Optimal $(\alpha\,,\,\mathbf{d})$-multi-completion of $\mathbf{d}$-designs

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Abstract

Given finite sequences $\mathbf{d} = (d_j)_{j\in\mathbb{I}_m} \in \mathbb{N}^m$ and $\alpha = (\alpha_i)_{i\in\mathbb{I}_n} \in \mathbb{R}^n_{>0}$ of dimensions and weights (where $\mathbb{I}_k = \{1,\ldots,k\}$, for $k \in \mathbb{N}$), we consider the set $\mathcal{D}(\alpha\,,\,\mathbf{d})$ of $(\alpha\,,\,\mathbf{d})$-designs, i.e. m-tuples $\Phi = (\mathcal{F}_j)_{j\in\mathbb{I}_m}$ such that each $\mathcal{F}_j = \{f_{i,j}\}_{i\in\mathbb{I}_n} \in (\mathbb{C}^{d_j})^n$ and

$$\sum_{j\in\mathbb{I}_m} \|f_{i,j}\|^2 = \alpha_i \quad \text{for} \quad i \in \mathbb{I}_n\,.$$

In this work we solve the optimal $(\alpha\,,\,\mathbf{d})$-completion problem of an initial $\mathbf{d}$-design $\Phi^0 = (\mathcal{F}_j^0)_{j\in\mathbb{I}_m}$ with $\mathcal{F}_j^0 \in (\mathbb{C}^{d_j})^k$, for $j \in \mathbb{I}_m$. Explicitly, given an strictly convex function $\varphi : [0,\infty) \to [0,\infty)$, we compute the $(\alpha\,,\,\mathbf{d})$-designs $\Phi_\varphi^{\rm op}$ that are (local) minimizers of the joint convex potential

$$\mathrm{P}_\varphi(\Phi^0\,,\,\Phi) = \sum_{j\in\mathbb{I}_m} \mathrm{tr}(\varphi[S_{(\mathcal{F}_j^0\,,\,\mathcal{F}_j)}])$$

of the multi-completions $(\Phi^0\,,\,\Phi)$, among all $(\alpha\,,\,\mathbf{d})$-designs $\Phi = (\mathcal{F}_j)_{j\in\mathbb{I}_m}$; here $S_{(\mathcal{F}_j^0\,,\,\mathcal{F}_j)}$ denotes the frame operator of the completed sequence $(\mathcal{F}_j^0\,,\,\mathcal{F}_j) \in (\mathbb{C}^{d_j})^{k+n}$, for $j \in \mathbb{I}_m$. We obtain the geometrical and spectral features of these optimal $(\alpha\,,\,\mathbf{d})$-multi-completions. We further show that the optimal $(\alpha\,,\,\mathbf{d})$-designs $\Phi_\varphi^{\rm op}$ as above do not depend on φ. We also consider some reformulations and applications of our main results in different contexts in frame theory. Finally, we describe a fast finite step algorithm for computing optimal multi-completions that becomes relevant for the applications of our results and present some numerical examples of optimal multi-completions with prescribed weights.

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Contents

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1 Introduction

Let $\mathcal{F} = \{f_j\}_{j\in\mathbb{I}_n}$ be a finite sequence of vectors in $\mathbb{C}^d$, where $\mathbb{I}_n = \{1,\ldots,n\}$. Recall that $\mathcal{F}$ is a frame for $\mathbb{C}^d$ if it linearly generates $\mathbb{C}^d$. In this case, we can consider encoding/decoding schemes of arbitrary vectors $f \in \mathbb{C}^d$ in terms of $\mathcal{F}$; indeed, if $\mathcal{F}$ is a frame and we denote by $S_\mathcal{F} \in \mathcal{M}_d(\mathbb{C})^+$ the so-called frame operator of $\mathcal{F}$ given by

$$S_\mathcal{F}\, g = \sum_{j\in\mathbb{I}_n} \langle g\,,\, f_j\rangle\; f_j \quad \text{for} \quad g \in \mathbb{C}^d\,,$$

then $S_\mathcal{F}$ is invertible, and we get the canonical reconstruction formula (see [12, 14])

$$f = \sum_{j\in\mathbb{I}_n} \langle f\,,\, S_\mathcal{F}^{-1}\, f_j\rangle\; f_j\,.$$

Recently in [4] we have considered a frame theoretic model of multitasking devices with energy restrictions, leading to the notion of $(\alpha\,,\,\mathbf{d})$-designs. Explicitly, given finite sequences (of dimensions) $\mathbf{d} = (d_j)_{j\in\mathbb{I}_m} \in \mathbb{N}^m$ and (of weights) $\alpha = (\alpha_i)_{i\in\mathbb{I}_n} \in \mathbb{R}^n_{>0}$, we consider the set $\mathcal{D}(\alpha\,,\,\mathbf{d})$ of $(\alpha\,,\,\mathbf{d})$-designs, i.e. m-tuples $\Phi = (\mathcal{F}_j)_{j\in\mathbb{I}_m}$ such that each $\mathcal{F}_j = \{f_{i,j}\}_{i\in\mathbb{I}_n}$ is a finite sequence in $\mathbb{C}^{d_j}$, and

$$\sum_{j\in\mathbb{I}_m} \|f_{i,j}\|^2 = \alpha_i \quad \text{for} \quad i \in \mathbb{I}_n\,. \tag{1}$$

Notice that the restrictions on the norms above involve vectors in the (possibly different) spaces $f_{i,j} \in \mathbb{C}^{d_j}$, for $j \in \mathbb{I}_m$ (and fixed $i \in \mathbb{I}_n$). Assuming that each family $\mathcal{F}_j = \{f_{i,j}\}_{i\in\mathbb{I}_n}$ is a system of generators for $\mathbb{C}^{d_j}$, for $j \in \mathbb{I}_m$, the $(\alpha\,,\,\mathbf{d})$-design Φ can be considered as a family of encoding-decoding schemes, that run in parallel, to be applied by a multitasking device with some sort of energy restriction (e.g. due to isolation, or devices that are far from energy networks). In this case, we want to control the overall energy needed (in each step of the encoding-decoding scheme) to apply simultaneously the m linear schemes, through the restrictions in Eq. (1). The different dimensions $\mathbf{d} = (d_j)_{j\in\mathbb{I}_m}$ play a key role, related to the need of better precision, accuracy or robustness of the different data to be encoded.

We can pose several well known problems from finite frame theory in the context of $(\alpha\,,\,\mathbf{d})$-designs; it turns out that these problems do not reduce to their counterparts for finite frames. Indeed, in [4] we solved the $(\alpha\,,\,\mathbf{d})$-design problem, in which we obtained necessary and sufficient conditions on the spectra of the positive operators $S_j \in \mathcal{M}_{d_j}(\mathbb{C})^+$, $j \in \mathbb{I}_m$, for the existence of an $(\alpha\,,\,\mathbf{d})$-design $\Phi =$

$(\mathcal{F}_j)_{j\in\mathbb{I}_m}$ such that $S_{\mathcal{F}_j} = S_j$, for $j \in \mathbb{I}_m$. Our characterization is based on majorization relations, and it extends the well known solution of the classical frame design problem (see [8, 9, 11, 15, 16]). We also solved the problem of optimal $(\alpha\,,\,\mathbf{d})$-designs, that is, $(\alpha\,,\,\mathbf{d})$-designs that minimize the joint convex potential induced by a strictly convex function, extending results from finite frames (see [1, 10, 23]).

In this work we extend our previous results on optimal $(\alpha\,,\,\mathbf{d})$-designs to the context of optimal $(\alpha\,,\,\mathbf{d})$-completions (or simply multi-completions) of an initial $\mathbf{d}$-design. Indeed, given an initial $\mathbf{d}$-design $\Phi^0 = (\mathcal{F}_j^0)_{j\in\mathbb{I}_m}$, with $\mathcal{F}_j^0 \in (\mathbb{C}^{d_j})^k$ for $j \in \mathbb{I}_m$, we consider the problem of computing $(\alpha\,,\,\mathbf{d})$-designs $\Phi = (\mathcal{F}_j)_{j\in\mathbb{I}_m}$ in such a way that the completed $\mathbf{d}$-design $(\Phi^0\,,\,\Phi) = ((\mathcal{F}_j^0\,,\,\mathcal{F}_j))_{j\in\mathbb{I}_m}$ induce more stable encoding/decoding schemes. In order to give a quantitative measure of stability, we introduce the joint convex potential of the multi-completion $(\Phi^0\,,\,\Phi)$ induced by a strictly convex function $\varphi : [0,\infty) \to [0,\infty)$ as follows:

$$\mathrm{P}_\varphi(\Phi^0\,,\,\Phi) = \sum_{j\in\mathbb{I}_m} \mathrm{tr}(\varphi[S_{(\mathcal{F}_j^0\,,\mathcal{F}_j)}])\,, \tag{2}$$

where $S_{(\mathcal{F}_j^0\,,\mathcal{F}_j)} = S_{\mathcal{F}_j^0} + S_{\mathcal{F}_j}$ denotes the frame operator of the sequence $(\mathcal{F}_j^0\,,\,\mathcal{F}_j) \in (\mathbb{C}^{d_j})^{k+n}$, for $j \in \mathbb{I}_m$. It is well known that minimizers of convex potentials induce more stable schemes. Thus, in the present work we compute the geometrical and spectral features of those $(\alpha\,,\,\mathbf{d})$-designs $\Phi_\varphi^{\mathrm{op}}$ that minimize the joint convex potential $\mathrm{P}_\varphi(\Phi^0\,,\,\Phi)$ among $(\alpha\,,\,\mathbf{d})$-designs Φ. Moreover, by considering a natural notion of distance between $(\alpha\,,\,\mathbf{d})$-designs, we show that local minimizers of $\mathrm{P}_\varphi(\Phi^0\,,\,\Phi)$ are actually global minimizers. We further show that the optimal $(\alpha\,,\,\mathbf{d})$-designs $\Phi_\varphi^{\mathrm{op}}$ as above do not depend on the strictly convex function φ. Thus, our results extend previous results for frame completions (see [17, 18, 21, 22, 24, 25]).

There exist several areas in frame theory where these results about multi-completion problems can be applied. Among such areas we mention: finitely shift generated sequences in shift-invariant spaces, distributed sensor allocation, tight multi-completions problems with prescribed weights and multivariate matrix approximation. We include a detailed discussion of problems in these areas that serve as different motivations (based on equivalent reformulations of the multi-completion problem) as well as natural contexts for applications of our results (see Section 5.1).

Our approach to the multi-completion problem with prescribed weights, which is based on [4, 21], is constructive; indeed, we describe a fast finite step algorithm that produces $(\alpha\,,\,\mathbf{d})$-designs Φ^{op} such that the multi-completion $(\Phi^0\,,\,\Phi^{\mathrm{op}})$ minimizes the joint convex potential induced by every convex function φ as above. Therefore, the present results also become relevant for applications in the reformulations of the optimal multi-completion problem considered above.

The paper is organized as follows. In the first part of Section 2 we describe the notion of $(\alpha\,,\,\mathbf{d})$-design, and recall a characterization of the so-called admissible pairs from [4], that plays a key role in our work. Then, we introduce the notions of multi-completions and joint convex potentials and we describe the main problem in the paper, namely the computation of (local) minimizers of the joint convex potential of $(\Phi^0\,,\,\Phi)$ among $(\alpha\,,\,\mathbf{d})$-designs Φ. The section ends with the statement of our main result on multi-completions. In Section 3 we obtain several features of $(\alpha\,,\,\mathbf{d})$-designs $\tilde{\Phi}$ that are local minimizers for the joint convex potential P_φ of the multi-completion (Φ^0, Φ), for a fixed initial $\mathbf{d}$-design Φ^0 and a strictly convex function φ. It turns out that the spectra of the frame operators of the completed family in $(\Phi^0, \tilde{\Phi})$ are encoded in a single vector, that we denote $\mathbf{c}$. Section 4 focuses on the proof of the main result. In particular, we prove that all local minimizers have the same spectral structure, and that this structure does not depend on the strictly convex function φ. This section also contains a detailed analysis of the construction of the vector $\mathbf{c}$ which shall be useful for the development of an algorithm that implements optimal multi-completions. In Section 5.1 we describe in detail some applications of our main results in other areas of frame theory; in Section

5.2 we outline a fast finite step algorithm that computes optimal multi-completions with prescribed weights of an initial **d**-design, together with some numerical examples of multi-completions obtained in terms of the mentioned algorithm. Finally, in Section 6 (Appendix), we state some known results used throughout the paper.

2 Multi-completions with prescribed weights

In this Section we describe the notion of $(\alpha\,,\,\mathbf{d})$-design, and recall a characterization of the so-called admissible pairs from [4], that plays a key role in our work. Then, we introduce the multi-completions and describe the main problems considered in this work. The section ends with the statement of our main result on multi-completions. Next, we describe some basic notation and notions used throughout the rest of the paper.

Notation and terminology. We let $\mathcal{M}_{k,d}(\mathcal{S})$ be the set of $k \times d$ matrices with coefficients in $\mathcal{S} \subset \mathbb{C}$ and write $\mathcal{M}_{d,d}(\mathbb{C}) = \mathcal{M}_d(\mathbb{C})$ for the algebra of $d \times d$ complex matrices. We denote by $\mathcal{H}(d) \subset \mathcal{M}_d(\mathbb{C})$ the real subspace of selfadjoint matrices and by $\mathcal{M}_d(\mathbb{C})^+ \subset \mathcal{H}(d)$ the cone of positive semidefinite matrices. We let $\mathcal{U}(d) \subset \mathcal{M}_d(\mathbb{C})$ denote the group of unitary matrices. For $d \in \mathbb{N}$, let $\mathbb{I}_d = \{1, \ldots, d\}$ and let $\mathbb{1}_d = (1)_{i\in\mathbb{I}_d} \in \mathbb{R}^d$ be the vector with all its entries equal to 1.

Given $x = (x_i)_{i\in\mathbb{I}_d} \in \mathbb{R}^d$ we denote by $x^\downarrow = (x_i^\downarrow)_{i\in\mathbb{I}_d}$ (respectively $x^\uparrow = (x_i^\uparrow)_{i\in\mathbb{I}_d}$) the vector obtained by rearranging the entries of x in non-increasing (respectively non-decreasing) order. We denote by $(\mathbb{R}^d)^\downarrow = \{x^\downarrow :\ x \in \mathbb{R}^d\}$, $(\mathbb{R}^d_{\geq 0})^\downarrow = \{x^\downarrow :\ x \in \mathbb{R}^d_{\geq 0}\}$ and analogously for $(\mathbb{R}^d)^\uparrow$ and $(\mathbb{R}^d_{\geq 0})^\uparrow$.

Given $S \in \mathcal{M}_d(\mathbb{C})$ we let $R(S) \subset \mathbb{C}^d$ denote the range (or image) of S and $\operatorname{rk}(S)$ denote the rank of S, i.e. the dimension of $R(S)$. Given a matrix $A \in \mathcal{H}(d)$ we denote by $\lambda(A) = \lambda^\downarrow(A) = (\lambda_i(A))_{i\in\mathbb{I}_d} \in (\mathbb{R}^d)^\downarrow$ the eigenvalues of A counting multiplicities and arranged in non-increasing order, and by $\lambda^\uparrow(A)$ the same vector but arranged in non-decreasing order. On the other hand, we denote by $\sigma(A) \subset \mathbb{R}$ its spectrum, i.e. the set of eigenvalues of A. If $x, y \in \mathbb{C}^d$ we denote by $x \otimes y \in \mathcal{M}_d(\mathbb{C})$ the rank-one matrix given by $(x \otimes y)\, z = \langle z\,,\, y\rangle\, x$, for $z \in \mathbb{C}^d$.

Given a finite sequence $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n}$ in $\mathbb{C}^d$ then $S_\mathcal{F} \in \mathcal{M}_d(\mathbb{C})^+$ will denote the frame operator of $\mathcal{F}$, that is given by

$$S_\mathcal{F}\, f = \sum_{i\in\mathbb{I}_n} \langle f, f_i\rangle f_i = \sum_{i\in\mathbb{I}_n} (f_i \otimes f_i)\, f \quad \text{for} \quad f \in \mathbb{C}^d\,.$$

We say that $\mathcal{F}$ is a frame for $\mathbb{C}^d$ if it spans $\mathbb{C}^d$; equivalently, $\mathcal{F}$ is a frame for $\mathbb{C}^d$ if $S_\mathcal{F}$ is a positive invertible operator acting on $\mathbb{C}^d$.

Next we recall the notion of majorization between real vectors.

Definition 2.1. Let $x, y \in \mathbb{R}^d$. We say that x is *submajorized* by y, and write $x \prec_w y$, if

$$\sum_{i\in\mathbb{I}_j} x_i^\downarrow \leq \sum_{i\in\mathbb{I}_j} y_i^\downarrow \quad \text{for every} \quad 1 \leq j \leq d\,.$$

If $x \prec_w y$ and

$$\operatorname{tr} x = \sum_{i\in\mathbb{I}_d} x_i = \sum_{i\in\mathbb{I}_d} y_i = \operatorname{tr} y$$

then we say that x is *majorized* by y, and write $x \prec y$. In case $x \prec y$ and $x^\downarrow \neq y^\downarrow$ we say that x is *strictly* majorized by y.

We extend these notions to pairs of vectors of different sizes and non-negative entries as follows: let $x \in \mathbb{R}^n_{\geq 0}$ and $y \in \mathbb{R}^d_{\geq 0}$ with $n > d$. We say that x is submajorized (respectively majorized) by y if

$$x \prec_w y \oplus 0_{n-d} = (y_1, \ldots, y_d, 0, \ldots, 0) \in \mathbb{R}^n \quad (\text{respectively} \quad x \prec y \oplus 0_{n-d} \in \mathbb{R}^n\ )$$

in the sense defined above; in this case, we simply write $x \prec_w y$ (respectively $x \prec y$). △

Majorization is a partial pre-order relation in $\mathbb{R}^d$ that arises naturally in matrix analysis, and that will play a central role throughout our work. We describe several results related to this notion in the Appendix (Section 6).

2.1 Preliminaries on $(\alpha\,,\,\mathbf{d})$-designs

We begin this section by recalling the notion of $(\alpha\,,\,\mathbf{d})$-design introduced in [4].

Definition 2.2 (From [4])**.** Let $\alpha = (\alpha_i)_{i\in\mathbb{I}_n} \in \mathbb{R}^n_{>0}$ and $\mathbf{d} = (d_j)_{j\in\mathbb{I}_m} \in (\mathbb{N}^m)^\downarrow$. An $(\alpha\,,\,\mathbf{d})$-*design* is an m-tuple $\Phi = (\mathcal{F}_j)_{j\in\mathbb{I}_m}$ where each $\mathcal{F}_j = \{f_{i,j}\}_{i\in\mathbb{I}_n}$ is a finite sequence in $\mathbb{C}^{d_j}$ and such that

$$\sum_{j\in\mathbb{I}_m} \|f_{i,j}\|^2 = \alpha_i, \ \text{ for } \ i \in \mathbb{I}_n \,.$$

We denote by $\mathcal{D}(\alpha\,,\,\mathbf{d})$ the set of all $(\alpha\,,\,\mathbf{d})$-designs. △

In what follows we consider the following metric between n-tuples in $\mathbb{C}^d$: given $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n}$, $\mathcal{G} = \{g_i\}_{i\in\mathbb{I}_n} \in (\mathbb{C}^d)^n$ we let

$$m_d^2(\mathcal{F}\,,\,\mathcal{G}) = \sum_{i\in\mathbb{I}_n} \|f_i - g_i\|^2 \,. \tag{3}$$

Notice that $(\mathbb{C}^d)^n$ endowed with $m_d(\cdot,\cdot)$ becomes a (product) metric space.

Remark 2.3. Let $\mathbf{d} = (d_j)_{j\in\mathbb{I}_m} \in (\mathbb{N}^m)^\downarrow$ and $\alpha = (\alpha_i)_{i\in\mathbb{I}_n} \in (\mathbb{R}^n_{>0})^\downarrow$ be such that $d_1 \leq n$.

1. We consider the set of $(\alpha\,,\,m)$-weight partitions given by

$$P_{\alpha\,,\,m} = \{A = (a_{ij})_{i\in\mathbb{I}_n, j\in\mathbb{I}_m} \in \mathcal{M}_{n,m}(\mathbb{C}) \,:\, a_{ij} \geq 0 \quad \text{and} \sum_{j\in\mathbb{I}_m} a_{ij} = \alpha_i \quad \text{for} \quad i \in \mathbb{I}_n \,\} \,.$$

2. A sequence $\Phi = (\mathcal{F}_j)_{j\in\mathbb{I}_m} \in \mathcal{D}(\alpha\,,\,\mathbf{d})$ if and only if its matrix of weights

$$A = \Big\{ \|f_{i,j}\|^2 \Big\}_{i\in\mathbb{I}_n\,,\,j\in\mathbb{I}_m} \quad \text{belongs to } P_{\alpha\,,\,m} \,.$$

   We endow $\mathcal{D}(\alpha\,,\,d)$ with the metric (see Eq. (3)):

$$m(\Phi\,,\,\Phi') = \sum_{j\in\mathbb{I}_m} m_{d_j}(\mathcal{F}_j\,,\,\mathcal{F}'_j) = \sum_{j\in\mathbb{I}_m} \Bigg(\sum_{i\in\mathbb{I}_n} \|f_{i,j} - f'_{i,j}\|^2 \Bigg)^{1/2} \,, \tag{4}$$

   where $\Phi = (\mathcal{F}_j)_{j\in\mathbb{I}_m}\,,\ \Phi' = (\mathcal{F}'_j)_{j\in\mathbb{I}_m} \in \mathcal{D}(\alpha\,,\,\mathbf{d})$. △

In what follows we will need the following notions and results from [4].

Definition 2.4. Let $\mathbf{d} = (d_j)_{j\in\mathbb{I}_m} \in (\mathbb{N}^m)^\downarrow$ and $\alpha = (\alpha_i)_{i\in\mathbb{I}_n} \in (\mathbb{R}^n_{>0})^\downarrow$ be such that $d_1 \leq n$. Let

$$\mu_j = (\mu_{i,j})_{i\in\mathbb{I}_{d_j}} \in (\mathbb{R}^{d_j}_{\geq 0})^\downarrow \quad \text{for} \quad j \in \mathbb{I}_m \quad \text{and set} \quad \mathcal{M} := \{\mu_j\}_{j\in\mathbb{I}_m} \,.$$

We say that the pair $(\alpha\,,\,\mathcal{M})$ is **admissible** if there exists $\Phi = (\mathcal{F}_j)_{j\in\mathbb{I}_m} \in \mathcal{D}(\alpha\,,\,\mathbf{d})$ such that

$$\lambda(S_{\mathcal{F}_j}) = \mu_j \quad \text{for every} \quad j \in \mathbb{I}_m \,.$$

In this case, we denote $\mathcal{M} = \mathcal{M}_\Phi$. △

Theorem 2.5 (From [4]). Let $\mathbf{d} = (d_j)_{j\in\mathbb{I}_m} \in (\mathbb{N}^m)^\downarrow$ and let $\mathcal{M} = \{\mu_j\}_{j\in\mathbb{I}_m}$, where $\mu_j \in (\mathbb{R}^{d_j}_{\geq 0})^\downarrow$, for $j \in \mathbb{I}_m$. Let $n \geq d_1$ and let $\alpha = (\alpha_i)_{i\in\mathbb{I}_n} \in (\mathbb{R}^n_{>0})^\downarrow$. Set

$$\sigma_{\mathcal{M}} := \sum_{j\in\mathbb{I}_m} (\mu_j \oplus 0_{n-d_j}) \in (\mathbb{R}^n_{\geq 0})^\downarrow .$$

Then, the pair $(\alpha\,,\,\mathcal{M})$ is admissible if and only if $\alpha \prec \sigma_{\mathcal{M}}$.

Remark 2.6. With the notation in Theorem 2.5, assume that $\alpha \prec \sigma_{\mathcal{M}}$, so the pair $(\alpha\,,\,\mathcal{M})$ is admissible. In this case, we can construct $\Phi = (\mathcal{F}_j)_{j\in\mathbb{I}_m} \in \mathcal{D}(\alpha\,,\,\mathbf{d})$ such that its matrix of weights

$$A = \Big\{\|f_{i,j}\|^2\Big\}_{i\in\mathbb{I}_n\,,\,j\in\mathbb{I}_m} \in P_{\alpha\,,\,m}\ .$$

is given by $A = D\cdot\Sigma_{\mathcal{M}}$, where $\Sigma_{\mathcal{M}}$ is the $n\times m$ matrix whose j-th column is $(\mu_j \oplus 0_{n-d_j}) \in (\mathbb{R}^n_{\geq 0})^\downarrow$, for $1 \leq j \leq m$, and D is any doubly stochastic matrix of size n (i.e. D has non-negative entries and each row sum and column sum equals 1) such that $\alpha = D\cdot\sigma_{\mathcal{M}}$ (see [4]). △

2.2 Multi-completions: design problems and main results

Notation 2.7. In what follows we consider

1. $\mathbf{d} = (d_j)_{j\in\mathbb{I}_m} \in (\mathbb{N}^m)^\downarrow$ and a finite sequence $\alpha = (\alpha_i)_{i\in\mathbb{I}_n} \in (\mathbb{R}^n_{>0})^\downarrow$, such that $d_1 \leq n$.
2. $\Phi^0 = (\mathcal{F}^0_j)_{j\in\mathbb{I}_m}$ a (fixed) $\mathbf{d}$-design, i.e. such that

$$\mathcal{F}^0_j = \{f^0_{i,j}\}_{i\in\mathbb{I}_k} \in (\mathbb{C}^{d_j})^k \quad \text{for} \quad j \in \mathbb{I}_m\,.$$

In this case, we say that the triple $(\Phi^0\,,\,\alpha\,,\,\mathbf{d})$ is the initial data for a multi-completion problem. △

We consider the sets $\operatorname{Conv}(\mathbb{R}_{\geq 0}) = \{\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}\ :\ \varphi \text{ is a convex function }\}$ and

$$\operatorname{Conv_s}(\mathbb{R}_{\geq 0}) = \{\varphi \in \operatorname{Conv}(\mathbb{R}_{\geq 0}) : \varphi \ \text{ is strictly convex }\}\,.$$

Definition 2.8. Consider Notation 2.7. In what follows we study

1. the set of $(\alpha\,,\,\mathbf{d})$-completions of Φ^0, denoted $\mathcal{C}(\Phi^0\,,\,\alpha\,,\,\mathbf{d})$, given by

$$\mathcal{C}(\Phi^0\,,\,\alpha\,,\,\mathbf{d}) = \{((\mathcal{F}^0_j\,,\,\mathcal{F}_j))_{j\in\mathbb{I}_m} :\ \Phi = (\mathcal{F}_j)_{j\in\mathbb{I}_m} \in \mathcal{D}(\alpha\,,\,\mathbf{d})\}\,.$$

   In this case, we write $(\Phi^0\,,\,\Phi) = ((\mathcal{F}^0_j\,,\,\mathcal{F}_j))_{j\in\mathbb{I}_m} \in \mathcal{C}(\Phi^0\,,\,\alpha\,,\,\mathbf{d})$.

2. Given a strictly convex function $\varphi \in \operatorname{Conv_s}(\mathbb{R}_{\geq 0})$, we consider $\Psi_\varphi : \mathcal{D}(\alpha\,,\,\mathbf{d}) \to \mathbb{R}_{\geq 0}$, given by

$$\Psi_\varphi(\Phi) = \mathrm{P}_\varphi(\Phi^0\,,\,\Phi) = \sum_{j\in\mathbb{I}_m} \operatorname{tr}(\varphi[S_{(\mathcal{F}^0_j\,,\,\mathcal{F}_j)}]) = \sum_{j\in\mathbb{I}_m}\sum_{i\in\mathbb{I}_{d_j}} \varphi[\lambda_i(S_{(\mathcal{F}^0_j\,,\,\mathcal{F}_j)})]\,, \tag{5}$$

   where $S_{(\mathcal{F}^0_j\,,\,\mathcal{F}_j)} = S_{\mathcal{F}^0_j} + S_{\mathcal{F}_j}$ denotes the frame operator of the sequence $(\mathcal{F}^0_j\,,\,\mathcal{F}_j) \in (\mathbb{C}^{d_j})^{k+n}$, for $j \in \mathbb{I}_m$. That is, $\Psi_\varphi(\Phi)$ is the *joint* convex potential P_φ of the completed sequence $(\Phi^0\,,\,\Phi) = ((\mathcal{F}^0_j\,,\,\mathcal{F}_j))_{j\in\mathbb{I}_m} \in \mathcal{C}(\Phi^0\,,\,\alpha\,,\,\mathbf{d})$.

△

Problems 2.9. With the notation in Definition 2.8 above, we are interested in the following problems:

P1 Compute those $(\alpha\,,\,\mathbf{d})$-designs Φ that are (local) minimizers of Ψ_φ on $\mathcal{D}(\alpha\,,\,\mathbf{d})$.

P2 Compute

$$\mu_\varphi(\alpha\,,\,\mathbf{d}) = \min\{\Psi_\varphi(\Phi) : \Phi \in \mathcal{D}(\alpha\,,\,\mathbf{d})\} = \min\{\mathrm{P}_\varphi(\Phi^0\,,\,\Phi) : (\Phi^0\,,\,\Phi) \in \mathcal{C}(\Phi^0\,,\,\alpha\,,\,\mathbf{d})\}\ .$$

P3 Find an algorithmic procedure to compute

$$\Phi_\varphi^{\rm op} \in \mathcal{D}(\alpha\,,\,\mathbf{d}) \ : \ \Psi_\varphi(\Phi^{\rm op}) = \mu_\varphi(\alpha\,,\,\mathbf{d})\ .$$

We call such $\Phi_\varphi^{\rm op} \in \mathcal{D}(\alpha\,,\,\mathbf{d})$ an *optimal* $(\alpha\,,\,\mathbf{d})$ *multi-completion* of the initial family Φ^0.

P4 Determine whether the set of optimal $(\alpha\,,\,\mathbf{d})$ multi-completions depends on the strictly convex function φ.

□

Next we state our main result regarding optimal $(\alpha\,,\,\mathbf{d})$ multi-completions; we delay its proof until Section 4.1. In order to simplify the statement of the next result we use the following notations:

1. Given $p \in \mathbb{N}$, constants $c_1 > \cdots > c_p > 0$ and indices $0 = i_0 < i_1 < \ldots < i_p = n$, with $i_{p-1} < d_1$ we construct the *associated vector* $\mathbf{c} \in (\mathbb{R}_{>0}^{d_1})^\downarrow$ (to these constants and indices) as follows: we let $r_j = i_j - i_{j-1}$ for $j \in \mathbb{I}_{p-1}$ and $r_p = d_1 - i_{p-1}$ and we set
$$\mathbf{c} := (c_1\,\mathbb{1}_{r_1}\,,\,\ldots\,,\,c_p\,\mathbb{1}_{r_p}) \in (\mathbb{R}_{>0}^{d_1})^\downarrow\,. \tag{6}$$

2. Given vectors $\gamma = (\gamma_i)_{i\in\mathbb{I}_s} \in \mathbb{R}^s$ and $\delta = (\delta_i)_{i\in\mathbb{I}_t} \in \mathbb{R}^t$ we let $\max\{\gamma\,,\,\delta\} \in \mathbb{R}^u$, where $u = \min\{s\,,\,t\}$, be given by
$$\max\{\gamma\,,\,\delta\} = (\max\{\gamma_1\,,\,\delta_1\},\ldots,\max\{\gamma_u\,,\,\delta_u\}) \in \mathbb{R}^u\,.$$
That is, we compare the first $u = \min\{s\,,\,t\}$ coordinates of γ with those of δ.

Theorem 2.10. *Let $(\Phi^0 = (\mathcal{F}_j^0)_{j\in\mathbb{I}_m}\,,\,\alpha\,,\,\mathbf{d})$ be the initial data for a multi-completion problem, with $n \geq d_1$. Let $\lambda_j = (\lambda_{i,j})_{i\in\mathbb{I}_{d_j}} = \lambda^\uparrow(S_{\mathcal{F}_j^0}) \in (\mathbb{R}_{\geq 0}^{d_j})^\uparrow$, for $j \in \mathbb{I}_m$.*

Then there exist $p \in \mathbb{N}$, constants $c_1 > \cdots > c_p > 0$, indices $0 = i_0 < i_1 < \ldots < i_p = n$, with $i_{p-1} < d_1$ and the associated vector $\mathbf{c} = (c_1\,\mathbb{1}_{r_1}\,,\,\ldots\,,\,c_p\,\mathbb{1}_{r_p}) \in (\mathbb{R}_{>0}^{d_1})^\downarrow$ as in Eq. (6) such that, if we consider the vectors $\nu_j = \max\{\lambda_j\,,\,\mathbf{c}\} \in \mathbb{R}_{>0}^{d_j}$ for any $j \in \mathbb{I}_m$, they have the following property:

For every completion $\tilde\Phi = (\tilde{\mathcal{F}}_j)_{j\in\mathbb{I}_m} \in \mathcal{D}(\alpha\,,\,\mathbf{d})$ and any map $\varphi \in \mathrm{Conv_s}(\mathbb{R}_{\geq 0})$,

$$\tilde\Phi \ \text{ is a local minimizer of } \ \Psi_\varphi \ \text{ on } \ \mathcal{D}(\alpha\,,\,\mathbf{d}) \quad \Longleftrightarrow \quad \lambda(S_{(\mathcal{F}_j^0\,,\,\tilde{\mathcal{F}}_j)}) = \nu_j^\downarrow \ \ , \ \ j \in \mathbb{I}_m\,.$$

In this case,

1. *$\tilde\Phi$ is a global minimizer of Ψ_ϕ, for every $\phi \in \mathrm{Conv_s}(\mathbb{R}_{\geq 0})$.*
2. *For $j \in \mathbb{I}_m$, $\min\sigma(S_{(\mathcal{F}_j^0\,,\,\tilde{\mathcal{F}}_j)}) > 0$; in particular, $(\mathcal{F}_j^0\,,\,\tilde{\mathcal{F}}_j)$ is a frame for $\mathbb{C}^{d_j}$.*
3. *For $j \in \mathbb{I}_m$ such that $\tilde{\mathcal{F}}_j$ has non zero vectors, there exists $\mathcal{B}_j = \{v_{i,j}\}_{i\in\mathbb{I}_{d_j}}$ an ONB of $\mathbb{C}^{d_j}$ such that*
$$S_{\mathcal{F}_j^0} = \sum_{i\in\mathbb{I}_{d_j}} \lambda_{i,j}\ v_{i,j}\otimes v_{i,j} \qquad \textit{and}$$
$$S_{\tilde{\mathcal{F}}_j} = \sum_{\ell=1}^{p_j-1}\ \sum_{i=i_{\ell-1}+1}^{i_\ell} (c_\ell - \lambda_{i,j})\ v_{i,j}\otimes v_{i,j} + \sum_{i=i_{p_j-1}+1}^{d_j} (c_{p_j} - \lambda_{i,j})^+\ v_{i,j}\otimes v_{i,j}\,,$$
where $1 \leq p_j \leq p$ is uniquely determined by the condition: $c_{p_j} = (\mathbf{c})_{\ell_j}$, where $\ell_j := \max\{k \in \mathbb{I}_{d_j} : (\mathbf{c})_k > \lambda_{k,j}\}$. □

The vectors $\nu_j^\downarrow$ ($j \in \mathbb{I}_m$) of Theorem 2.10 are unique and they characterize the spectra of every optimal multi-completion, with respect to any map $\varphi \in \text{Conv}_s(\mathbb{R}_{\geq 0})$. On the other hand, these vectors typically have several entries in common by their construction using the formulae $\nu_j = \max\{\lambda_j\,,\,\mathbf{c}\}$ for $j \in \mathbb{I}_m$. This last fact shows the key role of the vector $\mathbf{c}$ in this setting.

We remark that the constants $c_1 > \cdots > c_p > 0$, the indices $0 = i_0 < i_1 < \ldots < i_p = n$ (that are related to a fundamental partition of the set $\mathbb{I}_n$), the integers $r_1\,,\,\ldots\,,\,r_p$ and hence the vector (of levels) $\mathbf{c}$ in Theorem 2.10, can be computed in terms of a (rather simple and fast) finite step algorithm. The input of this algorithm is the initial data $\lambda_j = (\lambda_{i,j})_{i\in\mathbb{I}_{d_j}} = \lambda^\uparrow(S_{\mathcal{F}_j^0}) \in (\mathbb{R}_{\geq 0}^{d_j})^\uparrow$, for $j \in \mathbb{I}_m$, and $\alpha \in (\mathbb{R}_{>0}^n)^\downarrow$ (see Section 4.2). Moreover, using the previous vectors and applying well-known algorithms (such as generalized Bendel-Mickey algorithms, see [15]) we can finally compute concrete optimal $(\alpha\,,\,\mathbf{d})$-multi-completions $\tilde{\Phi}$ (see Section 5.2).

As mentioned above, we prove Theorem 2.10 in Section 4.1, where we also show how this result allows one to solve the optimal multi-completion problem when $n < d_1 = \max\{d_j \;:\; j \in \mathbb{I}_m\}$. We will also show how Theorem 2.10 can be used to give explicit solutions to P1-P4 in Problems 2.9.

3 On the structure of local minimizers of Ψ_φ on $\mathcal{D}(\alpha\,,\,\mathbf{d})$

In this section we analyze the geometric and spectral properties of local minimizers of $\Psi_\varphi : \mathcal{D}(\alpha\,,\,\mathbf{d}) \to \mathbb{R}_{\geq 0}$, defined as in Eq. (5), for a strictly convex function $\varphi \in \text{Conv}_s(\mathbb{R}_{\geq 0})$. This analysis will reveal the special role played by a certain vector $\mathbf{c}$, which will enable us to characterize the spectra of local minimizers of Ψ_φ in the next section.

3.1 Spectral and geometrical structure of local minimizers: first features

We begin our analysis of local minimizers of Ψ_φ. First we fix the notation and terminology used throughout the rest of the paper.

Notation 3.1. Let $(\Phi^0\,,\,\alpha\,,\,\mathbf{d})$ be the initial data for a multi-completion problem, with $n \geq d_1$. Let

$$\lambda_j = (\lambda_{i,j})_{i\in\mathbb{I}_{d_j}} = \lambda^\uparrow(S_{\mathcal{F}_j^0}) \in (\mathbb{R}_{\geq 0}^{d_j})^\uparrow \quad , \text{ for } \quad j \in \mathbb{I}_m\,.$$

In what follows we consider:

1. $\Psi_\varphi : \mathcal{D}(\alpha\,,\,\mathbf{d}) \to \mathbb{R}_{\geq 0}$, for a strictly convex function $\varphi \in \text{Conv}_s(\mathbb{R}_{\geq 0})$, as described in Eq. (5).
2. $\tilde{\Phi} = (\tilde{\mathcal{F}}_j)_{j\in\mathbb{I}_m} \in \mathcal{D}(\alpha\,,\,\mathbf{d})$, where $\tilde{\mathcal{F}}_j = \{\tilde{f}_{i,j}\}_{i\in\mathbb{I}_n}$ for $j \in \mathbb{I}_m$, a local minimizer of Ψ_φ.
3. In this case, we let $\omega_j = (\|\tilde{f}_{i,j}\|^2)_{i\in\mathbb{I}_n}$, for $j \in \mathbb{I}_m$. Notice that the vectors ω_j are not necessarily arranged in decreasing order and could be, eventually, the zero vector. △

In what follows we obtain some spectral and geometric properties of $\tilde{\Phi}$. In order to do this we introduce the following sets: given $\eta = (\eta_i)_{i\in\mathbb{I}_n} \in \mathbb{R}_{\geq 0}^n$ and $d \in \mathbb{N}$ with $d \leq n$, the η-torus in $\mathbb{C}^d$ is the set:

$$\mathcal{B}_{\eta\,,\,d} = \{\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n} \in (\mathbb{C}^d)^n :\; \|f_i\|^2 = \eta_i\,,\; i \in \mathbb{I}_n\}\,. \tag{7}$$

That is, $\mathcal{B}_{\eta\,,\,d}$ is the Cartesian product of spheres of $\mathbb{C}^d$. In what follows we consider $\mathcal{B}_{\eta\,,\,d}$ endowed with the (product) metric $m_d(\cdot,\cdot)$ defined in Eq. (3).

Theorem 3.2. *Consider Notation 3.1. For each $j \in \mathbb{I}_m$, $\tilde{\mathcal{F}}_j$ is a local minimizer of $\Psi_{\varphi\,,\,j} : \mathcal{B}_{\omega_j\,,\,d_j} \to \mathbb{R}_{\geq 0}$ given by*

$$\Psi_{\varphi\,,\,j}(\mathcal{G}) = P_\varphi((\mathcal{F}_j^0\,,\,\mathcal{G})) = \text{tr}(\varphi(S_{\mathcal{F}_j^0} + S_{\mathcal{G}})) \quad \textit{for} \quad \mathcal{G} = \{g_i\}_{i\in\mathbb{I}_n} \in \mathcal{B}_{\omega_j\,,\,d_j}\,.$$

In particular, for $j \in \mathbb{I}_m$ such that $\tilde{\mathcal{F}}_j$ has non zero vectors (i.e. $\omega_j \neq 0_n$), there exists $\mathcal{B}_j = \{v_{i,j}\}_{i\in\mathbb{I}_{d_j}}$ an ONB for $\mathbb{C}^{d_j}$ such that

$$S_{\mathcal{F}_j^0} = \sum_{i\in\mathbb{I}_{d_j}} \lambda_{i,j}\ v_{i,j}\otimes v_{i,j} \quad \text{and} \quad S_{\tilde{\mathcal{F}}_j} = \sum_{i\in\mathbb{I}_{d_j}} \lambda_i(S_{\tilde{\mathcal{F}}_j})\ v_{i,j}\otimes v_{i,j}\,. \tag{8}$$

Moreover, if $r_j^* = \text{rk}(S_{\tilde{\mathcal{F}}_j}) \in \mathbb{I}_{d_j}$ and $c_{1,j} > \cdots c_{p_j,j} > 0$ are the distinct eigenvalues of the restriction of $S_{(\mathcal{F}_j^0\,,\,\tilde{\mathcal{F}}_j)} = S_{\mathcal{F}_j^0} + S_{\tilde{\mathcal{F}}_j}$ to (its invariant subspace) $R(S_{\tilde{\mathcal{F}}_j}) = \text{span}\{v_{i,j}\}_{i\in\mathbb{I}_{r_j^*}}$ then:

1. *Each non-zero $\tilde{f}_{i,j}$ is an eigenvector of $S_{(\mathcal{F}_j^0\,,\,\tilde{\mathcal{F}}_j)} = S_{\mathcal{F}_j^0} + S_{\tilde{\mathcal{F}}_j}$ such that, if $\tilde{f}_{h,j}$, $\tilde{f}_{\ell,j}$ correspond respectively to the eigenvalues $c_{i,j} > c_{k,j}$, then $\omega_{h,j} = \|\tilde{f}_{h,j}\|^2 > \|\tilde{f}_{\ell,j}\|^2 = \omega_{\ell,j}$.*
2. *For $1 \leq \ell \leq p_j - 1$, the vectors $\{\tilde{f}_{i,j}\}_{J_{\ell,j}}$ that are the non-zero eigenvectors of the eigenvalue c_ℓ form a basis of the eigenspace $W_{\ell,j}$ of the restriction of $S_{(\mathcal{F}_j^0\,,\,\tilde{\mathcal{F}}_j)}$ to $R(S_{\tilde{\mathcal{F}}_j})$ associated to c_ℓ.*

Proof. The fact that each $\tilde{\mathcal{F}}_j$ is a local minimizer of $\Psi_{\varphi\,,\,j}$ is an immediate consequence of Eq. (5) and the fact that $\tilde{\Phi}$ is a local minimizer of Ψ_φ. The rest of the claims follow from Theorem 6.10 and Remark 6.11 in the Appendix. □

Notation 3.3. Consider Notation 3.1 so that $\tilde{\Phi} = (\tilde{\mathcal{F}}_j)_{j\in\mathbb{I}_m} \in \mathcal{D}(\alpha\,,\,\mathbf{d})$ is a local minimizer of Ψ_φ, where $\tilde{\mathcal{F}}_j = \{\tilde{f}_{i,j}\}_{i\in\mathbb{I}_n}$ for $j \in \mathbb{I}_m$. For $j \in \mathbb{I}_m$ such that $\omega_j = (\|\tilde{f}_{i,j}\|^2)_{i\in\mathbb{I}_n} \neq 0_n$, in what follows we consider:

1. The distinct eigenvalues of the restriction of $S_{(\mathcal{F}_j^0\,,\,\tilde{\mathcal{F}}_j)}$ to its invariant subspace $R(S_{\tilde{\mathcal{F}}_j})$:
$$c_{1,j} > \cdots > c_{p_j,j} > 0$$
2. Let $W_{\ell,j}$ be the eigenspace of the restriction of $S_{(\mathcal{F}_j^0\,,\,\tilde{\mathcal{F}}_j)}$ to $R(S_{\tilde{\mathcal{F}}_j})$ associated to c_ℓ and let $r_{\ell,j} = \dim W_{\ell,j}$, for $\ell \in \mathbb{I}_{p_j}$. Notice that
$$\sum_{\ell\in\mathbb{I}_{p_j}} r_{\ell,j} = \text{rk}(S_{\tilde{\mathcal{F}}_j}) := r_j^* \quad \text{for} \quad j \in \mathbb{I}_m\,.$$
3. Let $J_{\ell,j} \subset \mathbb{I}_n$ be such that $\{\tilde{f}_{i,j}\}_{i\in J_{\ell,j}}$ are the non-zero vectors of $\tilde{\mathcal{F}}_j$ in $W_{\ell,j}$, for $\ell \in \mathbb{I}_{p_j}$. Notice that, by Theorem 3.2, $\#(J_{\ell,j}) \geq r_{\ell,j}$ for $\ell \in \mathbb{I}_{p_j}$ and $\#(J_{\ell,j}) = r_{\ell,j}$, for $1 \leq \ell \leq p_j - 1$.
4. Let $p_{\tilde{\Phi}} = p := \max\{p_j :\ j \in \mathbb{I}_m\} \geq 1$.

△

Proposition 3.4. *Consider Notation 3.3. For $j \in \mathbb{I}_m$ such that $\tilde{\mathcal{F}}_j$ has non zero vectors (i.e. $\omega_j \neq 0_n$), then $(\mathcal{F}_j^0\,,\,\tilde{\mathcal{F}}_j)$ is a frame for $\mathbb{C}^{d_j}$. In particular,*

1. *$c_{p_j,j} = \min \sigma(S_{(\mathcal{F}_j^0\,,\,\tilde{\mathcal{F}}_j)})$; moreover $c_{p_j,j} > \lambda_{r_j^*,j}$, where $r_j^* = \text{rk}(S_{\tilde{\mathcal{F}}_j}) \in \mathbb{I}_{d_j}$.*
2. *The spectrum of $\lambda(S_{(\mathcal{F}_j^0\,,\,\tilde{\mathcal{F}}_j)})$ can be described in the following way: if we let*
$$\mathbf{c}^j = (c_{1,j}\,\mathbb{1}_{r_{1,j}}, \cdots, c_{p_j,j}\,\mathbb{1}_{r_{p_j,j}}\,,\ c_{p_j,j}\,\mathbb{1}_{(d_j-r_j^*)}) \in (\mathbb{R}_{>0}^{d_j})^\downarrow \quad \textit{then}$$
$$\lambda(S_{(\mathcal{F}_j^0\,,\,\tilde{\mathcal{F}}_j)}) = \max\{\lambda_j, \mathbf{c}^j\}^\downarrow = (c_{1,j}\,\mathbb{1}_{r_{1,j}}\,,\ \ldots\,,\ c_{p_j,j}\,\mathbb{1}_{r_{p_j,j}}\,,\ \lambda_{r_j^*+1,j}\,,\ \ldots\,,\ \lambda_{d_j,j})^\downarrow \in (\mathbb{R}_{>0}^{d_j})^\downarrow. \tag{9}$$

Proof. Assume that $(\mathcal{F}_j^0\,,\,\tilde{\mathcal{F}}_j)$ is not a frame for $\mathbb{C}^{d_j}$. Since $n \geq d_1 \geq d_j$ then, by Theorem 3.2 and item 4d. in Theorem 6.10, we conclude that there exists $i_0 \in \mathbb{I}_n$ such that $\tilde{f}_{i_0,j} = 0$. Let $j \neq k \in \mathbb{I}_m$ be such that $\tilde{f}_{i_0,k} \neq 0$ and let $c_{\ell,k} > 0$ be such that $S_{(\mathcal{F}_k^0\,,\,\tilde{\mathcal{F}}_k)}\,\tilde{f}_{i_0,k} = c_{\ell,k}\,\tilde{f}_{i_0,k}$. Since $(\mathcal{F}_j^0\,,\,\tilde{\mathcal{F}}_j)$ is not a frame, there exists $h \in \mathbb{C}^{d_j}$ with $\|h\| = 1$, and such that $S_{(\mathcal{F}_j^0\,,\,\tilde{\mathcal{F}}_j)}\,h = 0$. We let

$$\tilde{f}_{i_0,j}(t) = t^{1/2}\,\|\tilde{f}_{i_0,k}\|\,h \quad \text{and} \quad \tilde{f}_{i_0,k}(t) = (1-t)^{1/2}\tilde{f}_{i_0,k}, \quad \text{for} \quad t \in [0,1]\,. \tag{10}$$

By construction, $\|\tilde{f}_{i_0,j}(t)\|^2 + \|\tilde{f}_{i_0,k}(t)\|^2 = \|\tilde{f}_{i_0,j}\|^2 + \|\tilde{f}_{i_0,k}\|^2$, for $t \in [0,1]$. Let $\tilde{\mathcal{F}}_j(t)$ and $\tilde{\mathcal{F}}_k(t)$ be obtained from $\tilde{\mathcal{F}}_j$ and $\tilde{\mathcal{F}}_k$ by replacing $\tilde{f}_{i_0,j}$ and $\tilde{f}_{i_0,k}$ by $\tilde{f}_{i_0,j}(t)$ and $\tilde{f}_{i_0,k}(t)$ respectively, for $t \in [0,1]$; similarly, let $\tilde{\Phi}(t) \in \mathcal{D}(\alpha\,,\,\mathbf{d})$ be obtained from $\tilde{\Phi}$ by replacing $\tilde{\mathcal{F}}_j$ and $\tilde{\mathcal{F}}_k$ by $\tilde{\mathcal{F}}_j(t)$ and $\tilde{\mathcal{F}}_k(t)$ respectively, for $t \in [0,1]$. Notice that, by construction, the distance $m(\tilde{\Phi}(t)\,,\,\tilde{\Phi}) \to 0$ when $t \to 0^+$. On the other hand,

$$S_{(\mathcal{F}_k^0,\,\tilde{\mathcal{F}}_k(t))} = S_{(\mathcal{F}_k^0,\,\tilde{\mathcal{F}}_k)} - t\,\|\tilde{f}_{i_0,k}\|^2\,\frac{\tilde{f}_{i_0,k} \otimes \tilde{f}_{i_0,k}}{\|\tilde{f}_{i_0,k}\|^2}\,,$$

and therefore, we have that $\lambda(S_{(\mathcal{F}_k^0,\,\tilde{\mathcal{F}}_k(t))})$ coincides with the vector

$$(c_{1,k}\,\mathbb{1}_{r_{1,k}}\,,\,\ldots,c_{\ell,k}\,\mathbb{1}_{r_{\ell,k}-1},\,c_{\ell,k} - t\,\|\tilde{f}_{i_0,k}\|^2,\,c_{\ell+1,k}\,\mathbb{1}_{r_{\ell+1,k}},\ldots,c_{p_k,k}\,\mathbb{1}_{r_{p_k,k}}\,,\,\lambda_{r_k^*+1,k},\ldots,\lambda_{d_k,k})^{\downarrow}$$

since $\tilde{f}_{i_0,k}$ is an eigenvector of $S_{(\mathcal{F}_k^0,\,\tilde{\mathcal{F}}_k)}$. Arguing as above, we get that

$$S_{(\mathcal{F}_j^0,\,\tilde{\mathcal{F}}_j(t))} = S_{(\mathcal{F}_j^0,\,\tilde{\mathcal{F}}_j)} + t\,\|\tilde{f}_{i_0,k}\|^2\,h \otimes h$$

so, we have that

$$\lambda(S_{(\mathcal{F}_j^0,\,\tilde{\mathcal{F}}_j(t))}) = (c_{1,j}\,\mathbb{1}_{r_{1,j}}\,,\,\ldots\,,\,c_{p_j,j}\,\mathbb{1}_{r_{p_j,j}}\,,\,t\,\|f_{i_0,k}\|^2\,,\,\lambda_{r_j^*+2,j},\ldots,\lambda_{d_j,j})^{\downarrow}\,,$$

where we used that h is an eigenvector of $S_{(\mathcal{F}_j^0,\,\tilde{\mathcal{F}}_j)}$ corresponding to the zero eigenvalue and that $\lambda_{r_j^*+1,j} = 0$, since $(\mathcal{F}_j^0\,,\,\tilde{\mathcal{F}}_j)$ is not a frame for $\mathbb{C}^{d_j}$. Now, for sufficiently small t we see that

$$(c_{\ell,k} - t\,\|\tilde{f}_{i_0,k}\|^2, t\,\|\tilde{f}_{i_0,k}\|^2) \prec (c_{\ell,k}, 0) = (c_{\ell,k}, \lambda_{r_j^*+1,j})$$

holds with strict majorization. Hence (see Remark 6.6) we conclude that sufficiently small $t \in (0,1]$ we have a strict majorization relation

$$(\lambda(S_{(\mathcal{F}_j^0,\,\tilde{\mathcal{F}}_j(t))})\,,\,\lambda(S_{(\mathcal{F}_k^0,\,\tilde{\mathcal{F}}_k(t))})) \prec (\lambda(S_{(\mathcal{F}_j^0,\,\tilde{\mathcal{F}}_j)})\,,\,\lambda(S_{(\mathcal{F}_k^0,\,\tilde{\mathcal{F}}_k)})) \in \mathbb{R}^{d_j+d_k}\,.$$

Thus, using the definition of Ψ_φ and Theorem 6.5, we get that $\Psi_\varphi(\tilde{\Phi}(t)) < \Psi_\varphi(\tilde{\Phi})$ for sufficiently small $t > 0$. This last fact contradicts our assumption that $\tilde{\Phi}$ is a local minimizer of Ψ_φ on $\mathcal{D}(\alpha\,,\,\mathbf{d})$. Therefore, $(\mathcal{F}_j^0\,,\,\tilde{\mathcal{F}}_j)$ is a frame for $\mathbb{C}^{d_j}$. The rest of the claims follow from Theorem 6.10 and Remark 6.11. □

Proposition 3.5. *Consider Notation 3.3. Then, for each pair $j, k \in \mathbb{I}_m$ such that $\omega_j\,,\,\omega_k \neq 0_n$ we have that $c_1 := c_{1,j} = c_{1,k}$. Moreover, if $p_k \geq 2$ then $r_{1,k} \geq r_{1,j}$ (in particular, $r_{1,k} = r_{1,j}$ if $p_k,\,p_j \geq 2$). In addition, for those $l \in \mathbb{I}_m$ for which $\omega_l = 0_n$ we have $0 < c_1 \leq \min \sigma(S_{\mathcal{F}_l^0})$.*

Proof. Suppose that $\omega_j \neq 0_n \neq \omega_k$, for some $k, j \in \mathbb{I}_m$, $k \neq j$. Assume, that $c_{1,j} > c_{1,k}$ and let $i_0 \in J_{1,j}$. We let

$$\tilde{f}_{i_0,j}(t) = (1-t)^{1/2}\tilde{f}_{i_0,j}\,, \quad \text{for} \quad t \in [0,1]\,, \tag{11}$$

and consider the following two cases.

Case 1: if $\tilde f_{i_0,k} \neq 0$, so that $\tilde f_{i_0,k} \in W_{\ell,k}$ for some $\ell \in \mathbb{I}_{p_k}$, we then define

$$\tilde f_{i_0,k}(t) = (t\,\frac{\|\tilde f_{i_0,j}\|^2}{\|\tilde f_{i_0,k}\|^2} + 1)^{1/2}\, \tilde f_{i_0,k}\ , \qquad \text{for} \qquad t \in [0,1]\,. \tag{12}$$

Case 2: if $\tilde f_{i_0,k} = 0$ we consider $h \in W_{1,k} \neq \{0\}$, with $\|h\| = 1$, and let

$$\tilde f_{i_0,k}(t) = t^{1/2}\, \|\tilde f_{i_0,j}\|\, h\ , \qquad \text{for} \qquad t \in [0,1]\,. \tag{13}$$

In any case, we get that $\|\tilde f_{i_0,j}(t)\|^2 + \|\tilde f_{i_0,k}(t)\|^2 = \|\tilde f_{i_0,j}\|^2 + \|\tilde f_{i_0,k}\|^2$, for $t \in [0,1]$. Consider $\tilde{\mathcal F}_j(t)$, $\tilde{\mathcal F}_k(t)$ and $\tilde\Phi(t)$ as in the proof of Proposition 3.4 above, for $t \in [0,1]$. With this notation and arguing as in the proof of Proposition 3.4, we see that for $t \in [0,1]$

$$(c_{1,j}\,\mathbb{1}_{r_{1,j}-1}\,,\ c_{1,j} - t\,\|\tilde f_{i_0,j}\|^2\,,\ c_{2,j}\,\mathbb{1}_{r_{2,j}}\,,\ \ldots\,,\ c_{p_j,j}\,\mathbb{1}_{r_{p_j,j}}\,,\ \lambda_{r_j^*+1,j}, \ldots, \lambda_{d_j,j})^{\downarrow} = \lambda(S_{(\mathcal F_j^0,\,\tilde{\mathcal F}_j(t))}) \tag{14}$$

since $\tilde f_{i_0,j}$ is an eigenvector of $S_{(\mathcal F_j^0,\,\tilde{\mathcal F}_j)}$ corresponding to the eigenvalue $c_{1,j}$. According to the previous two cases we have that:

In case 1: for $t \in [0,1]$, we can represent $\lambda(S_{(\mathcal F_k^0,\,\tilde{\mathcal F}_k(t))})$ as

$$(c_{1,k}\,\mathbb{1}_{r_{1,k}}\,,\ \ldots\,,\ c_{\ell,k}\,\mathbb{1}_{r_{\ell,k}-1}\,,\ c_{\ell,k} + t\,\|\tilde f_{i_0,j}\|^2\,,\ \ldots\,,\ c_{p_k,k}\,\mathbb{1}_{r_{p_k,k}}\,,\ \lambda_{r_k^*+1,k}, \ldots, \lambda_{d_k,k})^{\downarrow}\,, \tag{15}$$

where we used that $\tilde f_{i_0,k}$ is an eigenvector of $S_{(\mathcal F_k^0,\,\tilde{\mathcal F}_k)}$. Since $c_{\ell,k} \leq c_{1,k} < c_{1,j}$ we see that

$$(c_{1,j} - t\,\|\tilde f_{i_0,j}\|^2, c_{\ell,k} + t\,\|\tilde f_{i_0,j}\|^2) \prec (c_{1,j}, c_{\ell,k})$$

holds with strict majorization, for sufficiently small t. Hence (see Remark 6.6) we conclude that sufficiently small $t \in (0,1]$ we have a strict majorization relation

$$(\lambda(S_{(\mathcal F_j^0,\,\tilde{\mathcal F}_j(t))})\,,\ \lambda(S_{(\mathcal F_k^0,\,\tilde{\mathcal F}_k(t))})) \prec (\lambda(S_{(\mathcal F_j^0,\,\tilde{\mathcal F}_j)})\,,\ \lambda(S_{(\mathcal F_k^0,\,\tilde{\mathcal F}_k)})) \in \mathbb{R}^{d_j+d_k}\,. \tag{16}$$

Thus, using the definition of Ψ_φ and Theorem 6.5 we get that $\Psi_\varphi(\tilde\Phi(t)) < \Psi_\varphi(\tilde\Phi)$ for sufficiently small $t > 0$. This last fact contradicts our assumption that $\tilde\Phi$ is a local minimizer of Ψ_φ on $\mathcal D(\alpha\,,\,\mathbf d)$.

In case 2: arguing as above we get that for $t \in [0,1]$,

$$(c_{1,k}\,\mathbb{1}_{r_{1,k}-1}\,,\ c_{1,k} + t\,\|\tilde f_{i_0,j}\|^2\,,\ c_{2,k}\,\mathbb{1}_{r_{2,k}}\,,\ \ldots\,,\ c_{p_k,k}\,\mathbb{1}_{r_{p_k,k}}\,,\ \lambda_{r_k^*+1,k}, \ldots, \lambda_{d_k,k})^{\downarrow} = \lambda(S_{(\mathcal F_k^0,\,\tilde{\mathcal F}_k(t))}) \tag{17}$$

where we used that h is an eigenvector of $S_{(\mathcal F_k^0,\,\tilde{\mathcal F}_k)}$.

Since $c_{1,k} < c_{1,j}$ we see that $(c_{1,j} - t\,\|\tilde f_{i_0,j}\|^2, c_{1,k} + t\,\|\tilde f_{i_0,j}\|^2) \prec (c_{1,j}, c_{1,k})$ strictly, for sufficiently small t. Hence, we conclude that for sufficiently small $t \in (0,1)$, Eq. (16) holds with strict majorization. As before, this contradicts our assumption that $\tilde\Phi$ is a local minimizer of Ψ_φ on $\mathcal D(\alpha\,,\,\mathbf d)$.

Assume further that $p_k \geq 2$ and that $r_{1,k} < r_{1,j}$, for some $k \in \mathbb{I}_m$, $k \neq j$. In this case,

$$\dim W_{1,k} = r_{1,k} < \dim W_{1,j} \quad , \quad W_{1,k} = \operatorname{span}\{\tilde f_{i,k} :\ i \in J_{1,k}\} \quad , \quad W_{1,j} = \operatorname{span}\{\tilde f_{i,j} :\ i \in J_{1,j}\}\,.$$

Since $p_k \geq 2$ then $\{\tilde f_{i,k} :\ i \in J_{1,k}\}$ is a linearly independent family, by Theorem 3.2; hence, $r_{1,k} = \#(J_{1,k})$. On the other hand, since $\#(J_{1,j}) \geq r_{1,j}$ then $\#(J_{1,k}) < \#(J_{1,j})$. Hence, there exists $i_0 \in J_{1,j}$ such that $i_0 \notin J_{1,k}$. Then there exists $2 \leq \ell \leq p_k$ such that $i_0 \in J_{\ell,k}$ (and hence $\tilde f_{i_0,k} \neq 0$) or $\tilde f_{i_0,k} = 0$, and we can argue as above considering these two possible cases. Indeed, if $\tilde f_{i_0,k} \neq 0$ then we construct $\tilde f_{i_0,k}(t)$ as in Eq. (12); in case $\tilde f_{i_0,k} = 0$ then we choose $h \in W_{2,k} \neq \{0\}$ with $\|h\| = 1$ and construct $\tilde f_{i_0,k}(t)$ as in Eq. (13). In both cases, we conclude that Eq. (16) holds, with strict majorization. Again, this last fact contradicts our assumption that $\tilde{\Phi}$ is a local minimizer of Ψ_φ on $\mathcal{D}(\alpha\,,\,\mathbf{d})$

Finally, suppose now that $\omega_\ell = 0_n$ (i.e. $\tilde{\mathcal{F}}_\ell = \{0\}_{i\in\mathbb{I}_n}$) and $\lambda_{1,\ell} = \min\sigma(S_{\mathcal{F}^0_\ell}) < c_{1,j}$. Let us construct $\tilde{f}_{i_0,j}(t)$ and $\tilde{f}_{i_0,\ell}(t)$ according to Eqs. (11) and (13), where now h is an eigenvector of $S_{\mathcal{F}^0_\ell}$ corresponding to the eigenvalue $\lambda_{1,\ell} = \min\ \sigma(S_{\mathcal{F}^0_\ell})$. If we let $\tilde{\Phi}(t)$ be as it was defined above, then we get - with the same argument as before - that the strict majorization relation in Eq. (16) holds in this case. This last fact contradicts that $\tilde{\Phi}$ is a local minimizer of Ψ_φ. □

Proposition 3.6. *Consider Notation 3.3. Let $j \in \mathbb{I}_m$ be such that $\omega_j \neq 0_n$ and let $i_0 \in J_{1,j}$. Then, for every $k \in \mathbb{I}_m$ such that $p_k \geq 2$ we have $i_0 \in J_{1,k}$. In particular, $\tilde{f}_{i_0,k} \neq 0$.*

Proof. Let $k \in \mathbb{I}_m \setminus \{j\}$ be such that $p_k \geq 2$ and assume that $\tilde{f}_{i_0,k} = 0$. We consider the functions $\tilde{f}_{i_0,j}(t)$ and $\tilde{f}_{i_0,k}(t)$ for $t \in [0,1]$ as defined in Eqs. (11) and (13), where $h \in W_{2,k} \neq \{0\}$ is a unit norm vector. We further consider $\tilde{\mathcal{F}}_j(t)$, $\tilde{\mathcal{F}}_k(t)$ and $\tilde{\Phi}(t)$ as in the proof of Proposition 3.5 above; hence, the identities in Eqs. (14) and (15) -with $\ell = 2$ - hold in this case. Thus, we conclude that for sufficiently small $t \in (0,1]$ we have that $(c_{1,j} - t\,\|\tilde{f}_{i_0,j}\|^2, c_{2,k} + t\,\|\tilde{f}_{i_0,j}\|^2) \prec (c_{1,j}, c_{2,k})$ strictly and therefore, the strict majorization relation in Eq. (16) holds. This last fact contradicts our assumption that $\tilde{\Phi}$ is a local minimizer of Ψ_φ on $\mathcal{D}(\alpha\,,\,\mathbf{d})$. Therefore, $i_0 \in J_{l,k}$ for some $1 \leq \ell \leq p_k$.

Suppose that $i_0 \in J_{l,k}$ for $2 \leq \ell$. Thus,

$$S_{(\mathcal{F}^0_j,\,\tilde{\mathcal{F}}_j)}\,\tilde{f}_{i_0,j} = c_{1,j}\,\tilde{f}_{i_0,j} \neq 0 \quad \text{ and } \quad S_{(\mathcal{F}^0_k,\,\tilde{\mathcal{F}}_k)}\,\tilde{f}_{i_0,k} = c_{\ell,k}\,\tilde{f}_{i_0,k} \neq 0\,,$$

with $c_{\ell,k} < c_{1,k} = c_{1,j} = c_1$, by Proposition 3.5. We consider the functions $\tilde{f}_{i_0,j}(t)$ and $\tilde{f}_{i_0,k}(t)$ for $t \in [0,1]$ as defined in Eqs. (11) and (12). We further consider $\tilde{\mathcal{F}}_j(t)$, $\tilde{\mathcal{F}}_k(t)$ and $\tilde{\Phi}(t)$ as in the proof of Proposition 3.4 above, for $t \in [0,1]$; hence, the identities in Eqs. (14) and (15) hold in this case. Thus, we conclude that for sufficiently small $t \in (0,1]$ the strict majorization relation in Eq. (16) holds. This last fact contradicts our assumption that $\tilde{\Phi}$ is a local minimizer of Ψ_φ on $\mathcal{D}(\alpha\,,\,\mathbf{d})$. Then, we must have $i_0 \in J_{1,k}$. □

Corollary 3.7. *Consider Notation 3.3, so that $p_{\tilde{\Phi}} = p = \max\{p_j :\ j \in \mathbb{I}_m\} \geq 1$, and set*

$$J_\ell = \{i \in \mathbb{I}_n : \exists j \in \mathbb{I}_m\,,\ \ell \leq p_j\ /\ i \in J_{\ell,j}\} = \bigcup_{j\in\mathbb{I}_m\,,\,\ell\leq p_j} J_{\ell,j} \quad \text{for} \quad \ell \in \mathbb{I}_p\,. \tag{18}$$

Then, there exists $i_1 \in \mathbb{I}_n$ such that

$$J_1 = \{i :\ 1 \leq i \leq i_1\} \quad \text{and} \quad \bigcup_{2\leq\ell\leq p} J_\ell = \{i :\ i_1 + 1 \leq i \leq n\}\,. \tag{19}$$

In particular,

1. *$J_1 \cap (\bigcup_{2\leq\ell\leq p} J_\ell) = \emptyset$;*
2. *$J_{1,j} \subset J_1$ and $W_{1,j} = span\{\tilde{f}_{i,j} :\ 1 \leq i \leq i_1\}$ for $j \in \mathbb{I}_m$, such that $p_j \geq 1$.*
3. *If $j \in \mathbb{I}_m$ is such that $p_j \geq 2$ then $J_{1,j} = J_1$ and $\#(J_1) = r_{1,j}$.*
4. *If $j \in \mathbb{I}_m$ is such that $p_j = 1$ then $\tilde{f}_{i,j} = 0$ for $i_1 < i \leq n$.*

Proof. Assume that $1 \leq u \leq v \leq n$ so that $\alpha_u \geq \alpha_v$, $v \in J_1$ and $u \in J_\ell$, for some $2 \leq \ell \leq p$. By assumption, there exist $j_0,\, j_1 \in \mathbb{I}_m$ such that $u \in J_{\ell,j_0}$ and $v \in J_{1,j_1}$; in this case $\tilde{f}_{u,j_0} \neq 0$ and $p_{j_0} \geq 2$. Assume that $k \in \mathbb{I}_m$ is such that $\tilde{f}_{u,k} \neq 0$; in this case Proposition 3.6 shows that $u \notin J_{1,k}$ (otherwise, $u \in J_{1,j_0}$ which contradicts our previous assumption) and $p_k \geq 2$; hence, again by Proposition 3.6 we get that $\tilde{f}_{v,k} \neq 0$ and $v \in J_{1,k}$. Then, by item 1 in Theorem 3.2, we must have $\|\tilde{f}_{u,k}\|^2 < \|\tilde{f}_{v,k}\|^2$. Therefore,

$$\alpha_u = \sum_{k\in\mathbb{I}_m} \|\tilde{f}_{u,k}\|^2 = \sum_{k\in\mathbb{I}_m\,,\,\tilde{f}_{u,k}\neq 0} \|\tilde{f}_{u,k}\|^2 < \sum_{k\in\mathbb{I}_m} \|\tilde{f}_{v,k}\|^2 = \alpha_v \leq \alpha_u,$$

so we get a contradiction.

Hence, if $i_1 = \max\{i \in J_1 = \bigcup_{k\in\mathbb{I}_m} J_{1,k}\}$ then we have that $J_1 = \{i : 1 \leq i \leq i_1\}$. Indeed, if $1 \leq i \leq i_1$ there exists $j \in \mathbb{I}_m$ such that $\tilde{f}_{i,j} \neq 0$ (since $\alpha_i \neq 0$). By the previous arguments, $i \in J_{1,j} \subset J_1$. As a consequence, we see that for any $k \in \mathbb{I}_m$ with $p_k \geq 2$, $J_{\ell,k} \cap J_1 = \emptyset$, for $2 \leq \ell \leq p_j$. This clearly implies items 1 to 3. Item 4 is trivial by the fact that if $p_k = 1$ and $\tilde{f}_{i,k} \neq 0$, then $i \in J_{1,k} \subset J_1$. □

A closer look at Proposition 3.6 and Corollary 3.7 shows that all the vectors $\omega_j \neq 0$ must have their i_1 greatest entries located at the first i_1 coordinates. That is, a priori, when we apply Theorem 3.2 we do not have the weights ω_j necessarily arranged in a decreasing or increasing order, but it turns out that the fact that the $\mathbf{c}^j$ share their greatest coordinate c_1 leads to the previous particular feature of vectors ω_j. Hence, we have that $\mathbf{c}_1^j = c_1$ for all $j \in \mathbb{I}_m$ and all the $\omega_j \neq 0$ have the first i_1 entries associated to this constant. Now, we proceed to extend these results to the next values of $\mathbf{c}_i^j$ for $2 \leq i$. In order to do this, we use an inductive argument.

Theorem 3.8. *Consider Notation 3.3. There exist constants $c_1 > \ldots > c_p > 0$ and indices $i_0 := 0 < i_1 < i_2 < \ldots < i_p = n$ such that, for $j \in \mathbb{I}_m$ such that $p_j \neq 0$ then:*

1. *$c_{\ell,j} = c_\ell$, for $\ell \in \mathbb{I}_{p_j}$.*
2. *We have that*
$$J_\ell = \{i : \; i_{\ell-1} + 1 \leq i \leq i_\ell\} \quad \text{for} \quad \ell \in \mathbb{I}_p\,. \tag{20}$$
Moreover, $J_{\ell,j} = J_\ell$, for $\ell \in \mathbb{I}_{p_j-1}$, and $J_{p_j,j} \subset J_{p_j}$.
3. *In particular $r_{\ell,j} = i_\ell - i_{\ell-1}$, for $\ell \in \mathbb{I}_{p_j-1}$.*
4. *If $i_{p_j} < i \leq n$ then $\tilde{f}_{i,j} = 0$.*
5. *Hence, if $p = 1$ then $i_{p-1} = i_0 = 0 < d_1$; otherwise, $p \geq 2$ and $i_{p-1} = r_1 + \ldots + r_{p-1} < d_1$.*

Proof. The proof of the first four items is by induction in $p_{\tilde{\Phi}} = p$. In case that $p = 1$, the result follows immediately from Proposition 3.5 and Corollary 3.7. Notice that, in this case, we clearly have $J_1 = \mathbb{I}_n$.

Now, suppose that $p \geq 2$. Then, by Proposition 3.5 we see that there exists $c_1 > 0$ such that, for every $j \in \mathbb{I}_m$ such that $\omega_j \neq 0_n$, $c_{1,j} = c_1$. Moreover, by Corollary 3.7, there is an index $1 \leq i_1 \leq n$ such that $W_{1,j} = \text{span}\{\tilde{f}_{i,j}\}_{i\in\mathbb{I}_{i_1}} = \text{span}\{v_{i,j}\}_{i\in\mathbb{I}_{i_1}}$ for every $j \in \mathbb{I}_m$ such that $p_j \geq 2$. On the other side, for those $j \in \mathbb{I}_m$ such that $p_j = 1$, we have that $\tilde{f}_{i,j} = 0$ if $i \geq i_1 + 1$.

Let $M' = \{j \in \mathbb{I}_m \;,\; p_j \geq 2\}$ and $m' = \#M'$. Notice that, by Proposition 3.5, for $j \in M'$ we have that $r_1 = r_{1,j} = \dim W_{1,j}$; moreover, according to Corollary 3.7, for those indices we also have that $J_{1,j} = J_1 = \{1 \leq i \leq i_1\}$. Recall that, since $\tilde{\mathcal{F}}_j$ is a local minimizer of $\Psi_{\phi,j}$, Theorem 3.2 shows that $\text{span}\{\tilde{f}_{i,j}\}_{i\in J_1}$ is orthogonal to $\text{span}\{\tilde{f}_{i,j}\}_{i\in\mathbb{I}_n\setminus J_1}$ (when $p_j = 1$ the later space is the zero space).

Let $\mathbf{d}' = d'_1 > \cdots > d'_{m'}$ be the (properly renamed) dimensions of the subspaces $V_j = W_{1,j}^\perp$, $j \in M'$ (notice that these dimensions have the form $d_j - r_1$, for $j \in M'$). Consider the set of weights $\alpha' = (\alpha_{i+i_1})_{i\in\mathbb{I}_{n-i_1}}$ and let $(\Phi^0)' = \{(\mathcal{F}_j^0)'\} = \{P_{V_j} \cdot \mathcal{F}_j^0\}_{j\in\mathbb{I}_{m'}}$, i.e. we are considering as initial vectors (to be completed), the orthogonal projections of $\mathcal{F}_j^0 = \{f_{i,j}^0\}_{i\in\mathbb{I}_k}$ onto V_j, $j \in \mathbb{I}_{m'}$. Here, by abuse of notation, we consider the elements on V_j as elements of $\mathbb{C}^{d'_j}$ by means of an isometric isomorphism. Therefore, we have a new set of initial data: $((\Phi^0)', \alpha', \mathbf{d}')$.

Consider the family $\tilde{\Phi}' = \{\tilde{\mathcal{F}}'_j\}_{j\in\mathbb{I}_{m'}} \in \mathcal{D}(\alpha', \mathbf{d}')$, where $\tilde{\mathcal{F}}'_j = \{\tilde{f}_{i+i_1,j}\}_{i\in\mathbb{I}_{n-i_1}}$, $j \in \mathbb{I}_{m'}$. By Theorem 3.2, the subspace V_j reduces both $S_{(\mathcal{F}^0_j,\tilde{\mathcal{F}}_j)}$ and $S_{\tilde{\mathcal{F}}_j}$, for $j \in \mathbb{I}_m$; then,

$$\sum_{j\in M'} \operatorname{tr}(\varphi[S_{(\mathcal{F}^0_j,\tilde{\mathcal{F}}_j)}]) = \sum_{j\in M'} \operatorname{tr}(\varphi[S_{((\mathcal{F}^0_j)',\tilde{\mathcal{F}}'_j)}]) + \sum_{j\in M'} \operatorname{tr}(\varphi[S_{(\mathcal{F}^0_j,\tilde{\mathcal{F}}_j)}|_{V_j^\perp}])\,.$$

We claim that $\tilde{\Phi}' \in \mathcal{D}(\alpha', \mathbf{d}')$ is a local minimizer for Ψ'_φ in $\mathcal{D}(\alpha', \mathbf{d}')$, where

$$\Psi'_\varphi(\Phi') = \sum_{j\in M'} \operatorname{tr}(\varphi[S_{((\mathcal{F}^0_j)',\mathcal{G}'_j)}]) \quad \text{for} \quad \Phi' = \{\mathcal{G}'_j\}_{j\in\mathbb{I}_{m'}} \in \mathcal{D}(\alpha', \mathbf{d}')\,.$$

Indeed, let $\Phi' = \{\mathcal{G}'_j\}_{j\in\mathbb{I}_{m'}} \in \mathcal{D}(\alpha', \mathbf{d}')$ be a family (arbitrarily) close to $\tilde{\Phi}'$ such that $\Psi'_\varphi(\Phi') < \Psi'_\varphi(\tilde{\Phi}')$. Then, the family $\Phi = \{\mathcal{G}_j\}_{j\in\mathbb{I}_m}$, obtained by appending the vectors $\{\tilde{f}_{i,j}\}_{i\in\mathbb{I}_{i_1}}$ to $\mathcal{G}'_j$, when $j \in M'$ jointly with $\{\mathcal{F}_j\}_{j:p_j\le 1}$, is in $\mathcal{D}(\alpha, \mathbf{d})$ and it is (arbitrarily) close to $\tilde{\Phi}$. By construction we have that

$$\sum_{j\in M'} \operatorname{tr}(\varphi[S_{(\mathcal{F}^0_j,\mathcal{G}_j)}]) = \sum_{j\in M'} \operatorname{tr}(\varphi[S_{((\mathcal{F}^0_j)',\mathcal{G}'_j)}]) + \sum_{j\in M'} \operatorname{tr}(\varphi[S_{(\mathcal{F}^0_j,\tilde{\mathcal{F}}_j)}|_{V_j^\perp}])\,.$$

Using the previous identities, and that $\mathbb{I}_m = \{j \ : \ p_j \le 1\} \cup M'$ is a (weak) partition, we get

$$\begin{aligned}
\Psi_\varphi(\Phi) &= \sum_{j:p_j\le 1} \operatorname{tr}(\varphi[S_{(\mathcal{F}^0_j,\tilde{\mathcal{F}}_j)}]) + \sum_{j\in M'} \operatorname{tr}(\varphi[S_{((\mathcal{F}^0_j)',\mathcal{G}'_j)}]) + \sum_{j\in M'} \operatorname{tr}(\varphi[S_{(\mathcal{F}^0_j,\tilde{\mathcal{F}}_j)}|_{V_j^\perp}]) \\
&< \sum_{j:p_j\le 1} \operatorname{tr}(\varphi[S_{(\mathcal{F}^0_j,\tilde{\mathcal{F}}_j)}]) + \sum_{j\in M'} \operatorname{tr}(\varphi[S_{((\mathcal{F}^0_j)',\tilde{\mathcal{F}}'_j)}]) + \sum_{j\in M'} \operatorname{tr}(\varphi[S_{(\mathcal{F}^0_j,\tilde{\mathcal{F}}_j)}|_{V_j^\perp}]) \\
&= \sum_{j:p_j\le 1} \operatorname{tr}(\varphi[S_{(\mathcal{F}^0_j,\tilde{\mathcal{F}}_j)}]) + \sum_{j\in M'} \operatorname{tr}(\varphi[S_{(\mathcal{F}^0_j,\tilde{\mathcal{F}}_j)}]) = \Psi_\varphi(\tilde{\Phi})
\end{aligned}$$

which contradicts the local minimality of $\tilde{\Phi}$ in $\mathcal{D}(\alpha, \mathbf{d})$ (where we have used that the spectrum of the restriction of $\mathcal{S}_{(\mathcal{F}^0_j,\tilde{\mathcal{F}}_j)}$ to V_j is the spectrum of $\mathcal{S}_{((\mathcal{F}^0_j)',\tilde{\mathcal{F}}'_j)}$ for $j \in M'$). In particular, $p' = p_{\tilde{\Phi}'} = p-1$ so we can proceed inductively to complete the proof: notice, for example, that the largest constant c'_1 of $\tilde{\Phi}'$ is c_2 and we can apply Proposition 3.5 and Corollary 3.7 to the local minimizer $\tilde{\Phi}' \in \mathcal{D}(\alpha', \mathbf{d}')$ and conclude the desired properties about the constant c_2 and the index i_2.

Finally we show item 5 : in case $p = 1$ then we have set $i_0 := 0$. In case $p \ge 2$, let $j \in \mathbb{I}_m$ be such that $p_j = p$; hence, by item 3 above, $\dim W_{\ell,j} = r_{\ell_j} = i_\ell - i_{\ell-1}$, for $\ell \in \mathbb{I}_{p-1}$. Further, $W_{p,j} \neq \{0\}$ and therefore, $r_{p,j} = \dim W_{p,j} \ge 1$. Since the subspaces $W_{\ell,j} \subset \mathbb{C}^{d_j}$ are mutually orthogonal we now see that $\sum_{\ell\in\mathbb{I}_p} r_{\ell,j} \le d_j$ and hence,

$$i_{p-1} = \sum_{\ell\in\mathbb{I}_{p-1}} i_\ell - i_{\ell-1} = \sum_{\ell\in\mathbb{I}_{p-1}} r_{\ell,j} < d_j \le d_1\,.$$

□

In order to simplify the notation in the statements below, given a vector $(\delta_i)_{i\in\mathbb{I}_n} \in \mathbb{R}^n$ and indices $1 \le s \le t \le n$ in what follows we consider $(\delta_i)_{i=s}^t = (\delta_s, \ldots, \delta_t) \in \mathbb{R}^{t-s+1}$.

Corollary 3.9. *Consider the notation of Theorem 3.8. Let $r_\ell = i_\ell - i_{\ell-1}$ for $\ell \in \mathbb{I}_{p-1}$, let $r_p = d_1 - i_{p-1} \ge 1$ and set*

$$\mathbf{c} = (c_1 \mathbb{1}_{r_1}\,,\, \ldots\,,\, c_p \mathbb{1}_{r_p}) \in (\mathbb{R}^{d_1}_{>0})^{\downarrow}\,.$$

1. *If $j \in \mathbb{I}_m$ is such that $p_j \geq 1$ and $\nu_j \in \mathbb{R}_{\geq 0}^{d_j}$ is given by*

$$\nu_j = \max\{\mathbf{c}, \lambda_j\} = (c_1 \mathbb{1}_{r_1}, \ldots, c_{p_j-1} \mathbb{1}_{r_{p_j-1}}, (\max\{c_{p_j}, \lambda_{i,j}\})_{i_{p_j-1}+1 \leq i \leq d_j}) \tag{21}$$

then $\nu_j^\downarrow = \lambda(S_{(\mathcal{F}_j^0, \tilde{\mathcal{F}}_j)})$; in case $p_j = 0$, then $\tilde{\mathcal{F}}_j = \{0\}_{i \in \mathbb{I}_n}$.

2. *If $j \in \mathbb{I}_m$ is such that $p_j \geq 1$ then*

$$\lambda(S_{\tilde{\mathcal{F}}_j}) = \bigoplus_{\ell=1}^{p_j-1} (c_\ell - \lambda_{i,j})_{i=i_{\ell-1}+1}^{i_\ell} \oplus ((c_{p_j} - \lambda_{i,j})^+)_{i=i_{p_j-1}+1}^{d_j} \in (\mathbb{R}_{\geq 0}^{d_j})^\downarrow . \tag{22}$$

Proof. Item 1 follows from Eq. (9) together with Theorem 3.8. On the other hand, let $j \in \mathbb{I}_m$ be such that $p_j \geq 1$; then, Eq. (8) in Theorem 3.2 shows that $\lambda_{i,j} + \lambda_i(S_{\tilde{\mathcal{F}}_j}) = c_\ell$, for $i_{\ell-1} + 1 \leq i \leq i_\ell$, for $\ell \in \mathbb{I}_{p_j-1}$ and $\lambda_{i,j} + \lambda_i(S_{\tilde{\mathcal{F}}_j}) = c_{p_j}$, for $i_{p_j-1} + 1 \leq i \leq r_j^*$, where $r_j^* = \operatorname{rk}(S_{\tilde{\mathcal{F}}_j})$; moreover, recall that in this case $c_{p_j} = \min \sigma(S_{(\mathcal{F}_j^0, \tilde{\mathcal{F}}_j)})$. If $p_j = 0$ then $\tilde{\mathcal{F}}_j = \{0\}_{i \in \mathbb{I}_n}$, by definition of p_j, so $S_{\tilde{\mathcal{F}}_j} = 0$.

The representation in Eq. (22) (including the claim that the vector is downward ordered) now follows from the previous identities together with the fact that $c_1 > \ldots > c_p > 0$ and $(\lambda_{i,j})_{i \in \mathbb{I}_{d_j}} \in (\mathbb{R}_{\geq 0}^{d_j})^\uparrow$. □

3.2 Characterization of the vector c.

We can combine now Theorem 3.2 and Theorem 2.5 to obtain a more precise description of the spectra $\{\lambda(S_{\tilde{\mathcal{F}}_j})\}_{j \in \mathbb{I}_m}$ in terms of majorization (see Definition 2.1).

Lemma 3.10. *Consider the notation of Theorem 3.8. In particular, let $\mathbf{c} = (c_1 \mathbb{1}_{r_1}, \ldots, c_p \mathbb{1}_{r_p}) \in \mathbb{R}^{d_1}$, with $c_1 > \cdots > c_p > 0$ and $0 = i_0 < i_1 < \cdots < i_p = n$ such that $r_k = i_k - i_{k-1}$, for $k \in \mathbb{I}_{p-1}$, and $r_p = d_1 - i_{p-1}$. Then,*

1. *For each $k \in \mathbb{I}_{p-1}$,*

$$(\alpha_i)_{i=i_{k-1}+1}^{i_k} \prec \left(\sum_{j\,:\, i \leq d_j} (c_k - \lambda_{i,j})^+ \right)_{i=i_{k-1}+1}^{i_k} . \tag{23}$$

2. *If $k = p$, so $i_p = n$*

$$(\alpha_i)_{i=i_{p-1}+1}^{n} \prec \left(\sum_{j\,:\, i \leq d_j} (c_p - \lambda_{i,j})^+ \right)_{i=i_{p-1}+1}^{d_1} . \tag{24}$$

Proof. Let $k \in \mathbb{I}_{p-1}$ and set $m' = \#\{j \in \mathbb{I}_m : i_{k-1} < d_j\}$. For each $j \in \mathbb{I}_m$ such that $i_{k-1} < d_j$, we define $d'_j = \min\{r_k, (d_j - i_{k-1})\}$.

By Theorems 3.2 and 3.8 we have that (after a proper renaming of the subscripts) if $\mathbf{d}' = (d'_j)_{j \in \mathbb{I}_{m'}}$ and $\alpha' = (\alpha_i)_{i_{k-1}+1}^{i_k}$, then the m'-tuple $\tilde{\Phi}' = \{\tilde{\mathcal{F}}'_j\}_{j \in \mathbb{I}_{m'}}$ where $\tilde{\mathcal{F}}'_j = \{\tilde{f}_{i,j}\}_{i=i_{k-1}+1}^{i_k}$, is an $(\alpha', \mathbf{d}')$-design. Then, the first item follows from Theorem 2.5 since $\lambda(S_{\tilde{\mathcal{F}}'_j}) = ((c_k - \lambda_{i,j})^+)_{i=i_{k-1}+1}^{i_k}$ by Theorem 3.2 and Corollary 3.9. In case $k = p$ then $i_{p-1} < d_1$ and the assertion in item 2 can be proved similarly. □

Remark 3.11. Consider the notation of Theorem 3.8. A careful look at the formulae in Lemma 3.10 tells us that the constants $c_1 > \ldots > c_p > 0$ are uniquely determined by the indices $0 = i_0 < i_1 <$

$\cdots < i_p = n$ and the initial data: $\lambda^\uparrow(S_{\mathcal{F}_j^0}) = (\lambda_{i,j})_{i\in\mathbb{I}_{d_j}} \in (\mathbb{R}^{d_j}_{\geq 0})^\uparrow$, $j \in \mathbb{I}_m$, and $\alpha = (\alpha_i)_{i\in\mathbb{I}_n} \in (\mathbb{R}^n_{>0})^\downarrow$. Indeed, if we denote

$$A_k = \sum_{j=i_{k-1}+1}^{i_k} \alpha_j\,, \quad \text{for} \quad k \in \mathbb{I}_p$$

we must have: for $k \in \mathbb{I}_{p-1}$,

$$\sum_{i=i_{k-1}+1}^{i_k} \sum_{\{j\,:\,i\leq d_j\}} (c_k - \lambda_{i,j})^+ = A_k \quad \text{ and } \quad \sum_{i=i_{p-1}+1}^{d_1} \sum_{\{j\,:\,i\leq d_j\}} (c_k - \lambda_{i,j})^+ = A_p\,.$$

Now, consider the functions $b_k(x) : [a_k\,,\,\infty) \to \mathbb{R}_{\geq 0}$, for $k \in \mathbb{I}_{p-1}$ and $b_p(x) : [a_p\,,\,\infty) \to \mathbb{R}_{\geq 0}$ by

$$b_k(x) = \sum_{i=i_{k-1}+1}^{i_k} \sum_{\{j\,:\,i\leq d_j\}} (x - \lambda_{i,j})^+ \quad \text{ and } \quad b_p(x) = \sum_{i=i_{p-1}+1}^{d_1} \sum_{\{j\,:\,i\leq d_j\}} (x - \lambda_{i,j})^+$$

where $a_h = \min\{\lambda_{i_{h-1}+1,j} \;:\; i_{h-1}+1 \leq d_j\} \geq 0$ for $h \in \mathbb{I}_p$. Then, for $k \in \mathbb{I}_p$ we have that

1. $b_k(x) : [a_k\,,\,\infty) \to \mathbb{R}_{\geq 0}$ is a piece-wise linear, continuous and strictly increasing function;
2. $b_k(a_k) = 0$, $\lim_{x\to+\infty} b_k(x) = +\infty$.

Hence, c_k is determined as the unique solution of the equation $b_k(x) = A_k > 0$, with $x \geq a_k$, for $k \in \mathbb{I}_p$. It is worth pointing out that this equation can be solved by a simple and fast algorithm whose input are the eigenvalues $\lambda_{i,j}$ for $i \in \mathbb{I}_{d_j}$, $j \in \mathbb{I}_m$. $\triangle$

The following result establishes the uniqueness of the indices $0 = i_0 < i_1 < \cdots < i_p = n$ as above.

Theorem 3.12. *Let* $0 = i_0 < i_1 < \cdots < i_p = n$ *and* $\mathbf{c} = (c_1\mathbb{1}_{r_1}\,,\,\ldots\,,\,c_p\mathbb{1}_{r_p})$ *be as in Theorem 3.8. Let* $0 = j_0 < j_1 < \cdots < j_q = n$ *be indices, with* $j_{q-1} < d_1$, $e_1 > \cdots > e_q > 0$, *and set* $\mathbf{e} = (e_1\mathbb{1}_{s_1}\,,\,\ldots\,,\,e_q\mathbb{1}_{s_q}) \in \mathbb{R}^{d_1}$, *with* $s_k = j_k - j_{k-1}$, *for* $k \in \mathbb{I}_{q-1}$ *and* $s_q = d_1 - j_{q-1} \geq 1$. *Suppose that*

1. *For each* $k \in \mathbb{I}_{q-1}$,
$$(\alpha_i)_{i=j_{k-1}+1}^{j_k} \prec \left(\sum_{j\,:\,i\leq d_j} (e_k - \lambda_{i,j})^+\right)_{i=j_{k-1}+1}^{j_k}.$$
2. *For* $k = q$, *so* $j_q = n$
$$(\alpha_i)_{i=j_{q-1}+1}^{n} \prec \left(\sum_{j\,:\,i\leq d_j} (e_q - \lambda_{i,j})^+\right)_{i=j_{q-1}+1}^{d_1}.$$

Then, $p = q$, $e_k = c_k$, $i_k = j_k$ *and* $r_k = s_k$ *for every* $k \in \mathbb{I}_p$. *In particular,* $\mathbf{e} = \mathbf{c}$.

Proof. Let us assume, without loss of generality, that $j_{q-1} \leq i_{p-1}$. Since $i_p = j_q = n$ and

$$(\alpha_i)_{i=j_{q-1}+1}^{n} \prec \left(\sum_{j\,:\,i\leq d_j} (e_q - \lambda_{i,j})^+\right)_{i=j_{q-1}+1}^{d_1},$$

we have

$$\sum_{i=i_{p-1}+1}^{n} \alpha_i \geq \sum_{i=i_{p-1}+1}^{d_1} \sum_{\{j\,:\,i\leq d_j\}} (e_q - \lambda_{i,j})^+ . \tag{25}$$

On the other side, we have that

$$\sum_{i=i_{p-1}+1}^{n} \alpha_i = \sum_{i=i_{p-1}+1}^{d_1} \sum_{\{j\,:\,i\leq d_j\}} (c_p - \lambda_{i,j})^+ . \tag{26}$$

From (25) and (26) we deduce that $c_p \geq e_q$. In particular, $c_k > e_q$, $k \in \mathbb{I}_{p-1}$.

Now, suppose that $j_{q-1} < i_{p-1}$; in this case $p \geq 2$, and let $0 \leq \ell \leq p-2$ be such that $i_\ell \leq j_{q-1} < i_{\ell+1} \leq i_{p-1}$. Then,

$$\begin{aligned} \sum_{i=j_{q-1}+1}^{n} \alpha_i &= \sum_{i=j_{q-1}+1}^{d_1} \sum_{\{j\,:\,i\leq d_j\}} (e_q - \lambda_{i,j})^+ \\ &\overset{(*)}{<} \sum_{i=j_{q-1}+1}^{i_{\ell+1}} \sum_{\{j\,:\,i\leq d_j\}} (c_\ell - \lambda_{i,j})^+ + \sum_{h=\ell+1}^{p-2} \left(\sum_{i=i_h+1}^{i_{h+1}} \sum_{\{j\,:\,i\leq d_j\}} (c_h - \lambda_{i,j})^+ \right) + \sum_{i=i_{p-1}+1}^{d_1} \sum_{\{j\,:\,i\leq d_j\}} (c_h - \lambda_{i,j})^+ \\ &\overset{(**)}{\leq} \sum_{i=j_{q-1}+1}^{i_{\ell+1}} \alpha_i + \sum_{i=i_{\ell+1}+1}^{i_{p-1}} \alpha_i + \sum_{i=i_{p-1}+1}^{n} \alpha_i = \sum_{i=j_{q-1}+1}^{n} \alpha_i \ , \end{aligned}$$

where the strict inequality $(*)$ follows from the fact that $c_p \geq e_q$, $c_k > e_q$ for $k \in \mathbb{I}_{p-1}$ and $\ell \leq p-2$ (so that $c_\ell > \lambda_{i,1}$ for $i_\ell + 1 \leq i \leq i_{\ell+1}$, since $i \leq i_{\ell+1} < d_1$) and the inequality $(**)$ follows from the majorization relations given by Lemma 3.10. Indeed, in order to show $(**)$ notice that the inequalities hold in each term: in the first term we use the majorization of Eq. (23) with $k-1=\ell$, with the inequality reversed since we are summing the last entries of the vectors ($i_\ell \leq j_{q-1} < i_{\ell+1}$, see Remark 6.1); in the second and third terms we have equalities since they are traces (complete sums) and we use the majorization in Eqs. (23) and (24). The previous contradiction shows that we should have $j_{q-1} = i_{p-1}$. Thus, we also have $r_p = s_q$. Notice that, by Remark 3.11, the respective constants e_q and c_p must coincide too. Therefore, in case that $p=1$, then $q=1$ and we are done.

If $p \geq 2$ (and $q \geq 2$), we can proceed inductively using the vectors $\mathbf{c} = (c_1 \mathbb{1}_{r_1}\, , \, \ldots \, , \, c_{p-1} \mathbb{1}_{r_{p-1}})$ and $\mathbf{e} = (e_1 \mathbb{1}_{s_1}\, , \, \ldots \, , \, e_{q-1} \mathbb{1}_{s_{q-1}})$ along with $(\alpha_i)_{i=1}^{i_{p-1}}$ and $(\lambda_{i,j})_{i=1}^{i_{p-1}}$ for those $j \in \mathbb{I}_m$ such that $i_{p-1} \leq d_j$, in order to finish the proof (notice that these vectors satisfy the hypothesis of the statement). □

4 Proof of the main results and computation of optimal multi-completions

We begin this section by proving Theorem 2.10, which is our main result. Then, we show that the parameters of the vector $\mathbf{c}$ constructed in the previous section can be computed in terms of a simple and fast procedure, that depends on the initial data α and Φ^0. This last result will be used in Section 5.2 to obtain a simple and fast algorithm for the construction of concrete optimal $(\alpha\, , \, \mathbf{d})$-multi-completions of Φ^0.

4.1 Proof of Theorem 2.10

Next, we re-state Theorem 2.10 for convenience and give a detailed proof of this result.

Theorem 2.10. Let $(\Phi^0 = (\mathcal{F}_j^0)_{j\in\mathbb{I}_m}\, , \, \alpha\, , \, \mathbf{d})$ be the initial data for a multi-completion problem, with $n \geq d_1$. Let $\lambda_j = (\lambda_{i,j})_{i\in\mathbb{I}_{d_j}} = \lambda^\uparrow(S_{\mathcal{F}_j^0}) \in (\mathbb{R}_{\geq 0}^{d_j})^\uparrow$, for $j \in \mathbb{I}_m$.

Then there exist $p \in \mathbb{N}$, constants $c_1 > \cdots > c_p > 0$, indices $0 = i_0 < i_1 < \ldots < i_p = n$, with $i_{p-1} < d_1$ and the associated vector $\mathbf{c} = (c_1 \mathbb{1}_{r_1}, \ldots, c_p \mathbb{1}_{r_p}) \in (\mathbb{R}^{d_1}_{>0})^\downarrow$ as in Eq. (6) such that, if we consider the vectors $\nu_j = \max\{\lambda_j, \mathbf{c}\} \in \mathbb{R}^{d_j}_{>0}$ for any $j \in \mathbb{I}_m$, they have the following property:

For every completion $\tilde{\Phi} = (\tilde{\mathcal{F}}_j)_{j\in\mathbb{I}_m} \in \mathcal{D}(\alpha, \mathbf{d})$ and any map $\varphi \in \mathrm{Conv_s}(\mathbb{R}_{\geq 0})$,

$$\tilde{\Phi} \text{ is a local minimizer of } \Psi_\varphi \text{ on } \mathcal{D}(\alpha, \mathbf{d}) \quad \iff \quad \lambda(S_{(\mathcal{F}_j^0, \tilde{\mathcal{F}}_j)}) = \nu_j^\downarrow \ , \ \ j \in \mathbb{I}_m \, .$$

In this case,

1. $\tilde{\Phi}$ is a global minimizer of Ψ_ϕ, for every $\phi \in \mathrm{Conv_s}(\mathbb{R}_{\geq 0})$.
2. For $j \in \mathbb{I}_m$, $\min \sigma(S_{(\mathcal{F}_j^0, \tilde{\mathcal{F}}_j)}) > 0$; in particular, $(\mathcal{F}_j^0, \tilde{\mathcal{F}}_j)$ is a frame for $\mathbb{C}^{d_j}$.
3. For $j \in \mathbb{I}_m$ such that $\tilde{\mathcal{F}}_j$ has non zero vectors, there exists $\mathcal{B}_j = \{v_{i,j}\}_{i\in\mathbb{I}_{d_j}}$ an ONB of $\mathbb{C}^{d_j}$ such that
$$S_{\mathcal{F}_j^0} = \sum_{i\in\mathbb{I}_{d_j}} \lambda_{i,j} \, v_{i,j} \otimes v_{i,j} \quad \text{and}$$
$$S_{\tilde{\mathcal{F}}_j} = \sum_{\ell=1}^{p_j-1} \sum_{i=i_{\ell-1}+1}^{i_\ell} (c_\ell - \lambda_{i,j}) \, v_{i,j} \otimes v_{i,j} + \sum_{i=i_{p_j-1}+1}^{d_j} (c_{p_j} - \lambda_{i,j})^+ \, v_{i,j} \otimes v_{i,j} \, ,$$
where $1 \leq p_j \leq p$ is uniquely determined by the condition: $c_{p_j} = (\mathbf{c})_{\ell_j}$, where $\ell_j := \max\{k \in \mathbb{I}_{d_j} : (\mathbf{c})_k > \lambda_{k,j}\}$.

Proof. Let $\tilde{\Phi} = \{\tilde{\mathcal{F}}_j\}_{j\in\mathbb{I}_m}$ be a local minimizer of Ψ_φ on $\mathcal{D}(\alpha, \mathbf{d})$. By Theorem 3.8 and Corollary 3.9 there exists a vector $\mathbf{c} \in (\mathbb{R}^{d_1}_{>0})^\downarrow$ such that, if we construct $\nu_j = \max\{\lambda_j, \mathbf{c}\} \in \mathbb{R}^{d_j}_{>0}$ (entry-wise maximum) as above, we have $\lambda(S_{(\mathcal{F}_j^0, \tilde{\mathcal{F}}_j)}) = \nu_j$, for $j \in \mathbb{I}_m$. Moreover, by Lemma 3.10 and Theorem 3.12 such vector $\mathbf{c}$ is unique and independent of the function $\varphi \in \mathrm{Conv_s}(\mathbb{R}_{\geq 0})$, so $\tilde{\Phi}$ is actually a global minimizer of Ψ_φ and it is also a global minimizer of Ψ_ϕ, for every $\phi \in \mathrm{Conv_s}(\mathbb{R}_{\geq 0})$. On the other hand, any $\Phi = \{\mathcal{F}_j\}_{j\in\mathbb{I}_m} \in \mathcal{D}(\alpha, \mathbf{d})$ with $\lambda(S_{(\mathcal{F}_j^0, \mathcal{F}_j)}) = \nu_j$, for $j \in \mathbb{I}_m$, satisfies that $\Psi_\varphi(\Phi) = \Psi_\varphi(\tilde{\Phi})$ and hence is a global minimizer of Ψ_φ. By Propositions 3.4 and 3.5 the sequences $(\mathcal{F}_j^0, \tilde{\mathcal{F}}_j)$ are frames in $\mathbb{C}^{d_j}$ (when $\tilde{\mathcal{F}}_j$ has non-zero vectors as well as when $\tilde{\mathcal{F}}_j = \{0\}_{j\in\mathbb{I}_n}$) for $j \in \mathbb{I}_m$. The rest of the claims follow from Theorem 3.2. □

Notice that Theorem 2.10 only applies when the number n of weights satisfies $n \geq d_1 = \max\{d_j : j \in \mathbb{I}_m\}$. In the next remark we explain how we can handle the case when $n < d_1$.

Remark 4.1. Let $(\Phi^0 = (\mathcal{F}_j^0)_{j\in\mathbb{I}_m}, \alpha, \mathbf{d})$ be the initial data for a multi-completion problem and assume that $n < d_1$. In this case, we set

$$j_0 = \max\{j \in \mathbb{I}_m \ : \ d_j > n\} \geq 1 \, .$$

Let $\lambda_j = (\lambda_{i,j})_{i\in\mathbb{I}_{d_j}} = \lambda^\uparrow(S_{\mathcal{F}_j^0}) \in (\mathbb{R}^{d_j}_{\geq 0})^\uparrow$, for $j \in \mathbb{I}_m$. Consider $\Phi = (\mathcal{F}_j)_{j\in\mathbb{I}_m} \in \mathcal{D}(\alpha, \mathbf{d})$ and notice that by Lidskii's inequality (Theorem 6.2) we get that

$$\lambda(S_{\mathcal{F}_j^0})^\uparrow + \lambda(S_{\mathcal{F}_j}) = \lambda_j + \lambda(S_{\mathcal{F}_j}) \prec \lambda(S_{\mathcal{F}_j^0} + S_{\mathcal{F}_j}) \, .$$

In particular, for $1 \leq j \leq j_0$ we have that $\lambda_i(S_{\mathcal{F}_j}) = 0$ for $n+1 \leq i \leq d_j$ and hence,

$$\lambda_j + \lambda(S_{\mathcal{F}_j}) = (\lambda_{1,j} + \lambda_1(S_{\mathcal{F}_j}), \ldots, \lambda_{n,j} + \lambda_n(S_{\mathcal{F}_j}), \underbrace{\lambda_{n+1,j}, \ldots, \lambda_{d_j,j}}_{d_j - n}) \prec \lambda(S_{\mathcal{F}_j^0} + S_{\mathcal{F}_j}) \, . \tag{27}$$

The previous facts motivate considering the following *reduced multi-completion problem*: for $1 \leq j \leq j_0$ let $\mathcal{B}_j = \{v_{i,j}\}_{i\in\mathbb{I}_{d_j}}$ be an ONB of $\mathbb{C}^{d_j}$ such that

$$S_{\mathcal{F}_j^0} = \sum_{i\in\mathbb{I}_{d_j}} \lambda_{i,j}\ v_{i,j}\otimes v_{i,j}$$

and set $\mathcal{V}_j = \text{Span}\{v_{i,j}\ :\ 1\leq i\leq n\}\subset\mathbb{C}^{d_j}$. Hence, if we let $P_j\in\mathcal{M}_{d_j}(\mathbb{C})^+$ denote the orthogonal projection onto $\mathcal{V}_j$ we consider $P_j\mathcal{F}_j^0 = \{P_j\, f_{i,j}^0\}_{i\in\mathbb{I}_k}\in(\mathcal{V}_j)^k$. By construction we see that

$$S_{P_j\,\mathcal{F}_j^0} = \sum_{i\in\mathbb{I}_n} \lambda_{i,j}\ v_{i,j}\otimes v_{i,j}\,.$$

For $1\leq j\leq j_0$ we now replace the initial finite sequence $\mathcal{F}_j^0$ and the subspace $\mathbb{C}^{d_j}$ by the finite sequence $P_j\mathcal{F}_j^0$ and the subspace $\mathcal{V}_j$, with $\dim\mathcal{V}_j = n$ (so that we can identify $\mathcal{V}_j\approx\mathbb{C}^n$). In this case, we consider the reduced multi-completion problem of the initial family $\tilde{\Phi}_0 = ((P_j\,\mathcal{F}_j^0)_{j=1}^{j_0}\,,\,(\mathcal{F}_j^0)_{j=j_0+1}^m)$ with weights given by $\alpha\in(\mathbb{R}_{>0}^n)^\downarrow$ and with the completing families $\Phi' = (\mathcal{F}_j')_{j\in\mathbb{I}_m}$ satisfying the weight restrictions and such that

$$\mathcal{F}_j' = \{f_{i,j}'\}_{i\in\mathbb{I}_n}\in(\mathcal{V}_j)^n\approx(\mathbb{C}^n)^n \quad\text{ for }\quad 1\leq j\leq j_0$$

and $\mathcal{F}_j' = \{f_{i,j}'\}_{i\in\mathbb{I}_n}\in(\mathbb{C}^{d_j})^n$, for $j_0+1\leq j\leq m$. Notice that the vector of dimensions of this reduced multi-completion problem is given by $\mathbf{d}' = (d_j')_{j\in\mathbb{I}_m} := ((n)_{j=1}^{j_0}, (d_j)_{j=j_0+1}^m)$, so formally $d_1'\leq n$. In this setting, given $\varphi\in\text{Conv}_\text{s}(\mathbb{R}_{\geq 0})$, we minimize the associated joint convex potential for the reduced multi-completion problem, given by

$$\Psi_\text{red}(\Phi') \overset{\text{def}}{=} \sum_{j=1}^{j_0}\sum_{i\in\mathbb{I}_n}\varphi(\lambda_i(S_{P_j\,\mathcal{F}_j^0}+S_{\mathcal{F}_j'})) + \sum_{j=j_0+1}^{m}\sum_{i\in\mathbb{I}_{d_j}}\varphi(\lambda_i(S_{\mathcal{F}_j^0}+S_{\mathcal{F}_j'}))\,, \tag{28}$$

where we have taken into consideration $d_j' = \dim\mathcal{V}_j = n$ in first j_0 terms above (notice that by construction we get that $\lambda_i(S_{P_j\,\mathcal{F}_j^0}+S_{\mathcal{F}_j'}) = 0$, for $n+1\leq i\leq d_j$, for $1\leq j\leq j_0$).

Notice that for $\Phi = (\mathcal{F}_j)_{j\in\mathbb{I}_m}\in\mathcal{D}(\alpha,\mathbf{d})$ as before and $1\leq j\leq j_0$, we can consider a unitary operator $U_j\in\mathcal{U}(d_j)$ such that U_j maps the subspace $\text{Span}(\mathcal{F}_j)$ into $\mathcal{V}_j$ in such a way that

$$S_{U_j\mathcal{F}_j} = U_jS_{\mathcal{F}_j}U^* = \sum_{i\in\mathbb{I}_n}\lambda_j(S_{\mathcal{F}_j})\,v_{i,j}\otimes v_{i,j} \tag{29}$$

where $U_j\mathcal{F}_j = \{U_j\,f_{i,j}\}_{i\in\mathbb{I}_n}$ (we can always do this, since $\dim\mathcal{V}_j = n$). Hence, $U_j\mathcal{F}_j\in(\mathcal{V}_j)^n$ is such that

$$\lambda(S_{P_j\,\mathcal{F}_j^0}+S_{U_j\,\mathcal{F}_j}) = (\lambda_{1,j}+\lambda_1(S_{\mathcal{F}_j}),\ldots,\lambda_{n,j}+\lambda_n(S_{\mathcal{F}_j}),\underbrace{0,\ldots,0}_{d_j-n}) \tag{30}$$

Taking into account Eqs. (27), (29) and (30) we see that for $\varphi\in\text{Conv}_\text{s}(\mathbb{R}_{\geq 0})$ and $1\leq j\leq j_0$ then

$$\text{tr}(\varphi[S_{\mathcal{F}_j^0}+S_{U_j\,\mathcal{F}_j}]) = \sum_{i\in\mathbb{I}_n}\varphi(\lambda_i[S_{P_j\,\mathcal{F}_j^0}+S_{U_j\,\mathcal{F}_j}]) + \sum_{i=n+1}^{d_j}\varphi(\lambda_{i,j}) \leq \text{tr}(\varphi[S_{(\mathcal{F}_j^0\,,\,\mathcal{F}_j)}])\,.$$

Hence, the family $\Phi' = ((U_j\,\mathcal{F}_j)_{j=1}^{j_0}\,,\,(\mathcal{F}_j)_{j=j_0+1}^m)$ is not only an $(\alpha\,,\,\mathbf{d})$-design (in the usual sense) but also an admissible completing family for the reduced multi-completion problem; moreover, the previous facts together with Eq. (28) show that

$$\Psi(\Phi') = \Psi_\text{red}(\Phi') + \sum_{j=1}^{j_0}\sum_{i=n+1}^{d_j}\varphi(\lambda_{i,j}) \leq \Psi(\Phi)\,.$$

We remark that our methods can be now applied to the reduced multi-completion problem (that is, minimizing $\Psi_{\rm red}$ with initial data given by $\tilde\Phi_0$, α and $\mathbf{d}'$, so that the corresponding eigenvalue lists are $(\lambda_{i,j})_{i\in\mathbb{I}_n}$, for $1\leq j\leq j_0$, and $(\lambda_{i,j})_{i\in\mathbb{I}_{d_j}}$ for $j_0+1\leq j\leq m$, with $d_1'\leq n$). If $\Phi_{\rm red}^{\rm op}$ is an optimal solution of the reduced multi-completion problem then the previous remarks show that $\Phi_{\rm red}^{\rm op}\in\mathcal{D}(\alpha\,,\,\mathbf{d})$ is also a minimizer of the joint convex potential Ψ_ϕ on $\mathcal{D}(\alpha\,,\,\mathbf{d})$, for every $\phi\in{\rm Conv_s}(\mathbb{R}_{\geq 0})$. Using these remarks together with Theorem 2.10 we can obtain a description of these optimal families $\Phi_{\rm red}^{\rm op}$. We remark that in the general case, the completed families $(\Phi_0\,,\,\Phi_{\rm red}^{\rm op})$ are not formed by frames in the respective spaces $\mathbb{C}^{d_j}$ for $1\leq j\leq j_0$. △

4.2 On the construction of the vector c

Let $(\Phi^0=(\tilde{\mathcal{F}}_j^0)_{j\in\mathbb{I}_m}\,,\,\alpha\,,\,\mathbf{d})$ be the initial data for a multi-completion problem. Let

$$\lambda_j=(\lambda_{i,j})_{i\in\mathbb{I}_{d_j}}=\lambda^\uparrow(S_{\mathcal{F}_j^0})\in(\mathbb{R}_{\geq 0}^{d_j})^\uparrow\ ,\qquad\text{for}\qquad j\in\mathbb{I}_m\,.$$

In order to clarify the definitions below, throughout this section we will adopt the following notation: given $k\in\mathbb{I}_{d_1}$ then we denote $k^*=k$ if $k\in\mathbb{I}_{d_1-1}$ and $k^*=n$ if $k=d_1$.

Define i_1 in the following way:

$$i_1=\max\left\{k^*:k\in\mathbb{I}_{d_1}\,,\ (\alpha_i)_{i=1}^{k^*}\prec\left(\sum_{j\,:\,i\leq d_j}(c_{1,k}^*-\lambda_{i,j})^+\right)_{i=1}^{k}\right\}\ ,$$

where $c_{1,k}^*$ is the unique constant that satisfies the tracial condition:

$$\sum_{i=1}^{k^*}\alpha_i=\sum_{i=1}^{k}\sum_{j\,:\,i\leq d_j}(c_{1,k}^*-\lambda_{i,j})^+\qquad\text{for }k\in\mathbb{I}_{d_1}$$

(see Remark 3.11 for an explicit computation of this constant). Note that, in the case $k=d_1$ so $k^*=n$, the majorization in the definition of i_1 above compares vectors of different sizes (see Definition 2.1).

Suppose that $i_1<n$ (and hence $i_1<d_1$, by definition of i_1). Define i_2 as

$$i_2=\max\left\{k^*:\ i_1+1\leq k\leq d_1\ ,\ (\alpha_i)_{i=i_1+1}^{k^*}\prec\left(\sum_{j\,:\,i\leq d_j}(c_{2,k}^*-\lambda_{i,j})^+\right)_{i=i_1+1}^{k}\right\},$$

where as before we set $c_{2,k}^*$ as the unique constant that satisfies the tracial condition:

$$\sum_{i_1+1}^{k^*}\alpha_i=\sum_{i_1+1}^{k}\sum_{j\,:\,i\leq d_j}(c_{2,k}^*-\lambda_{i,j})^+$$

Lemma 4.2. *With the previous notation, suppose that $i_1<n$ (and hence $i_1<d_1$) and let i_2 be defined as before. Let $c_1=c_{1,i_1}^*$ and $c_2=c_{2,i_2}^*$. Then, $c_1>c_2$.*

Proof. Notice that, by construction, $i_2=k^*$ for some $i_1+1\leq k\leq d_1$. Hence, in this case, we have that $k=\min\{i_2\,,\,d_1\}$.

Suppose that $c_1\leq c_2$. Let $\tilde c$ be the unique positive constant such that

$$\sum_{i=1}^{i_2}\alpha_i=\sum_{i=1}^{\min\{i_2\,,\,d_1\}}\left(\sum_{j\,:\,i\leq d_j}(\tilde c-\lambda_{i,j})^+\right),$$

We claim that $c_1 \le \tilde{c} \le c_2$. Indeed, notice that by construction we have that

$$\sum_{i=1}^{i_1} \alpha_i = \sum_{i=1}^{i_1} \left(\sum_{j\,:\,i\le d_j} (c_1 - \lambda_{i,j})^+ \right) \quad \text{and}$$

$$\sum_{i=i_1+1}^{i_2} \alpha_i = \sum_{i=i_1+1}^{\min\{i_2\,,\,d_1\}} \left(\sum_{j\,:\,i\le d_j} (c_2 - \lambda_{i,j})^+ \right) .$$

If we assume that $\tilde{c} < c_1 \le c_2$, then

$$\begin{aligned}\sum_{i=1}^{i_2} \alpha_i &= \sum_{i=1}^{\min\{i_2\,,\,d_1\}} \left(\sum_{j\,:\,i\le d_j} (\tilde{c} - \lambda_{i,j})^+ \right) \\ &< \sum_{i=1}^{i_1} \left(\sum_{j\,:\,i\le d_j} (c_1 - \lambda_{i,j})^+ \right) + \sum_{i=i_1+1}^{\min\{i_2\,,\,d_1\}} \left(\sum_{j\,:\,i\le d_j} (c_2 - \lambda_{i,j})^+ \right) \\ &= \sum_{i=1}^{i_1} \alpha_i + \sum_{i=i_1+1}^{i_2} \alpha_i \,,\end{aligned}$$

which is a contradiction. A similar argument allows us to show that $\tilde{c}$ can not be greater than c_2, i.e. $\tilde{c} \le c_2$. Therefore, since $(\alpha_i)_{i=1}^{i_1} \prec \left(\sum_{j\,:\,i\le d_j} (c_1 - \lambda_{i,j})^+\right)_{i=1}^{i_1}$, and $c_1 \le \tilde{c}$ then, for any $1 \le s \le i_1 < d_1$ we have that

$$\sum_{i=1}^{s} \left(\sum_{j\,:\,i\le d_j} (\tilde{c} - \lambda_{i,j})^+ \right) \ge \sum_{i=1}^{s} \left(\sum_{j\,:\,i\le d_j} (c_1 - \lambda_{i,j})^+ \right) \ge \sum_{i=1}^{s} \alpha_i \,. \tag{31}$$

Similarly, for $i_1 + 1 \le s \le i_2$ (see Remark 6.1)

$$\sum_{i=s}^{i_2} \alpha_i \ge \sum_{i=s}^{\min\{i_2\,,\,d_1\}} \left(\sum_{j\,:\,i\le d_j} (c_2 - \lambda_{i,j})^+ \right) \ge \sum_{i=s}^{\min\{i_2\,,\,d_1\}} \left(\sum_{j\,:\,i\le d_j} (\tilde{c} - \lambda_{i,j})^+ \right) . \tag{32}$$

Notice that, (31) along with (32) and definition of $\tilde{c}$ yield

$$(\alpha_i)_{i=1}^{i_2} \prec \left(\sum_{j\,:\,i\le d_j} (\tilde{c} - \lambda_{i,j})^+ \right)_{i=1}^{\min\{i_2\,,\,d_1\}} ,$$

which contradicts the definition of i_1. Then, we must have $c_1 > c_2$.

□

Remark 4.3. The construction of the *unique* partition $i_1 < i_2 < \cdots < i_p = n$ that defines the vector $\mathbf{c}$ (see Theorem 3.12) can be obtained by iterating the above process. Notice that, in each step such that $i_k < d_1$, the constant c_k is strictly greater than the next constant c_{k+1}. △

5 Applications and numerical examples

In this section, we consider some applications of our main results on optimal multi-completions in different contexts of frame theory. We describe some natural problems (related to finitely generated shift-invariant subspaces, distributed sensor allocation, existence of tight multi-completions of initial designs and simultaneous matrix approximations) and show that they can be translated into the context of multi-completion problems with prescribed weights. We also outline an algorithmic procedure that computes optimal multi-completions with prescribed weights and we include numerical examples obtained by the implementation of this procedure.

5.1 Applications of multi-completion problems

In what follows we describe some relations between multi-completion problems and other problems in frame theory. These relations serve as different motivations (based on reformulations of the multi-completion problem) and provide natural contexts for applications of our main results.

Convex potentials in finitely shift generated spaces. Consider $L^2(\mathbb{R}^d)$ (with respect to Lebesgue measure) as a separable and complex Hilbert space. Recall that a closed subspace $\mathcal{V} \subseteq L^2(\mathbb{R}^d)$ is *shift-invariant* (SI) if $f \in \mathcal{V}$ implies $T_\ell f \in \mathcal{V}$ for any $\ell \in \mathbb{Z}^d$, where $T_y f(x) = f(x-y)$ is the translation by $y \in \mathbb{R}^d$. For example, take a subset $\mathcal{G} \subset L^2(\mathbb{R}^d)$ and set

$$\mathcal{S}(\mathcal{G}) = \overline{\text{span}}\,\{T_\ell f : \ f \in \mathcal{G}\,,\ \ell \in \mathbb{Z}^d\}\,.$$

Then, $\mathcal{S}(\mathcal{G})$ is a SI subspace called the (closed) *SI subspace generated by* $\mathcal{G}$. We say that $\mathcal{V}$ is *finitely shift generated* (FSI) if there exists a finite family $\mathcal{G} = \{g_i\}_{i\in\mathbb{I}_m} \subset L^2(\mathbb{R}^d)$ such that $\mathcal{V} = \mathcal{S}(\mathcal{G})$.

Let $\mathcal{V} = \mathcal{S}(\mathcal{G}) \subset L^2(\mathbb{R}^d)$ be an FSI subspace as above. Given a finite sequence $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n}$ in $\mathcal{V}$, we consider the sequence of integer translates of $\mathcal{F}$ given by $E(\mathcal{F}) = \{T_\ell\, f_i\}_{(\ell,\,i)\,\in\,\mathbb{Z}^d\times\mathbb{I}_n}$. In this context, there is a fundamental decomposition theory (see [7] for a detailed exposition of the technical results) that allows us to study the properties of shift-generated sequences. Indeed, let $\mathbb{T}^d = [0,1)^d$ and consider $\Gamma : L^2(\mathbb{R}^d) \to L^2(\mathbb{T}^d, \ell^2(\mathbb{Z}^d))$ defined for $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ by

$$\Gamma f : \mathbb{T}^d \to \ell^2(\mathbb{Z}^d)\ , \quad \Gamma f(x) = (\hat{f}(x+\ell))_{\ell\in\mathbb{Z}^d} \quad \text{for} \quad x \in \mathbb{T}^d\,,$$

where $\hat{f}$ denotes the Fourier transform. Then, Γ extends uniquely to an isometric isomorphism between $L^2(\mathbb{R}^d)$ and $L^2(\mathbb{T}^d, \ell^2(\mathbb{Z}^d))$. For $x \in \mathbb{T}^d$ we let $J_\mathcal{V}(x)$ be the finite dimensional subspace of $\ell^2(\mathbb{Z}^d)$ given by $J_\mathcal{V}(x) = \text{span}\,\{\Gamma g_i(x) : \ i \in \mathbb{I}_m\} \subset \ell^2(\mathbb{Z}^d)$. Then, we get the representation

$$\mathcal{V} = \{f \in L^2(\mathbb{R}^d) : \Gamma f(x) \in J_\mathcal{V}(x) \ \text{ for a.e. } \ x \in \mathbb{T}^d\}.$$

We denote by $\text{Spec}_j(\mathcal{V}) = \{x \in \mathbb{T}^d : \dim J_\mathcal{V}(x) = j\}$, for $j \in \mathbb{I}_m$. Let $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n}$ be a finite family in $\mathcal{V}$ such that $E(\mathcal{F})$ is a Bessel sequence in $L^2(\mathbb{R}^d)$, and let $S = S_{E(\mathcal{F})} \in B(\mathcal{H})$ be the (well defined, bounded, positive semidefinite) frame operator of $E(\mathcal{F})$. Then, there exists a (uniquely determined, essentially bounded) measurable field of operators $\mathbb{T}^d \ni x \mapsto \hat{S}_x \in B(J_\mathcal{V}(x))$ such that

$$\hat{S}_x\big(\Gamma f(x)\big) = \Gamma(S\,f)(x) \quad \text{ for } \ f \in \mathcal{V}\,,\, x \in \mathbb{T}^d\,.$$

It turns out that the properties of $E(\mathcal{F})$ can be understood in terms of the measurable field $\hat{S}_{(\cdot)}$. For example, $E(\mathcal{F})$ is a Parseval frame for $\mathcal{V}$ (i.e. S coincides with the orthogonal projection onto $\mathcal{V}$) if and only if $\hat{S}_x = I_{J_\mathcal{V}(x)}$ (where $I_{J_\mathcal{V}(x)}$ is the identity operator acting on $J_\mathcal{V}(x)$) for a.e. $x \in \mathbb{T}^d$. More generally, $E(\mathcal{F})$ is a frame for $\mathcal{V}$ if there exist constants $A,\, B > 0$ such that $A\, I_{J_\mathcal{V}(x)} \leq \hat{S}_x \leq B\, I_{J_\mathcal{V}(x)}$ for a.e. $x \in \mathbb{T}^d$. Now, given an strictly convex function $\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $\varphi(0) = 0$ we let the *convex potential* of a Bessel sequence $E(\mathcal{F})$ associated with φ, denoted $\text{P}_\varphi(E(\mathcal{F}))$, be given by (see [2])

$$\text{P}_\varphi(E(\mathcal{F})) = \int_{\mathbb{T}^d} \text{tr}(\varphi(\hat{S}_x))\ dx\,,$$

where the trace is well defined since $\hat{S}_x$ acts on the finite dimensional space $J_\mathcal{V}(x)$, $x \in \mathbb{T}^d$. Fix a finite sequence $\mathcal{F}_0$ in $\mathcal{V}$ such that $E(\mathcal{F}_0)$ is a Bessel sequence. Let $(\alpha_i)_{i\in\mathbb{I}_n} \in \mathbb{R}^n_{>0}$ for some $n \geq m$ and denote by $\mathcal{D}(\alpha, \mathcal{V})$ the set of all $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n}$ in $\mathcal{V}$ such that $E(\mathcal{F})$ is a Bessel sequence and $\|f_i\|^2_{L^2(\mathbb{R}^d)} = \alpha_i$, for $i \in \mathbb{I}_n$. Then, it is natural to consider those sequences $E(\mathcal{F}^{\text{op}})$ for $\mathcal{F}^{\text{op}} \in \mathcal{D}(\alpha, \mathcal{V})$ that minimize the convex potential of the completed sequence $\text{P}_\varphi(E(\mathcal{F}_0), E(\mathcal{F}))$, among all $\mathcal{F} \in \mathcal{D}(\alpha, \mathcal{V})$. In the particular case $\mathcal{F}_0 = \{0\}$ then (see [3]) it turns out that there are indeed sequences $\mathcal{F}^{\text{op}} \in \mathcal{D}(\alpha, \mathcal{V})$ such that $(E(\mathcal{F}_0), E(\mathcal{F}^{\text{op}}))$ minimizes the convex potential among all such sequences:

moreover, such optimal completions have a *discrete structure* (see Theorems 4.7 and 4.8. and 4.13 in [3]). For the sake of simplicity, consider the case where the Lebesgue measure of each $\text{Spec}_j(\mathcal{V})$ is $1/m$, for $j \in \mathbb{I}_m$ (the case where these measures are rational can be treated in a similar way). In this case, by [3, Theorem 3.4], for each $1 \leq j \leq m$ there exists a frame $\mathcal{H}_j = \{h_{i,j}\}_{i \in \mathbb{I}_n}$ for $\mathbb{C}^j$ and a measurable field of isometric isometries $\text{Spec}_j(\mathcal{V}) \ni x \mapsto U_x \in B(\mathbb{C}^j, J_{\mathcal{V}}(x))$ for $j \in \mathbb{I}_m$ such that: if $f_i^{\text{op}} \in \mathcal{V}$ is uniquely determined by the condition

$$\Gamma f_i^{\text{op}}(x) = U_x \, h_{i,j} \quad \text{for} \quad x \in \text{Spec}_j(\mathcal{V}) \,,\, j \in \mathbb{I}_m \,,\, i \in \mathbb{I}_n \,,$$

then, $\mathcal{F}^{\text{op}} = \{f_i^{\text{op}}\}_{i \in \mathbb{I}_n} \in \mathcal{D}(\alpha, \mathcal{V})$ is such that $(E(\mathcal{F}_0), E(\mathcal{F}^{\text{op}}))$ minimizes the convex potential among all completions $(E(\mathcal{F}_0), E(\mathcal{F}))$ for $\mathcal{F} \in \mathcal{D}(\alpha, \mathcal{V})$. In this case, if $\hat{S}^{\text{op}}$ denotes the measurable field associated to $E(\mathcal{F}^{\text{op}})$, then $\hat{S}^{\text{op}}_x = U_x \, S_{\mathcal{H}_j} \, U_x^* \in B(J_{\mathcal{V}}(x))$, for $x \in \text{Spec}_j(\mathcal{V})$, $j \in \mathbb{I}_m$. Since the Lebesgue measure of $\text{Spec}_j(\mathcal{V})$ is $1/m$, for $j \in \mathbb{I}_m$, and $\{\text{Spec}_j(\mathcal{V})\}_{j \in \mathbb{I}_m}$ is a partition of $\mathbb{T}^d$, we have that (recall that by hypothesis $E(\mathcal{F}_0) = \{0\}$)

$$\text{P}_\varphi(E(\mathcal{F}_0), E(\mathcal{F}^{\text{op}})) = \int_{\mathbb{T}^d} \text{tr}(\varphi(\hat{S}^{\text{op}}_x)) \; dx = \sum_{j=1}^m \frac{1}{m} \; \text{tr}(\varphi(S_{\mathcal{H}_j})) = \frac{1}{m} \cdot \text{P}_\varphi((\mathcal{H}_j)_{j \in \mathbb{I}_m}) \,.$$

where the last member is (a multiple of) the joint convex potential of the $\mathbf{d}$-design $(\mathcal{H}_j)_{j \in \mathbb{I}_m}$ (here $\mathbf{d} = (m, \ldots, 1) \in \mathbb{N}^m$). Moreover, since Γ is an isometric isomorphism

$$\alpha_i = \|f_i^{\text{op}}\|^2 = \int_{\mathbb{T}^d} \|\Gamma f_i^{\text{op}}(x)\|^2 \; dx = \frac{1}{m} \sum_{j=1} \|h_{i,j}\|^2 \quad \text{for} \quad i \in \mathbb{I}_n \,.$$

Therefore, optimal $(\alpha, \mathbf{d})$-designs $(\mathcal{H}_j)_{j \in \mathbb{I}_m}$ give rise to optimal shift generated frames in $\mathcal{D}(\alpha, \mathcal{V})$ in this case. In the general case in which $\mathcal{F}_0 = \{f_i^0\}_{i \in \mathbb{I}_k} \neq \{0\}$ we conjecture a similar result: that is, that the structure of optimal completions can be described in terms of discrete structures related to multi-completion problems with prescribed weights. This points out a possible context for applications of optimal multi-completions and a direction for further research related to finitely generated shift-invariant spaces. These problems will be considered elsewhere.

Distributed sensor allocation. Consider a set of sensors that are to be distributed in groups. Assume further that some of these sensors are already placed and we are asked to find optimal locations to place the rest. Typically, the location of these sensors is restricted. In some cases the locations are restricted to specific (finite) places and we are interested in promoting stable encoding-decoding schemes (see [26]); there are also situations in which we are interested in promoting sparse representations of signals (see [13]). In some other cases, the location of sensors can be restricted to some specific subspaces. We model this last situation as follows: consider a finite family $\{W_j\}_{j \in \mathbb{I}_m}$ of subspaces of $\mathbb{C}^d$, such that $\dim W_j = d_j$, for $j \in \mathbb{I}_m$, and such that the algebraic sum $\mathcal{W}_1 + \ldots + \mathcal{W}_m = \mathbb{C}^d$. This last condition is a necessary condition for the existence of a distribution of sensors that will give rise to a frame for $\mathbb{C}^d$.

We assume that we are given initial sensors corresponding to vectors $\mathcal{F}_j^0 = \{f_{i,j}^0\}_{i \in \mathbb{I}_k}$ in $\mathcal{W}_j$, $1 \leq j \leq m$ and weights $\alpha = (\alpha_i)_{i=1}^n \in \mathbb{R}^n_{>0}$, such that $n \geq \max\{d_j \,:\, 1 \leq j \leq m\}$. In this context we are asked to design grouped finite sequences $\mathcal{F}_j = \{f_{i,j}\}_{i \in \mathbb{I}_n}$ in $\mathcal{W}_j$ for $j \in \mathbb{I}_m$, that further satisfy $\sum_{j \in \mathbb{I}_m} \|f_{i,j}\|^2 = \alpha_i$, for $i \in \mathbb{I}_n$. This last condition can be seen as a balancing condition (or a link) between the different families; it can also be interpreted as a geometrical restriction (in terms of distances to a fixed source) imposed to the distribution of the i-th sensors of each family i.e. $\{f_{i,j}\}_{j \in \mathbb{I}_m}$. We further search for those designs that induce optimally stable families $\mathcal{F} := (\mathcal{F}_j^0, \mathcal{F}_j)_{j \in \mathbb{I}_m}$ which are sequences of $(k+n) \cdot m$ vectors in $\mathbb{C}^d$. This problem can be also considered within the context of finite g-frame theory (see [28]), in which we also make use of a frame of subspaces (i.e. $\{W_j\}_{j \in \mathbb{I}_m}$) as support subspaces for a g-frame completion with weight restrictions.

With the previous level of generality, the analysis of these distributed families of vectors (grouped according to the subspaces $\{\mathcal{W}_j\}_{j\in\mathbb{I}_m}$ and satisfying the previous balancing conditions) is quite challenging from a mathematical point of view. Notice that not only the spectral and geometrical properties of each of the families $(\mathcal{F}_j^0, \mathcal{F}_j)$, for $j \in \mathbb{I}_m$, come into play; also, the relative position (geometry) of the subspaces $\{\mathcal{W}_j\}_{j\in\mathbb{I}_m}$ play a crucial role. For example, notice that the properties of the family $\mathcal{F} = (\mathcal{F}_j^0, \mathcal{F}_j)_{j\in\mathbb{I}_m}$ may change under unitary rotations $(\mathcal{F}_j^0, \mathcal{F}_j) \mapsto U_j \cdot (\mathcal{F}_j^0, \mathcal{F}_j)$, where U_j is a unitary operator acting on $\mathcal{W}_j$ and $U_j \cdot (\mathcal{F}_j^0, \mathcal{F}_j)$ is obtained from the entry-wise action of U_j on every vector in $(\mathcal{F}_j^0, \mathcal{F}_j)$, for $j \in \mathbb{I}_m$.

As a first step towards the study of this problem, it is natural to consider some restrictions in the previous model: if we further require that the (support) subspaces $\{\mathcal{W}_j\}_{j\in\mathbb{I}_m}$ are *mutually orthogonal* we get that

$$S_{\mathcal{F}} = \sum_{j\in\mathbb{I}_m} S_{(\mathcal{F}_j^0,\mathcal{F}_j)} = \bigoplus_{j\in\mathbb{I}_m} S_{(\mathcal{F}_j^0,\mathcal{F}_j)}$$

where the previous notation emphasizes the fact that the positive semidefinite operators $S_{(\mathcal{F}_j^0,\mathcal{F}_j)}$ have mutually orthogonal ranges. This restricted model corresponds to orthogonal decompositions of the signal space (e.g. a partition of the spectrum space of signals under a unitary transformation) that are relevant in applications. In this case, if we measure the stability of the distributed and balanced family $\mathcal{F}$ in terms of the convex potential $\mathrm{P}_\varphi(\mathcal{F})$ induced by a convex function $\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ i.e.

$$\mathrm{P}_\varphi(\mathcal{F}) = \operatorname{tr}(\varphi(S_{\mathcal{F}})) = \operatorname{tr}(\varphi[\bigoplus_{j\in\mathbb{I}_m} S_{(\mathcal{F}_j^0,\mathcal{F}_j)}]) = \operatorname{tr}(\bigoplus_{j\in\mathbb{I}_m} \varphi[S_{(\mathcal{F}_j^0,\mathcal{F}_j)}]) = \sum_{j\in\mathbb{I}_m} \operatorname{tr}(\varphi[S_{(\mathcal{F}_j^0,\mathcal{F}_j)}])\,.$$

Notice that, with some minor abuse of notation, the convex potential $\mathrm{P}_\varphi(\mathcal{F})$ coincides with the joint convex potential $\mathrm{P}_\varphi(\Phi^0\,,\,\Phi)$ (see Eq. (2)) of the completed $\mathbf{d}$-design $(\Phi^0\,,\,\Phi) = ((\mathcal{F}_j^0\,,\,\mathcal{F}_j))_{j\in\mathbb{I}_m}$ corresponding to the $(\alpha\,,\,\mathbf{d})$-design $\Phi = (\mathcal{F}_j)_{j\in\mathbb{I}_m}$ (where $\mathbf{d} = (d_j)_{j\in\mathbb{I}_m}$ are the dimensions of the support subspaces and we have identified $\mathcal{W}_j$ with $\mathbb{C}^{d_j}$, for $j \in \mathbb{I}_m$). In this case, our main results completely characterize the *optimal distributed sensor allocation with a balancing condition.* We point out that more general cases (e.g. when the subspaces are obtained from actions of unitary groups on a fixed space) will be considered elsewhere.

Existence of tight multi-completions with prescribed weights. Consider an initial family $\mathcal{F}_0 = \{f_i^0\}_{i\in\mathbb{I}_k}$ of vectors in $\mathbb{C}^d$. In [16] the authors posed the problem of computing the unit norm tight frame completions of $\mathcal{F}_0$ i.e., those families $\mathcal{F} = \{f_i\}_{i=1}^n$ such that $\|f_i\| = 1$, for $i \in \mathbb{I}_n$, and such that the completed family $(\mathcal{F}_0, \mathcal{F})$ is a c-tight frame (that is $S_{(\mathcal{F}_0,\mathcal{F})} = c\,I$). This problem was later solved in [22]. A generalized version of this type of problems has been recently considered in [5] from the perspective of matroid theory.

Given $\mathcal{F}_0 = \{f_i^0\}_{i\in\mathbb{I}_k}$ as above, assume that there exists a family $\mathcal{F} = \{f_i\}_{i=1}^n$ in $\mathbb{C}^d$ such that the completion $(\mathcal{F}_0, \mathcal{F})$ is a tight frame for $\mathbb{C}^d$. Then, if we let $\|f_i\|^2 = \alpha_i$, for $i \in \mathbb{I}_n$, it turns out that $(\mathcal{F}_0, \mathcal{F})$ minimizes every convex potential P_φ defined in the set of (α, d)-completions of $\mathcal{F}_0$ (in this case, we consider $m = 1$). Moreover, since the spectral structure of minimizers of convex potentials P_φ associated to strictly convex functions φ is uniquely determined (see [25]) then, every (α, d)-completion $(\mathcal{F}_0, \mathcal{G})$ that minimizes P_φ is also a tight frame completion of $\mathcal{F}_0$ (with norms prescribed by $(\alpha_i)_{i\in\mathbb{I}_n}$). That is, strictly convex potentials detect the existence of tight completions.

We point out that, as a particular case, our results also provide a characterization of the existence of tight $(\alpha, \mathbf{d})$-completions: that is, given an initial $\mathbf{d}$-design $\Phi^0 = (\mathcal{F}_j^0)_{j\in\mathbb{I}_m}$ we characterize the existence of $(\alpha\,,\,\mathbf{d})$-designs $\Phi = (\mathcal{F}_j)_{j\in\mathbb{I}_m}$ such that the completed $\mathbf{d}$-design $(\Phi^0\,,\,\Phi) = ((\mathcal{F}_j^0\,,\,\mathcal{F}_j))_{j\in\mathbb{I}_m}$ is a c-tight multi-frame i.e. each $(\mathcal{F}_j^0\,,\,\mathcal{F}_j)$ is a c-tight frame for $\mathbb{C}^{d_j}$ for some $c > 0$, for each $j \in \mathbb{I}_m$ (with the same constant $c > 0$).

Corollary 5.1. *Let* $(\Phi^0=(\mathcal{F}_j^0)_{j\in\mathbb{I}_m}\,,\,\alpha\,,\,\mathbf{d})$ *be the initial data for a multi-completion problem, with* $n\geq d_1$. *Let* $\lambda_j=(\lambda_{i,j})_{i\in\mathbb{I}_{d_j}}=\lambda^\uparrow(S_{\mathcal{F}_j^0})\in(\mathbb{R}_{\geq 0}^{d_j})^\uparrow$, *for* $j\in\mathbb{I}_m$. *Then, there exists a* $\Phi^{\rm op}\in\mathcal{D}(\alpha\,,\,\mathbf{d})$ *such that* $(\Phi^0\,,\,\Phi^{\rm op})$ *is a tight multi-frame if and only if*

$$\max\{\lambda_{d_j,j}\,:\ j\in\mathbb{I}_m\}\leq c:=\frac{\sum_{j\in\mathbb{I}_m}\operatorname{tr}(S_{\mathcal{F}_j^0})+\operatorname{tr}\alpha}{\operatorname{tr}\mathbf{d}}\quad \text{and}\quad (\alpha_i)_{i\in\mathbb{I}_n}\prec\left(\sum_{j\,:\,i\leq d_j}c-\lambda_{i,j}\right)_{i=1}^{d_1}\,,$$

where $\operatorname{tr}\alpha=\sum_{i\in\mathbb{I}_n}\alpha_i$ *and* $\operatorname{tr}\mathbf{d}=\sum_{j\in\mathbb{I}_m}d_j$. *In this case, given* $\varphi\in\operatorname{Conv_s}(\mathbb{R}_{\geq 0})$ *then:*

$$\tilde\Phi\in\mathcal{D}(\alpha\,,\,\mathbf{d})\ \text{ is a local minimizer of } \Psi_\varphi\iff(\Phi^0\,,\,\tilde\Phi)\ \text{ is a c-tight multi-frame}\,.$$

Proof. Notice that $\Phi^{\rm op}=(\mathcal{F}_j^{\rm op})_{j\in\mathbb{I}_m}\in\mathcal{D}(\alpha\,,\,\mathbf{d})$ is a c-tight multi-completion of Φ^0 if and only if $S_{\mathcal{F}_j^0}+S_{\mathcal{F}_j^{\rm op}}=c\,I_{d_j}$ or equivalently

$$S_{\mathcal{F}_j^{\rm op}}=c\,I_{d_j}-S_{\mathcal{F}_j^0}\in\mathcal{M}_{d_j}(\mathbb{C})^+\quad\text{for}\quad j\in\mathbb{I}_m\,.$$

Clearly, a necessary condition is that $c\geq\max\{\lambda_{d_j,j}\,:\ j\in\mathbb{I}_m\}$. Notice that from the previous identity we get that $\lambda(S_{\mathcal{F}_j^{\rm op}})=(c-\lambda_{i,j})_{i\in\mathbb{I}_{d_j}}$, for $j\in\mathbb{I}_m$. Since these eigenvalues come from an $(\alpha\,,\,\mathbf{d})$-design, Theorem 2.5 shows that the majorization relation in the statement above holds. Notice that in this case, we should have

$$\begin{aligned}c\cdot\operatorname{tr}\mathbf{d}&=\sum_{j\in\mathbb{I}_m}(\operatorname{tr}(S_{\mathcal{F}_j^0})+\operatorname{tr}(S_{\mathcal{F}_j^{\rm op}}))=\sum_{j\in\mathbb{I}_m}\operatorname{tr}(S_{\mathcal{F}_j^0})+\sum_{j\in\mathbb{I}_m}\sum_{i\in\mathbb{I}_n}\|f_{i,j}^{\rm op}\|^2\\&=\sum_{j\in\mathbb{I}_m}\operatorname{tr}(S_{\mathcal{F}_j^0})+\sum_{i\in\mathbb{I}_n}\sum_{j\in\mathbb{I}_m}\|f_{i,j}^{\rm op}\|^2=\sum_{j\in\mathbb{I}_m}\operatorname{tr}(S_{\mathcal{F}_j^0})+\sum_{i\in\mathbb{I}_n}\alpha_i\,.\end{aligned}\tag{33}$$

Hence, we now see that

$$c=\frac{\sum_{j\in\mathbb{I}_m}\operatorname{tr}(S_{\mathcal{F}_j^0})+\operatorname{tr}\alpha}{\operatorname{tr}\mathbf{d}}\,.$$

The converse can be shown in a similar way (using Theorem 2.5).

Assume that there exists a c-tight multi-completion $\Phi^{\rm op}\in\mathcal{D}(\alpha\,,\,\mathbf{d})$ of Φ^0 as above. For any $\Phi=(\mathcal{F}_j)\in\mathcal{D}(\alpha\,,\,\mathbf{d})$, if we set

$$\Lambda_\Phi=\bigoplus_{j\in\mathbb{I}_m}\lambda(S_{(\mathcal{F}_j^0\,,\,\mathcal{F}_j)})\in\mathbb{R}^{\operatorname{tr}\mathbf{d}}\quad\text{then}\quad c\cdot\mathbb{1}_{\operatorname{tr}\mathbf{d}}=\bigoplus_{j\in\mathbb{I}_m}\lambda(S_{(\mathcal{F}_j^0\,,\,\mathcal{F}_j^{\rm op})})\prec\Lambda_\phi\,.$$

Indeed, by Remark 6.4 we only need to check that $\operatorname{tr}(\Lambda_\Phi)=c\cdot\operatorname{tr}\mathbf{d}$: but this last fact can be checked with an identity similar to that in Eq. (33). Hence, by Theorem 6.5 we have that for any $\varphi\in\operatorname{Conv_s}(\mathbb{R}_{\geq 0})$

$$\Psi_\varphi(\Phi)=\operatorname{tr}(\varphi(\Lambda_\Phi))\geq\varphi(c)\cdot\operatorname{tr}\mathbf{d}=\Psi_\varphi(\Phi^{\rm op})\,.$$

That is, $\Phi^{\rm op}$ is a global minimizer of Ψ_φ. By Theorem 2.10 we get that in this case

$$\mathbf{c}=c\cdot\mathbb{1}_{d_1}\in(\mathbb{R}_{>0}^{d_1})^\downarrow\,.$$

Hence, again by Theorem 2.10, any local minimizer $\tilde\Phi=(\tilde{\mathcal{F}}_j)_{j\in\mathbb{I}_m}\in\mathcal{D}(\alpha\,,\,\mathbf{d})$ of Ψ_φ is such that $\lambda(S_{(\mathcal{F}_j^0\,,\,\tilde{\mathcal{F}}_j)})=c\cdot\mathbb{1}_{d_j}$, i.e. $S_{(\mathcal{F}_j^0\,,\,\tilde{\mathcal{F}}_j)}=c\cdot I_{d_j}$, for $j\in\mathbb{I}_m$; therefore $\tilde\Phi$ is also a c-tight $(\alpha\,,\,\mathbf{d})$ multi-completion of Φ^0. □

We point out that the fact that c-tight $(\alpha\,,\,\mathbf{d})$ multi-completions, whenever they exist (or optimal multi-completions in the general case), can be detected as *local minimizers* of the strictly convex potentials Ψ_φ (for $\varphi \in \mathrm{Conv_s}(\mathbb{R}_{\geq 0})$) play a central role in numerical computations of such completions. Indeed, in some situations we are interested in *improving* a encoding/decoding scheme $(\Phi^0\,,\,\Phi)$ by replacing $\Phi \in \mathcal{D}(\alpha\,,\,\mathbf{d})$ by some other $\tilde{\Phi} \in \mathcal{D}(\alpha\,,\,\mathbf{d})$ that is close to the initial Φ: the previous result shows that a natural approach to deal with this situation would be to consider gradient descent algorithms for Ψ_φ around Φ. A similar approach (based on alternating projection methods) has already been considered in [20] (for the case $m = 1$ that is, for frame completions).

Procrustes type problems and frame completions. Consider a $S \in \mathcal{M}_d(\mathbb{C})^+$ and fix $\alpha = (\alpha_i)_{i\in\mathbb{I}_m} \in (\mathbb{R}^n_{>0})^\downarrow$. In [27] N. Strawn considered the following setting: let $\mathcal{D}(\alpha, d)$ denote the finite sequences $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n} \in (\mathbb{C}^d)^n$ such that $\|f_i\|^2 = \alpha_i$, $i \in \mathbb{I}_n$ (i.e. we consider (α, d)-designs with $m = 1$). We endow $\mathcal{D}(\alpha, d)$ with the product metric (i.e. the metric as a subset of $(\mathbb{C}^d)^n$); let $\Theta : \mathcal{D}(\alpha, d) \to \mathbb{R}_{\geq 0}$, be given by $\Theta(\mathcal{F}) = \|S - S_\mathcal{F}\|_F$, where $\|\cdot\|_F$ denotes the Frobenius norm. Based on theoretical results and numerical experiments, Strawn conjectured that local minimizers of Θ were actually global minimizers: the problem becomes relevant in applied situations in which numerical methods based on gradient descent are used to obtain approximate minimizers of Θ (indeed, this was one of Strawn's original motivations). Strawn's conjecture can be seen as a matrix Procrustes type problem, that is an optimal approximation problem through structured matrices (see [19]).

In [21] it was shown that Strawn's conjecture can be *translated* into a (seemingly unrelated) problem about the structure of local minimizers of the joint frame potential $\mathrm{P}_\varphi(\mathcal{F}_0, \mathcal{F})$ where $\mathcal{F}_0 = \{f_i^0\}_{i\in\mathbb{I}_k} \in (\mathbb{C}^d)^k$ is fixed, $\varphi(x) = x^2$ for $x \in \mathbb{R}_0$ and $\mathcal{F}$ varies in $\mathcal{D}(\alpha, d)$. As a consequence, in [21] Strawn's conjecture was settled in the affirmative and the spectral and geometrical structures of the minimizers of the function Θ defined above (called the *frame operator distance*) were explicitly computed. The results in our present work extend the previous results on the structure of local minimizers from [21] to the context of multi-completions (with respect to *any* strictly convex function φ). Therefore, our present results solve a corresponding multivariate Strawn's problem (i.e. a multivariate matrix Procrustes type problem) and provide new insights related to the (differential) geometric aspect of multivariate frame operator distance problems. A detailed analysis of this problem will be considered elsewhere.

5.2 Numerical examples

Let $(\Phi^0 = (\tilde{\mathcal{F}}_j^0)_{j\in\mathbb{I}_m}\,,\,\alpha\,,\,\mathbf{d})$ be the initial data for a multi-completion problem. It is clear that the parameters corresponding to an optimal $(\alpha\,,\,\mathbf{d})$ multi-completion (that is, the indices $i_0 := 0 < \ldots < i_{p-1} < i_p = n$ and the positive constants $c_1 > \ldots > c_p > 0$) can be constructed by an implementation of an algorithm following the lines provided in Remark 4.3. This allows us to construct the vector $\mathbf{c}$ as in Theorem 2.10.

Remark 5.2 (Concrete computation of optimal multi-completions)**.** Once the vector $\mathbf{c}$ (and therefore the ν_j's) as in Theorem 2.10 is constructed, an optimal multi-completion $\Phi = \{\mathcal{F}_j\}_{j\in\mathbb{I}_m}$ can be computed in the following way: construct the vectors

$$\mu_j = \bigoplus_{\ell=1}^{p_j-1} (c_\ell - \lambda_{i,j})_{i=i_{\ell-1}+1}^{i_\ell} \oplus ((c_{p_j} - \lambda_{i,j})^+)_{i=i_{p_j-1}+1}^{d_j} \in (\mathbb{R}^{d_j}_{\geq 0})^\downarrow\,, \quad \text{for} \quad j \in \mathbb{I}_m\,.$$

Let $\mathcal{M} = \{\mu_j\}_{j\in\mathbb{I}_m}$; notice that by Theorem 2.10 and Corollary 3.9, $(\alpha\,,\,\mathcal{M})$ is an admissible pair (since it corresponds to the $(\alpha\,,\,\mathbf{d})$-design that is an optimal multi-completion of the initial $\mathbf{d}$-design Φ^0). By Theorem 2.5, we see that if we let $\sigma = \sum_{j\in\mathbb{I}_m}(\mu_j \oplus 0_{n-d_j}) \in (\mathbb{R}^n_{\geq 0})^\downarrow$, then $\alpha \prec \sigma$. In this case, it is possible to construct a $n \times n$ doubly stochastic matrix D in terms of the so-called T-transforms (see [6, Theorem II.1.10]) in a simple and fast way, such that $D\,\sigma = \alpha$.

Let $\omega_j = (\omega_{i,j})_{i\in\mathbb{I}_n} = D\,(\mu_j \oplus 0_{n-d_j}) \in \mathbb{R}^n_{\geq 0}$, for $j \in \mathbb{I}_m$. Then, by construction, the majorization relations $\omega_j \prec \mu_j$ hold, for $j \in \mathbb{I}_m$. Moreover, we also get that

$$\sum_{j\in\mathbb{I}_m} \omega_j = \sum_{j\in\mathbb{I}_m} D\,(\mu_j \oplus 0_{n-d_j}) = D\,\sigma = \alpha\,. \tag{34}$$

For each $j \in \mathbb{I}_m$ such that $\mu_j = 0$, set $\mathcal{F}_j = \{0\}_{i\in\mathbb{I}_n}$. For each $j \in \mathbb{I}_m$ such that $\mu_j \neq 0$ (or equivalently, $\omega_j \neq 0$) let $\lambda_j = (\lambda_{i,j})_{i\in\mathbb{I}_{d_j}} = \lambda^\uparrow(S_{\mathcal{F}_j^0}) \in (\mathbb{R}^{d_j}_{\geq 0})^\uparrow$, for $j \in \mathbb{I}_m$, and consider $\mathcal{B}_j = \{v_{i,j}\}_{i\in\mathbb{I}_{d_j}}$ an ONB of $\mathbb{C}^{d_j}$ such that

$$S_{\mathcal{F}_j^0} = \sum_{i\in\mathbb{I}_{d_j}} \lambda_{i,j}\ v_{i,j} \otimes v_{i,j}\,.$$

Since $\omega_j \prec \mu_j \oplus 0_{n-d_j} = (\mu_{i,j})_{i\in\mathbb{I}_{d_j}} \oplus 0_{n-d_j}$ then (see Theorem 6.7) we proceed with the application of a one-sided Bendel-Mickey type algorithm and construct in a concrete way $\mathcal{F}_j = \{f_{i,j}\}_{i\in\mathbb{I}_n} \in \mathbb{C}^{d_j}$ such that

$$\|f_{i,j}\|^2 = \omega_{i,j}\ ,\ i \in \mathbb{I}_n \quad \text{and} \quad S_{\mathcal{F}_j} = \sum_{\ell=1}^{p_j-1} \sum_{i=i_{\ell-1}+1}^{i_\ell} \mu_{i,j}\ v_{i,j} \otimes v_{i,j} + \sum_{i=i_{p_{j-1}}+1}^{d_j} \mu_{i,j}\ v_{i,j} \otimes v_{i,j}\,.$$

Notice that by Eq. (34)

$$\sum_{i\in\mathbb{I}_n} \|f_{i,j}\|^2 = \sum_{i\in\mathbb{I}_n} \omega_{i,j} = \alpha_j$$

and by construction, $\lambda_j + \mu_j = \nu_j$. Therefore $\lambda(S_{(\mathcal{F}_j^0,\mathcal{F}_j)}) = \nu_j$, for $j \in \mathbb{I}_m$, where ν_j is as in Theorem 2.10; thus, this result shows that $\Phi = (\mathcal{F}_j)_{j\in\mathbb{I}_m}$ is an optimal $(\alpha\,,\,\mathbf{d})$-multi-completion of Φ^0. $\triangle$

An outline of the algorithmic procedure would be as follows:

Algorithm 5.3. IMPUT DATA: $\Phi^0 = \{\mathcal{F}_j^0\}_{j\in\mathbb{I}_m}$ and $\alpha = (\alpha_i)_{i\in I_n} \in (\mathbb{R}^n_{>0})^\downarrow$

STEP 1 Extract the dimensions $\mathbf{d} = (d_j)_{j\in\mathbb{I}_m}$ (arranged in decreasing order) and compute the spectra $\lambda_j = (\lambda_{i,j})_{i\in\mathbb{I}_{d_j}} = \lambda^\uparrow(S_{\mathcal{F}_j^0}) \in (\mathbb{R}^{d_j}_{\geq 0})^\uparrow$, from the input data. Check if $n \geq d_1$.

STEP 2 Set $s = 1$ and $r = d_1$. Compute the block constant c as it is described in Remark 3.11. Employing an auxiliary routine, test if there is majorization between $(\alpha_i)_{i=s}^r$ (resp $(\alpha_i)_{i=s}^n$ if $r = d_1$) and the vector $\left(\sum_{j\,:\,i\leq d_j}(c-\lambda_{i,j})^+\right)_{i=s}^r$ (resp. $\left(\sum_{j\,:\,i\leq d_j}(c-\lambda_{i,j})^+\right)_{i=s}^{d_1} \oplus 0_{n-d_1}$).

While there is no majorization between these vectors, set $r = r-1$ and repeat the process until the first index i_1 is found (according to the notation used in Theorem 2.10). If i_1 is d_1, set $\mathbf{c} = c\mathbb{1}_{d_1}$ and go to STEP 3.

If $i_1 < d_1$ set $s = i_1 + 1$ and repeat STEP 2 to find i_2. Continue the process until $s = d_1$. This iteration produces the vector $\mathbf{c} = (c_1\,\mathbb{1}_{r_1}\,,\,\dots\,,\,c_p\,\mathbb{1}_{r_p}) \in (\mathbb{R}^{d_1}_{>0})^\downarrow$.

STEP 3 With the help of an additional routine that computes the T-transforms, obtain an $n\times n$ doubly stochastic matrix D that implements $\alpha \prec \sigma$ (see Remark 5.2 for the discussion and notations). This step produces a partition matrix, by computing the weights ω_j for each completion.

STEP 4 With the spectra μ_j and the weights ω_j as initial data, calculate the desired optimal completions by applying standard algorithms (one-sided generalized Bendel-Mickey, for example).

This algorithm was implemented to produce the following examples:

Example 5.4. Consider the initial set $\{\mathcal{F}_j^0\}_{j\in\mathbb{I}_3}$ $(m=3)$ such that

$$\mathcal{F}_1^0 = (\,\sqrt{7.5}\,e_1^1,\ \sqrt{\tfrac{3}{2}}\,e_2^1,\ \sqrt{2.5}\,e_3^1,\ e_4^1\,) \subset (\mathbb{C}^7)^4$$

$$\mathcal{F}_2^0 = (\,\sqrt{10}\,e_1^2,\ \sqrt{10}\,e_1^2,\ \frac{1}{2}\,e_2^2,\ \frac{1}{2}\,e_2^2\,) \subset (\mathbb{C}^5)^4$$

and

$$\mathcal{F}_3^0 = (\,2\,e_1^3,\ \sqrt{\tfrac{3}{2}}\,e_2^3,\ \frac{\sqrt{2}}{2}\,e_3^3,\ \frac{\sqrt{2}}{2}\,e_3^3\,) \subset (\mathbb{C}^3)^4$$

where $\mathbf{d} = (7,5,3)$ and each $\{e_i^j\}$ is the canonical orthonormal basis on $\mathbb{C}^{d_j}$, for $j = 1,2,3$.

In particular, $\lambda_1 = (0,0,0,1,2.5,3,7.5)$, $\lambda_3 = (0,0,0,0.5,20)$ and $\lambda_3 = (1,1.5,4)$.

Now, if we take $\alpha = (100,\ 90,\ 39,\ 38,\ 7,\ 5,\ 2.7,\ 2.5,\ 2,\ 1.5,\ 1,\ 0.6,\ 0.5) \in (\mathbb{R}^{13}_{>0})^{\downarrow}$ as a set of 13 weights for the multi-completion, our previously described procedure yields the following vector $\mathbf{c} \in (\mathbb{R}^7_{>0})^{\downarrow}$:

$$\mathbf{c} = (33.6667,\ 30.5,\ 16.5,\ 16.5,\ 11.9333,\ 11.9333,\ 11.9333)$$

and a partition matrix

$$A = \begin{bmatrix} 33.6667 & 33.6667 & 32.6667 \\ 30.5 & 30.5 & 29 \\ 16.0357 & 16.2679 & 6.6964 \\ 15.9643 & 16.2321 & 5.8036 \\ 6.4846 & 0.5154 & 0 \\ 3.2226 & 1.7774 & 0 \\ 1.4107 & 1.2893 & 0 \\ 1.3062 & 1.1938 & 0 \\ 1.045 & 0.955 & 0 \\ 0.7837 & 0.7163 & 0 \\ 0.5225 & 0.4775 & 0 \\ 0.3135 & 0.2865 & 0 \\ 0.2612 & 0.2388 & 0 \end{bmatrix}$$

from which optimal multi-completions can be computed using standard algorithms. △

Example 5.5. Considering the same notation as in the previous example, let $\Phi^0 = \{\mathcal{F}_j^0\}_{j\in\mathbb{I}_3}$ be given in matrix form by

$$\mathcal{F}_1^0 = \begin{bmatrix} 0.3066 & 1.6919 & -1.14 & 0.0488 \\ 0.9339 & -0.4353 & -0.2197 & 0.2354 \\ -1.8151 & 0.8134 & 0.3742 & 0.2428 \\ 1.7690 & 1.0168 & 0.8745 & -0.045 \\ -0.4706 & 0.7223 & 0.8595 & 0.0609 \\ 1.1678 & -0.0164 & 0.0839 & 0.2206 \\ -0.1574 & 0.48 & 0.042 & -0.3589 \end{bmatrix}$$

$$\mathcal{F}_2^0 = \begin{bmatrix} -2.723 & -0.068 & -0.5242 \\ -2.2341 & -0.5975 & 0.2401 \\ -1.5660 & 0.7992 & 0.0219 \\ 2.2048 & -0.1835 & -0.4038 \\ 0.5298 & 0.2569 & 0.0631 \end{bmatrix}$$

and

$$\mathcal{F}_3^0 = \begin{bmatrix} -0.8048 & -0.9958 & -0.1026 \\ 1.0153 & -0.5127 & -0.4653 \\ 0.5669 & -0.4955 & 0.6877 \end{bmatrix}$$

Now, for the weights $\alpha = (20,\ 19.5,\ 10,\ 5,\ 4.5,\ 3,\ 2.4,\ 2\,) \in (\mathbb{R}^8_{>0})^{\downarrow}$ the algorithm provides the set of constants $\mathbf{c} \in (\mathbb{R}^7_{>0})^{\downarrow}$:

$$\mathbf{c} = (6.95,\ 6.95,\ 5.6143,\ 5.6143,\ 5.6143,\ 5.6143,\ 5.6143\,)$$

the partition of weights is given by

$$A = \begin{bmatrix} 6.95 & 6.95 & 6.1 \\ 6.95 & 6.95 & 5.6 \\ 5.3257 & 4.5371 & 0.1373 \\ 3.2312 & 1.051 & 0.7178 \\ 1.9449 & 1.5183 & 1.0368 \\ 1.2792 & 1.0225 & 0.6983 \\ 1.0234 & 0.818 & 0.5586 \\ 0.8528 & 0.6817 & 0.4655 \end{bmatrix}$$

Which in turn allows us to construct the optimal multi-completion (written in the respective canonical basis $\{e_i^j\}$):

$$\tilde{\mathcal{F}}_1 = \begin{bmatrix} -0.401 & -0.0192 & 0.1339 & 0.2513 & -1.0368 & -0.2548 & 0.1414 & 0.1411 \\ -0.1054 & -1.803 & -1.1057 & 0.809 & -0.1431 & 0.1769 & 0.4553 & 0.0759 \\ 1.1322 & 0.0888 & -0.7415 & 0.7137 & 0.2382 & 0.6032 & 0.4016 & 0.1246 \\ 0.1003 & 0.6515 & -0.8319 & -0.2946 & 0.6227 & 0.4544 & -0.1658 & -0.3316 \\ -1.355 & -1.1143 & 0.9675 & 0.0617 & 0.6327 & 0.5607 & 0.0347 & 0.2647 \\ 1.4233 & -0.0427 & 1.37 & 0.7071 & 0.0683 & 0.3434 & 0.3979 & 0.6416 \\ 1.2745 & -1.4223 & 0.1734 & -1.1889 & -0.0023 & -0.4246 & -0.6691 & -0.4690 \end{bmatrix}$$

$$\tilde{\mathcal{F}}_2 = \begin{bmatrix} -0.4454 & 0.5706 & -1.57 & -0.0664 & -0.0798 & -0.0655 & -0.0586 & -0.5888 \\ -0.412 & -1.4104 & 0.7847 & -0.5841 & -0.702 & -0.5761 & -0.5153 & 0.0874 \\ 0.4572 & -1.3589 & -0.0177 & 0.7812 & 0.9389 & 0.7705 & 0.6892 & 0.2554 \\ -0.0363 & -1.6673 & -1.1957 & -0.1793 & -0.2156 & -0.1769 & -0.1582 & -0.4916 \\ -2.5242 & -0.0926 & 0.1629 & 0.2511 & 0.3018 & 0.2477 & 0.2215 & 0.143 \end{bmatrix}$$

and

$$\tilde{\mathcal{F}}_3 = \begin{bmatrix} -1.0983 & 1.5782 & -0.2108 & -0.4821 & -0.5795 & -0.4756 & -0.4254 & -0.3883 \\ -1.6765 & 0.2788 & 0.266 & 0.6082 & 0.731 & 0.5999 & 0.5366 & 0.4898 \\ 1.4433 & 1.7411 & 0.1485 & 0.3396 & 0.4082 & 0.335 & 0.2996 & 0.2735 \end{bmatrix}$$

As it was mentioned in STEP 3 in the description of the algorithm, the weight partition is computed from a doubly stochastic matrix D. In the case where the polytope of doubly stochastic matrices that implement the majorization $\alpha \prec \sigma$ (See Remark 5.2) is not a singleton, different doubly stochastic matrices produce different partition matrices for optimal multi-completions.

In this example, by applying a different technique that produces an orthostochastic matrix, we also have the following partition of weights:

$$\tilde{A} = \begin{bmatrix} 6.95 & 6.95 & 6.1 \\ 6.95 & 6.95 & 5.6 \\ 3.9158 & 3.5662 & 2.518 \\ 2.9185 & 2.0815 & 0 \\ 3.32 & 1.18 & 0 \\ 1.7709 & 1.2291 & 0 \\ 0.9486 & 0.8586 & 0.5927 \\ 0.7832 & 0.7132 & 0.5036 \end{bmatrix}$$

6 Appendix

In this section we describe several properties of the majorization pre-order between vectors in $\mathbb{C}^d$ and optimal frame completions, for the convenience of the reader.

6.1 Majorization theory in $\mathbb{R}^d$

Recall that given vectors $x,\, y \in \mathbb{R}^d$ we say that x is majorized by y if

$$\sum_{i\in\mathbb{I}_j} x_i^\downarrow \leq \sum_{i\in\mathbb{I}_j} y_i^\downarrow \quad \text{for every} \quad 1\leq j\leq d-1 \quad \text{and} \quad \operatorname{tr}(x) := \sum_{i\in\mathbb{I}_d} x_i = \sum_{i\in\mathbb{I}_d} y_i =: \operatorname{tr}(y)\,.$$

In case $x \prec y$ and $x^\downarrow \neq y^\downarrow$ we say that x is *strictly* majorized by y.

Recall that given a pair of vectors of different sizes and non-negative entries we extend the notion of majorization as follows: if $x \in \mathbb{R}^n_{\geq 0}$ and $y \in \mathbb{R}^d_{\geq 0}$ with $n > d$, we say that x is majorized by y if $x \prec y \oplus 0_{n-d} = (y_1, \ldots, y_d, 0, \ldots, 0) \in \mathbb{R}^n$, in the sense defined above. In this case, we simply write $x \prec y$ (see Definition 2.1).

Remark 6.1. Let $x,\, y \in \mathbb{R}^d$ be such that $x \prec y$. Then, using the definition of majorization and the equality of traces we conclude that

$$\sum_{i=j}^{d} x_i^\downarrow \geq \sum_{i=j}^{d} y_i^\downarrow \quad \text{for} \quad j\in\mathbb{I}_d\,.$$

△

The following result is Lidskii's additive inequality for selfadjoint matrices.

Theorem 6.2 (See [6]). *Let $S_1,\, S_2 \in \mathcal{H}(d)$. Then $\lambda(S_1)^\uparrow + \lambda(S_2) \prec \lambda(S_1+S_2)$.* □

It is well known that majorization in $\mathbb{R}^d$ is related to the class $\mathcal{DS}(d)$ of doubly stochastic matrices i.e., formed by $D \in \mathcal{M}_d(\mathbb{C})$ with real non-negative entries such that each row sum and column sum equals one.

Theorem 6.3 (See [6]). Let $x,\, y \in \mathbb{R}^d$. Then

$$x \prec y \quad \iff \quad \text{there exists} \quad D \in \mathcal{DS}(d) \quad \text{such that} \quad x = Dy\,. \qquad \square$$

Remark 6.4. Let $x \in \mathbb{R}^d$ and let $c \in \mathbb{R}$ be such that $c\cdot d = \operatorname{tr} x$. Then, $c\cdot \mathbb{1}_d \prec x$, where $\mathbb{1}_d \in \mathbb{R}^d$ is the vector with entries equal to 1. Indeed, let $D = (\,\frac{1}{d}\,)_{i,j=1}^d \in \mathcal{M}_d(\mathbb{C})$. Then $D \in \mathcal{DS}(d)$ and we have that $Dx = \frac{\operatorname{tr}(x)}{d}\,\mathbb{1}_d = c\cdot\mathbb{1}_d$. Hence, by Theorem 6.3 we see that $c\cdot\mathbb{1}_d \prec x$. △

Majorization is intimately related to tracial inequalities of convex functions. The following result summarizes these relations (see for example [6]):

Theorem 6.5. Let $x,\, y \in \mathbb{R}^d$. If $\varphi: I \to \mathbb{R}$ is a convex function defined on an interval $I \subseteq \mathbb{R}$ such that $x,\, y \in I^d$ then:

1. If $x \prec y$, then $\operatorname{tr}\varphi(x) \overset{\text{def}}{=} \sum_{i\in\mathbb{I}_d} \varphi(x_i) \leq \sum_{i\in\mathbb{I}_d} \varphi(y_i) = \operatorname{tr}\varphi(y)$.
2. If $x \prec y$ and φ is a strictly convex function such that $\operatorname{tr}\,\varphi(x) = \operatorname{tr}\,\varphi(y)$ then, $x^\downarrow = y^\downarrow$.

Hence, if x is strictly majorized by y and φ is a strictly convex function then $\operatorname{tr}\,\varphi(x) < \operatorname{tr}\,\varphi(y)$.

□

Remark 6.6. The following argument is used several times in our work: if $x_1,\, y_1 \in \mathbb{R}^{d_1}$ and $x_2,\, y_2 \in \mathbb{R}^{d_2}$ are such that $x_i \prec y_i$, for $i = 1, 2$, then $(x_1\,,\, x_2) \prec (y_1\,,\, y_2) \in \mathbb{R}^{d_1+d_2}$; this is a consequence of Theorem 6.3 and the fact that the direct sum $D_1 \oplus D_2 \in \mathcal{DS}(d_1+d_2)$ whenever $D_1 \in \mathcal{DS}(d_1)$ and $D_2 \in \mathcal{DS}(d_2)$.

Moreover, if x_1 is strictly majorized by y_1 then $(x_1\,,\, x_2)$ is strictly majorized $(y_1\,,\, y_2)$; indeed, by Theorem 6.5, if $\varphi \in \operatorname{Conv_s}(\mathbb{R}_{\geq 0})$ then

$$\operatorname{tr}\varphi((x_1\,,\, x_2)) = \operatorname{tr}\varphi(x_1) + \operatorname{tr}\varphi(x_2) < \operatorname{tr}\varphi(y_1) + \operatorname{tr}\varphi(y_2) = \operatorname{tr}\varphi((y_1\,,\, y_2))\,.$$

△

6.2 Completion problems with prescribed norms

Recall that given $\alpha = (\alpha_i)_{i\in\mathbb{I}_n} \in \mathbb{R}^n_{\geq 0}$ and $d \in \mathbb{N}$ with $d \leq n$, $\mathcal{B}_{\alpha\,,\,d}$ denotes the Cartesian product of n spheres of $\mathbb{C}^d$ of radius α_i, $i \in \mathbb{I}_n$ (see Eq. (7)). In what follows we consider $\mathcal{B}_{\alpha\,,\,d}$ endowed with the product metric $m_d(\cdot,\cdot)$. (see Eq. (3)).

We first consider the following result that describes the solution to the classical frame design problem. Notice Theorem 2.5 extends this result to multi-designs.

Theorem 6.7 (See [8]). *Let $\alpha = (\alpha_i)_{i\in\mathbb{I}_n} \in \mathbb{R}^n_{\geq 0}$ and $\lambda = (\lambda_i)_{i\in\mathbb{I}_d} \in (\mathbb{R}^d_{\geq 0})^\downarrow$, with $n \geq d$. Then there exists $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n} \in \mathcal{B}(\alpha\,,\,d)$ with $\lambda_i(S_\mathcal{F}) = \lambda_i$, for $i \in \mathbb{I}_d$, if and only if $\alpha \prec \lambda$.* □

In order to describe the general setting for the optimal frame completion problems we consider the so-called convex potentials.

Definition 6.8. Following [23] we consider the convex potential P_φ associated with $\varphi \in \mathrm{Conv}(\mathbb{R}_{\geq 0})$, given by

$$\mathrm{P}_\varphi(\mathcal{F}) \;\; = \mathrm{tr}\;\varphi(S_\mathcal{F}) = \textstyle\sum_{i\in\mathbb{I}_d} \varphi(\lambda_i(S_\mathcal{F})\,) \quad \text{for} \quad \mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n} \in (\mathbb{C}^d)^n \ ,$$

where the matrix $\varphi(S_\mathcal{F})$ is defined by means of the usual functional calculus. △

Notice that the *frame potential* introduced by Benedetto and Fickus (see [1, 10]) is the convex potential corresponding the strictly convex function $\varphi(x) = x^2$, for $x \geq 0$.

Definition 6.9. *Given an initial family $\mathcal{F}_0 = \{f_i^0\}_{i\in\mathbb{I}_k} \in (\mathbb{C}^d)^k$ and a finite sequence $\alpha = (\alpha_i)_{i\in\mathbb{I}_n} \in \mathbb{R}^n_{\geq 0}$ we consider:*

1. *The set of completions of $\mathcal{F}_0$ with norms prescribed by α, denoted $\mathcal{C}(\alpha\,,\,d)$, given by*

$$\mathcal{C}(\mathcal{F}_0\,,\,\alpha\,,\,d) = \{(\mathcal{F}_0\,,\,\mathcal{F}) \in (\mathbb{C}^d)^{(k+n)} : \ \mathcal{F} \in \mathcal{B}_{\alpha\,,\,d}\}\,.$$

2. *Given $\varphi \in \mathrm{Conv}(\mathbb{R}_{\geq 0})$ we consider the function $\Psi_\varphi : \mathcal{B}_{\alpha\,,\,d} \to \mathbb{R}_{\geq 0}$ given by*

$$\Psi_\varphi(\mathcal{F}) = P_\varphi(\mathcal{F}_0\,,\,\mathcal{F}) = \mathrm{tr}(\varphi(S_{(\mathcal{F}_0\,,\,\mathcal{F})}))\,.$$

That is, $\Psi_\varphi(\mathcal{F})$ is the convex potential P_φ of the completed sequence $(\mathcal{F}_0\,,\,\mathcal{F}) \in \mathcal{C}(\mathcal{F}_0\,,\,\alpha\,,\,d)$.

△

With the notation of Definition 6.9 above, it is well known that minimizers of Ψ_φ within $\mathcal{B}_{\alpha\,,d}$ correspond to completions $(\mathcal{F}_0\,,\,\mathcal{F}) \in \mathcal{C}(\mathcal{F}_0\,,\,\alpha\,,\,d)$ that induce optimal encoding/decoding schemes (e.g. in terms of numerical performance) within the set of completions with norms prescribed by α. The following result summarizes several properties of (local) optimal frame completions (see [21] and [24, 25]).

Theorem 6.10 ([21, 25]). Let $\mathcal{F}_0 = \{f_i^0\}_{i\in\mathbb{I}_k} \in (\mathbb{C}^d)^k$, $\alpha = (\alpha_i)_{i\in\mathbb{I}_n} \in \mathbb{R}^n_{\geq 0}$, $\alpha \neq 0$. Let $S_0 = S_{\mathcal{F}_0} \in \mathcal{M}_d(\mathbb{C})^+$ and let $\lambda = (\lambda_i)_{i\in\mathbb{I}_d} \in (\mathbb{R}^d_{\geq 0})^\uparrow$ be the eigenvalues of S_0, counting multiplicities. Then there exists $\nu = \nu(\mathcal{F}_0\,,\,\alpha) \in \mathbb{R}^d_{\geq 0}$ such that: for $\mathcal{F} \in \mathcal{B}_{\alpha\,,\,d}$ and $\varphi \in \mathrm{Conv_s}(\mathbb{R}_{\geq 0})$,

$$\mathcal{F} \;\text{ is a local minimizer of }\; \Psi_\varphi \;\text{ on }\; \mathcal{B}_{\alpha\,,\,d} \qquad \Longleftrightarrow \; \lambda(S_{(\mathcal{F}_0\,,\,\mathcal{F})}) = \nu^\downarrow\,.$$

In this case,

1. $\mathcal{F}$ is a global minimizer of Ψ_ϕ, for every $\phi \in \mathrm{Conv_s}(\mathbb{R}_{\geq 0})$.

2. There exists $\mathcal{B} = \{v_i\}_{i\in\mathbb{I}_d}$ an ONB of $\mathbb{C}^d$ such that

$$S_0 = \sum_{i\in\mathbb{I}_d} \lambda_i \; v_i \otimes v_i \quad \text{ and } \quad S_{\mathcal{F}} = \sum_{i\in\mathbb{I}_d} \lambda_i(S_{\mathcal{F}}) \; v_i \otimes v_i \, ,$$

where $\lambda(S_{\mathcal{F}}) = (\lambda_i(S_{\mathcal{F}}))_{i\in\mathbb{I}_d} \in (\mathbb{R}^d_{\geq 0})^{\downarrow}$.

3. The subspace $W = R(S_{\mathcal{F}})$ reduces $S_0 + S_{\mathcal{F}} \in \mathcal{M}_d(\mathbb{C})^+$; hence $A = (S_0 + S_{\mathcal{F}})|_W \in L(W)^+$.

4. All vectors f_i are eigenvectors of A; thus if we let $c_1 > \ldots > c_p > 0$ be such that $\sigma(A) = \{c_1, \ldots, c_p\}$ and we set $r^* = \dim W = \text{rk}(S_{\mathcal{F}}) \in \mathbb{I}_d$,

$$J_j = \{\ell \in \mathbb{I}_n : \; f_\ell \neq 0 \, , \; A\, f_\ell = c_j \, f_\ell\} \quad \text{ and } \quad K_j = \{\ell \in \mathbb{I}_{r^*} : \lambda_\ell + \lambda_\ell(S_{\mathcal{F}}) = c_j\} \quad \text{ for } \quad j \in \mathbb{I}_p \, ,$$

then:

   (a) $\{K_j\}_{j\in\mathbb{I}_p}$ is a partition of $\mathbb{I}_{r^*}$.
   (b) If $1 \leq j < k \leq p$ and we chose $i \in J_j$, $\ell \in J_k$ then, $\|f_i\|^2 > \|f_\ell\|^2$.
   (c) $\{f_i\}_{i\in J_j}$ is a linearly independent set, for $j \in \mathbb{I}_{p-1}$.
   (d) If $\#\{i \in \mathbb{I}_n \; : \; \alpha_i \neq 0\} + \#\{i \in \mathbb{I}_d \; : \; \lambda_i \neq 0\} \geq d$ then $(\mathcal{F}_0 \, , \mathcal{F})$ is a frame for $\mathbb{C}^d$.
   (e) If $(\mathcal{F}_0 \, , \mathcal{F})$ is a frame for $\mathbb{C}^d$ then $c_p \leq c$, for $c \in \sigma(S_0 + S_{\mathcal{F}})$; moreover, $c_p > \lambda_{r^*}$.

Proof. All items above follow from the main results from [21] and [25]. Nevertheless, we include a few comments related to the fact that the entries in the sequence of norms $\alpha = (\alpha_i)_{i\in\mathbb{I}_n}$ are not necessarily arranged in non-increasing order. For example, by [21, Proposition 3.10] (see also the detailed proof of [25, Proposition 4.5]) we see that if $i, \, \ell \in \mathbb{I}_n$ are such that $f_i \neq 0$, $f_\ell \neq 0$, $(S_0 + S_{\mathcal{F}})f_i = c_j \, f_i$, $(S_0 + S_{\mathcal{F}})f_\ell = c_k \, f_\ell$ with $c_j > c_k$ (that follows from the assumption that $1 \leq j < k \leq p$ in the statement above) then $\|f_i\|^2 - \|f_\ell\|^2 \geq c_j - c_k > 0$ and hence, $\|f_i\|^2 > \|f_\ell\|^2$. This last fact shows the validity of item (b) above. The other items can be proved by taking into account similar considerations. □

Remark 6.11. With the notation and terminology of Theorem 6.10, assume that $(\mathcal{F}_0 \, , \mathcal{F})$ is a frame for $\mathbb{C}^d$. Then, the vector $\nu \in \mathbb{R}^d_{>0}$, which characterizes the spectrum of optimal completions, can be described as follows:

$$\nu = (c_1 \, \mathbb{1}_{r_1} \, , \; \ldots \; , \, c_p \, \mathbb{1}_{r_p} \, , \, \lambda_{r^*+1} \, , \; \ldots \; , \, \lambda_d) \, ,$$

where $r_j = |K_j|$, for $j \in \mathbb{I}_p$, so that $\sum_{j\in\mathbb{I}_p} r_j = r^* \in \mathbb{I}_d$. In case $r^* < d$ then, $c_p \leq \lambda_{r^*+1}$.

On the other hand, if we let $W_j = \text{span}\{f_i : \; i \in J_j\}$ for $j \in \mathbb{I}_p$ then,

$$W = \bigoplus_{j\in\mathbb{I}_p} W_j \quad \text{ and } \quad W_j = \text{span}\{v_i : \; i \in K_j\} \quad \text{ for } \quad j \in \mathbb{I}_p \, .$$

In particular $\dim W_j = r_j$, for $j \in \mathbb{I}_p$. Moreover, by item 4c. in Theorem 6.10, we see that $r_j = \dim W_j = |J_j|$, for $j \in \mathbb{I}_{p-1}$. △

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