcal{N}} \hat{+} (\{0\} \times \mathcal{S}))^* = P_{\mathcal{N}^\perp,\mathcal{M}^\perp} \cap (\mathcal{S}^\perp \times \mathcal{H}).$$

Proof. Apply (2.3) and Proposition 5.1. □

By the above proposition, we get that the adjoint of a super-idempotent is always sub-idempotent. However, a similar statement is no longer valid if we interchange the sub- and super-idempotent condition.

Proposition 5.9.

$$(P_{\mathcal{M},\mathcal{N}} \cap (\mathcal{S} \times \mathcal{H}))^* = P_{\mathcal{N}^\perp,\mathcal{M}^\perp} \hat{+} (\{0\} \times \mathcal{S}^\perp)$$

if and only if

$$\overline{\mathcal{M}} + \overline{\mathcal{N}} + \overline{\mathcal{S}} \text{ is closed and } \overline{P_{\mathcal{M},\mathcal{N}} \cap (\mathcal{S} \times \mathcal{H})} = P_{\overline{\mathcal{M}},\overline{\mathcal{N}}} \cap (\overline{\mathcal{S}} \times \mathcal{H}).$$

Proof. Set $R := P_{\mathcal{M},\mathcal{N}} \cap (\mathcal{S} \times \mathcal{H})$. If $R^* = P_{\mathcal{N}^\perp,\mathcal{M}^\perp} \hat{+} (\{0\} \times \mathcal{S}^\perp)$ then, by Corollary 5.4, $\overline{\mathcal{M}} + \overline{\mathcal{N}} + \overline{\mathcal{S}}$ is closed. Also, $\overline{R} = R^{**} = P_{\overline{\mathcal{M}},\overline{\mathcal{N}}} \cap (\overline{\mathcal{S}} \times \mathcal{H})$.

Conversely, if $\overline{R} = P_{\overline{\mathcal{M}},\overline{\mathcal{N}}} \cap (\overline{\mathcal{S}} \times \mathcal{H})$ then, by Proposition 5.1,

$$R^* = \overline{P_{\mathcal{N}^\perp,\mathcal{M}^\perp} \hat{+} (\{0\} \times \mathcal{S}^\perp)} = P_{\mathcal{N}^\perp,\mathcal{M}^\perp} \hat{+} (\{0\} \times \mathcal{S}^\perp),$$

because $\overline{\mathcal{M}} + \overline{\mathcal{N}} + \overline{\mathcal{S}}$ is closed. □

Corollary 5.10. *Let $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{S}$ be operator ranges of $\mathcal{H}$ such that $\mathcal{M} + \mathcal{N} + \mathcal{S}$ is closed. Then*

$$(P_{\mathcal{M},\mathcal{N}} \cap (\mathcal{S} \times \mathcal{H}))^* = P_{\mathcal{N}^\perp,\mathcal{M}^\perp} \hat{+} (\{0\} \times \mathcal{S}^\perp).$$

Proof. Apply Lemma 5.5 and Proposition 5.9. □

Corollary 5.11. *Let $E \in \mathrm{lr}(\mathcal{H})$ be sub-idempotent. Then E^* is super-idempotent if and only if $\overline{\mathrm{ran}}\, E + \overline{\mathrm{ran}}\,(I-E) + \overline{\mathrm{dom}}\, E$ is closed and $\overline{E} = P_{\overline{\mathrm{ran}}\, E, \overline{\mathrm{ran}}\,(I-E)} \cap (\overline{\mathrm{dom}}\, E \times \mathcal{H})$.*

Proof. If E^* is super-idempotent then, by Proposition 3.13,

$$E^* = P_{\ker(I-E^*),\ker E^*} \hat{+} (\{0\} \times \mathrm{mul}\, E^*) = P_{\mathrm{ran}(I-E)^\perp, \mathrm{ran}\, E^\perp} \hat{+} (\{0\} \times \mathrm{dom}\, E^\perp).$$

Then, by Lemma 5.3, $\overline{\mathrm{ran}}\, E + \overline{\mathrm{ran}}\,(I-E) + \overline{\mathrm{dom}}\, E$ is closed and, by (2.3) and Proposition 5.1, $\overline{E} = P_{\overline{\mathrm{ran}}\, E, \overline{\mathrm{ran}}\,(I-E)} \cap (\overline{\mathrm{dom}}\, E \times \mathcal{H})$. Conversely, since E is sub-idempotent, by Proposition 3.10, $E = P_{\mathrm{ran}\, E, \mathrm{ran}(I-E)} \cap (\mathrm{dom}\, E \times \mathcal{H})$. Then, by Proposition 5.9, $E^* = P_{\mathrm{ran}(I-E)^\perp, \mathrm{ran}\, E^\perp} \hat{+} (\{0\} \times \mathrm{dom}\, E^\perp)$. Then, by Proposition 3.13, E^* is super-idempotent. □

As a corollary of Proposition 5.9 we get the following characterization of those idempotents admitting an idempotent adjoint.

Theorem 5.12. *Let $E = P_{\mathcal{M},\mathcal{N},\mathcal{S}}$. Then $E^* \in \mathrm{Id}(\mathcal{H})$ if and only if*

$$\overline{E} = P_{\overline{\mathcal{M}+\mathcal{S}},\overline{\mathcal{N}+\mathcal{S}}} \cap ((\overline{\mathcal{M}+\mathcal{N}}) \times \mathcal{H})$$

and $\overline{\mathcal{M}+\mathcal{S}} + \overline{\mathcal{N}+\mathcal{S}} + \overline{\mathcal{M}+\mathcal{N}}$ is closed.

Proof. The result follows applying Proposition 5.9 to $E = P_{\mathcal{M}+\mathcal{S},\mathcal{N}+\mathcal{S}} \cap ((\mathcal{M}+\mathcal{N}) \times \mathcal{H})$. □

Proposition 5.13. *Let $E = P_{\mathcal{M},\mathcal{N},\mathcal{S}}$. Then the following are equivalent:*

i) $E^* \in \mathrm{Id}(\mathcal{H})$;
ii) $E^* = P_{\mathcal{N}^\perp\cap\mathcal{S}^\perp,\mathcal{M}^\perp\cap\mathcal{S}^\perp,\mathcal{M}^\perp\cap\mathcal{N}^\perp}$;
iii) $(\mathcal{N}^\perp + \mathcal{M}^\perp) \cap \mathcal{S}^\perp = \mathcal{N}^\perp \cap \mathcal{S}^\perp + \mathcal{M}^\perp \cap \mathcal{S}^\perp$.

Proof. Since $P_{\ker(I-E^*),\ker E^*} \hat{+} (\{0\} \times \mathrm{mul}\, E^*) = P_{\mathcal{N}^\perp\cap\mathcal{S}^\perp,\mathcal{M}^\perp\cap\mathcal{S}^\perp,\mathcal{M}^\perp\cap\mathcal{N}^\perp}$ and E^* is sub-idempotent, the result follows from Proposition 3.17 and Corollary 4.3. □

Corollary 5.14. *Let $E = P_{\mathcal{M},\mathcal{N},\mathcal{S}}$. Then the following are equivalent:*

i) E^* *and* $\overline{E} \in \mathrm{Id}(\mathcal{H})$;
ii) $(\mathcal{N}^\perp + \mathcal{M}^\perp) \cap \mathcal{S}^\perp = \mathcal{N}^\perp \cap \mathcal{S}^\perp + \mathcal{M}^\perp \cap \mathcal{S}^\perp$ *and*
$(\overline{\mathcal{M}+\mathcal{S}}+\overline{\mathcal{N}+\mathcal{S}})\cap\overline{\mathcal{M}+\mathcal{N}} = \overline{\mathcal{M}+\mathcal{S}}\cap\overline{\mathcal{M}+\mathcal{N}}+\overline{\mathcal{N}+\mathcal{S}}\cap\overline{\mathcal{M}+\mathcal{N}}$;
iii) $E^* = P_{\mathcal{N}^\perp\cap\mathcal{S}^\perp,\mathcal{M}^\perp\cap\mathcal{S}^\perp,\mathcal{M}^\perp\cap\mathcal{N}^\perp}$ *and*
$\overline{E} = P_{\overline{\mathcal{M}+\mathcal{S}}\cap\overline{\mathcal{M}+\mathcal{N}},\overline{\mathcal{N}+\mathcal{S}}\cap\overline{\mathcal{M}+\mathcal{N}},\overline{\mathcal{M}+\mathcal{S}}\cap\overline{\mathcal{N}+\mathcal{S}}}$.

Using results about the adjoint of linear relations [16] and operator ranges [12], we give examples of closed idempotent linear relations with idempotent adjoint and idempotent linear relations admitting idempotent adjoint and idempotent closure.

Proposition 5.15. *Let $E = P_{\mathcal{M},\mathcal{N},\mathcal{S}}$ such that $\mathcal{M}+\mathcal{S}$ and $\mathcal{N}$ are closed. If*

$$\mathcal{N}^\perp \cap \mathcal{S}^\perp + \mathcal{N}^\perp \cap \mathcal{M}^\perp = \mathcal{N}^\perp$$

then E is closed and $E^* \in \mathrm{Id}(\mathcal{H})$.

Proof. It follows by Lemma 2.8 applied to $A = P_{\mathcal{N}^\perp\cap\mathcal{S}^\perp,\mathcal{M}^\perp\cap\mathcal{S}^\perp,\mathcal{M}^\perp\cap\mathcal{N}^\perp}$ and $B = P_{\mathcal{M},\mathcal{N},\mathcal{S}}$. □

Proposition 5.16. *Let $E = P_{\mathcal{M},\mathcal{N},\mathcal{S}}$ where $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{S}$ are operator ranges. If $\mathcal{M}+\mathcal{N}+\mathcal{S}$ is closed then $E^* \in \mathrm{Id}(\mathcal{H})$ and $\overline{E} \in \mathrm{Id}(\mathcal{H})$.*

Proof. If $\mathcal{M}+\mathcal{N}+\mathcal{S}$ is closed, since $\mathcal{S}=(\mathcal{M}+\mathcal{S})\cap(\mathcal{N}+\mathcal{S})$, applying Proposition 2.3 to $\mathcal{M}+\mathcal{S}$ and $\mathcal{N}+\mathcal{S}$, we get that

$$\mathcal{S}^\perp=\mathcal{M}^\perp\cap\mathcal{S}^\perp+\mathcal{N}^\perp\cap\mathcal{S}^\perp. \tag{5.1}$$

Then $(\mathcal{M}^\perp+\mathcal{N}^\perp)\cap\mathcal{S}^\perp\subseteq\mathcal{S}^\perp=\mathcal{M}^\perp\cap\mathcal{S}^\perp+\mathcal{N}^\perp\cap\mathcal{S}^\perp$. Therefore, $(\mathcal{M}^\perp+\mathcal{N}^\perp)\cap\mathcal{S}^\perp=\mathcal{M}^\perp\cap\mathcal{S}^\perp+\mathcal{N}^\perp\cap\mathcal{S}^\perp$ and, by Proposition 5.13, $E^*\in\mathrm{Id}(\mathcal{H})$.

Applying again Proposition 2.3 to $\mathcal{M}+\mathcal{N}$ and $\mathcal{S}$, it follows that

$$((\mathcal{M}+\mathcal{N})\cap\mathcal{S})^\perp=\mathcal{M}^\perp\cap\mathcal{N}^\perp+\mathcal{S}^\perp.$$

So that $\mathcal{M}^\perp\cap\mathcal{N}^\perp+\mathcal{S}^\perp$ is closed. Therefore, by (5.1), $\mathcal{M}^\perp\cap\mathcal{N}^\perp+\mathcal{M}^\perp\cap\mathcal{S}^\perp+\mathcal{N}^\perp\cap\mathcal{S}^\perp$ is closed. Then, from the same arguments in the first part of the proof applied to E^*, it follows that $\overline{E}\in\mathrm{Id}(\mathcal{H})$. □

Remark 5.17. By Proposition 5.16, if $\mathcal{H}$ is finite-dimensional then the adjoint of every idempotent linear relation is idempotent.

Now, we are in a position to give an example of an idempotent E such that $E^*\notin\mathrm{Id}(\mathcal{H})$.

Example 5.18. Let $\mathcal{X}$ be an infinite dimensional Hilbert space and set $\mathcal{H}:=\mathcal{X}\times\mathcal{X}\times\mathcal{X}$. Take $\mathcal{M}:=\mathcal{X}\times\{0\}\times\{0\}$ and $\mathcal{N}:=\{0\}\times\mathcal{X}\times\{0\}$. Let $\mathcal{Z}:=\{0\}\times\{0\}\times\mathcal{X}$ and $\mathcal{W}:=\{(x,x,0):x\in\mathcal{X}\}\subseteq\mathcal{M}\dot{+}\mathcal{N}$. Following similar arguments as those found in [9, page 28], we can construct a closed subspace $\mathcal{S}$ such that $\mathcal{S}\cap\mathcal{W}=\{0\}$ and $c_0(\mathcal{S},\mathcal{W})=1$, so that $\mathcal{S}\dot{+}\mathcal{W}$ is not closed and $\mathcal{S}\subseteq\ \mathcal{W}+\mathcal{Z}:=\Pi$.

The subspace $\mathcal{M}\dot{+}\mathcal{N}=\mathcal{X}\times\mathcal{X}\times\{0\}$ is closed and $\mathcal{M}\dot{+}\Pi=\mathcal{H}$. In fact, if $x\in\mathcal{M}\cap\Pi$ then there exists $\alpha\in\mathcal{X}$ such that $x=(\alpha,0,0)\in\Pi$. But also $x=(\beta,\beta,\gamma)$ with $\beta,\gamma\in\mathcal{X}$. So that $\alpha=\beta=0=\gamma$ and $x=0$. Also, given $x=(\alpha,\beta,\gamma)\in\mathcal{H}$ then $x=(\alpha-\beta,0,0)+(\beta,\beta,\gamma)\in\mathcal{M}\dot{+}\Pi$.

Now, let us see that $\mathcal{M}\dot{+}\mathcal{S}$ is closed. In fact, $\mathcal{M}\cap\mathcal{S}\subseteq\mathcal{M}\cap\Pi=\{0\}$. Then $\mathcal{M}\dot{+}\mathcal{S}\subseteq\mathcal{M}\dot{+}\Pi$ so that,

$$c_0(\mathcal{M},\mathcal{S})\leq c_0(\mathcal{M},\Pi)<1.$$

Hence $\mathcal{M}\dot{+}\mathcal{S}$ is closed. In a similar way, $\mathcal{N}\dot{+}\mathcal{S}$ is closed.

Also $\mathcal{M},\mathcal{N}$ and $\mathcal{S}$ satisfy the IC: in fact $(\mathcal{M}\dot{+}\mathcal{N})\cap\mathcal{S}=(\mathcal{M}\dot{+}\mathcal{N})\cap\mathcal{S}\cap\Pi=\mathcal{W}\cap\mathcal{S}=\{0\}=\mathcal{M}\cap\mathcal{N}$, where we used that $(\mathcal{M}\dot{+}\mathcal{N})\cap\Pi=\mathcal{W}$. Set $E:=P_{\mathcal{M},\mathcal{N},\mathcal{S}}$. Since $\mathcal{M},\mathcal{N},\mathcal{S}$ are closed and $\mathcal{M}^\perp+\mathcal{N}^\perp+\mathcal{S}^\perp=\mathcal{H}$, E is closed, by Proposition 5.6. Now, since $\mathcal{S}\dot{+}\mathcal{W}\subseteq\mathcal{S}\dot{+}(\mathcal{M}\dot{+}\mathcal{N})$, it follows that $1=c_0(\mathcal{S},\mathcal{W})\leq c_0(\mathcal{S},\mathcal{M}\dot{+}\mathcal{N})$. Then $c_0(\mathcal{S},\mathcal{M}\dot{+}\mathcal{N})=1$ and $\mathcal{M}+\mathcal{N}+\mathcal{S}$ is not closed. But, since $\mathcal{M}\dot{+}\mathcal{N}$, $\mathcal{M}\dot{+}\mathcal{S}$ and $\mathcal{N}\dot{+}\mathcal{S}$ are closed, if $E^*\in\mathrm{Id}(\mathcal{H})$ then, by Proposition 5.13, $E^*=P_{\mathcal{N}^\perp\cap\mathcal{S}^\perp,\mathcal{M}^\perp\cap\mathcal{S}^\perp,\mathcal{M}^\perp\cap\mathcal{N}^\perp}$ and, by Proposition

5.6, $\overline{\mathcal{N}+\mathcal{S}}+\overline{\mathcal{M}+\mathcal{S}}+\overline{\mathcal{M}+\mathcal{N}}=\mathcal{M}+\mathcal{N}+\mathcal{S}$ is closed, which is absurd. Hence $E^* \notin \mathrm{Id}(\mathcal{H})$.

The same example provides an idempotent F such that $\overline{F} \notin \mathrm{Id}(\mathcal{H})$ and also a linear relation $T \notin \mathrm{Id}(\mathcal{H})$ such that $T^* \in \mathrm{Id}(\mathcal{H})$.

Example 5.19. Let $\mathcal{H}$ and $\mathcal{M},\mathcal{N},\mathcal{S}$ be as in Example 5.18. Set $E := P_{\mathcal{M},\mathcal{N},\mathcal{S}}$ and $F := P_{\mathcal{N}^\perp\cap\mathcal{S}^\perp,\mathcal{M}^\perp\cap\mathcal{S}^\perp,\mathcal{M}^\perp\cap\mathcal{N}^\perp}$. Then $E, F \in \mathrm{Id}(\mathcal{H})$ and since $\mathcal{M}\dot{+}\mathcal{N}$, $\mathcal{M}\dot{+}\mathcal{S}$ and $\mathcal{N}\dot{+}\mathcal{S}$ are closed,

$$E = P_{\mathcal{M}+\mathcal{S},\mathcal{N}+\mathcal{S}} \cap ((\mathcal{M}+\mathcal{N})\times\mathcal{H}) = P_{\overline{\mathcal{M}+\mathcal{S}},\overline{\mathcal{N}+\mathcal{S}}} \cap (\overline{\mathcal{M}+\mathcal{N}}\times\mathcal{H}) = F^*.$$

Hence $E^* = \overline{F}$. By Example 5.18, $\overline{F} \notin \mathrm{Id}(\mathcal{H})$.

On the other hand, set $T := \overline{F}$ then $T \notin \mathrm{Id}(\mathcal{H})$ and $T^* = F^* = E \in \mathrm{Id}(\mathcal{H})$.

If $E = P_{\mathcal{M},\mathcal{N},\mathcal{S}}$ then the inclusions

$$P_{\overline{\mathcal{M}},\overline{\mathcal{N}}} \hat{+} (\{0\}\times\overline{\mathcal{S}}) \subseteq \overline{E} \subseteq P_{\overline{\mathcal{M}+\mathcal{S}},\overline{\mathcal{N}+\mathcal{S}}} \cap (\overline{\mathcal{M}+\mathcal{N}}\times\mathcal{H}) \tag{5.2}$$

always hold with equalities when E is a semi-projection. In the next proposition we provide conditions to get equalities in (5.2).

Proposition 5.20. *Let $E = P_{\mathcal{M},\mathcal{N},\mathcal{S}}$. Then:*

1. *If $E^* \in \mathrm{Id}(\mathcal{H})$ then $\overline{E} = P_{\overline{\mathcal{M}+\mathcal{S}},\overline{\mathcal{N}+\mathcal{S}}} \cap (\overline{\mathcal{M}+\mathcal{N}}\times\mathcal{H})$.*
2. *$\overline{E} = P_{\overline{\mathcal{M}},\overline{\mathcal{N}}} \hat{+} (\{0\}\times\overline{\mathcal{S}})$ if and only if $\mathcal{M}^\perp+\mathcal{N}^\perp+\mathcal{S}^\perp$ is closed.*
3. *If $E^* \in \mathrm{Id}(\mathcal{H})$ and $\mathcal{M}^\perp+\mathcal{N}^\perp+\mathcal{S}^\perp$ is closed then there is equality in (5.2) and*
$$\overline{E} = P_{\overline{\mathcal{M}}+\overline{\mathcal{N}}\cap\overline{\mathcal{S}},\overline{\mathcal{N}}+\overline{\mathcal{M}}\cap\overline{\mathcal{S}},\overline{\mathcal{S}}+\overline{\mathcal{M}}\cap\overline{\mathcal{N}}}.$$
Moreover,
$$\overline{E} = P_{\overline{\mathcal{M}}+\overline{\mathcal{S}},\overline{\mathcal{N}}+\overline{\mathcal{S}}} \cap ((\overline{\mathcal{M}}+\overline{\mathcal{N}})\times\mathcal{H}).$$

Proof. 1: Use Theorem 5.12.

2: Use Corollary 5.4.

3: From 1 and 2, $\overline{E}$ is sub- and super-idempotent. Then $\overline{E} \in \mathrm{Id}(\mathcal{H})$ and Corollary 5.14 gives the formula for $\overline{E}$. Finally, using Lemma 3.4 we get that $\ker\overline{E} = \overline{\mathcal{N}}+\overline{\mathcal{M}}\cap\overline{\mathcal{S}}$, $\ker(I-\overline{E}) = \overline{\mathcal{M}}+\overline{\mathcal{N}}\cap\overline{\mathcal{S}}$ and $\mathrm{mul}\,\overline{E} = \overline{\mathcal{S}}+\overline{\mathcal{M}}\cap\overline{\mathcal{N}}$, $\mathrm{ran}\,\overline{E} = \overline{\mathcal{M}}+\overline{\mathcal{S}}$, $\mathrm{ran}(I-\overline{E}) = \overline{\mathcal{N}}+\overline{\mathcal{S}}$ and $\mathrm{dom}\,\overline{E} = \overline{\mathcal{M}}+\overline{\mathcal{N}}$. ☐

If E is a closed semi-projection then E^* is a semi-projection, and $\mathrm{ran}\,E$ and $\mathrm{ran}(I-E) = \ker E$ are both closed. In general, this is no longer true for closed idempotents. In what follows, we characterize those closed idempotents E with $\mathrm{ran}\,E$ and $\mathrm{ran}(I-E)$ closed such that E^* is idempotent. In this case, by Theorem 2.6, $\mathrm{ran}\,E^*$ and $\mathrm{ran}(I-E^*)$ are closed.

Proposition 5.21. *Consider $E = P_{\mathcal{M},\mathcal{N},\mathcal{S}}$ such that E is closed. Then the following are equivalent:*

i) $E^* \in \mathrm{Id}(\mathcal{H})$, $\operatorname{ran} E$ *and* $\operatorname{ran}(I-E)$ *are closed;*
ii) $\overline{\mathcal{M}+\mathcal{N}}+\mathcal{S}$ *is closed and* $\overline{\mathcal{M}+\mathcal{N}}\cap\mathcal{S} = \mathcal{M}\cap\mathcal{N}$.

Proof. If $i)$ holds, by Corollaries 4.1 and 4.4, $\mathcal{M}+\mathcal{S}$ and $\mathcal{N}+\mathcal{S}$ are closed and, applying Theorem 5.12, $\overline{\mathcal{M}+\mathcal{N}}+\mathcal{S} = \overline{\mathcal{M}+\mathcal{N}}+\overline{\mathcal{N}+\mathcal{S}}+\overline{\mathcal{M}+\mathcal{S}}$ is closed. On the other hand, by Theorem 5.12 again, $E = \overline{E} = P_{\mathcal{M}+\mathcal{S},\mathcal{N}+\mathcal{S}} \cap (\overline{\mathcal{M}+\mathcal{N}}\times\mathcal{H})$ then $\mathcal{M}+\mathcal{N} = \operatorname{dom} E = (\mathcal{M}+\mathcal{N}+\mathcal{S})\cap\overline{\mathcal{M}+\mathcal{N}} = \mathcal{M}+\mathcal{N}+\mathcal{S}\cap\overline{\mathcal{M}+\mathcal{N}}$, so that $\mathcal{S}\cap\overline{\mathcal{M}+\mathcal{N}} \subseteq \mathcal{M}+\mathcal{N}$. Then $\mathcal{S}\cap\overline{\mathcal{M}+\mathcal{N}} = \mathcal{S}\cap(\mathcal{M}+\mathcal{N}) = \mathcal{M}\cap\mathcal{N}$, where we used that $\mathcal{M},\mathcal{N}$ and $\mathcal{S}$ satisfy the IC.

Conversely, if $ii)$ holds then $\overline{\mathcal{M}+\mathcal{N}}\cap\mathcal{S} = \mathcal{M}\cap\mathcal{N} = \mathcal{M}\cap\mathcal{S} = \mathcal{N}\cap\mathcal{S}$, because $\mathcal{M},\mathcal{N}$ and $\mathcal{S}$ satisfy the IC. Then, by Lemma 2.2 and Theorem 2.1,

$$c(\mathcal{M},\mathcal{S}) \leq c(\overline{\mathcal{M}+\mathcal{N}},\mathcal{S}) < 1.$$

So that, by Theorem 2.1, $\operatorname{ran} E = \mathcal{M}+\mathcal{S}$ is closed. Likewise, $\operatorname{ran}(I-E) = \mathcal{N}+\mathcal{S}$ is closed. Therefore, $\overline{\mathcal{M}+\mathcal{S}}+\overline{\mathcal{N}+\mathcal{S}}+\overline{\mathcal{M}+\mathcal{N}} = \overline{\mathcal{M}+\mathcal{N}}+\mathcal{S}$ is closed. Also, since $E \subseteq P_{\mathcal{M}+\mathcal{S},\mathcal{N}+\mathcal{S}}\cap(\overline{\mathcal{M}+\mathcal{N}}\times\mathcal{H})$, $\operatorname{dom}(P_{\mathcal{M}+\mathcal{S},\mathcal{N}+\mathcal{S}}\cap(\overline{\mathcal{M}+\mathcal{N}}\times\mathcal{H})) = \mathcal{M}+\mathcal{N}+\mathcal{S}\cap\overline{\mathcal{M}+\mathcal{N}} = \mathcal{M}+\mathcal{N} = \operatorname{dom} E$ and $\operatorname{mul}(P_{\mathcal{M}+\mathcal{S},\mathcal{N}+\mathcal{S}}\cap(\overline{\mathcal{M}+\mathcal{N}}\times\mathcal{H})) = \mathcal{M}\cap\mathcal{S}+\mathcal{N}\cap\mathcal{S} = \mathcal{S} = \operatorname{mul} E$, using Theorem 5.12, it follows that $E^* \in \mathrm{Id}(\mathcal{H})$. $\square$

Theorem 5.22. *Let $E = P_{\mathcal{M},\mathcal{N},\mathcal{S}}$. Then the following are equivalent:*

i) $E^* \in \mathrm{Id}(\mathcal{H})$, $\overline{E} = P_{\overline{\mathcal{M}},\overline{\mathcal{N}}} \hat{+} (\{0\}\times\overline{\mathcal{S}})$, $\operatorname{ran}\overline{E}$ *and* $\operatorname{ran}(I-\overline{E})$ *are closed;*
ii) $\mathcal{M}^\perp+\mathcal{N}^\perp+\mathcal{S}^\perp$ *and* $\overline{\mathcal{M}+\mathcal{N}}+\overline{\mathcal{S}}$ *are closed and* $\overline{\mathcal{M}+\mathcal{N}}\cap\overline{\mathcal{S}} = \overline{\mathcal{M}}\cap\overline{\mathcal{S}}+\overline{\mathcal{N}}\cap\overline{\mathcal{S}}$.

In this case, $\overline{E}\in\mathrm{Id}(\mathcal{H})$ *and*

$$\overline{E} = P_{\overline{\mathcal{M}}+\overline{\mathcal{N}}\cap\overline{\mathcal{S}},\overline{\mathcal{N}}+\overline{\mathcal{M}}\cap\overline{\mathcal{S}},\overline{\mathcal{S}}+\overline{\mathcal{M}}\cap\overline{\mathcal{N}}}. \tag{5.3}$$

Proof. Assume that item $i)$ holds. Then $\overline{E}$ is super-idempotent and, by Lemma 5.3, $\mathcal{M}^\perp+\mathcal{N}^\perp+\mathcal{S}^\perp$ is closed. Since $E^*\in\mathrm{Id}(\mathcal{H})$ then $\overline{E} = (E^*)^*$ is sub-idempotent and hence $\overline{E}\in\mathrm{Id}(\mathcal{H})$. Therefore $\overline{E} = P_{\ker(I-\overline{E}),\ker\overline{E},\operatorname{mul}\overline{E}}$ and applying Lemma 3.4, formula (5.3) follows.

By Lemma 3.4, $\operatorname{ran}\overline{E} = \overline{\mathcal{M}}+\overline{\mathcal{S}}$ and $\operatorname{ran}(I-\overline{E}) = \overline{\mathcal{N}}+\overline{\mathcal{S}}$. Then $\overline{\mathcal{M}}+\overline{\mathcal{S}}$ and $\overline{\mathcal{N}}+\overline{\mathcal{S}}$ are closed and, by Propositions 5.13 and 5.6 applied to $\overline{E}$, $\overline{\mathcal{N}+\mathcal{S}}+\overline{\mathcal{M}+\mathcal{S}}+\overline{\mathcal{M}+\mathcal{N}} = \overline{\mathcal{N}}+\overline{\mathcal{S}}+\overline{\mathcal{M}}+\overline{\mathcal{M}+\mathcal{N}} = \overline{\mathcal{S}}+\overline{\mathcal{M}+\mathcal{N}}$ is closed. On the other hand, by Corollary 5.14,

$$\overline{E} = P_{\overline{\mathcal{M}+\mathcal{S}}\cap\overline{\mathcal{M}+\mathcal{N}},\ \overline{\mathcal{N}+\mathcal{S}}\cap\overline{\mathcal{M}+\mathcal{N}},\ \overline{\mathcal{N}+\mathcal{S}}\cap\overline{\mathcal{M}+\mathcal{S}}}.$$

Then

$$\begin{aligned}\overline{\mathcal{M}+\mathcal{N}}\cap\overline{\mathcal{S}} &= \overline{\mathcal{M}+\mathcal{N}}\cap\overline{\mathcal{M}+\mathcal{S}}\cap\overline{\mathcal{N}+\mathcal{S}}\cap\overline{\mathcal{S}} = \ker(I-\overline{E})\cap\ker\overline{E}\cap\overline{\mathcal{S}}\\ &= (\overline{\mathcal{M}}+\overline{\mathcal{N}}\cap\overline{\mathcal{S}})\cap(\overline{\mathcal{N}}+\overline{\mathcal{M}}\cap\overline{\mathcal{S}})\cap\overline{\mathcal{S}}\\ &= \overline{\mathcal{M}}\cap\overline{\mathcal{S}}+\overline{\mathcal{N}}\cap\overline{\mathcal{S}}.\end{aligned}$$

Conversely, assume that item $ii)$ holds. By Lemma 5.3,

$$\overline{E} = P_{\overline{\mathcal{M}},\overline{\mathcal{N}}}\ \hat{+}\ (\{0\}\times\overline{\mathcal{S}}).$$

Since $\overline{\mathcal{M}+\mathcal{N}}\cap\overline{\mathcal{S}} = \overline{\mathcal{M}}\cap\overline{\mathcal{S}}+\overline{\mathcal{N}}\cap\overline{\mathcal{S}}$, it follows that $\overline{\mathcal{M}+\mathcal{N}}\cap\overline{\mathcal{S}} = (\overline{\mathcal{M}}+\overline{\mathcal{N}})\cap\overline{\mathcal{S}}$. From this fact it can be seen that

$$\ker(I-\overline{E}) = \overline{\mathcal{M}}+\overline{\mathcal{N}}\cap\overline{\mathcal{S}},\ \ker\overline{E} = \overline{\mathcal{N}}+\overline{\mathcal{M}}\cap\overline{\mathcal{S}}\ \text{and}\ \operatorname{mul}\overline{E} = \overline{\mathcal{S}}+\overline{\mathcal{M}}\cap\overline{\mathcal{N}}$$

satisfy the IC. Then $\overline{E}\in\mathrm{Id}(\mathcal{H})$.

Since $\operatorname{mul}\overline{E}$ is closed, it follows that

$$\begin{aligned}\overline{\ker(I-\overline{E})+\ker\overline{E}}+\overline{\operatorname{mul}\overline{E}} &= \overline{\mathcal{M}+\mathcal{N}}+\overline{\mathcal{S}}+\overline{\mathcal{M}}\cap\overline{\mathcal{N}}\\ &= \overline{\mathcal{M}+\mathcal{N}}+\overline{\mathcal{S}}\end{aligned}$$

is closed. In a similar fashion, it can be seen that

$$\overline{\ker(I-\overline{E})+\ker\overline{E}}\cap\overline{\operatorname{mul}\overline{E}} = \ker(I-\overline{E})\cap\ker\overline{E}.$$

Then, by Proposition 5.21 applied to $\overline{E}$, $E^*\in\mathrm{Id}(\mathcal{H})$ and $\operatorname{ran}\overline{E}$, $\operatorname{ran}(I-\overline{E})$ are closed. □

Corollary 5.23. *Let* $E = P_{\mathcal{M},\mathcal{N},\mathcal{S}}$ *such that* $\overline{\mathcal{M}+\mathcal{N}}\cap\overline{\mathcal{S}} = \overline{\mathcal{M}}\cap\overline{\mathcal{N}}$, $\overline{\mathcal{M}+\mathcal{N}}+\overline{\mathcal{S}}$ *and* $\mathcal{M}^\perp+\mathcal{S}^\perp+\mathcal{N}^\perp$ *are closed. Then:*

1. $E^*\in\mathrm{Id}(\mathcal{H})$;
2. $\overline{E} = P_{\overline{\mathcal{M}},\overline{\mathcal{N}},\overline{\mathcal{S}}}$;
3. $\operatorname{ran}\overline{E}$ *and* $\operatorname{ran}(I-\overline{E})$ *are closed.*

Proof. Since $\overline{\mathcal{M}+\mathcal{N}}\cap\overline{\mathcal{S}} = \overline{\mathcal{M}}\cap\overline{\mathcal{N}} = \overline{\mathcal{M}}\cap\overline{\mathcal{N}}\cap\overline{\mathcal{S}}$ and also $(\overline{\mathcal{M}}\cap\overline{\mathcal{S}}+\overline{\mathcal{N}}\cap\overline{\mathcal{S}})\subseteq\overline{\mathcal{M}+\mathcal{N}}\cap\overline{\mathcal{S}} = \overline{\mathcal{M}}\cap\overline{\mathcal{N}}$, then $\overline{\mathcal{M}}\cap\overline{\mathcal{N}} = \overline{\mathcal{N}}\cap\overline{\mathcal{S}} = \overline{\mathcal{M}}\cap\overline{\mathcal{S}}$. We then apply Theorem 5.22 to get that $E^*\in\mathrm{Id}(\mathcal{H})$, $\overline{E} = P_{\overline{\mathcal{M}},\overline{\mathcal{N}},\overline{\mathcal{S}}+\overline{\mathcal{M}}\cap\overline{\mathcal{N}}} = P_{\overline{\mathcal{M}},\overline{\mathcal{N}},\overline{\mathcal{S}}}$ and $\operatorname{ran}\overline{E}$ and $\operatorname{ran}(I-\overline{E})$ are closed. □

Acknowledgements. We thank Professor S. V. de Snoo for carefully reading our manuscript and giving us constructive comments.

We thank the anonymous referee.

M. L. Arias was supported in part by UBACyT (20020190100330BA) and FonCyT (PICT 2017-0883). A. Maestripieri was supported in part by the Interdisciplinary Center for Applied Mathematics at Virginia Tech. M. Contino, A. Maestripieri and S. Marcantognini were supported by CONICET PIP 11220200102127CO.

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