THE KARCHER MEAN OF EQUILATERAL TRIANGLES

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Abstract. In this paper we present a class of Riemannian equilateral triangles ΔABC of the Cartan-Hadamard Riemannian manifold of $N \times N$ positive definite matrices of determinant 1 whose Karcher mean of vertices is of the form

$$\Lambda(A, B, C) = \frac{A + B + C}{\sqrt[N]{\det(A + B + C)}}.$$

The obtained class of triples (A, B, C) forms an analytic manifold with an analytic parameterization over $\mathrm{SL}(N, \mathbb{C})^{\pm} \times \mathbb{R}$. The determinantal identity $\det(A + B + C) = \det(A^{-1} + B^{-1} + C^{-1})$ and an extended arithmetic-Karcher-harmonic mean inequalities is derived for such triples.

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1. Introduction

Let $\mathbb{P}_N$ be the space of $N \times N$ positive definite Hermitian matrices equipped with the Riemannian metric $\delta(A, B) = || \log A^{-1}B||_2$, where $\|X\|_2 := \sqrt{\mathrm{tr}\, X^2}$ for Hermitian matrices X. The Karcher mean (alternatively, Riemannian mean or Cartan mean) on $\mathbb{P}_N$ is uniquely defined by

$$\Lambda(A_1, \dots, A_n) := \underset{X \in \mathbb{P}_N}{\arg\min} \sum_{k=1}^{n} \delta^2(X, A_k)$$

and is characterized by being the unique positive definite solution of the Karcher equation $\sum_{j=1}^{n} \log(X^{-1/2} A_j X^{-1/2}) = 0$. It is well known that $\Lambda(A, B) = A\#B := A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$, the geometric mean of A and B, and is the unique midpoint between A and B for the Riemannian metric δ. See [2, 20, 12, 13] for basic properties of the Karcher mean including the Ando-Li-Mathias ten axioms for matrix geometric means. By joint homogeneity, we can restrict our concern to positive definite matrices of determinant 1.

Let $\mathbf{H}_N \subset \mathbb{P}_N$ be the geodesic submanifold of positive definite matrices of determinant 1. For the 2×2 setting, $\mathbf{H}_2$ is a model of hyperbolic geometry and the Karcher mean of triples in $\mathbf{H}_2^3$ has recently been intensely investigated [5, 6, 8, 9, 14, 15]. For $A, B \in \mathbf{H}_2$, the Karcher mean $A\#B = \Lambda(A, B)$ has the following linear form [2, 4]

$$\Lambda(A,B) = \frac{A+B}{\sqrt{\det(A+B)}}. \tag{1.1}$$

Surprisingly this linear representation arises for Riemannian equilateral triangles [8]: the triangle ΔABC in $\mathbf{H}_2$ is equilateral if and only if

$$\Lambda(A,B,C) = \frac{A+B+C}{\sqrt{\det(A+B+C)}}. \tag{1.2}$$

Linear independency of the vertices of equilateral triangles and non-collinearity in $\mathbf{H}_2$ play key roles in the proof: see Remark 2.2 for details. The linear formula (1.2) was first obtained by Bhatia and Jain for the family of exponentials of Pauli matrices [5]. It turns out [15] that the three medians of a triangle ΔABC in $\mathbf{H}_2$ always meet at the right hand side of (1.2), called the center of mass or centroid of the triangle, from which two notions of the centroid in the hyperbolic geometry and the Karcher mean agree *only* for equilateral triangles. Applying to the equilateral triangle $\Delta A^{-1}B^{-1}C^{-1}$, where inversion acts as an isometry on $\mathbf{H}_2$, together with the determinantal identity $\det(A+B+C) = \det(A^{-1}+B^{-1}+C^{-1})$ on $\mathbf{H}_2$, leads to the following special form of the arithmetic-Karcher-harmonic mean inequalities for the equilateral triangle ΔABC :

$$\mathcal{H}(A,B,C) = \frac{\mathcal{A}(A,B,C)}{\det \mathcal{A}(A,B,C)} \leq \Lambda(A,B,C) = \frac{\mathcal{A}(A,B,C)}{\sqrt{\det \mathcal{A}(A,B,C)}} \leq \mathcal{A}(A,B,C),$$

where $\mathcal{A}$ and $\mathcal{H}$ are the arithmetic and harmonic means respectively [15]. In particular, the harmonic mean is a *scalar multiple* of the arithmetic mean.

For general $N \geq 3$, one may wonder whether the Karcher mean of the vertices of an equilateral triangle in $\mathbf{H}_N$ is of the form

$$\Lambda(A,B,C) = \frac{A+B+C}{\sqrt[N]{\det(A+B+C)}}, \tag{1.3}$$

which is equivalent to $\Lambda(A,B,C) = \alpha(A+B+C)$ for some $\alpha > 0$ due to the determinantal identity of the Karcher mean. However this fails for general $N \geq 3$, see Example 3.9. Alternatively this raises the classification problem of equilateral triangles in $\mathbf{H}_N$ having the Karcher mean formula (1.3). From the result on 2×2 matrices and the self-duality of the Karcher mean, $\Lambda(A,B,C)^{-1} = \Lambda(A^{-1}, B^{-1}, C^{-1})$, it is natural to consider the class $\mathcal{K}_N$ of equilateral triangles ΔABC in $\mathbf{H}_N$ such that both (A,B,C) and (A^{-1}, B^{-1}, C^{-1}) satisfy (1.3). This set is invariant under inversion, $(A,B,C) \mapsto (A^{-1}, B^{-1}, C^{-1})$.

In this paper we consider the special class $\mathcal{E}_N$ of triples $(A,B,C) \in \mathbf{H}_N^3$ such that

$$\sigma(A^{-1}B) = \sigma(A^{-1}C) = \sigma(B^{-1}C) = \{\lambda, \lambda^{-1}\}$$

for some $\lambda > 1$, where $\sigma(X)$ denotes the spectrum of X. It is shown that $\mathcal{E}_N \neq \emptyset$ if and only if N is an even number, and that A, B, C are not collinear and linearly independent for every $(A,B,C) \in \mathcal{E}_N$. Obviously $\mathcal{E}_N$ is invariant under inversion. The sets $\mathcal{E}_N$ and $\mathcal{K}_N$ are invariant under the action of the Lie group $\mathrm{SL}(N,\mathbb{C})^{\pm} = \det^{-1}(\{\pm 1\})$ via the congruence transformations:

$$M.(A,B,C) := (MAM^*, MBM^*, MCM^*).$$

It is shown that $\mathcal{E}_N$ forms an analytic manifold via an analytic parameterization over $\mathrm{SL}(N,\mathbb{C})^{\pm} \times (1,\infty)$.

The main result of this paper is the following.

Theorem 1.1. *For every even integer N, $\mathcal{E}_N \subset \mathcal{K}_N$ and $\mathcal{E}_N$ is diffeomorphic to $\left(\mathrm{SL}(N,\mathbb{C})^{\pm}/I_2 \otimes \mathbb{U}_{\frac{N}{2}}\right) \times (1,\infty)$, where $\mathbb{U}_k$ denotes the Lie group of $k \times k$ unitary matrices and $\otimes$ denotes the tensor product operation.*

In Section 4, we establish the determinantal identity $\det(A+B+C) = \det(A^{-1} + B^{-1} + C^{-1})$ on $\mathcal{E}_N$ and obtain a one parameter monotonic family

$$\frac{\frac{1}{3}(A+B+C)}{\sqrt[p]{\det\left(\frac{1}{3}(A+B+C)\right)}}, \qquad 0 < p \leq \infty$$

interpolating the harmonic mean at $p = \frac{N}{2}$, the Karcher mean at $p = N$, and the arithmetic mean at $p = \infty$. In Section 5, we introduce a method of constructing triples in $\mathbf{H}_N$ having the Karcher mean formula (1.3) via positive linear maps.

2. Invariant sets of equilateral triangles

Let $A, B, C \in \mathbf{H}_2$. The spectrum of $A^{-1}B$ is $\{\lambda, \lambda^{-1}\}$ for some $\lambda \geq 1$, because $\det(A^{-1}B) = 1$, and the Riemannian distance between A and B is $\delta(A, B) = \sqrt{2}\log\lambda$. Then $\delta(A, B) = \delta(A, C) = \delta(B, C)$ if and only if

$$\sigma(A^{-1}B) = \sigma(A^{-1}C) = \sigma(B^{-1}C) = \{\lambda, \lambda^{-1}\} \tag{2.4}$$

for some $\lambda > 1$. This implies that the set $\mathcal{E}_2$ coincides with the set of all equilateral triangles in $\mathbf{H}_2$. In terms of the trace functional, $\delta(A, B) = \delta(A, C)$ if and only if $\mathrm{tr}(AB^{-1}) = \mathrm{tr}(AC^{-1})$. We recall the main theorem of [8].

Theorem 2.1. *Let $A, B, C \in \mathbf{H}_2$.*

(1) (2.4) *implies* (1.3). *The converse holds if A, B, C are linearly independent.*
(2) *If A, B, C are linearly dependent, then* (1.3) *holds true if and only if one of A, B, C is the geometric mean of the other two.*

Remark 2.2. [Riemannian equilateral triangles and linear independency] Note that the unique (up to parametrization) Riemannian geodesic between A and B is of the form $t \mapsto A\#_t B := A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}$ and the vertices of a Riemannian triangle in $\mathbf{H}_N$ (by definition) are not collinear in the sense of Riemannian sense. Then the theorem in above actually classifies triangles in $\mathbf{H}_2$ having the Karcher mean formula (1.3) on vertices: *a triangle ΔABC in $\mathbf{H}_2$ is equilateral if and only if the Karcher mean $\Lambda(A, B, C)$ satisfies (1.3).* See also [15] via an approach of hyperbolic geometry. In particular, the vertices of an Riemannian equilateral triangle in $\mathbf{H}_2$ are *linearly independent* in the Euclidean space of 2×2 Hermitian matrices. It is clearly of interest in matrix analysis and geometry to see whether the linear independency of vertices of Riemannian equilateral triangles in $\mathbf{H}_N$ holds for general $N \geq 3$. We leave this problem to the interested reader.

Recall that

$$\mathcal{E}_N = \left\{(A, B, C) \in \mathbf{H}_N^3 : \sigma(A^{-1}B) = \sigma(A^{-1}C) = \sigma(B^{-1}C) = \{\lambda, \lambda^{-1}\}, \lambda > 1\right\}.$$

Remark 2.3. For $A, B, C \in \mathbb{P}_N$, $\left(\frac{A}{\sqrt[N]{\det(A)}}, \frac{B}{\sqrt[N]{\det(B)}}, \frac{C}{\sqrt[N]{\det(C)}}\right) \in \mathcal{E}_N$ if and only if

$$\begin{aligned} \sigma(A^{-1}B) &= \sqrt[N]{\det(AB^{-1})}\left\{\lambda, \lambda^{-1}\right\}, \\ \sigma(A^{-1}C) &= \sqrt[N]{\det(AC^{-1})}\left\{\lambda, \lambda^{-1}\right\}, \\ \sigma(B^{-1}C) &= \sqrt[N]{\det(BC^{-1})}\left\{\lambda, \lambda^{-1}\right\}, \end{aligned}$$

for some $\lambda > 1$.

Let $A, B \in \mathbf{H}_N$ such that $\sigma(A^{-1}B) = \{\lambda, \lambda^{-1}\}, \lambda > 1$. Since $\det(A^{-1}B) = 1$, the multiplicity of λ is equal to that of λ^{-1}. This implies that N is even number and the positive definite matrix $A^{-1/2}BA^{-1/2} \in \mathbf{H}_N$ is unitary similar to the diagonal matrix

$$D_N(\lambda) := \begin{bmatrix} \lambda I_{\frac{N}{2}} & 0 \\ 0 & \lambda^{-1} I_{\frac{N}{2}} \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \otimes I_{\frac{N}{2}}. \tag{2.5}$$

It turns out in [19] that for $A, B \in \mathbf{H}_N$, $\sigma(A^{-1}B) = \{\lambda, \lambda^{-1}\}$ for some $\lambda \geq 1$ if and only if

$$A \# B = \frac{A+B}{\sqrt[N]{\det(A+B)}}. \tag{2.6}$$

From this together with the case of 2×2, we expect the Karcher mean formula (1.3) for $(A, B, C) \in \mathcal{E}_N$.

Remark 2.4. The set of pairs (A, B) such that $\sigma(A^{-1}B) = \{\lambda, \lambda^{-1}\}$ has recently appeared in the study of Finsler structures on $\mathbb{P}_N$ or positive cones of C^* algebras equipped with the Thompson metric $d(A, B) = ||\log A^{-1}B||$. For such a pair (A, B), the curve $t \mapsto A\#_t B = A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}$ is a *unique* minimal geodesic between A and B for the Thompson metric $d_T(A, B) = ||\log AB^{-1}||$. See [18, 16, 17].

We list some basic properties of $\mathcal{E}_N$.

Proposition 2.5. *We have that $\mathcal{E}_N \neq \emptyset$ if and only if N is even. For $(A, B, C) \in \mathcal{E}_N$,*

1. $\det(A + B) = \det(A + C) = \det(B + C)$.
2. $A\#B = a(A + B)$, $A\#C = a(A + C)$, $B\#C = a(B + C)$, *where* $a = \frac{\sqrt{\lambda}}{1+\lambda}$ *and* $\lambda > 1$ *is an eigenvalue of* $A^{-1}B$.
3. *A, B, C are not collinear in the sense of Riemannian. In particular, ΔABC is equilateral.*

(4) *A, B, C are linearly independent.*
(5) $(A^{-1}, B^{-1}, C^{-1}) \in \mathcal{E}_N$.
(6) *$M.(A, B, C) \in \mathcal{E}_N$ for every $M \in \mathrm{SL}^{\pm}(N, \mathbb{C})$.*

Proof. Let $N = 2k$ and let

$$D = \mathrm{diag}(2, 1/2) \otimes I_k, \qquad W = \begin{bmatrix} \frac{5}{3} & -\frac{1}{3}\sqrt{\frac{7}{2}} \\ -\frac{1}{3}\sqrt{\frac{7}{2}} & \frac{5}{6} \end{bmatrix} \otimes I_k.$$

Then $\sigma(D) = \sigma(W) = \sigma(DW^{-1}) = \{2, 1/2\}$ and hence $(I, D, W) \in \mathcal{E}_N$. Indeed, the eigenvalues of

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} \otimes I_k, \qquad b \in \mathbb{R}$$

are $\frac{1}{2}\left(a + c \pm \sqrt{(a-c)^2 + 4b^2}\right)$. See Lemma 3.1 for general case. This shows that $\mathcal{E}_N \neq \emptyset$. (1) is straightforward. (2) follows from (2.6). (3) Suppose that A, B, C are collinear. We may assume that $A = B\#_t C$ for some $0 < t < 1$. Then $t\delta(B, C) = \delta(B\#_t C, C) = \delta(A, C) = \delta(B, C)$, which is impossible from $B \neq C$. (4) Suppose that $C = aA + bB$ for some $a, b \in \mathbb{R}$. Then $ab \neq 0$, since any two of A, B, C are linearly independent. If $a, b < 0$, then $-C$ is positive definite, which is impossible. Hence we have either $a, b > 0$ or $ab < 0$. Without loss generality, we may assume that $a, b > 0$. Indeed if $a < 0$, then $B = b^{-1}C + (-ab^{-1})A$ and hence we can replace C with B. Pick a unitary matrix U so that $UC^{-1/2}AC^{-1/2}U^* = D := D_N(\lambda), \lambda > 1$. It then follows from $I = aC^{-1/2}AC^{-1/2} + bC^{-1/2}BC^{-1/2}$ that $I = aD + bUC^{-1/2}BC^{-1/2}U^*$. That is, $(1/b)(I - aD) = UC^{-1/2}BC^{-1/2}U^*$. Comparing eigenvalues of $(1/b)(I - aD)$ and $U^*C^{-1/2}BC^{-1/2}U$, which has the eigenvalues λ and λ^{-1} by assumption, leads to

$$(1/b)(1 - a\lambda) = \lambda, \qquad (1/b)(1 - a\lambda^{-1}) = \lambda^{-1}$$

or

$$(1/b)(1 - a\lambda) = \lambda^{-1}, \qquad (1/b)(1 - a\lambda^{-1}) = \lambda.$$

The first case leads to $1 = (a+b)\lambda = (a+b)\lambda^{-1}$ from which $\lambda = \lambda^{-1}$, since $a + b \neq 0$. This is impossible due to $\lambda \neq 1$. The second case leads to $a = b$ from $\lambda \neq 1$ and hence

$C = a(A+B)$. Taking determinant yields $a = \frac{1}{\sqrt[N]{\det(A+B)}}$ and hence by (2.6)

$$C = \frac{A+B}{\sqrt[N]{\det(A+B)}} = A\#B.$$

By (4), this contradicts to (3). (5) and (6) are straightforward. □

Remark 2.6. It is shown in [14] that the set of linearly independent triples in $\mathbf{H}_2$ is closed under inversion; $\{A,B,C\} \subset \mathbf{H}_2$ is linearly independent if and only if $\{A^{-1},B^{-1},C^{-1}\}$ also is. This fails for general $N \geq 3$. For example, $A = I_3, B = \operatorname{diag}\left(2,1,\frac{1}{2}\right), C = \operatorname{diag}\left(\frac{2}{3+2\sqrt{2}}, \frac{1}{2}, 3+2\sqrt{2}\right)$.

Recall the set $\mathcal{K}_N \subset \mathbf{H}_N^3$ from Introduction.

Definition 2.7. Denoted by $\mathcal{K}_N^\dagger \subset \mathcal{K}_N$ the set of triples $(A,B,C) \in \mathcal{K}_N$ such that both (A,B,C) and (A^{-1},B^{-1},C^{-1}) are linearly independent.

By Theorem 2.1, $\mathcal{E}_2 = \mathcal{K}_2 = \mathcal{K}_2^\dagger$. For general $N > 2$, the sets $\mathcal{E}_N, \mathcal{K}_N$ and $\mathcal{K}_N^\dagger$ seem to be distinct and new in the context of Riemannian geometry and matrix analysis. These are invariant under inversion and congruence transformations $(A,B,C) \mapsto M.(A,B,C) = (MAM^*, MBM^*, MBM^*)$, $M \in \mathrm{SL}^{\pm}(N,\mathbb{C})$, due to that of the Karcher mean.

Finally we discuss about $\mathcal{K}_N^\dagger \neq \emptyset$ for $N \geq 2$. This is true for $N = 2$ and for every even number N, by assuming Theorem 1.1. Let $a > 1$ and

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} a & 0 & 0 \\ 0 & a^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a^{-1} & 0 \\ 0 & 0 & a \end{bmatrix}.$$

Then $\sigma(AB^{-1}) = \sigma(AC^{-1}) = \sigma(BC^{-1}) = \{a, a^{-1}, 1\}$ and hence $\delta(A,B) = \delta(A,C) = \delta(B,C)$. One can directly see that $\{A,B,C\}$ and $\{A^{-1},B^{-1},C^{-1}\}$ are linearly independent. Moreover

$$\begin{aligned} \Lambda(A,B,C) &= (ABC)^{\frac{1}{3}} = \begin{bmatrix} a^{1/3} & 0 & 0 \\ 0 & a^{-2/3} & 0 \\ 0 & 0 & a^{1/3} \end{bmatrix} \\ &= \frac{\sqrt[3]{a}}{a+2}(A+B+C) = \frac{A+B+C}{\sqrt[3]{\det(A+B+C)}}. \end{aligned}$$

Replacing a with a^{-1} yields

$$\Lambda(A^{-1}, B^{-1}, C^{-1}) = \frac{A^{-1} + B^{-1} + C^{-1}}{\sqrt[3]{\det(A^{-1} + B^{-1} + C^{-1})}}.$$

This shows that $\mathcal{K}_3^\dagger \neq \emptyset$. We believe that $\mathcal{K}_N^\dagger \neq \emptyset$ for all $N \geq 2$ and leave the proof to the reader.

3. Proof of the main theorem

Let $N = 2k$ be an even number. The Lie group $\mathrm{SL}^{\pm}(N, \mathbb{C})$ acts on $\mathcal{E}_N$ via the congruence transformations. We recall a reduction process on the triples [5, 14]. Let $(A, B, C) \in \mathcal{E} = \mathcal{E}_N$ be fixed. Let λ, λ^{-1} be eigenvalues of $A^{-1/2}BA^{-1/2}$. Since $A \neq B$, we may assume that $\lambda > 1$. Pick a unitary matrix U such that

$$A^{-1/2}BA^{-1/2} = UDU^*, \qquad D := D_N(\lambda)$$

and set $W := U^*A^{-1/2}CA^{-1/2}U$. Then $(A, B, C) = M.(I, D, W)$, where $M := A^{1/2}U$, from

$$\begin{aligned}(A, B, C) &= \left(A^{1/2}IA^{1/2}, A^{1/2}(A^{-1/2}BA^{-1/2})A^{1/2}, A^{1/2}(A^{-1/2}CA^{-1/2})A^{1/2}\right) \\ &= A^{1/2}\left(I, UDU^*, A^{-1/2}CA^{-1/2}\right)A^{1/2} = A^{1/2}\left(I, UDU^*, UWU^*\right)A^{1/2} \\ &= A^{1/2}U(I, D, W)U^*A^{1/2} = M.(I, D, W).\end{aligned}$$

Since $M \in \mathrm{SL}(N, \mathbb{C})^{\pm}$, we have from Proposition 2.5 (7) that

$$(I, D = D_N(\lambda), W) \in \mathcal{E}_N. \tag{3.7}$$

That is, $\sigma(D) = \sigma(W) = \sigma(DW^{-1}) = \{\lambda, \lambda^{-1}\}$.

Next, we describe the positive definite matrix W in (3.7) explicitly. Since $\sigma(W) = \{\lambda, \lambda^{-1}\} = \sigma(D)$, we can find a unitary matrix V such that $W = VDV^*$. By CS-decomposition of V (cf. Theorem VII. 1. 6 of [3]),

$$V = V_1 \begin{bmatrix} C & S \\ -S & C \end{bmatrix} V_2 \tag{3.8}$$

where $V_1 = \mathrm{diag}(V_{11}, V_{12})$ and $V_2 = \mathrm{diag}(V_{21}, V_{22})$ are unitary matrices such that V_{ij} are $k \times k$ for $i, j = 1, 2$ and C, S are nonnegative diagonal matrices, with diagonal entries

$$0 \le c_1 \le \ldots \le c_k \le 1, \qquad 1 \ge s_1 \ge \ldots \ge s_k \ge 0 \tag{3.9}$$

respectively, and

$$C^2 + S^2 = I. \tag{3.10}$$

It follows from $W = VDV^*$ and $V_2 D V_2^* = D$ that

$$\begin{aligned} W &= V_1 \begin{bmatrix} C & S \\ -S & C \end{bmatrix} V_2 D V_2^* \begin{bmatrix} C & -S \\ S & C \end{bmatrix} V_1^* = V_1 \begin{bmatrix} C & S \\ -S & C \end{bmatrix} D \begin{bmatrix} C & -S \\ S & C \end{bmatrix} V_1^* \\ &= V_1 \begin{bmatrix} C & S \\ -S & C \end{bmatrix} \begin{bmatrix} \lambda I_N & 0 \\ 0 & \lambda^{-1} I_N \end{bmatrix} \begin{bmatrix} C & -S \\ S & C \end{bmatrix} V_1^* \\ &= V_1 \underbrace{\begin{bmatrix} \lambda C^2 + \lambda^{-1} S^2 & (\lambda^{-1} - \lambda) SC \\ (\lambda^{-1} - \lambda) SC & \lambda S^2 + \lambda^{-1} C^2 \end{bmatrix}}_{:= W(C,S;\lambda)} V_1^*. \end{aligned}$$

We then have from $V_1^* D V_1 = D$ that

$$\begin{aligned} (A, B, C) &= A^{1/2} U.(I, D, W) = A^{1/2} U.\,(I, D = D_N(\lambda), V_1 W(C, S; \lambda) V_1^*) \\ &= A^{1/2} U V_1.\,(I, V_1^* D V_1, W(C, S; \lambda)) \\ &= A^{1/2} U V_1.\,(I, D, W(C, S; \lambda)) \end{aligned} \tag{3.11}$$

and hence $(I, D = D_N(\lambda), W(C, S; \lambda)) \in \mathcal{E}_N$. We note that $W(C, S; \lambda)$ is a partitioned matrix whose blocks are diagonal matrices, and its construction depends only on the identity $\sigma(W) = \sigma(D) = \{\lambda, \lambda^{-1}\}$. Note that $(\lambda^{-1} - \lambda)SC$, the entry of $(1, 2)$ block of $W(C, S; \lambda)$, is nonpositive from $\lambda > 1$.

The other condition $\sigma(D^{-1}W) = \{\lambda, \lambda^{-1}\}$ together with the following lemma is crucial for the following description of $W(C, S; \lambda)$:

$$W(C, S; \lambda) = W_\lambda \otimes I_k, \tag{3.12}$$

where

$$W_\lambda := \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

and

$$a = \frac{\lambda^2+1}{\lambda+1}, \qquad b = -\frac{(\lambda-1)\sqrt{\lambda^2+\lambda+1}}{\sqrt{\lambda}(\lambda+1)}, \qquad c = \frac{\lambda^2+1}{\lambda(\lambda+1)}.$$

Lemma 3.1. *Let* $M = \begin{bmatrix} X & Y \\ Y & Z \end{bmatrix}$ *be a* $2n \times 2n$ *matrix where each block is* $n \times n$ *real diagonal, i.e,* $X = \operatorname{diag}(a_1, \ldots, a_n), Y = \operatorname{diag}(b_1, \ldots, b_n), Z = \operatorname{diag}(c_1, \ldots, c_n)$ *and* $a_j, b_j, c_j \in \mathbb{R}$ *for* $j = 1, \ldots, n$. *Then*

$$\sigma(M) = \left\{ \frac{a_j + c_j \pm \sqrt{(a_j - c_j)^2 + 4b_j^2}}{2}, j = 1, \ldots, n \right\}. \tag{3.13}$$

If further $a_j, c_j > 0$ *for all* j *(for instance,* M *is positive definite) and* $\sigma(M) = \sigma(D_{2n}(\lambda)^{-1}M) = \{\lambda, \lambda^{-1}\}$ *for some* $\lambda > 1$*, then for all* $j = 1, \ldots, n$,

$$a_j = \frac{\lambda^2+1}{\lambda+1}, \qquad b_j \in \left\{ \pm\frac{(\lambda-1)\sqrt{\lambda^2+\lambda+1}}{\sqrt{\lambda}(\lambda+1)} \right\}, \qquad c_j = \frac{\lambda^2+1}{\lambda(\lambda+1)}.$$

.

Proof. The first assertion is direct from

$$\det(xI_{2n} - M) = \det \begin{bmatrix} xI_n - X & -Y \\ -Y & xI_n - Z \end{bmatrix} = \det((xI_n - Z)(xI_n - X) - Y^2)$$
$$= \prod_{j=1}^{n} [(x - c_j)(x - a_j) - b_j^2] = 0.$$

For the second assertion we will separate by cases. First note that

$$D^{-1}M = \begin{bmatrix} \lambda^{-1}X & Y \\ Y & \lambda Z \end{bmatrix}, \qquad D := D_{2n}(\lambda).$$

By the first assertion, we have that

$$\sigma(D^{-1}M) = \left\{ \frac{a_j\lambda^{-1} + c_j\lambda \pm \sqrt{(a_j\lambda^{-1} - c_j\lambda)^2 + 4b_j^2}}{2}, j = 1, \ldots, n \right\}. \tag{3.14}$$

(1) We will show that $b_j \neq 0$ for all j. Suppose that $b_k = 0$ for some k. We divide two cases; $a_k = c_k$ or $a_k \neq c_k$. Suppose that $a_k \neq c_k$. By (3.13) and $\sigma(M) = \{\lambda, \lambda^{-1}\}$,

$$\frac{a_k + c_k + |a_k - c_k|}{2} = \lambda, \qquad \frac{a_k + c_k - |a_k - c_k|}{2} = \lambda^{-1}.$$

Also by $\sigma(D^{-1}M) = \{\lambda, \lambda^{-1}\}$ and (3.14),

$$\frac{a_k\lambda^{-1} + c_k\lambda + |a_k\lambda^{-1} - c_k\lambda|}{2} = \lambda;$$
$$\frac{a_k\lambda^{-1} + c_k\lambda - |a_k\lambda^{-1} - c_k\lambda|}{2} = \lambda^{-1}.$$

One sees easily that $\lambda = 1$, which is impossible. Similarly the case $a_k = c_k$ gives us a contradiction.

(2) Suppose that $a_k = c_k$ for some k. By (1), $b_k \neq 0$ and hence

$$a_k + |b_k| = \lambda, \qquad a_k - |b_k| = \lambda^{-1}$$

and

$$\frac{a_k(\lambda^{-1} + \lambda) + 2|b_k|}{2} = \lambda, \qquad \frac{a_k(\lambda^{-1} + \lambda) - 2|b_k|}{2} = \lambda^{-1}.$$

This leads to $\lambda = 1$. Therefore $a_j \neq c_j$ for all j.

(3) By (1) and (2), $b_j \neq 0$ and $a_j \neq c_j$, for all $j = 1, \ldots, N$. We then have that

$$\frac{a_j + c_j + \sqrt{(a_j - c_j)^2 + 4b_j^2}}{2} = \lambda;$$
$$\frac{a_j + c_j - \sqrt{(a_j - c_j)^2 + 4b_j^2}}{2} = \lambda^{-1}.$$

Also by $\sigma(D^{-1}M) = \{\lambda, \lambda^{-1}\}$ and (3.14),

$$\frac{a_j\lambda^{-1} + c_j\lambda + \sqrt{(a_j\lambda^{-1} - c_j\lambda)^2 + 4b_j^2}}{2} = \lambda;$$
$$\frac{a_j\lambda^{-1} + c_j\lambda - \sqrt{(a_j\lambda^{-1} - c_j\lambda)^2 + 4b_j^2}}{2} = \lambda^{-1}.$$

The only possible solution of the previous four equalities are of the form

$$a_j = \frac{\lambda^2 + 1}{\lambda + 1}, b_j = \pm\frac{(\lambda - 1)\sqrt{\lambda^2 + \lambda + 1}}{\sqrt{\lambda}(\lambda + 1)}, c_j = \frac{\lambda^2 + 1}{\lambda(\lambda + 1)},$$

for all $j = 1, \ldots, n$. Indeed, summing up the first two equalities and the second two we get that $a_j + c_j = \lambda + \lambda^{-1} = a_j\lambda^{-1} + c_j\lambda$, from which $a_j(1 - \lambda^{-1}) + c_j(1 - \lambda) = 0$

and so $a_j = c_j \frac{\lambda-1}{1-\lambda^{-1}} = c_j\lambda$. We then have that $a_j = \frac{\lambda^2+1}{\lambda+1}$ and $c_j = \frac{\lambda^2+1}{\lambda(\lambda+1)}$. On the other hand by subtracting the first two equalities we get that

$$\sqrt{(a_j - c_j)^2 + 4b_j^2} = \lambda - \lambda^{-1},$$

from which $2|b_j| = \sqrt{(\lambda-\lambda^{-1})^2 - (a_j - c_j)^2}$. Note that

$$\begin{aligned}
\sqrt{(\lambda-\lambda^{-1})^2 - (a_j - c_j)^2} &= \sqrt{(\lambda-\lambda^{-1})^2 - \left(\frac{\lambda^2+1}{\lambda+1} - \frac{\lambda^2+1}{\lambda(\lambda+1)}\right)^2} \\
&= \sqrt{(\lambda-\lambda^{-1})^2 - \left(\frac{(\lambda^2+1)(\lambda-1)}{\lambda(\lambda+1)}\right)^2} \\
&= \frac{1}{\lambda(\lambda+1)}\sqrt{[(\lambda^2-1)(\lambda+1)]^2 - [(\lambda^2+1)(\lambda-1)]^2} \\
&= \frac{1}{\lambda(\lambda+1)}\sqrt{[(\lambda^3-1)+(\lambda^2-\lambda)]^2 - [(\lambda^3-1)-(\lambda^2-\lambda)]^2} \\
&= \frac{2}{\lambda(\lambda+1)}\sqrt{(\lambda^3-1)(\lambda^2-\lambda)} \\
&= \frac{2}{\lambda(\lambda+1)}\sqrt{(\lambda-1)(\lambda^2+\lambda+1)\lambda(\lambda-1)} \\
&= \frac{2(\lambda-1)}{\sqrt{\lambda}(\lambda+1)}\sqrt{(\lambda^2+\lambda+1)}.
\end{aligned}$$

This completes the proof. □

Remark 3.2. Note that the 2×2 matrix $W_\lambda = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ in (3.12) belongs to $\mathbf{H}_2$ such that (1) $a > c$, (2) $b < 0$ and (3) $\sigma(W_\lambda) = \{\lambda, \lambda^{-1}\}$. Applying the reduction process for $N = 2$ leads to

$$\gamma(\lambda) := (I_2, D_1(\lambda), W_\lambda) \in \mathcal{E}_2, \qquad \lambda > 1.$$

This defines a one-parameter family in $\mathcal{E}_2$.

We then finally conclude that

$$\begin{aligned}(A,B,C) &= L.\,(I,D,W(C,S;\lambda)) \qquad (L:=A^{1/2}UV_1, D=D_N(\lambda))\\ &= L.\left(I_2\otimes I_k, \begin{bmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{bmatrix}\otimes I_k, W_\lambda\otimes I_k\right)\\ &= L.\,(I_2\otimes I_k, D_1(\lambda)\otimes I_k, W_\lambda\otimes I_k)\\ &= L.\,(I_2, D_1(\lambda), W_\lambda)\otimes I_k,\end{aligned}$$

where $(X,Y,Z)\otimes K := (X\otimes K, Y\otimes K, Z\otimes K)$. In particular,

$$(I_2, D_1(\lambda), W_\lambda)\otimes I_k \in \mathcal{E}_N, \qquad (N=2k).$$

Definition 3.3. Define $\gamma_N : (1,\infty)\to\mathcal{E}_N$ by

$$\gamma_N(\lambda) = \gamma(\lambda)\otimes I_k = (I_2, D_1(\lambda), W_\lambda)\otimes I_k, \qquad (N=2k).$$

The analytic map γ_N can be extended on $(0,\infty)$ with $\gamma_N(1)=(I,I,I)$.

We have shown that the map

$$\mathrm{SL}(N,\mathbb{C})^{\pm}\times(1,\infty)\to\mathcal{E}_N, \qquad (M,\lambda)\mapsto M.\gamma_N(\lambda)$$

is surjective, which is an analytic parametrization of $\mathcal{E}_N$.

Next, we will describe the quotient structure of $\mathcal{E}_N$ from the preceding parameterization. To do this, let $\mathbb{U}_k\oplus\mathbb{U}_k$ be the subgroup of the unitary group $\mathbb{U}_N, N=2k$, consisting of block diagonal matrices

$$\begin{bmatrix}V_1 & 0\\ 0 & V_2\end{bmatrix}, \qquad V_1, V_2\in\mathbb{U}_k.$$

It contains the Lie subgroup

$$I_2\otimes\mathbb{U}_k = \left\{\begin{bmatrix}V & 0\\ 0 & V\end{bmatrix} : V\in\mathbb{U}_k\right\}\subset\mathbb{U}_k\oplus\mathbb{U}_k.$$

Lemma 3.4. *For $U\in\mathbb{U}_N, N=2k$, and $\lambda>1$, $U.D_N(\lambda)=D_N(\lambda)$ if and only if $U\in\mathbb{U}_k\oplus\mathbb{U}_k$. Moreover for $U\in\mathbb{U}_k\oplus\mathbb{U}_k$, $U.\,(W_\lambda\otimes I_k)=W_\lambda\otimes I_k$ if and only if $U\in I_2\otimes\mathbb{U}_k$.*

Proof. Suppose that $UD_N(\lambda)U^* = D_N(\lambda)$. Let $U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}$, where U_{ij} are $k \times k$ matrices. Then

$$\begin{bmatrix} U_{11}U_{11}^* + U_{12}U_{12}^* & U_{11}U_{21}^* + U_{12}U_{22}^* \\ U_{21}U_{11}^* + U_{22}U_{12}^* & U_{21}U_{21}^* + U_{22}U_{22}^* \end{bmatrix} = I_N$$

from $UU^* = I_N$, and

$$\begin{bmatrix} \lambda U_{11}U_{11}^* + \lambda^{-1}U_{12}U_{12}^* & \lambda U_{11}U_{21}^* + \lambda^{-1}U_{12}U_{22}^* \\ \lambda U_{21}U_{11}^* + \lambda^{-1}U_{22}U_{12}^* & \lambda U_{21}U_{21}^* + \lambda^{-1}U_{22}U_{22}^* \end{bmatrix} = D_N(\lambda)$$

from $U.D_N(\lambda) = D_N(\lambda)$. From the $(1,1)$ entries, we see that

$$\lambda U_{11}U_{11}^* + \lambda^{-1}U_{12}U_{12}^* \;\; = \;\; \lambda I_k = \lambda(U_{11}U_{11}^* + U_{12}U_{12}^*)$$

from which $U_{12}U_{12}^* = \lambda^2 U_{12}U_{12}^*$. Since $\lambda \neq 1$, $U_{12}U_{12}^* = 0$ and hence $U_{11}U_{11}^* = I_k$. Similarly $U_{21}U_{21}^* = 0$ and $U_{22}U_{22}^* = I_k$.

From the $(1,2)$ entries,

$$U_{11}U_{21}^* + U_{12}U_{22}^* = 0 = \lambda U_{11}U_{21}^* + \lambda^{-1}U_{12}U_{22}^*$$

from which $(\lambda - 1)U_{11}U_{21}^* = (1 - \lambda^{-1})U_{12}U_{22}^*$, that is $\lambda U_{11}U_{21}^* = U_{12}U_{22}^*$. This implies that $\lambda U_{21}U_{11}^* + \lambda^{-1}U_{22}U_{12} = U_{22}U_{12} + U_{21}U_{11}^*$ from which $\lambda U_{21}U_{11}^* = U_{22}U_{12}^* = -U_{21}U_{11}^*$. Hence $U_{21}U_{11}^* = 0$. Since U_{11} is unitary, $U_{21} = 0$ and hence $U_{12} = 0$.

Next, the block unitary matrix $U = \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix}$ fixes $W_\lambda \otimes I_k$. Letting $W_\lambda = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ leads to $bV_1V_2^* = bI_k$. Since $b \neq 0$, we have that $V_1 = W_2$. □

We consider the homogeneous space

$$\mathbf{E}_N := \mathrm{SL}(N, \mathbb{C})^{\pm}/I_2 \otimes \mathbb{U}_k, \qquad (N = 2k).$$

Theorem 3.5. *The map* $\Gamma : \mathbf{E}_N \times (1, \infty) \to \mathcal{E}_N$ *defined by* $\Gamma([M], \lambda) = M.\gamma_N(\lambda)$ *is bijective.*

Proof. Let $M_1, M_2 \in \mathcal{S}(N, \mathbb{C})^{\pm}$ such that $[M_1] = [M_2]$, that is, $M_1 = M_2K$ for some $K \in I_2 \otimes \mathbb{U}_k$. For every $\lambda > 1$,

$$\begin{aligned} M_1.\,(I.D_N(\lambda), W_\lambda \otimes I_k) &= M_2K.\,(I.D_N(\lambda), W_\lambda \otimes I_k) \\ &= M_2\,(KK^*, KD_N(\lambda)K^*, K\,(W_\lambda \otimes I_k)\,K^*) \\ &= M_1.\,(I.D_N(\lambda), W_\lambda \otimes I_k) \end{aligned}$$

where the last equality follows from $K \in I_2 \otimes \mathbb{U}_{\frac{N}{2}}$. This proves that the map Γ is well-defined. The map Γ is surjective since the map $(M, \lambda) \mapsto M.\gamma_N(\lambda)$ is surjective on $\mathrm{SL}(N, \mathbb{C})^{\pm} \times (1, \infty)$. Suppose that $\Gamma([M_1], \lambda) = \Gamma([M_2], \beta)$. Then $M_1 M_1^* = M_2 M_2^*$ which implies that $K := M_2^{-1} M_1$ is unitary. By the second coordinates, $K D_N(\lambda) K^* = D_N(\beta)$ and hence $\{\lambda, \lambda^{-1}\} = \{\beta, \beta^{-1}\}$. This implies that $\lambda = \beta$ and $K \in \mathbb{U}_k \oplus \mathbb{U}_k$, by Lemma 3.4. By the third coordinates, the unitary matrix K fixes $W_\lambda \otimes I_k$. By Lemma 3.4, $K \in I_2 \otimes \mathbb{U}_k$. This implies that $[M_1] = [M_2]$ and hence Γ is injective. □

Remark 3.6. The set $\mathcal{E}_N$ has an analytic manifold structure form the preceding theorem. As $\mathcal{E}_N \subset \mathbf{H}_N^3$, equipped with the product Riemannian structure from that of $\mathbf{H}_N$, it is not obvious whether it becomes a submanifold $\mathbf{H}_N^3$. In the proof of the preceding theorem, we have proved that the set $\{(A, B) \in \mathbf{H}_N : \sigma(A^{-1}B) = \{\lambda, \lambda^{-1}\}, \lambda > 1\}$ can be identified with the analytic manifold $\left(\mathrm{SL}(N, \mathbb{C})^{\pm} / \mathbb{U}_{\frac{N}{2}} \oplus \mathbb{U}_{\frac{N}{2}}\right) \times (1, \infty)$.

The first part of the main theorem 1.1 is immediate from the following.

Theorem 3.7. *We have that* $\mathcal{E}_N = \mathrm{SL}(N, \mathbb{C})^{\pm}.\gamma_N((1, \infty)) \subset \mathcal{K}_N^{\dagger}$.

Proof. Since the set $\mathcal{E}_N$ is closed under inversion, it is enough to show that

$$\Lambda(A, B, C) = \frac{A + B + C}{\sqrt[N]{\det(A + B + C)}}$$

for every $(A, B, C) \in \mathcal{E}_N$. Let $(A, B, C) \in \mathcal{E}_N$. Then $(A, B, C) = M.(I_2, D_1(\lambda), W_\lambda) \otimes I_{\frac{N}{2}}$ for some $\lambda > 1$ and $M \in \mathrm{SL}^{\pm}$. By the determinantal identity, $\det(X \otimes Y) = \det(X)^n \det(Y)^m$ for $m \times m$ matrix X and $n \times n$ matrix Y (see [1, 3] for further basic properties of tensor product), we have that

$$\begin{aligned}
\det(A + B + C) &= \det\left(M\left(I_2 \otimes I_{\frac{N}{2}} + D_1(\lambda) \otimes I_{\frac{N}{2}} + W_\lambda \otimes I_{\frac{N}{2}}\right) M^*\right) \\
&= \det\left(I_2 \otimes I_{\frac{N}{2}} + D_1(\lambda) \otimes I_{\frac{N}{2}} + W_\lambda \otimes I_{\frac{N}{2}}\right) \\
&= \det\left[(I_2 + D_1(\lambda) + W_\lambda) \otimes I_{\frac{N}{2}}\right] \\
&= \det\left(I_2 + D_1(\lambda) + W_\lambda\right)^{\frac{N}{2}}
\end{aligned}$$

from which $\sqrt[N]{\det(A + B + C)} = \sqrt{\det(I_2 + D_1(\lambda) + W_\lambda)}$.

The map $X \mapsto X \otimes I_{\frac{N}{2}}$ on $\mathbf{H}_2$ preserves the Karcher mean (see Example 5.2 for general case):

$$\begin{aligned}\Lambda\left(X \otimes I_{\frac{N}{2}}, Y \otimes I_{\frac{N}{2}}, Z \otimes I_{\frac{N}{2}}\right) &= \Lambda(X,Y,Z) \otimes \Lambda\left(I_{\frac{N}{2}}, I_{\frac{N}{2}}, I_{\frac{N}{2}}\right) \\ &= \Lambda(X,Y,Z) \otimes I_{\frac{N}{2}}.\end{aligned}$$

Then

$$\begin{aligned}\Lambda(A,B,C) &= \Lambda\left(M.(I_2, D_1(\lambda), W_\lambda) \otimes I_{\frac{N}{2}}\right) = M.\Lambda\left((I_2, D_1(\lambda), W_\lambda) \otimes I_{\frac{N}{2}}\right) \\ &= M\left[\Lambda\left(I_2, D_1(\lambda), W_\lambda\right) \otimes I_{\frac{N}{2}}\right] M^* \\ &= M\left[\left(\frac{I_2 + D_1(\lambda) + W_\lambda}{\sqrt{\det(I_2 + D_1(\lambda) + W_\lambda)}}\right) \otimes I_{\frac{N}{2}}\right] M^* \\ &= M\left[\left(\frac{I_2 \otimes I_{\frac{N}{2}} + D_1(\lambda) \otimes I_{\frac{N}{2}} + W_\lambda \otimes I_{\frac{N}{2}}}{\sqrt{\det(I_2 + D_1(\lambda) + W_\lambda)}}\right)\right] M^* \\ &= \frac{A+B+C}{\sqrt{\det(I_2 + D_1(\lambda) + W_\lambda)}} \\ &= \frac{A+B+C}{\sqrt[N]{\det(A+B+C)}}.\end{aligned}$$

This completes the proof. $\square$

Example 3.8. For $N = 4$ and $\lambda = 2$,

$$W_2 = \frac{1}{3}\begin{bmatrix} 5 & -\frac{\sqrt{7}}{\sqrt{2}} \\ -\frac{\sqrt{7}}{\sqrt{2}} & \frac{5}{2} \end{bmatrix}$$

and $\gamma_4(2) = (I, D, W)$ where

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}, W = \frac{1}{3}\begin{bmatrix} 5 & 0 & -\sqrt{\frac{7}{2}} & 0 \\ 0 & 5 & 0 & -\sqrt{\frac{7}{2}} \\ -\sqrt{\frac{7}{2}} & 0 & \frac{5}{2} & 0 \\ 0 & -\sqrt{\frac{7}{2}} & 0 & \frac{5}{2} \end{bmatrix}.$$

Let

$$W' = \frac{1}{3}\begin{bmatrix} 5 & 0 & \frac{\sqrt{7}}{2} & \frac{\sqrt{7}}{2} \\ 0 & 5 & -\frac{\sqrt{7}}{2} & \frac{\sqrt{7}}{2} \\ \frac{\sqrt{7}}{2} & -\frac{\sqrt{7}}{2} & \frac{5}{2} & 0 \\ \frac{\sqrt{7}}{2} & \frac{\sqrt{7}}{2} & 0 & \frac{5}{2} \end{bmatrix} \in \mathbf{H}_4.$$

Then $\sigma(D) = \sigma(W') = \sigma(D^{-1}W') = \{2, 1/2\}$ and hence $(I, D, W') \in \mathcal{E}_4$. One see that

$$(I, D, W') = U.\gamma_4(2) = U.(I, D, W),$$

where

$$U = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Example 3.9. For $N \geq 3$ and $\lambda > 1$ let

$$A = I_N, B = \operatorname{diag}(\lambda, \lambda^{-1}, \underbrace{1, \dots, 1}_{N-2}), C = \operatorname{diag}(\lambda, 1, \lambda^{-1}, \underbrace{1, \dots, 1}_{N-3}).$$

Then $\sigma(AB^{-1}) = \sigma(AC^{-1}) = \sigma(BC^{-1}) = \{\lambda, \lambda^{-1}, 1\}$ where the multiplicity of 1 is $N-2$. This implies that $\delta(A, B) = \delta(A, C) = \delta(B, C)$. It is easy to see that A, B, C are linearly independent and not collinear. Since A, B, C are commuting, $\Lambda(A, B, C) = (ABC)^{\frac{1}{3}}$ and is different from $\frac{A+B+C}{\sqrt[N]{\det(A+B+C)}}$:

$$(ABC)^{\frac{1}{3}} = \operatorname{diag}(\lambda^{2/3}, \lambda^{-1/3}, \lambda^{-1/3}, \underbrace{1, \dots, 1}_{N-3});$$

$$\frac{A+B+C}{\sqrt[N]{\det(A+B+C)}} = \frac{\operatorname{diag}(1+2\lambda, 2+\lambda^{-1}, 2+\lambda^{-1}, 3, \dots, 3))}{\sqrt[N]{(1+2\lambda)(2+\lambda^{-1})^2 3^{N-3}}}.$$

The following result for $N = 2$ appears in [15]. One can alternatively prove that using the formula $\det(I + A) = 2 + \operatorname{tr}(A)$ for every $A \in \mathbf{H}_2$; for $A, B, C \in \mathbf{H}_2$, $\det(A + B) = \det(A^{-1} + B^{-1})$ and

$$\begin{aligned} \det(A+B+C) &= \det(A^{-1}+B^{-1}+C^{-1}) \\ &= 3 + \operatorname{tr}(AB^{-1}) + \operatorname{tr}(AC^{-1}) + \operatorname{tr}(AC^{-1}). \end{aligned} \tag{3.15}$$

Corollary 3.10. *For every* $(A, B, C) \in \mathcal{E}_N$,

$$\det(A + B + C) = \det(A^{-1} + B^{-1} + C^{-1}).$$

Proof. It follows from the reduction process $(A, B, C) = M.\gamma_N(\lambda)$ and the self-duality of tensor product $(X \otimes Y)^{-1} = X^{-1} \otimes Y^{-1}$:

$$(A^{-1}, B^{-1}, C^{-1}) = (M^*)^{-1}.\gamma(\lambda)^{-1} \otimes I_{\frac{N}{2}}$$

and hence

$$\begin{aligned} \det(A^{-1} + B^{-1} + C^{-1}) &= \det(I_2 + D_1(\lambda)^{-1} + W_\lambda^{-1})^{\frac{N}{2}} \\ &= \det(I_2 + D_1(\lambda) + W_\lambda)^{\frac{N}{2}} \\ &= \det(A + B + C). \end{aligned}$$

□

Note that for every $(A, B, C) \in \mathcal{E}_N$,

$$\det(A + B + C) = \big(3(\lambda + \lambda^{-1} + 1)\big)^{\frac{N}{2}},$$

that is,

$$\sqrt[N]{\det(A + B + C)} = \sqrt{3(\lambda + \lambda^{-1} + 1)}. \tag{3.16}$$

where λ is uniquely determined by $(A, B, C) = M.\gamma_N(\lambda)$.

Remark 3.11. [Isosceles triangles] The reduction argument (3.11) is applicable for every isosceles triangle ΔABC with $\delta(A, B) = \delta(A, C) = \{\lambda, \lambda^{-1}\}$, in which case the matrix

$$W(C, S; \lambda) = \begin{bmatrix} \lambda C^2 + \lambda^{-1} S^2 & (\lambda^{-1} - \lambda) SC \\ (\lambda^{-1} - \lambda) SC & \lambda S^2 + \lambda^{-1} C^2 \end{bmatrix} \in \mathbf{H}_N,$$

depends on C, S, and λ. The one parameter family $W(C, S; \lambda)$ over λ but fixed diagonal matrices C and S satisfying $C^2 + S^2 = I$ is apparently new in matrix analysis. A detailed study for the Karcher mean of 2×2 isosceles triangles appears in [9].

4. Inequalities

In this section we are interested in matrix mean inequalities from the Karcher mean formula (1.3) for $(A,B,C) \in \mathcal{K}_N$. Recall the Löwner ordering for Hermitian matrices: $X \le Y$ if and only if $Y - X$ is positive semidefinite. Denote by $\mathcal{A}$ and $\mathcal{H}$ the arithmetic and harmonic means:

$$\mathcal{A}(A,B,C) = \frac{A+B+C}{3}, \qquad \mathcal{H}(A,B,C) = \left[\frac{A^{-1}+B^{-1}+C^{-1}}{3}\right]^{-1}.$$

The arithmetic-Karcher-harmonic mean inequalities hold true for every $(A,B,C) \in \mathbb{P}_N$:

$$\mathcal{H}(A,B,C) \le \Lambda(A,B,C) \le \mathcal{A}(A,B,C).$$

Let $(A,B,C) \in \mathcal{K}_N$. Since $(A^{-1},B^{-1},C^{-1}) \in \mathcal{K}_N$,

$$\begin{aligned} \left[\frac{A+B+C}{\sqrt[N]{\det(A+B+C)}}\right]^{-1} &= \Lambda(A,B,C)^{-1} = \Lambda(A^{-1},B^{-1},C^{-1}) \\ &= \frac{A^{-1}+B^{-1}+C^{-1}}{\sqrt[N]{\det(A^{-1}+B^{-1}+C^{-1})}}, \end{aligned}$$

from which

$$(A+B+C)(A^{-1}+B^{-1}+C^{-1}) = \alpha(A,B,C)I_N, \tag{4.17}$$

and

$$\mathcal{A}(A,B,C) = \frac{\alpha(A,B,C)}{9}\mathcal{H}(A,B,C), \tag{4.18}$$

where

$$\alpha(A,B,C) := \sqrt[N]{\det(A+B+C)(A^{-1}+B^{-1}+C^{-1})}. \tag{4.19}$$

It follows from (4.17) that

$$\begin{aligned} & A(B^{-1}+C^{-1}) + B(A^{-1}+C^{-1}) + C(A^{-1}+B^{-1}) \\ = \ & (B^{-1}+C^{-1})A + (A^{-1}+C^{-1})B + (A^{-1}+B^{-1})C. \end{aligned}$$

We have from (1.3) and (4.18) that

$$\Lambda(A,B,C) = \frac{\mathcal{A}(A,B,C)}{\sqrt[N]{\det \mathcal{A}(A,B,C)}}$$

and

$$\begin{aligned}\mathcal{H}(A,B,C) &= \frac{3}{\alpha(A,B,C)}\mathcal{A}(A,B,C)\\ &= \frac{3}{\alpha(A,B,C)}\sqrt[p]{\det\mathcal{A}(A,B,C)}\frac{\mathcal{A}(A,B,C)}{\sqrt[p]{\det\mathcal{A}(A,B,C)}}\\ &= 3^{2-\frac{N}{p}}\frac{\det(A+B+C)^{\frac{1}{p}-\frac{1}{N}}}{\det(A^{-1}+B^{-1}+C^{-1})^{\frac{1}{N}}}\frac{\mathcal{A}(A,B,C)}{\sqrt[p]{\det\mathcal{A}(A,B,C)}}\end{aligned}$$

for every $p>0$.

Definition 4.1. For $(A,B,C)\in\mathcal{K}_N$, define

$$\Lambda_p(A,B,C) := \frac{\mathcal{A}(A,B,C)}{\sqrt[p]{\det(\mathcal{A}(A,B,C))}}, \qquad 0<p\leq\infty.$$

Letting $\Lambda_p=\Lambda_p(A,B,C)$ we have that $\Lambda_N=\Lambda, \Lambda_\infty=\mathcal{A}$ and $\Lambda_p\leq\Lambda_q$ for $p\leq q$, from $\det(\mathcal{A}(A,B,C))=\frac{1}{3^N}\det(A+B+C)\geq\frac{1}{3^N}3^N=1$, where the last inequality follows from the Minkowski's determinantal inequality.

By the arithmetic-Karcher-harmonic mean inequalities, one would ask whether $\mathcal{H}(A,B,C)=\Lambda_p(A,B,C)$ for some $p<N$.

Theorem 4.2. *Let $(A,B,C)\in\mathcal{K}_N$. If $\det(A+B+C)=\det(A^{-1}+B^{-1}+C^{-1})$, then for $N/2\leq p\leq N\leq q$,*

$$\begin{aligned}\mathcal{H}(A,B,C) &= \Lambda_{\frac{N}{2}}(A,B,C)=\frac{\mathcal{A}(A,B,C)}{\sqrt[\frac{N}{2}]{\det(\mathcal{A}(A,B,C))}}\leq\frac{\mathcal{A}(A,B,C)}{\sqrt[p]{\det(\mathcal{A}(A,B,C))}}\\ &\leq \Lambda(A,B,C)=\frac{\mathcal{A}(A,B,C)}{\sqrt[N]{\det(\mathcal{A}(A,B,C))}}\leq\frac{\mathcal{A}(A,B,C)}{\sqrt[q]{\det(\mathcal{A}(A,B,C))}}\\ &\leq \mathcal{A}(A,B,C).\end{aligned}$$

Proof. We have that

$$\mathcal{H}(A,B,C) = 3^{2-\frac{N}{p}}\det(A+B+C)^{\frac{1}{p}-\frac{2}{N}}\frac{\mathcal{A}(A,B,C)}{\sqrt[p]{\det\mathcal{A}(A,B,C)}}$$

for every $p>0$. For $p=N/2$, $\Lambda_{\frac{N}{2}}=\mathcal{H}$. □

Remark 4.3. Theorem 4.2 holds without the equilateral condition on (A,B,C). It holds true on $\mathcal{E}_N$ by Corollary 3.10. For $N=2$, it holds for every k-tuple in $\mathbf{H}_2$ [15].

5. Positive linear maps

Let $\phi : \mathbf{H}_m \to \mathbf{H}_n$ be a continuous map preserving the geometric mean:

$$\phi(A\#B) = \phi(A)\#\phi(B), \qquad A, B \in \mathbf{H}_m.$$

By continuity and the fact $(A\#_s B)\#(A\#_t B) = A\#_{\frac{s+t}{2}} B$, we have that $\phi(A\#_t B) = \phi(A)\#_t\phi(B)$ for all $t \in [0,1]$. Then Holbrook's no dice approximation of the Karcher mean [10] ensures

$$\phi(\Lambda(A_1, \ldots, A_k)) = \Lambda(\phi(A_1), \ldots, \phi(A_k)) \tag{5.20}$$

for $A_j \in \mathbf{H}_m, j = 1, \ldots, k$.

We further assume that the map ϕ extends to a positive homogeneous and additive map $\Phi : \mathbb{P}_m \to \mathbb{H}_n$, where $\mathbb{H}_n$ is the Euclidean space of $n \times n$ Hermitian matrices:

$$\Phi(tA) = t\Phi(A), \qquad \Phi(A + B) = \Phi(A) + \Phi(B), \qquad t > 0.$$

Since

$$\Phi(A) = \Phi\left(\sqrt[m]{\det(A)}\frac{A}{\sqrt[m]{\det(A)}}\right) = \sqrt[m]{\det(A)}\phi\left(\frac{A}{\sqrt[m]{\det(A)}}\right),$$

Φ maps $\mathbb{P}_m$ into $\mathbb{P}_n$. It then extends to a linear map $\Phi : \mathbb{H}_m \to \mathbb{H}_n$ sending the positive cone $\mathbb{P}_m$ into $\mathbb{P}_n$, so called a strictly positive linear map. An important property of the positive linear map Φ is the following determinantal identity:

$$\sqrt[n]{\det(\Phi(A))} = \sqrt[m]{\det(A)}, \qquad A \in \mathbb{P}_m. \tag{5.21}$$

Indeed,

$$\begin{aligned}\det(\Phi(A)) &= \det\left(\sqrt[m]{\det(A)}\phi\left(\frac{A}{\sqrt[m]{\det(A)}}\right)\right)\\ &= \det(A)^{\frac{n}{m}}\det\left(\phi\left(\frac{A}{\sqrt[m]{\det(A)}}\right)\right) = \det(A)^{\frac{n}{m}}.\end{aligned}$$

Example 5.1. Every strictly positive C^*-homomorphism $\Phi : \mathbb{H}_m \to \mathbb{H}_n$ preserves the geometric mean. To see this, note that $\Phi(X^2) = \Phi(X)^2$ for all $X \in \mathbb{H}_m$ implies that $\Phi(A^{-1}) = \Phi(A)^{-1}$ for all $A \in \mathbb{P}_m$ by Kadison's inequality. By the arithmetic-harmonic mean iteration of A and B, we have that $A\#B = \lim_{k\to\infty} A_k = \lim_{k\to\infty} B_k$, where

$$A_0 := A, \ \ B_0 := B, \ \ A_{k+1} := \frac{A_k + B_k}{2}, \ \ B_{k+1} := \left[\frac{A_k^{-1} + B_k^{-1}}{2}\right]^{-1}.$$

Applying to $\Phi(A)$ and $\Phi(B)$ yields $\Phi(A)\#\Phi(B) = \lim_{k\to\infty} X_k = \lim_{k\to\infty} Y_k$, where $X_0 := \Phi(A) = \Phi(A_0), Y_0 := \Phi(B) = \Phi(B_0)$ and

$$X_{k+1} := \frac{X_k + Y_k}{2}, \quad Y_{k+1} := \left[\frac{X_k^{-1} + Y_k^{-1}}{2}\right]^{-1}.$$

It follows from induction and invariancy of Φ under inversion that $X_k = \Phi(A_k)$ and $Y_k = \Phi(B_k)$ for all k. This implies that

$$\begin{aligned} \Phi(A\#B) &= \lim_{k\to\infty} \Phi(A_{k+1}) = \lim_{k\to\infty} \Phi\left(\frac{A_k + B_k}{2}\right) = \lim_{k\to\infty} \frac{\Phi(A_k) + \Phi(B_k)}{2} \\ &= \lim_{k\to\infty} \frac{X_k + Y_k}{2} = \lim_{k\to\infty} X_{k+1} = \Phi(A)\#\Phi(B). \end{aligned}$$

However, $\Phi(\mathbf{H}_m) \subset \mathbf{H}_n$ fails in general, due to (5.21). It is unknown to us a classification of strictly positive C^*-homomorphisms satisfying (5.21) for $m \neq n$.

Example 5.2. Let $D \in \mathbf{H}_n$ and let $\Phi : \mathbb{H}_m \to \mathbb{H}_{mn}$ defined by $\Phi(X) = X \otimes D$. By basic properties of tensor product, Φ is injective and a strictly positive linear map sending $\mathbf{H}_m$ into $\mathbf{H}_{mn}$. Moreover it satisfies (5.21) and preserves the geometric mean [1], $\Phi(A\#B) = \Phi(A)\#\Phi(B)$. We then have from (5.20) that

$$\Lambda(A \otimes D, B \otimes D, C \otimes D) = \Lambda(A, B, C) \otimes D = \Lambda(A, B, C) \otimes \Lambda(D, D, D).$$

In a similar way, $\Lambda(A\otimes D, B\otimes E, C\otimes F) = \Lambda(A, B, C)\otimes\Lambda(D, E, F)$. See Proposition 4.4 of [11].

Denote $\Phi(A, B, C) := (\Phi(A), \Phi(B), \Phi(C))$ for $(A, B, C) \in \mathbb{P}_m^3$. Let $\mathcal{T}_m$ be the set of triples in $\mathbf{H}_m$ satisfying (1.3) and $\mathcal{T}_m^\dagger \subset \mathcal{T}_m$ the subclass of linearly independent triples.

Proposition 5.3. *Let $\phi : \mathbf{H}_m \to \mathbf{H}_n$ be a continuous map preserving the geometric mean and admitting a strictly positive linear extension $\Phi : \mathbb{H}_m \to \mathbb{H}_n$. Then ϕ sends $\mathcal{T}_m$ into $\mathcal{T}_n$. If Φ is injective, it sends $\mathcal{T}_m^\dagger$ into $\mathcal{T}_n^\dagger$.*

Proof. We have from (5.21) and (5.20) that for every $(A,B,C)\in\mathcal{T}_m$,

$$\det(\phi(A)+\phi(B)+\phi(C)) = \det(\Phi(A+B+C)) = \det(A+B+C)^{\frac{n}{m}}$$

and

$$\begin{aligned}\Lambda(\phi(A,B,C)) &= \Lambda(\phi(A),\phi(B),\phi(C)) = \phi(\Lambda(A,B,C))\\ &= \phi\left(\frac{A+B+C}{\sqrt[m]{\det(A+B+C)}}\right)\\ &= \frac{1}{\sqrt[m]{\det(A+B+C)}}\Phi(A+B+C)\\ &= \frac{\phi(A)+\phi(B)+\phi(C)}{\sqrt[n]{\det(\phi(A)+\phi(B)+\phi(C))}}.\end{aligned}$$

This proves that $\phi(\mathcal{T}_m)\in\mathcal{T}_n$. The remaining part of proof is immediate. □

By Example 5.2,

Theorem 5.4. *For every $D\in\mathbf{H}_n$, $\mathcal{T}_m^{\dagger}\otimes D\subset\mathcal{T}_{mn}^{\dagger}$.*

We note that $\mathcal{E}_2\otimes D$ is not contained in $\mathcal{E}_{2n}$ for $D\neq I_n$.

6. Final remarks

Let E be a Hilbert space and let $\mathbb{P}=\mathbb{P}(E)$ be the convex cone of positive invertible operators on E. The Thompson metric on $\mathbb{P}$ is defined by $d(A,B)=||\log A^{-1}B||$, where $||X||$ is the operator norm. The Karcher mean $\Lambda(A_1,\dots,A_k)$ is defined as a unique positive invertible solution of the Karcher equation [13]

$$\sum_{j=1}^{n}\log(X^{-1/2}A_jX^{-1/2})=0.$$

For $A,B\in\mathbb{P}$, the geometric mean curve $t\mapsto A\#_tB := A^{1/2}(A^{-1/2}BA^{-1/2})^tA^{1/2}$ is a minimal geodesic between A and B for the Thompson metric [7, 21, 22]. However there are infinitely many minimal geodesics in general: see [21, 22]. It turns out [18, 16, 17] for linearly independent A and B, that there exists a unique minimal geodesic between A and B if and only if $\sigma(A^{-1}B)=\{\lambda,\lambda^{-1}\}$ for some $\lambda>1$, in which case by using Nussbaum's construction of minimal geodesic,

$$A\#_tB = L_{1-t}(\lambda)A+L_t(\lambda)B, \qquad 0\le t\le 1,$$

where $L_t(\lambda) = \frac{\lambda^t - \lambda^{-t}}{\lambda - \lambda^{-1}}$. It follows from $L_{1/2}(\lambda) = \sqrt{\lambda}(\lambda+1)^{-1}$ that

$$A \# B = \sqrt{\lambda}(\lambda+1)^{-1}(A+B) = \frac{A+B}{\sqrt{\lambda}+\sqrt{\lambda^{-1}}}$$

for $\sigma(A^{-1}B) = \{\lambda, \lambda^{-1}\}$. For finite dimensional case, we have seen that under $\det(A) = \det(B) = 1$,

$$A \# B = \frac{A+B}{\sqrt[N]{\det(A+B)}} = \frac{A+B}{\sqrt[N]{\det(I+A^{-1}B)}} = \frac{A+B}{\sqrt{\lambda}+\sqrt{\lambda^{-1}}}$$

from

$$\sqrt[N]{\det(A+B)} = \sqrt[N]{\det(I+A^{-1}B)} = \sqrt{\lambda}+\sqrt{\lambda^{-1}}.$$

In other words, the geometric mean is a scalar multiple of the arithmetic mean.

Next, let us consider $(A, B, C) \in \mathbb{P}(E)^3$ satisfying

$$\sigma(A^{-1}B) = \sigma(A^{-1}C) = \sigma(B^{-1}C) = \{\lambda, \lambda^{-1}\}, \qquad \lambda > 1 \tag{6.22}$$

and the Karcher mean $\Lambda(A, B, C)$ is of the form

$$\Lambda(A, B, C) = \frac{A+B+C}{\sqrt{3(\lambda+\lambda^{-1}+1)}}. \tag{6.23}$$

Since $\lambda + \lambda^{-1} > 2$, the Karcher mean is a scalar multiple of the arithmetic mean:

$$\Lambda(A, B, C) = a\left[\frac{A+B+C}{3}\right], \qquad a < 1.$$

It is then of interest to find an additional condition on (A, B, C) satisfying (6.22) under which (6.23) holds.

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