Operators which preserve a positive definite inner product

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Abstract

Let $\mathcal{H}$ be a Hilbert space, A a positive definite operator in $\mathcal{H}$ and $\langle f, g\rangle_A = \langle Af, g\rangle$, $f, g \in \mathcal{H}$, the A-inner product. This paper studies the geometry of the set

$$\mathcal{I}_A^a := \{ \text{ adjointable isometries for } \langle\ ,\ \rangle_A\}.$$

It is proved that $\mathcal{I}_A^a$ is a submanifold of the Banach algebra of adjointable operators, and a homogeneous space of the group of invertible operators in $\mathcal{H}$, which are unitaries for the A-inner product. Smooth curves in $\mathcal{I}_A^a$ with given initial conditions, which are minimal for the metric induced by $\langle\ ,\ \rangle_A$, are presented. This result depends on an adaptation of M.G. Krein's method for the lifting of symmetric contractions, in order that it works also for symmetrizable transformations (i.e., operators which are selfadjoint for the A-inner product).

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1 Introduction

Let A be a positive contraction in the Hilbert space $(\mathcal{H}, \langle\ \ ,\ \ \rangle)$ with trivial nullspace $N(A) = \{0\}$. Denote by $\langle\ \ ,\ \ \rangle_A$ the inner product defined by A:

$$\langle f, g\rangle_A = \langle Af, g\rangle,\ \ f, g \in \mathcal{H}.$$

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This paper considers the operators which preserve this inner product, which will be called A-isometries. That is, a bounded operator $T \in \mathcal{B}(\mathcal{H})$ is called an A-isometry if

$$\langle ATf, Tg\rangle = \langle Af, g\rangle,$$

or equivalently, $T^*AT = A$. The focus will be on A-isometries which additionally admit an adjoint for the A-inner product: there exists $S \in \mathcal{B}(\mathcal{H})$ such that $\langle Tf, g\rangle_A = \langle f, Sg\rangle_A$, for $f, g \in \mathcal{H}$. In general, an operator B in $\mathcal{H}$ will be called A-*adjointable* (or just *adjointable*) if it admits an adjoint for the A-inner product, which shall be denoted by $B^\sharp$. Denote by

$$\mathcal{I}_A := \{T \in \mathcal{B}(\mathcal{H}) : T \text{ is } A - \text{isometric}\}$$

and

$$\mathcal{I}_A^a := \{T \in \mathcal{I}_A : T \text{ is } A - \text{adjointable}\} \tag{1}$$

It will be shown that there are A-isometries which are not A-adjointable. It is the latter set $\mathcal{I}_A^a$ which admits a differentiable structure, and which is a homogeneous space of a Banach-Lie group. Moreover, endowed with a natural Finsler metric (namely, the one induced by the A-inner product), curves which have minimal length for given initial conditions are computed.

The contents of the paper are the following. In Section 2 the basic facts needed are stated, on operators in Hilbert spaces with two norms (see [16], [18], [12], [15]; see also [6], [13] for the case of Krein spaces, or [7] for unbounded operators). Examples of adjointable and non adjointable A-isometries are presented. Also Theorem 2.6 is proved, characterizing adjointable A-isometries. A result by R. Douglas [14] is used, on existence of solutions of the operator equation $AX = B$. In Section 3 the Wold decomposition of A-isometries is briefly analized. Section 4 contains Theorem 4.4, stating that $\mathcal{I}_A^a$ is a C^∞-submanifold of the $*$-Banach algebra of adjointable operators (with a suitable norm), and a homogeneous space of the Banach-Lie group of A-unitary operators (see the definition below). Section 5 presents an adaptation of Krein's method for the lifting of symmetric transformations with norm constraints, to work in the context of symmetrizable transformations (Lemma 5.1). We believe that this section is of independent interest. This result is used in Section 6 to prove Theorem 6.1, which computes minimal curves in $\mathcal{I}_A^a$ satisfying given initial conditions. Section 7 treats the action of the restricted group of A-unitaries, the orbits of the action of this group are characterized (Theorem 7.2), and it is proved that these are also C^∞-manifolds and homogeneous spaces (Proposition 7.3).

Let us finish this section with basic facts and notations. Denote by $\mathcal{L}$ the completion of the pre-Hilbert space $(\mathcal{H}, \langle\ ,\ \rangle_A)$. Note that $T \in \mathcal{I}_A$ lifts to an isometry $\mathbf{T}$ acting in $\mathcal{L}$ (denote the inner product on $\mathcal{L}$ by $\langle\ ,\ \rangle_\mathcal{L}$). Conversely, an isometry $\mathbf{V} \in \mathcal{B}(\mathcal{L})$ that leaves the dense subspace $\mathcal{H}$ invariant, i.e. $\mathbf{V}(\mathcal{H}) \subset \mathcal{H}$, induces an operator $V = \mathbf{V}|_\mathcal{H}$ in $\mathcal{H}$, which is an A-isometry.

This study relies on the theory of symmetrizable operators, (or more broadly, operators in Hilbert spaces with two norms), developed initially (and independently) by M.G. Krein [16], P.D. Lax [18] and J. Dieudonné [12], and extended afterwards by several authors, as mentioned above.

Upper case letters $T, S, X, G, \dots$ will denote operators acting in $\mathcal{H}$ (with adjoints in $\mathcal{H}$ denoted $T^*, S^*, X^*, G^*, \dots$). Their eventual liftings to $\mathcal{L}$ will be denoted with capital **bold** letters: $\mathbf{T}, \mathbf{S}, \mathbf{X}, \mathbf{G}, \dots$, and adjoints $\mathbf{T}^*, \mathbf{S}^*, \mathbf{X}^*, \mathbf{G}^*, \dots$ (in $\mathcal{L}$). That is, which adjoint is refered to, will depend on the context. Vectors in $\mathcal{H}$ will be denoted f, g, h and vectors in $\mathcal{L}$ with Greek letters φ, γ, η.

A special class of A-isometries, is given by the A-unitary operators: $G \in \mathcal{B}(\mathcal{H})$ is called A-unitary if it is an A-isometry which is invertible in $\mathcal{H}$. Denote by

$$\mathcal{U}_A = \{G \in \mathcal{B}(\mathcal{H}) : G \text{ is invertible and } G^*AG = A\},$$

the group of A-unitaries. Note that A-unitaries are adjointable. In [3] it was shown that $\mathcal{U}_A$ is a C^∞ Banach-Lie group. Clearly, $\mathcal{U}_{\mathcal{A}}$ acts on $\mathcal{I}_A$ and on $\mathcal{I}_A^a$ by left multiplication:

$$G \cdot T = GT \in \mathcal{I}_A \ , \quad \text{if } G \in \mathcal{U}_A \text{ and } T \in \mathcal{I}_A,$$

and clearly GT is adjointable if T is adjointable. Again, $G \in \mathcal{U}_A$ lifts to a unitary operator $\mathbf{G}$ such that $\mathbf{G}(\mathcal{H}) = \mathcal{H}$.

2 Symmetrizable operators

An operator B acting in $\mathcal{H}$ will be called A-symmetric (or symmetrizable) if it is symmetric for the A-inner product. Let us recall the following fact, adapted from their original broader context to our case:

Theorem 2.1. (See M.G.Krein [17], P.D. Lax [18], J. Dieudonné [12]) *Let B, C be bounded operators in $\mathcal{H}$ such that*

$$\langle Bf, g\rangle_A = \langle f, Cg\rangle_A, \quad \textit{for all } f, g \in \mathcal{H}.$$

Then they can be lifted to bounded operators $\mathbf{B}, \mathbf{C}$ in $\mathcal{L}$ such that $\mathbf{B}^* = \mathbf{C}$.

Denote by

$$\mathcal{B}_A(\mathcal{H}) = \{B \in \mathcal{B}(\mathcal{H}) : B \text{ is } A-\text{adjointable}\}. \tag{2}$$

If $B \in \mathcal{B}_A(\mathcal{H})$, denote by $B^\sharp$ its A-adjoint in $\mathcal{H}$.

If A is not invertible, $\mathcal{B}_A(\mathcal{H})$ is not closed in $\mathcal{B}(\mathcal{H})$, neither is the A-adjoint map $B \mapsto B^\sharp$ a continuous map (in the norm topology). $\mathcal{B}_A(\mathcal{H})$ is endowed with he norm

$$|B| := \max\{\|B\|, \|B^\sharp\|\}.$$

Then $(\mathcal{B}_A(\mathcal{H}), |\ \ |)$ becomes an involutive Banach algebra.

Remark 2.2. It is not difficult to see that if B is A-symmetric, then $\|\mathbf{B}\| \leq \|B\|$. Also note that A itself is A-symmetric, and that its lifting $\mathbf{A}$ remains positive definite.

Remark 2.3. A closed linear subspace $\mathcal{S} \subset \mathcal{H}$ is called *compatible with* A, A-*compatible*, or shortly *compatible*, if it admits a complement which is orthogonal with respect to the inner product defined by A. In [4], the compatible Grassmannian was studied, namely

$$Gr_A = \{\mathcal{S} \subset \mathcal{H} : \mathcal{S} \text{ is closed and compatible with } A\}.$$

Since A has trivial nullspace, if S is compatible with A, then the complement is unique, and it is given by $A(\mathcal{S})^\perp$ (the orthogonal complement of $A(\mathcal{S})$ with respect to the usual inner product of $\mathcal{H}$). This allows one to identify each compatible subspace $\mathcal{S}$ with the idempotent $Q_\mathcal{S}$ with range $\mathcal{S}$ and nullspace $A(\mathcal{S})^\perp$. Thus the compatible Grassmannian may be regarded as the following set

$$\begin{aligned} Gr_A &= \{Q \in \mathcal{B}(\mathcal{H}) : Q^2 = Q, Q^*A = AQ\} \\ &= \{Q \in \mathcal{B}(\mathcal{H}) : Q^2 = Q \text{ is symmetrizable}\}. \end{aligned}$$

In other words, $\mathbf{Q} = P_{\overline{R(Q)}}$. Consider in Gr_A the topology inherited from the norm of $\mathcal{B}_A(\mathcal{H})$. The proof of the afore-mentioned facts and examples of compatible and non-compatible subspaces can be found in [9], [10], where a systematic study of compatible subspaces was done. The notion of compatible subspaces goes back to A. Sard [23], who introduced an equivalent definition under a different terminology, to give an operator theoretic approach to problems in approximation theory (see [8]). In [4] it was shown that Gr_A is a complemented submanifold of $\mathcal{B}_A(\mathcal{H})$, and a homogeneous space of the group $\mathcal{U}_A$ under the action

$$G \cdot \mathcal{S} = G(\mathcal{S})$$

or equivalently

$$G \cdot Q_\mathcal{S} = GQ_\mathcal{S}G^{-1}, \quad G \in \mathcal{U}_A, \ \mathcal{S} \in Gr_A.$$

An A-isometry may not be an adjointable operator.

Examples 2.4.

1. Let $\mathcal{H}$ be the Dirichlet space of the unit disk $\mathbb{D}$,

$$\mathcal{H} = \{f : \mathbb{D} \to \mathbb{C} : f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n, \text{ with } \sum_{n=0}^{\infty} (n+1)|\hat{f}(n)|^2 < \infty\}$$

with its usual inner product $\langle f, g\rangle = \sum_{n=0}^{\infty}(n+1)\hat{f}(n)\overline{\hat{g}(n)}$. Let $\mathcal{L} = H^2(\mathbb{D})$ the Hardy space of $\mathbb{D}$,

$$\mathcal{L} = \{f : \mathbb{D} \to \mathbb{C} : f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n, \text{ with } \sum_{n=0}^{\infty} |\hat{f}(n)|^2 < \infty\},$$

with its usual inner product $\langle f, g\rangle = \sum_{n=0}^{\infty} \hat{f}(n)\overline{\hat{g}(n)}$. So that $A \in \mathcal{B}(\mathcal{H})$ is given by

$$Af(z) = \sum_{n=0}^{\infty} \frac{1}{n+1} \hat{f}(n) z^n.$$

Note that A is compact. Let $T = M_z \in \mathcal{B}(\mathcal{H})$ be the operator $M_z f(z) = zf(z)$. Clearly T lifts to the operator $\mathbf{T} = M_z$, the usual shift operator in $\mathcal{L}$. Then T is an adjointable A-isometry. In particular, see below (Theorem 2.6), this means that $R(T) = z\mathcal{H}$ is a compatible subspace.

2. Consider a slight modification of the above setting. Put $\mathcal{L} = \ell^2$ and $\mathcal{H} = \{(x_n) \in \ell^2 : \sum_{n=1}^{\infty} n|x_n|^2 < \infty\}$, with the inner product $\langle (x_n), (y_n)\rangle_{\mathcal{H}} = \sum_{n=1}^{\infty} n x_n \bar{y}_n$. Let e_n, $n \geq 1$ be the canonical orthonormal basis of ℓ^2. Let $\mathbb{A}$ the subset of the natural numbers which are squares of odd integers,

$$\mathbb{A} := \{(2k+1)^2 : k \geq 0\},$$

and denote its complement

$$\mathbb{A}^c = \{\sigma(1) < \sigma(2) < \sigma(3) < \dots\},$$

i.e., $\sigma : \mathbb{N} \to \mathbb{A}^c$ is the strictly increasing counting of $\mathbb{A}^c$. Let $\mathbf{U}$ be the unitary operator in ℓ^2 given by (for $x = (x_n) \in \ell^2$)

$$(\mathbf{U}x)_n = \begin{cases} x_{n^2} & \text{if } n \text{ is odd,} \\ x_{\sigma(n/2)} & \text{if } n \text{ is even.} \end{cases}$$

Its adjoint $\mathbf{U}^*$ is given by

$$\mathbf{U}^* e_n = \begin{cases} e_{n^2} & \text{if } n \text{ is odd,} \\ e_{\sigma(n/2)} & \text{if } n \text{ is even.} \end{cases}$$

We claim that $\mathbf{U}(\mathcal{H}) \subset \mathcal{H}$, but $\mathbf{U}^*(\mathcal{H})$ is not contained in $\mathcal{H}$. Which means that $\mathbf{U}|_{\mathcal{H}}$ induces an element $U \in \mathcal{I}_A$, but U is not A-adjointable. For the first assertion, note the fact that for all $k \geq 1$, $k \leq \sigma(k) \leq 2k$. Then, if $(x_n) \in \mathcal{H}$,

$$\begin{aligned} \langle \mathbf{U}x, \mathbf{U}x\rangle_{\mathcal{H}} &= \sum_{n \text{ odd}} n|x_{n^2}|^2 + \sum_{n \text{ even}} n|x_{\sigma(n/2)}|^2 \\ &\leq \sum_{n \text{ odd}} n^2|x_{n^2}|^2 + \sum_{n \text{ even}} 2\sigma(n/2)|x_{\sigma(n/2)}|^2 \\ &\leq 2\langle x, x\rangle_{\mathcal{H}}. \end{aligned}$$

To prove that $\mathcal{H}$ is not invariant for $\mathbf{U}^*$, pick $x_k = \frac{1}{k^{3/2}}$ if k is odd and $x_k = 0$ otherwise. Then clearly, $x = (x_n) \in \mathcal{H}$, and

$$(\mathbf{U}^* x)_n = \begin{cases} \frac{1}{n^{3/4}} & \text{if } n \in \mathbb{A}, \\ 0 & \text{if } n \in \mathbb{A}^c \end{cases}$$

Or equivalently, $(\mathbf{U}^*x)_{(2k+1)^2} = \frac{1}{(2k+1)^{3/2}}$ and equal to 0 in all other entries. Then

$$\langle \mathbf{U}^*x, \mathbf{U}^*x\rangle_{\mathcal{H}} = \sum_{k=0}(2k+1)^2\left(\frac{1}{(2k+1)^{3/2}}\right)^2 = \sum_{k=0}\frac{1}{(2k+1)} = +\infty.$$

Consider the subclass of A-adjointable A-isometries, or shortly, *adjointable isometries*. Operators $G \in \mathcal{U}_A$ are examples of adjointable isometries. It is proved below that adjointable isometries are those with compatible final spaces. It will be useful to recall the following result by R. Douglas [14]:

Remark 2.5. Douglas' theorem considers the existence of solutions of the operator equation $AX = B$ [14]: let $A, B \in \mathcal{B}(\mathcal{H})$, then the following conditions are equivalent:

1. there exists $X \in \mathcal{B}(\mathcal{H})$ such that $AX = B$;
2. $R(B) \subset R(A)$;
3. there exists $\lambda > 0$ such that $BB^* \leq \lambda AA^*$.

Theorem 2.6. *Let $T \in \mathcal{I}_A$. Denote by $\mathbf{T}$ its (isometric) lifting to $\mathcal{L}$. The following are equivalent:*

1. *T is A-adjointable;*
2. *$\mathbf{T}^*(\mathcal{H}) \subset \mathcal{H}$;*
3. *$R(T)$ is a compatible subspace.*
4. *$R(T^*A) = R(A)$;*
5. *there exists $\lambda > 0$ such that*
$$T^*A^2T \leq \lambda A^2.$$

Proof. The equivalence $1 \iff 2$ is clear: if T has an A-adjoint S, then $\mathbf{T}^*$ and S coincide on $\mathcal{H}$. Then $\mathbf{T}^*(\mathcal{H}) \subset \mathcal{H}$ and $Sh = \mathbf{T}^*h$ for all $h \in \mathcal{H}$. If $\mathbf{T}^*(\mathcal{H}) \subset \mathcal{H}$, then $\mathbf{T}^*|_{\mathcal{H}}$ is the A-adjoint of T.

$2 \Longrightarrow 3$: if 2 holds, the final projection $\mathbf{P} = \mathbf{T}\mathbf{T}^*$ of $\mathbf{T}$ leaves $\mathcal{H}$ invariant: $\mathbf{P}(\mathcal{H}) \subset \mathcal{H}$. Thus, $\mathbf{P}$ induces an (A-symmetric) idempotent $P = \mathbf{P}|_{\mathcal{H}}$ in $\mathcal{H}$. Clearly, $P = TS$, and $R(P) = R(T)$: $P = TS$ implies that $R(P) \subset R(T)$; on the other hand, $\mathbf{T}\mathbf{T}^*\mathbf{T} = \mathbf{T}$, so that, restricting to $\mathcal{H}$, $TST = PT = T$ and therefore $R(T) \subset R(P)$. Summarizing, $R(T)$ is the range of an A-symmetric projection, i.e., a compatible subspace.

$3 \Longrightarrow 1$: suppose that $R(T)$ is compatible and denote by P the unique A-symmetric idempotent with $R(P) = R(T)$. In particular, $R(T)$ is closed, and there exists $S \in \mathcal{B}(\mathcal{H})$ such that $TS = P$. Indeed, $T|_{N(T)^\perp} \to R(T)$ is an isomorphism between Banach spaces, thus it has a bounded inverse T'. Consider the direct sum decomposition $\mathcal{H} = R(P) \dot{+} N(P)$

and put $S(f+g) = T'f$ for $f \in R(P)$ and $g \in N(P)$. Then TS equals the identity in $R(P)$ and is zero in $N(P)$, i.e., $TS = P$. Note that $PT = T$, and that $P^*A = AP$. Then $T^*AT = A$ implies that

$$AS = T^*ATS = T^*AP = T^*P^*A = (PT)^*A = T^*A,$$

i.e., S is the A-adjoint of T.

The equivalence with the last two conditions follows using Douglas' result: T is A-adjointable if and only if there exists a solution X to the operator equation $AX = T^*A$ (i.e., $B = T^*A$), which occurs if and only if $R(T^*A) \subset R(A)$, or equivalently, there exists $\lambda > 0$ such that

$$BB^* = T^*A^2T \leq \lambda A^2.$$

The former condition $R(T^*A) \subset R(A)$ is in turn equivalent to

$$R(A) = R(T^*AT) \subset R(T^*A) \subset R(A),$$

i.e., $R(A) = R(T^*A)$. □

Remark 2.7. Note that if $T \in \mathcal{I}_A^a$, then T is injective and has closed range, i.e. T is bounded from below. Note the following example, of an operator in $\mathcal{I}_A$, with closed range, but whose range is not a compatible subspace of $\mathcal{H}$.

Example 2.8. Consider the Sobolev space

$$H^1(0,1) = \{f \in L^2(0,1) : \text{ there exists } f' \in L^2(0,1)\}$$

with its usual inner product

$$\langle f, g\rangle = \int_0^1 f(t)\bar{g}(t) + f'(t)\bar{g}'(t)dt.$$

Let $\mathcal{H} = H_0^1(0,1) \subset H^1(0,1)$ be the subspace obtained as the closure of the smooth functions in $(0,1)$ with compact support. Let $\mathcal{L} = L^2(0,1)$ with its usual inner product. Here the positive contraction A is given by is the solution operator of the Sturm-Liouville problem

$$\begin{cases} u - u'' = f \\ u(0) = u(1) = 0, \end{cases}$$

that is, $Af = u$, the unique solution u of the equation above for a given $f \in \mathcal{H}$. With the same argument as in [4] (Example 3.7), it can be shown that

$$\mathcal{S} = \{f \in \mathcal{H}_0^1(0,1) : f \equiv 0 \text{ in } [1/2, 1)\}$$

is a (closed) non compatible subspace of $\mathcal{H} \subset \mathcal{L}$. Let us sketch how this is proved. Let f_0 be a C^∞ function in $(0,1)$, of compact support, which equals 1 on an interval centered at

$t = \frac{1}{2}$. Then $f_0 = f_0\ \chi_{(0,1/2)} + f_0\ \chi_{[1/2,1)}$ is an orthogonal sum in $\mathcal{L}$, with $f_0\ \chi_{(0,1/2)} \in \overline{\mathcal{S}}$ (the closure of $\mathcal{S}$ in $\mathcal{L}$). Thus, if $\mathbf{P}_{\overline{\mathcal{S}}}$ denotes the orthogonal projection in $\mathcal{L}$ onto $\overline{\mathcal{S}}$, then $\mathbf{P}_{\overline{\mathcal{S}}}(f_0) = f_0\ \chi_{(0,1/2)}$, which does no belong to $\mathcal{H}$. That is, $\mathbf{P}_{\overline{\mathcal{S}}}$ does not map $\mathcal{H}$ into $\mathcal{H}$, and therefore $\mathcal{S}$ is not compatible in $\mathcal{H}$ (see Remark 2.3 above). Consider the operator $\mathbf{V}$ in $\mathcal{B}(\mathcal{L})$ given by

$$\mathbf{V}f(t) = \begin{cases} \sqrt{2}\ f(2t) & \text{if } t \in (0,1/2) \\ 0 & \text{if } t \in [1/2,1). \end{cases}$$

Note that

$$\|\mathbf{V}f\|_2 = \int_0^{1/2} |\mathbf{V}f(t)|^2 dt \overset{2t=s}{=} \int_0 (\sqrt{2})^2 |f(s)|^2 \frac{1}{2} ds = \|f\|^2,$$

i.e. $\mathbf{V}$ is an isometry of $\mathcal{L}$. Clearly $\mathbf{V}$ preserves smooth functions of compact support, $\mathbf{V}(\mathcal{H}) \subset \mathcal{H}$. Also it is clear that $\mathbf{V}(\mathcal{H})$ consists of all functions in $H^1(0,1)$ with compact support contained in $(0,1/2)$, i.e., $\mathbf{V}(\mathcal{H}) = \mathcal{S}$. Thus, $\mathbf{V}|_{\mathcal{H}}$ induces an element in $\mathcal{I}_A$, which has closed range, but is not adjointable.

Examples 2.9.

1. Example 2.4.1, where T is the shift operator and $\mathcal{H} \subset \mathcal{L}$ are, respectively, the Dirichlet and the Hardy space of the disk, can be generalized. Let $T \in \mathcal{I}_A$ such that $R(T)$ is closed and has finite co-dimension. Then $T \in \mathcal{I}_A^a$. Indeed, in [4] it was shown that closed subspaces with finite co-dimension are compatible.

2. There are also examples of $T \in \mathcal{I}_A^a$ where $R(T)$ has infinite co-dimension. For instance, consider the shift (of infinite multiplicity) $\mathbf{T}$ in $\mathcal{L} = \ell^2$, $\mathbf{T}e_n = e_{2n}$. Consider as before, $\mathcal{H} = \{(x_n) \in \ell^2 : \sum_{n=1}^{\infty} n|x_n|^2 < \infty\}$. Clearly $\mathbf{T}(\mathcal{H}) \subset \mathcal{H}$. Note that $(\mathbf{T}^*x)_n = x_{2n}$, and then

$$\langle \mathbf{T}^*x, \mathbf{T}^*x \rangle_{\mathcal{H}} = \sum_{n=1}^{\infty} n|x_{2n}|^2 < \sum_{n=1}^{\infty} 2n|x_{2n}|^2 \leq \langle x, x \rangle_{\mathcal{H}},$$

   i.e., $\mathbf{T}^*(\mathcal{H}) \subset \mathcal{H}$. Then $T = \mathbf{T}|_{\mathcal{H}} \in \mathcal{I}_A^a$.

Remark 2.10. Again, using Douglas' result, it holds that $T \in \mathcal{I}_A$ is always $A^{1/2}$-adjointable:

$$T^*(A^{1/2})^2 T = T^*AT = A = (A^{1/2})^2,$$

which is the second condition of Douglas for $\lambda = 1$.

Note the evident facts that if $T \in \mathcal{I}_A^a$, then $T^n \in \mathcal{I}_A^a$ for $n \geq 1$; also, $T^\sharp T = 1$ in $\mathcal{H}$.

3 Wold decomposition of A-adjointable isometries

Recall the Wold decomposition of an isometry (see for instance [19]): given an isometry $\mathbf{V}$ acting in $\mathcal{L}$, there exists an orthogonal decomposition

$$\mathcal{L} = \mathcal{L}_0 \oplus \mathcal{L}_1$$

such that $\mathcal{L}_0, \mathcal{L}_1$ reduce $\mathbf{V}$, $\mathbf{V}|_{\mathcal{L}_0}$ is a unitary operator and $\mathbf{V}|_{\mathcal{L}_1}$ is a shift ($\mathcal{L}_1 = \bigoplus_{n=0}^{\infty} \mathbf{V}^n \mathcal{L}_w$, where $\mathcal{L}_w = \mathcal{L} \ominus \mathbf{V}\mathcal{L}$ is the so called *wandering space* of $\mathbf{V}$). This decomposition is unique, $\mathcal{L}_1$ is determined by $\mathbf{V}$, as seen in the above formula; also $\mathcal{L}_0 = \bigcap_{n=0}^{\infty} \mathbf{V}^n \mathcal{L}$.

The next result shows that the Wold decomposition of $\mathbf{V}$ in $\mathcal{L}$ induces an analogous compatible decomposition for V in $\mathcal{H}$.

Theorem 3.1. *Let $V \in \mathcal{I}_A^a$ and $\mathbf{V}$ its lifting to an isometry of $\mathcal{L}$, then the Wold decomposition $\mathcal{L} = \mathcal{L}_0 \oplus \mathcal{L}_1$ of $\mathbf{V}$ induces a direct sum decomposition*

$$\mathcal{H} = \mathcal{H}_0 \dot{+} \mathcal{H}_1,$$

which is A-orthogonal, where the subspaces

$$\mathcal{H}_0 = \mathcal{L}_0 \cap \mathcal{H}, \ \mathcal{H}_1 = \mathcal{H} \cap \mathcal{L}_1,$$

are compatible subspaces. $V\mathcal{H}_0 = \mathcal{H}_0$, $V\mathcal{H}_1 \subset \mathcal{H}_1$, so that $V|_{\mathcal{H}_0}$ induces a bounded invertible operator in $\mathcal{H}_0$.

Proof. The wandering subspace $\mathcal{L}_w = \mathcal{L} \ominus \mathbf{V}\mathcal{L}$ is the orthogonal complement of the range of $\mathbf{V}$, therefore its intersection with $\mathcal{H}$ is the A-orthogonal complement $R(V)^{\perp_A}$ of $R(V)$ (which are both compatible subspaces of $\mathcal{H}$). Note that for $n \geq 1$,

$$V^n(R(V)^{\perp_A}) = \mathbf{V}^n(R(V)^{\perp_A}) \text{ is dense in } \mathbf{V}^n \mathcal{L}_w.$$

Thus the subspaces $V^n(R(V)^{\perp_A})$ are in direct sum for different n, and their sum

$$\bigoplus_{n=0}^{\infty} V^n(R(V)^{\perp_A}) \text{ is dense in } \bigoplus_{n=0}^{\infty} \mathbf{V}^n \mathcal{L}_w.$$

Note also that since $V^n \in \mathcal{I}_A^a$, $V^n(R(V)^{\perp_A})$ is a compatible subspace (in particular, closed). Therefore the A-orthogonal sum (and direct sum in $\mathcal{H}$) $\bigoplus_{n=0}^{\infty} V^n(R(V)^{\perp_A})$ is a compatible subspace.

Next, $\mathbf{V}\mathcal{L}_0 = \mathcal{L}_0$, and thus $V(\mathcal{H}_0) \subset \mathbf{V}\mathcal{L}_0 = \mathcal{L}_0$, together with $V(\mathcal{H}_0) \subset \mathcal{H}$, implies $V(\mathcal{H}_0) \subset \mathcal{H} \cap \mathcal{L}_0 = \mathcal{H}_0$. By the same reason, since $V^\sharp = \mathbf{V}^*|_{\mathcal{H}}$. Also $V^\sharp(\mathcal{H}_0) \subset \mathcal{H}_0$. Moreover,

$$\mathcal{H}_0 = V^\sharp V(\mathcal{H}_0) \subset V^\sharp(\mathcal{H}_0) \subset \mathcal{H}_0.$$

Since clearly $\mathcal{H}_0 \subset R(V)$, then $VV^\sharp = P_V$ (the A-symmetric idempotent in $\mathcal{H}$ with range equal to $R(V)$) satisfies $VV^\sharp \mathcal{H}_0 = \mathcal{H}_0$. Then

$$\mathcal{H}_0 = VV^\sharp(\mathcal{H}_0) \subset V(\mathcal{H}_0) \subset \mathcal{H}_0.$$

That is, $V(\mathcal{H}_0) = \mathcal{H}_0$, i.e. $V|_{\mathcal{H}_0}$ is invertible. □

Remark 3.2. In the above theorem it was shown that $V|_{\mathcal{H}_0}$ is invertible in $\mathcal{H}_0$. The operator A does not necessarily leave $\mathcal{H}_0$ invariant, nevertheless A induces an inner product in $\mathcal{H}_0$. For this induced inner product (which is implemented by the compression of A to $\mathcal{H}_0$: if $f, g \in \mathcal{H}_0$, $\langle f, g\rangle_A = \langle Af, g\rangle = \langle AP_{\mathcal{H}_0} f, P_{\mathcal{H}_0} g\rangle = \langle P_{\mathcal{H}_0} AP_{\mathcal{H}_0} f, g\rangle$), the restriction $V|_{\mathcal{H}_0}$ is a isometric and onto, i.e. $V|_{\mathcal{H}_0}$ belongs to the group $\mathcal{U}_{P_{\mathcal{H}_0} AP_{\mathcal{H}_0}}$ of the Hilbert space $\mathcal{H}_0$.

4 Regular structure of $\mathcal{I}_A^a$

In the introduction it was observed that $\mathcal{U}_A$ acts on $\mathcal{I}_A^a$: if $G \in \mathcal{U}_A$ and $T \in \mathcal{I}_A^a$, then $GT \in \mathcal{I}_A^a$, being a composition of A-adjointable isometries.

Let us recall in the next remark, certain facts on A-orthogonal projections (see [4])

Remark 4.1. Let

$$\mathcal{P}_A = \{P \in \mathcal{B}(\mathcal{H}) : P^2 = P, P \text{ is } A-\text{symmetrizable}\}.$$

Recall that $\mathcal{B}_A(\mathcal{H})$ denotes the algebra of operators acting in $\mathcal{H}$ that are A-adjointable. It is an involutive Banach algebra with the norm $|T| = \max\{\|T\|, \|T^\sharp\|\} = |T^\sharp| = |T|$. The set $\mathcal{P}_A$ is a complemented C^∞-submanifold of $\mathcal{B}_A(\mathcal{H})$, and a homogeneous space of $\mathcal{U}_A$ (which in turn is a Banach-Lie group and a C^∞ submanifold of $\mathcal{B}_A(\mathcal{H})$). This means that for each $P_0 \in \mathcal{P}_A$, the map $\pi_{P_0} : \mathcal{U}_A \to \mathcal{P}_A$, $\pi_{P_0}(G) = GP_0G^{-1}$, is a submersion: it is surjective onto the orbit of P_0 by the action. This orbit $\mathcal{O}_{P_0} := \{GP_0P^{-1} : G \in \mathcal{U}_A\}$ is a union of connected components of $\mathcal{P}_A$. In particular, π_{P_0} has smooth local cross sections. That is, there exists a radius $r_{P_0} > 0$ and a map σ_{P_0},

$$\sigma_{P_0} : \{P \in \mathcal{P}_A : |P - P_0| < r_{P_0}\} \subset \mathcal{P}_A \to \mathcal{U}_A,$$

with the following properties:

- σ_{P_0} is a C^∞ (local) cross section for π_{P_0}:

$$\pi_{P_0}(\sigma_{P_0}(P)) = \sigma_{P_0}(P)P_0\sigma_{P_0}(P)^{-1} = P, \text{ for } P \in \mathcal{P}_A \text{ with } |P - P_0| < r_{P_0};$$

- the map σ_{P_0} extends to an open ball *of* $\mathcal{B}_A(\mathcal{H})$ centered at P_0, as a C^∞ map with values in the invertible group of $\mathcal{B}_A(\mathcal{H})$;

- if one moves P and P_0, locally, the element $\sigma_{P_0}(P)$ is C^∞ in both variables.

Let us state the following lemmas, which will be useful to prove the the set $\mathcal{I}_A^a$ is a complemented submanifold of $\mathcal{B}_A(\mathcal{H})$.

The first result is contained in the appendix of the paper [21] by I. Raeburn, and is a consequence of the implicit function theorem in Banach spaces.

Lemma 4.2. *Let $\mathcal{U}$ be a Banach-Lie group acting smoothly on a Banach space X. For a fixed $x_0 \in X$, denote by $\pi_{x_0} : G \to X$ the smooth map $\pi_{x_0}(g) = g \cdot x_0$. Suppose that*

1. *π_{x_0} is an open mapping, regarded as a map from $\mathcal{U}$ onto the orbit $\{g \cdot x_0 : g \in \mathcal{U}\}$ of x_0 (with the relative topology of X).*
2. *The differential $d(\pi_{x_0})_1 : (T\mathcal{U})_1 \to X$ splits: its kernel and range are closed complemented subspaces.*

Then the orbit $\{g \cdot x_0 : g \in \mathcal{U}\}$ is a smooth submanifold of X, and the map

$$\pi_{x_0} : G \to \{g \cdot x_0 : g \in \mathcal{U}\}$$

is a smooth submersion.

The next result shows that if the cross section of π_{x_0} has an extension to a smooth map on an open subset of the ambient Banach space X, then the Lemma 4.2 applies.

Lemma 4.3. *Let $\mathcal{U}$ be a Banach-Lie group and X a Banach space on which $\mathcal{U}$ acts smoothly. Let $x_0 \in X$ be a fixed element. Suppose that the map*

$$\pi_{x_0} : \mathcal{U} \to \mathcal{O}_{x_0} := \{u \cdot x_0 : u \in \mathcal{U}\}$$

has a continuous cross section σ_{x_0} defined on a neighbourhood of x_0 in $\mathcal{O}_{x_0}$ in the relative topology induced by X. If σ_{x_0} can be extended to a smooth map in an open neighbourhood of x_0 in X, then $\mathcal{O}_{x_0}$ is a complemented smooth submanifold of X and the map π_{x_0} is a submersion.

Proof. Without loss of generality, we may suppose $\sigma_{x_0}(x_0) = 1$. The hypothesis that π_{x_0} has a continuous cross section implies that it is open as a map $\mathcal{U} \to \mathcal{O}_{x_0}$: one can obtain local cross sections for π_{x_0} at any point in $\mathcal{O}_{x_0}$ using the group translation. Let us show that $d(\pi_{x_0})_1$ splits. Indeed, σ_{x_0} extends to a smooth map defined on an open set $x_0 \in \mathcal{B} \subset X$, let us still denote this extension by σ_{x_0}. Then, the fact that σ_{x_0} is a cross section for π_{x_0}, implies that if $g \in \mathcal{B}' \subset \mathcal{U}$, for an appropriate open set $1 \in \mathcal{B}'$, namely

$$\mathcal{B}' = \{g \in \mathcal{U} : \pi_{x_0}(g) \in \mathcal{B}\},$$

we have that $\pi_{x_0}\sigma_{x_0}\pi_{x_0}(g) = \pi_{x_0}(g)$ for $g \in \mathcal{B}'$. Since this identity between smooth maps holds in a neighbourhood $\mathcal{B}'$ of 1 in $\mathcal{U}$, we may diferentiate boths sides at 1. Thus, if we denote $\mathfrak{u} := (T\mathcal{U})_1$, $\Pi := d(\pi_{x_0})_1 : \mathfrak{u} \to X$ and $\Sigma := d(\sigma_{x_0})_{x_0} : X \to \mathfrak{u}$, we have that

$$\Pi\Sigma\Pi = \Pi. \tag{3}$$

Equation (3) implies that $\Pi\Sigma \in \mathcal{B}(X)$ and $\Sigma\Pi \in \mathcal{B}(\mathfrak{u})$ are idempotent operators. Note that

$$R(\Pi) = R(\Pi\Sigma\Pi) \subset R(\Pi\Sigma) \subset R(\Pi)$$

and

$$N(\Pi) \subset N(\Sigma\Pi) \subset N(\Pi\Sigma\Pi).$$

Then $R(\Pi)$ and $N(\Pi)$ are complemented subspaces of X and $\mathfrak{u}$, respectively , i.e., $d(\pi_{x_0})_1$ splits. $\square$

Let us fix $T_0 \in \mathcal{I}_A^a$, and construct the extendable local cross sections of

$$\pi_{T_0} : \mathcal{U}_A \to \mathcal{I}_A^a, \ \pi_{T_0}(G) = GT_0. \tag{4}$$

For $T \in \mathcal{I}_A^a$, denote by $P_T := TT^\sharp$. Clearly P_T is an A-symmetric projection, which lifts to the orthogonal projection $\mathbf{P}_{R(\mathbf{T})}$ onto the range of $\mathbf{T}$ in $\mathcal{L}$.

If T is close to T_0, then P_T is close in P_{T_0}. Explicitly,

$$|P_T - P_{T_0}| = |TT^\sharp - T_0T_0^\sharp| \leq |TT^\sharp - TT_0^\sharp| + |TT_0^\sharp - T_0T_0^\sharp| \leq |T||T^\sharp - T_0^\sharp| + |T - T_0||T_0^\sharp|$$

$$= |T - T_0|(|T| + |T_0|) \leq |T - T_0|(|T - T_0| + 2|T_0|),$$

In particular, there exists δ_{T_0}, which depends on T_0, such that if $|T - T_0| < \delta_{T_0}$, then $|P_T - P_{T_0}| < r_{P_{T_0}}$ (the radius given in the above remark).

Suppose that $|T - T_0| < \delta_{T_0}$, and denote by $G_T = \sigma_{P_{T_0}}(P_T) \in \mathcal{U}_A$. Clearly G_T is a C^∞ map of T which satisfies that $G_T P_{T_0} G_{T_0}^{-1} = P_T$. Then $T' := G_T^{-1}T$ is an element of $\mathcal{I}_A^a$ which has final projection

$$T'T'^\sharp = G_T^{-1}TT^\sharp(G^{-1})^\sharp = G_T^{-1}P_TG_T = P_{T_0}.$$

Two elements T', T_0 of $\mathcal{I}_A^a$ with the same final projection P_{T_0} are conjugate by the action of $\mathcal{U}_A$: there exists $H = H_{T_0}(T') \in \mathcal{U}_A$ such that $HT_0 = T'$. Pick, for instance

$$H = T'T_0^\sharp + (1 - P_{T_0}).$$

Note that

$$\begin{aligned} HH^\sharp &= T'T_0^\sharp T_0 T'^\sharp + T'T_0^\sharp(1 - P_{T_0}) + (1 - P_{T_0})T_0T'^\sharp + (1 - P_{T_0}) \\ &= T'T'^\sharp + (1 - P_{T_0}) = P_{T_0} + 1 - P_{T_0} \\ &= 1, \end{aligned}$$

because $(1-P_{T_0})T' = T_0^\sharp(1-P_{T_0}) = 0$. Similarly $H^\sharp H = 1$. Moreover,

$$HT_0 = T'T_0^\sharp T_0 + (1-P_{T_0})T_0 = T'.$$

Clearly $H = H_{T_0}(T')$ is a C^∞ map in terms of T' and T_0, and thus a C^∞ map in terms of T.

Thus, $H \in \mathcal{U}_A$ and $HT_0 = G_T^{-1}T$, i.e., $T = G_T HT_0$. Define

$$\sigma_{T_0}(T) = G_T H = G_T(G_T^{-1}TT_0^\sharp + (1-P_{T_0}) = TT_0^\sharp + G_T(1-P_{T_0}), \tag{5}$$

for $T \in \mathcal{I}_A^a, \ \ |T - T_0| < \delta_{T_0}$.

Theorem 4.4. *Let $T_0 \in \mathcal{I}_A^a$. Then the orbit*

$$\mathcal{O}_{T_0} := \{GT_0 : G \in \mathcal{U}_A\}$$

is a C^∞ complemented submanifold of $\mathcal{B}_A(\mathcal{H})$, and the map

$$\pi_{T_0} : \mathcal{U}_A \to \mathcal{O}_{T_0}, \ \ \pi_{T_0}(G) = GT_0$$

is a C^∞ submersion. The orbit $\mathcal{O}_{T_0}$ is a union of connected components of $\mathcal{I}_A^a$, which implies that $\mathcal{I}_A^a$ is a C^∞ complemented submanifold of $\mathcal{B}_A(\mathcal{H})$

Proof. The fact that $\mathcal{O}_{T_0}$ is a submanifold follows using Lemma 4.3, noting that the cross section σ_{T_0} defined in (5) clearly extends to a C^∞ map defined on a neighbourhood of T_0 in $\mathcal{B}_A(\mathcal{H})$. Let us check that $\mathcal{O}_{T_0}$ is a union of connected components of $\mathcal{I}_A^a$. In [4] it was observed that the local cross section $P_T \mapsto G_T$ for π_{P_0} satisfies that that $G_{T_0} = 1$, and therefore G_T belongs to the connected component of the identity in $\mathcal{U}_A$ if T is close to T_0. This implies that if T is close enough to T_0, then $\sigma_{T_0}(T)$ belongs to the connected component of the identity of $\mathcal{U}_A$. This observation implies that any T close to T_0, can be connected to T_0 by means of a continuous path in the orbit. □

5 The lifting method of M.G. Krein for symmetrizable operators

In this section the lifting method of M.G. Krein [16] is considered (see also Section 125 of the classic book [22] for a detailed exposition). Krein shows that if $S : \mathcal{L}_0 \subset \mathcal{L} \to \mathcal{L}$ is a contractive symmetric operator, then there exists a symmetric lifting $S : \mathcal{L} \to \mathcal{L}$ which is also a contraction. Later on, other authors elaborated on this problem (see [24] where the parametrization of all liftings was obtained, or further [5] where the defect spaces are identified; in [11], [20] there are also descriptions of the liftings). In this section, following the ideas of M.G. Krein with slight modifications, it will be shown that an A-symmetric operator defined on a A-compatible subspace $\mathcal{H}_0 \subset \mathcal{H}$, which is contractive for the norm induced by $\langle \ , \ \rangle_A$, can be lifted to a contraction for this norm to the whole space $\mathcal{H}$. Or rather, and equivalently, we shall work in the bigger Hilbert space $\mathcal{L}$, lifting it to a symmetric contraction of $\mathcal{L}$ which leaves $\mathcal{H}$ invariant.

Lemma 5.1. *Let* $\mathbf{X}$ *be a selfadjoint operator in* $\mathcal{L}$ *such that* $\mathbf{X}(\mathcal{H}) \subset \mathcal{H}$, *and let* $\mathbf{P}$ *be a projection in* $\mathcal{L}$ *with range* $\mathcal{L}_0$, *such that* $P = \mathbf{P}|_{\mathcal{H}}$ *is an idempotent in* $\mathcal{H}$ *with range* $\mathcal{H}_0$. *Suppose that* $\|\mathbf{XP}\| = 1$. *Then there exists a selfadjoint operator* $\mathbf{Z}$ *in* $\mathcal{L}$ *such that* $\mathbf{Z}(\mathcal{H}) \subset \mathcal{H}$, $\mathbf{ZP} = \mathbf{XP}$ *and* $\|\mathbf{Z}\| = 1$.

Proof. For $m > 0$, let us denote by $\mathbf{X}_m = \frac{1}{m}\mathbf{X}$, m will be adjusted in the process. Denote by $\mathcal{H}_0 = \mathbf{P}(\mathcal{H})$. Let $\mathbf{B}_0 = \mathbf{P}\mathbf{X}_m\mathbf{P} : \mathcal{L}_0 \to \mathcal{L}_0$. Then $\mathbf{B}_0$ lifts to an operator in $\mathcal{L}$ with the same norm: $\bar{\mathbf{B}}_0 = \mathbf{P}\mathbf{X}_m : \mathcal{L} \to \mathcal{L}_0$. Note that $\bar{\mathbf{B}}_0$ leaves $\mathcal{H}$ invariant, because $\mathbf{X}_m$ and $\mathbf{P}$ do.

Next lift $\mathbf{B}_1 := \mathbf{P}^\perp\mathbf{X}_m\mathbf{P} : \mathcal{L}_0 \to \mathcal{L}_0^\perp$ to the whole space $\mathcal{L}$ (not enlarging the norm of $\mathbf{B}_1$ and leaving $\mathcal{H}$ invariant). Consider the inner product

$$[\varphi, \gamma] := \langle \varphi, \gamma \rangle_{\mathcal{L}} - \langle \bar{\mathbf{B}}_0\varphi, \bar{\mathbf{B}}_0\gamma \rangle_{\mathcal{L}}.$$

Note that

$$[\varphi, \gamma] := \langle \varphi, \gamma \rangle_{\mathcal{L}} - \langle \mathbf{P}\mathbf{X}_m\varphi, \mathbf{P}\mathbf{X}\gamma \rangle_{\mathcal{L}} = \langle (I - \mathbf{X}_m\mathbf{P}\mathbf{X}_m)\varphi, \gamma \rangle_{\mathcal{L}}.$$

Take $m > 0$ so that $\|\mathbf{X}_m\mathbf{P}\mathbf{X}_m\| < 1$, then $I - \mathbf{X}_m\mathbf{P}\mathbf{X}_m$ is (positive and) invertible. Then $(\mathcal{L}, [\ , \ ])$ is a Hilbert space, whose norm is equivalent to the original norm induced by $\langle \ , \ \rangle_{\mathcal{L}}$. The operator $\mathbf{B}_1$ is bounded by (for $\varphi = \mathbf{P}\varphi \in \mathcal{L}_0$)

$$\|\mathbf{B}_1\varphi\|^2 = \langle \mathbf{P}^\perp\mathbf{X}_m\mathbf{P}\varphi, \mathbf{P}^\perp\mathbf{X}_m\mathbf{P}\varphi \rangle_{\mathcal{L}} = \langle \mathbf{X}_m(1 - \mathbf{P})\mathbf{X}_m\mathbf{P}\varphi, \mathbf{P}\varphi \rangle_{\mathcal{L}}$$

$$= \langle \mathbf{X}_m^2\mathbf{P}\varphi, \mathbf{P}\varphi \rangle_{\mathcal{L}} - \langle \mathbf{X}_m\mathbf{P}\mathbf{X}_m\mathbf{P}\varphi, \mathbf{P}\varphi \rangle_{\mathcal{L}}.$$

Since $\|\mathbf{XP}\| = \frac{1}{m}$, the first term is bounded by

$$\langle \mathbf{X}_m^2\mathbf{P}\varphi, \mathbf{P}\varphi \rangle_{\mathcal{L}} = \langle \mathbf{P}\mathbf{X}_m^2\mathbf{P}\varphi, \varphi \rangle_{\mathcal{L}} \le \|\mathbf{P}\mathbf{X}_m^2\mathbf{P}\|\|\varphi\|^2 = \frac{1}{m^2}\|\varphi\|^2.$$

Pick m so that $m \ge 1$, one has that

$$\|\mathbf{B}_1\varphi\|^2 \le \frac{1}{m^2}\langle \varphi, \varphi \rangle_{\mathcal{L}} - \langle \mathbf{X}_m\mathbf{P}\mathbf{X}_m\varphi, \varphi \rangle_{\mathcal{L}} \le \frac{1}{m^2}\langle \varphi, \varphi \rangle_{\mathcal{L}} - \frac{1}{m^2}\langle \mathbf{X}_m\mathbf{P}\mathbf{X}_m\varphi, \varphi \rangle_{\mathcal{L}} = \frac{1}{m^2}[\varphi, \varphi].$$

Therefore $\mathbf{B}_1$ induces an operator $\mathcal{B} : (\mathcal{L}_0, [\ , \ ]) \to (\mathcal{L}_0^\perp, \langle \ , \ \rangle_{\mathcal{L}})$ with norm less than or equal to $\frac{1}{m}$. Since at the set level, this mapping coincides with $\mathbf{B}_1$, $\mathcal{B}$ maps $\mathcal{H}_0$ into $\mathcal{L}_0^\perp \cap \mathcal{H}$. Let $\Pi : (\mathcal{L}, [\ , \ ]) \to (\mathcal{L}_0, [\ , \ ])$ denote the $[\ , \ ]$-orthogonal projection. Since $[\ , \ ] = \langle (1 - \mathbf{X}_m\mathbf{P}\mathbf{X}_m) \ , \ \rangle_{\mathcal{L}}$, the $[\ , \ ]$ adjoint of an operator $\mathcal{T}$ acting in $(\mathcal{L}, [\ , \ ])$ is given by $\mathcal{T}^\sharp = (1 - \mathbf{X}_m\mathcal{T}^*\mathbf{X}_m)^{-1}\mathcal{T}^*(1 - \mathbf{X}_m\mathcal{T}^*\mathbf{X}_m)$, with $\mathcal{T}^*$ the $\langle \ , \ \rangle_{\mathcal{L}}$- adjoint of

$\mathcal{T}$. It is known (see for instance [1]), that the orthogonal projection onto the range of an idempotent Q in a Hilbert space, is given by the formula

$$P_{R(Q)} = Q(Q + Q^* - 1)^{-1}.$$

In our case, this implies that

$$\Pi = \mathbf{P}\{\mathbf{P} + (1 - \mathbf{X}_m\mathbf{P}\mathbf{X}_m)^{-1}\mathbf{P}(1 - \mathbf{X}_m\mathbf{P}\mathbf{X}_m) - 1\}^{-1}.$$

If one further adjusts $m > 0$, one has that $\Pi(\mathcal{H}) \subset \mathcal{H}$. To shorten the writing, denote by Q_m the idempotent $(1 - \mathbf{X}_m\mathbf{P}\mathbf{X}_m)^{-1}\mathbf{P}(1 - \mathbf{X}_m\mathbf{P}\mathbf{X}_m)$. First one needs that $Q_m(\mathcal{H}) \subset \mathcal{H}$. Since $1-\mathbf{X}_m\mathbf{P}\mathbf{X}_m$ leaves $\mathcal{H}$ invariant, it induces an operator in $\mathcal{H}$. If one further enlarges m so that $X_mPX_m = \mathbf{X}_m\mathbf{P}\mathbf{X}_m|_{\mathcal{H}}$ has norm strictly less than 1 in $\mathcal{B}(\mathcal{H})$, $1-X_mPX_m$ will be invertible in $\mathcal{B}(\mathcal{H})$, i.e. $(1-\mathbf{X}_m\mathbf{P}\mathbf{X}_m)^{-1}$ leaves $\mathcal{H}$ invariant. It follows that $Q_m(\mathcal{H}) \subset \mathcal{H}$.

Next, note the elementary identity

$$(\mathbf{P} + Q_m - 1)^2 = 1 - (\mathbf{P} - Q_m)^2, \tag{6}$$

which only uses the fact that $\mathbf{P}$ and Q_m are idempotents. In view of (6), in order to to have that $(\mathbf{P}+Q_m-1)^2$, and therefore also $\mathbf{P}+Q_m-1$, is invertible, it suffices to adjust m so that the norm of $\mathbf{P}-Q_m$ as an operator restricted to $\mathcal{H}$, is strictly less than 1. This clearly can be done, since $X_mPX_m \to 0$ in $\mathcal{B}(\mathcal{H})$ as $m \to +\infty$. It follows that $\Pi(\mathcal{H}) \subset \mathcal{H}$, for m sufficiently large.

Denote by $J : (\mathcal{L}, \langle\ ,\ \rangle_{\mathcal{L}}) \to (\mathcal{L}, [\ ,\ ])$ the identity mapping. Note that J is contractive:

$$[\varphi, \varphi] = \langle(1 - \mathbf{X}_m\mathbf{P}\mathbf{X}_m)\varphi, \varphi\rangle_{\mathcal{L}} \le \langle\varphi, \varphi\rangle_{\mathcal{L}},$$

because $\|\mathbf{X}_m\| < 1$. Clearly $J(\mathcal{H}) \subset \mathcal{H}$. Therefore, if one puts

$$\bar{\mathbf{B}}_1 := \mathcal{B}\Pi J : \mathcal{L} \to \mathcal{L}_0^{\perp},$$

where both spaces are considered with their original inner product $\langle\ ,\ \rangle_{\mathcal{L}}$, one has that (since Π and J are contractive),

$$\|\bar{\mathbf{B}}_1\| \le \frac{1}{m}.$$

Note also that all the operators involved in this product leave $\mathcal{H}$ invariant, thus $\bar{\mathbf{B}}_1(\mathcal{H}) \subset \mathcal{H}$. Put

$$\bar{\mathbf{B}} = \bar{\mathbf{B}}_0 + \bar{\mathbf{B}}_1.$$

It is easily verified that for $\varphi \in \mathcal{L}$ with $\|\varphi\| = 1$,

$$\|\bar{\mathbf{B}}\varphi\|^2 = \|\bar{\mathbf{B}}_0\varphi\|^2 + \|\bar{\mathbf{B}}_1\varphi\|^2 \le \frac{1}{m^2},$$

since $\|\bar{\mathbf{B}}_0\|, \|\bar{\mathbf{B}}_1\| \le \frac{1}{m}$, $R(\bar{\mathbf{B}}_0) \subset \mathcal{L}_0$ and $R(\bar{\mathbf{B}}_1) \subset \mathcal{L}_0^{\perp}$. Note that $\bar{\mathbf{B}}_0\mathbf{P} = \mathbf{P}\mathbf{X}_m\mathbf{P}$ and that if $\varphi_0 \in \mathcal{L}_0 = R(\mathbf{P})$,

$$\bar{\mathbf{B}}_1\varphi_0 = \mathcal{B}\Pi J\varphi_0 = \mathcal{B}\varphi_0 = \mathbf{B}_1\varphi_0,$$

because $\varphi_0 \in R(\Pi)$. Thus $\bar{\mathbf{B}}_1\mathbf{P} = \mathbf{B}_1\mathbf{P} = \mathbf{P}^\perp\mathbf{X}_m\mathbf{P}$. It follows that

$$\bar{\mathbf{B}}\mathbf{P} = \bar{\mathbf{B}}_0\mathbf{P} + \bar{\mathbf{B}}_1\mathbf{P} = \mathbf{P}\mathbf{X}_m\mathbf{P} + \mathbf{P}^\perp\mathbf{X}_m\mathbf{P} = \mathbf{X}_m\mathbf{P}.$$

Clearly $\bar{\mathbf{B}}(\mathcal{H}) \subset \mathcal{H}$. The last thing to fix is that $\bar{\mathbf{B}}$ is not selfadjoint.

Note that

$$\mathbf{P}\bar{\mathbf{B}} = \mathbf{P}\bar{\mathbf{B}}_0 = \mathbf{P}\mathbf{X}_m,$$

and then $\bar{\mathbf{B}}^*\mathbf{P} = \mathbf{X}_m\mathbf{P}$.

$$\mathbf{Z}_m := \frac{1}{2}(\bar{\mathbf{B}} + \bar{\mathbf{B}}^*)$$

does the feat: $\mathbf{Z}_m^* = \mathbf{Z}_m$, $\mathbf{Z}_m\mathbf{P} = \mathbf{X}_m\mathbf{P}$, $\|\mathbf{Z}_m\| \leq \|\bar{\mathbf{B}}\|$. In this case one needs further to verify that $\bar{\mathbf{B}}^*(\mathcal{H}) \subset \mathcal{H}$. Note that

$$\bar{\mathbf{B}}^* = \mathbf{X}_m\mathbf{P} + J^*\Pi^*\mathcal{B}^*,$$

with the adjoints taken in their respective spaces. Since Π is an orthogonal projection, $\Pi^* = \Pi$. Note that the adjoint of J is given by

$$\langle \varphi, J^*\eta\rangle_{\mathcal{L}} = [J\varphi, \eta] = \langle (1 - \mathbf{X}_m\mathbf{P}\mathbf{X}_m)\varphi, \eta\rangle_{\mathcal{L}} = \langle \varphi, (1 - \mathbf{X}_m\mathbf{P}\mathbf{X}_m)\eta\rangle_{\mathcal{L}},$$

i.e., $J^* : (\mathcal{L}, [\ \ ,\ \ ]) \to (\mathcal{L}, \langle\ \ ,\ \ \rangle_{\mathcal{L}})$ is $J^* = (1 - \mathbf{X}_m\mathbf{P}\mathbf{X}_m)$. The adjoint of

$$\mathcal{B} : (\mathcal{L}_0, [\ \ ,\ \ ]) \to (\mathcal{L}_0^\perp, \langle\ \ ,\ \ \rangle_{\mathcal{L}})$$

is given by

$$[\mathcal{B}^*\varphi, \eta] = \langle \varphi, \mathcal{B}\eta\rangle_{\mathcal{L}} = \langle \varphi, \mathbf{P}^\perp\mathbf{X}_m\mathbf{P}\eta\rangle_{\mathcal{L}} = \langle (1 - \mathbf{X}_m\mathbf{P}\mathbf{X}_m)^{-1}(1 - \mathbf{X}_m\mathbf{P}\mathbf{X}_m)\varphi, \mathbf{P}^\perp\mathbf{X}_m\mathbf{P}\eta\rangle_{\mathcal{L}}$$

$$= [\varphi, (1 - \mathbf{X}_m\mathbf{P}\mathbf{X}_m)^{-1}\mathbf{P}^\perp\mathbf{X}_m\mathbf{P}\eta],$$

i.e., $\mathcal{B}^* = (1 - \mathbf{X}_m\mathbf{P}\mathbf{X}_m)^{-1}\mathbf{P}^\perp\mathbf{X}_m\mathbf{P}$, which leaves $\mathcal{H}$ invariant for the appropriate election of m.

It follows that $\mathbf{Z}_m(\mathcal{H}) \subset \mathcal{H}$. Finally, m is disposed of: $\mathbf{Z} = m\mathbf{Z}_m$ is selfadjoint, satisfies $\mathbf{Z}\mathbf{P} = \mathbf{X}\mathbf{P}$, $\mathbf{Z}$ leaves $\mathcal{H}$ invariant, and

$$1 = \|\mathbf{Z}\mathbf{P}\| \leq \|\mathbf{Z}\| \leq 1,$$

i.e. $\|\mathbf{Z}\| = 1$. □

6 Minimality of curves in $\mathcal{I}_A^a$ with given initial conditions

In [2], a metric was introduced in the set of isometries of a Hilbert space. If $\mathcal{I}$ denotes the set of isometries in $\mathcal{L}$, a natural metric in $\mathcal{I}$ is given by

$$d(V, W) = \inf\{\ell(\gamma) : \gamma \text{ is a smooth curve in } \mathcal{I} \text{ joining } V \text{ and } W\},$$

where

$$\ell(\gamma) = \int_I \|\dot\gamma(t)\| dt.$$

Fix $V \in \mathcal{I}$. A vector $\mathcal{V}$ tangent to $\mathcal{I}$ at V is the velocity vector $\dot\gamma(0)$ of a curve $\gamma \subset \mathcal{I}$ with $\gamma(0) = V$. A smooth curve γ in $\mathcal{I}$ can be lifted to a smooth curve of unitary operators in $\mathcal{L}$: $\gamma(t) = \Gamma(t)V$ for Γ in $\mathcal{U}(\mathcal{L})$. It follows that $\mathcal{V} = iXV$, for $X^* = X \in \mathcal{B}(\mathcal{L})$. The main result established in [2] (Theorem 2.2) states the following:

Suppose that $\|\mathcal{V}\| = 1$, and let $P = VV^*$ be the final projection of V. If $Z^* = Z$ satisfies $\|Z\| = 1$ and $ZP = \mathcal{V}P$, then the curve

$$\delta(t) = e^{itZ}V$$

has minimal length along its path for $|t| \leq \pi$, among all smooth curves in $\mathcal{I}$ which join the same endpoints as δ.

Define the metric in $\mathcal{I}_A^a$, by considering the norm of $\mathcal{B}(\mathcal{L})$ at every tangent space: if $T \in \mathcal{I}_A^a$ and $\mathcal{V} \in (T\mathcal{I}_A^a)_T$, then

$$|\mathcal{V}| = \|\boldsymbol{\mathcal{V}}\|,$$

where $\boldsymbol{\mathcal{V}}$ is the lifting of $\mathcal{V}$ to $\mathcal{L}$. Formally, since one has considered the regular structure of $\mathcal{I}_A^a$ regarding it as a submanifold of $\mathcal{B}_A(\mathcal{H})$, one is introducing a (constant) distribution of norms in the tangent bundle, consisting of the norm defined by the A-inner product. Or, if one does not care about the regular structure of $\mathcal{I}_A^a$, one just regards $\mathcal{I}_A^a$ as a subset of the metric space $(\mathcal{I}, d)$.

A direct consequence of the result cited above and Lemma 5.1 is the following:

Theorem 6.1. *Let $T \in \mathcal{I}_A^a$, and $\mathcal{V} \in (T\mathcal{I}_A^a)_T$ with $|\mathcal{V}| = 1$. Then there exists a symmetrizable operator $Z \in \mathcal{B}_A(\mathcal{H})$ such that the curve*

$$\delta(t) = e^{it\mathbf{Z}}\mathbf{T}|_{\mathcal{H}} = e^{itZ}T \in \mathcal{I}_A^a$$

satisfies $\delta(0) = T$, $\dot\delta(0) = \mathcal{V}$ and δ has minimal length along its path for $|t| \leq \pi$, among all smooth curves in $\mathcal{I}_A^a$ which join the same endpoints as δ. In fact, δ is minimal also in the bigger manifold $\mathcal{I}$.

Proof. The tangent vector $\mathcal{V}$ is of the form $\mathcal{V} = iXT$, for some $X \in \mathcal{B}(\mathcal{H})$ symmetrizable operator with $|\mathcal{V}| = \|\mathbf{XT}\| = 1$. We claim that there exists $\mathbf{Z}^* = \mathbf{Z}$ with $\|\mathbf{Z}\| = 1$, $\mathbf{ZT} = \mathbf{XT}$ and $\mathbf{Z}(\mathcal{H}) \subset \mathcal{H}$. Indeed, note that $\mathbf{ZT} = \mathbf{XT}$ if and only if $\mathbf{ZP} = \mathbf{XP}$, for $\mathbf{P}$ the final projection of $\mathbf{T}$:

$$\mathbf{ZT} = \mathbf{XT} \Longrightarrow \mathbf{ZP} = \mathbf{ZTT}^* = \mathbf{XTT}^* = \mathbf{XP} \Longrightarrow \mathbf{ZPT} = \mathbf{ZT} = \mathbf{XPT} = \mathbf{XP}.$$

Also note that

$$\|\mathbf{XP}\| = \|\mathbf{XTT}^*\| \leq \|\mathbf{XT}\| = \|\mathbf{XTT}^*\mathbf{T}\| \leq \|\mathbf{XTT}^*\| = \|\mathbf{XP}\|,$$

i.e. $\|\mathbf{XP}\| = 1$. If $Z = \mathbf{Z}|_{\mathcal{H}}$, then $\delta(t) = e^{itZ}T$ has minimal length along its path, due to Theorem 2.2 in [2]. Clearly $\dot\delta(0) = ZT = XT = \mathcal{V}$. □

7 Components of $\mathcal{I}_A^a$ and the action of the restricted group

The connected components of the set of usual isometries in a Hilbert space are parametrized by the co-rank of the isometries: two isometries lie in the same connected component (i.e., are conjugate by the left action of the unitary group) if and only if they have the same co-rank $0 \leq n \leq +\infty$. This fact follows easily and is a consequence of the connectedness of the unitary group of a Hilbert space. We do not know if the group $\mathcal{U}_A$ is connected (or how the properties of the operator A determine the components of $\mathcal{U}_A$): we believe that it is an interesting problem. What does hold in general, both for usual or A-isometries, is that two isometries are conjugate by the group action if and only if their final projections are conjugate. The argument is essentially contained in the discussion before Theorem 4.4.

Proposition 7.1. *Let $T_1, T_2 \in \mathcal{I}_A^a$, with final A-orthogonal idempotents $P_1 = T_1T_1^\sharp$, $P_2 = T_2T_2^\sharp$.*

1. *There exists $G \in \mathcal{U}_A$ such that $GT_1 = T_2$ if and only if there exists $H \in \mathcal{U}_A$ such that $HP_1H^{-1} = P_2$.*
2. *T_1 and T_2 lie in the same connected component of $\mathcal{I}_A^a$ if and only if P_1 and P_2 lie in the same component of $\mathcal{P}_A$.*

Proof. If there exists $G \in \mathcal{I}_A^a$ such that $GT_1 = T_2$, then

$$P_2 = T_2T_2^\sharp = GT_1(GT_1)^\sharp = GT_1T_1^\sharp G^\sharp = GP_1G^{-1}.$$

Conversely, if there exists $H \in \mathcal{U}_A$ such that $HP_1H^{-1} = P_2$, then $T_1' = HT_1$ and T_2 are A-isometries with the same final space P_2. Consider the operator K given by $K = T_2(T_1')^\sharp + 1 - P_2$. Note that

$$KK^\sharp = (T_2(T_1')^\sharp + 1 - P_2)(T_1'T_2^\sharp + 1 - P_2) = T_2(T_1')^\sharp T_1'T_2^\sharp + 1 - P_2,$$

because $T_2(T_1')^\sharp(1 - P_2) = 0 = (1 - P_2)T_1'T_2^\sharp$. Since

$$T_2(T_1')^\sharp T_1'T_2^\sharp = T_2T_2^\sharp = P_2,$$

then $KK^\sharp = 1$. Similarly $K^\sharp K = 1$, i.e. $K \in \mathcal{U}_A$. Moreover

$$KT_1' = (T_2(T_1')^\sharp + 1 - P_2)T_1' = T_2(T_1')^\sharp T_1 = T_2.$$

Therefore, $T_2 = KT_1' = KHT_1$, with $KH \in \mathcal{U}_A$.

In order to prove part 2., one uses the fact that both

$$\pi_{T_1} : \mathcal{U}_A \to \{GT_1 : G \in \mathcal{U}_A\}, \ \pi_{T_1}(G) = GT_1$$

and

$$\pi_{P_1}: \mathcal{U}_A \to \{GP_1G^{-1} : G \in \mathcal{U}_A\},\ \pi_{P_1}(G) = GP_1G^{-1}$$

C^∞ submersions, and that the orbits $\{GT_1 : G \in \mathcal{U}_A\}$ and $\{GP_1G^{-1} : G \in \mathcal{U}_A\}$ are unions of connected components of $\mathcal{I}_A^a$ and $\mathcal{P}_A$, respectively. Therefore a continuous path $T(t) \in \mathcal{I}_A^a$ with $T(0) = T_1$ and $T(1) = T_2$ lifts to a continuous path $G(t) \in \mathcal{U}_A$, i.e., $T(t) = G(t)T_1$. Then $P(t) = G(t)P_1G^{-1}(t)$ is a continuous path in $\mathcal{P}_A$ joining P_1 and P_2.

Similarly, if $P(t)$ is a continuous path in $\mathcal{P}_A$, there exists a continuous path $H(t) \in \mathcal{U}_A$ such that $P(t) = H(t)P_1H^{-1}(t)$. Then as in the proof of part 1.,

$$K(t) = T_2(H(t)T_1)^\sharp + 1 - P_2 \in \mathcal{U}_A$$

is a continuous path, and $T(t) = K(t)H(t)T_1$ is a continuous path in $\mathcal{I}_A^a$, joining T_1 and T_2. □

In the remaining of this section, the action of the restricted group

$$\mathcal{U}_A^\infty := \{G \in \mathcal{U}_A : G - 1 \text{ is compact}\}$$

on $\mathcal{I}_A^a$ is considered. In [3] it was shown that $\mathcal{U}_A^\infty$ is a C^∞-Banach-Lie group, whose Banach-Lie algebra is

$$\mathfrak{u}_A^\infty = \{iX \in \mathcal{B}_A(\mathcal{H}) : X \text{ is symmetrizable and compact}\}.$$

Denote by $\mathcal{K}(\mathcal{H})$ the space of compact operators in $\mathcal{H}$. In [3] it was also proved that the usual exponential map $\exp(iX) = e^{iX}$,

$$\exp : \mathfrak{u}_A^\infty \to \mathcal{U}_A^\infty$$

is surjective. This implies, in particular, that $\mathcal{U}_A^\infty$ is connected. The first result characterizes the orbits of the restricted action. Note that these orbits are connected.

Theorem 7.2. *Let $T_0 \in \mathcal{I}_A^a$. Then*

$$\{GT_0 : G \in \mathcal{U}_A^\infty\} = \{T \in \mathcal{I}_A^a : T - T_0 \in \mathcal{K}(\mathcal{H})\}.$$

Proof. If $T = GT_0$ for $G \in \mathcal{U}_A^\infty$, then clearly $T \in \mathcal{I}_A^a$ and

$$T - T_0 = GT_0 - T_0 = (G-1)T_0 \in \mathcal{K}(\mathcal{H}).$$

Conversely, suppose that $T \in \mathcal{I}_A^a$ satisfies that $T - T_0$ is compact. Then, by a theorem of I.C. Gohberg and M.I. Zambickii [15] the lifting $\mathbf{T} - \mathbf{T}_0$ is compact in $\mathcal{L}$. Then $\mathbf{T}^*(\mathbf{T} - \mathbf{T}_0) = 1 - \mathbf{T}^*\mathbf{T}_0$ is compact, i.e. $\mathbf{T}^*\mathbf{T}_0$ is a Fredholm operator in $\mathcal{L}$ with zero index. Then, since $\mathbf{T}$ is an isometric isomorphism between $\mathcal{L}$ and $R(\mathbf{T}) = R(\mathbf{T}\mathbf{T}^*)$, and $\mathbf{T}_0^*$ is an isometric isomorphism between $R(\mathbf{T}_0) = R(\mathbf{T}_0\mathbf{T}_0^*)$ and $\mathcal{L}$, it follows that

$$\mathbf{T}\mathbf{T}^*\mathbf{T}_0\mathbf{T}_0^* : R(\mathbf{T}_0\mathbf{T}_0^*) \to R(\mathbf{T}\mathbf{T}^*)$$

is a Fredholm operator of zero index. In [4] (Theorem 6.3) it was shown this implies that $P_T = TT^\sharp$ and $P_{T_0} = T_0T_0^\sharp$ lie in the same connected component (same orbit) of the restricted Grassmannian: there exists $G \in \mathcal{U}_A^\infty$ such that $GP_{T_0}G^{-1} = P_T$. As in Proposition 7.1, let $K = T(GT_0)^\sharp + 1 - P_T \in \mathcal{U}_A$, which satisfies $KGT_0 = T$. We claim that $K \in \mathcal{U}_A^\infty$. Indeed,

$$K - 1 = T(GT_0)^\sharp - P_T = TT_0^\sharp G^{-1} - GT_0T_0^\sharp G^{-1} = (T - GT_0)T_0^\sharp G^{-1},$$

and since $G \in \mathcal{U}_A^\infty$ is of the form $G = 1 + C$ with $C \in \mathcal{K}(\mathcal{H})$,

$$T - GT_0 = T - (1 + C)T_0 = T - T_0 - CT_0 \in \mathcal{K}(\mathcal{H}).$$

□

With a similar argument as the one used to prove that $\mathcal{I}_A^a$ has differentiable structure, it can be shown that the orbit of T_0 under the action on $\mathcal{U}_A^\infty$ is a C^∞-manifold and a homogeneous space of this group.

Proposition 7.3. *Given a fixed $T_0 \in \mathcal{I}_A^a$, the set $\{T \in \mathcal{I}_A^a : T - T_0 \in \mathcal{K}(\mathcal{H})\}$ is a complemented C^∞-submanifold of the affine space $T_0 + \mathcal{B}_A(\mathcal{H}) \cap \mathcal{K}(\mathcal{H})$, and the onto map*

$$\pi_{T_0}^\infty : \mathcal{U}_A^\infty \to \{T \in \mathcal{I}_A^a : T - T_0 \in \mathcal{K}(\mathcal{H})\}, \ \pi_{T_0}^\infty(G) = GT_0$$

is a C^∞-submersion

Proof. We shall use Lemma 4.3. Consider the Banach algebra $\mathcal{B} = \mathbb{C}1 + \mathcal{B}_A(\mathcal{H}) \cap \mathcal{K}(\mathcal{H})$, with the norm $|B| = \max\{\|B\|, \|B\|^\sharp\}$. The group $\mathcal{U}_A^\infty$ is a subgroup of the Banach-Lie group of invertible operators in $\mathcal{B}$. The element

$$K = T(GT_0)^\sharp + 1 - P_T \in \mathcal{U}_A^\infty$$

defines a map - in fact a cross section - on a neighbourhood of T_0 in $\{T \in \mathcal{I}_A^a : T - T_0 \in \mathcal{K}(\mathcal{H})\}$, with values in $\mathcal{U}_A^\infty$. Indeed, the operator G is also a C^∞ map in the argument T (because of the regular structure of the restricted orbit of P_{T_0} under the action of $\mathcal{U}_A^\infty$ [4]). Note that the map $T \mapsto P_T = TT^\sharp$ is C^∞, it is the restriction (to the orbit of T_0) of the global C^∞ map $B \mapsto BB^\sharp$ of the ambient algebra $\mathcal{B}$. Clearly, then, the local cross section $T \mapsto KG$ extends to a C^∞ map defined on a neighbourhood of T_0 in $\mathcal{B}$.

The space on which $\mathcal{U}_A^\infty$ acts by left multiplication, namely $\{X \in \mathcal{B}_A(\mathcal{H}) : X - T_0 \in \mathcal{K}(\mathcal{H})\}$, is not a Banach space. It is the *affine* Banach space $T_0 + \mathcal{B}_A(\mathcal{H}) \cap \mathcal{K}(\mathcal{H})$. Therefore the argument proceeds, with slight modifications. □

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