The Poincaré space of a C*-algebra

E. Andruchow, G. Corach and L. Recht

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Abstract

This article surveys the results in three recent papers by the present authors, which study the metric, differential and projective geometry of the Poincaré half space and the Poincaré disk of a C*-algebra.

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1 Introduction

Given a C*-algebra $\mathcal{A}$, denote by $G_{\mathcal{A}}$ the group of invertible elements of $\mathcal{A}$ and by G^+ the positive elements of $G_{\mathcal{A}}$. We define the *Poincaré half space of* $\mathcal{A}$

$$\mathcal{H} = \{h \in \mathcal{A} : Im\ h = \frac{1}{2i}(h - h^*) \text{ is a positive invertible element of } \mathcal{A}\}.$$

and the *Poincaré disk* of $\mathcal{A}$

$$\mathcal{D} = \{z \in \mathcal{A} : \|z\| < 1\}.$$

In the papers [1], [2] and [3] we define the groups $\mathcal{U}_\rho, \mathcal{U}_{\rho'}$ which operate on $\mathcal{D}$ and $\mathcal{H}$, respectively, and define on $\mathcal{D}$ and $\mathcal{H}$ isomorphic structures of reductive homogeneous spaces with invariant Finsler metrics. Additionally both spaces carry Kähler structures (in a non commutative sense). We study metric and geometric properties of these spaces. For example the geodesics of the reductive structure of $\mathcal{H}$ (and $\mathcal{D}$) have minimal length among all smooth curves joining their endpoints. Moreover, the function $f(t)$ = distance between $\gamma(t)$ and $\delta(t)$, for two geodesics γ and δ, is convex. Therefore, $\mathcal{H}$ and $\mathcal{D}$ become non positively curved spaces in the sense of Busemann [18], [7], [10].

As part of their Kähler structure, $\mathcal{H}$ and $\mathcal{D}$ carry an invariant symplectic structure. We compute the moment map of this structure, always in a non commutative sense. In the presence of a trace in $\mathcal{A}$, we show that, for an appropriate subgroup of $\mathcal{U}_\rho$, the image of the moment map is a convex set, a result which resembles the classical theorem of Atiyah, Guillemin and Sternberg [4], [19] (see also [8]).

Finally, we describe the metric properties of $\mathcal{D}$, as a subset of the *projective line* of $\mathcal{A}$. Given $z_0, z_1 \in \mathcal{D}$, let $\delta : \mathbb{R} \to \mathcal{D}$ be the unique geodesic with $\delta(0) = z_0$ and $\delta(1) = z_1$, and its limit points

$$z_{+\infty} = \lim_{t\to+\infty} \delta(t)\ ,\quad z_{-\infty} = \lim_{t\to-\infty} \delta(t).$$

Using ideas of Zelikin [36], we define the cross ratio of the four-tuple $\{z_{-\infty}, z_0, z_1, z_{+\infty}\}$, which is an operator $cr(z_{-\infty}, z_0, z_1, z_{+\infty})$. On the other hand, we have the "initial velocity operator" $\dot{\delta}(0)$, whose norm is the distance between z_0 and z_1. We establish an identity which relates the distance operator $\dot{\delta}(0)$ and the cross ratio operator $cr(z_{-\infty}, z_0, z_1, z_{+\infty})$. This identity is an operator valued version of the classical formula involving the hyperbolic distance in the complex unit disk with the cross ratio in the complex projective line.

One of the motivations for the study of $\mathcal{H}$ is the following. We consider G^+ as the configuration space of a dynamical system. The elements of G^+ may be thought of as a familiy of equivalent inner products of a Hilbert space. The corresponding phase space is TG^+. Note that TG^+ consists of pairs of the form (a, X), with $a \in G^+$ and $X \in (TG^+)_a \simeq \mathcal{A}_h$, the selfadjoint elements of $\mathcal{A}$. TG^+ identifies with $\mathcal{H}$ by means of $(a, X) \longleftrightarrow X + ia$. This identification is not merely formal. The Liouville 1-form on $\mathcal{H}$ can be obtained, in the usual way, by identifying the tangent bundle TG^+ with the cotangent bundle T^*G^+, as explained in [2].

As a second motivation, let E be a complex vector bundle over a compact space M provided with a Hermitian structure. Consider the C*-algebra $\Gamma(\mathrm{End}(E))$ of continuous sections of the endomorphism bundle of E. The Poincaré space of $\Gamma(\mathrm{End}(E))$ consists of the elements of the form $X + ia$, where a is a Hermitian structure on E and X is infinitesimal deformation of a.

2 The geometry of G^+

Since one of the motivations of this work is the identification of $\mathcal{H}$ with the tangent bundle TG^+, we begin with a summary of results on the geometry of G^+. Moreover, the geometry of $\mathcal{D}$ and $\mathcal{H}$ depend on the geometry of G^+. The results of this section can be found in [13], [14]. For other examples of homogeneous spaces of groups of invertible elements in C*-algebras see [5], [16], [25], [26], [27], [35].

First we fix notation for this section. $\mathcal{A}$ is a unital C*-algebra, $\mathcal{A}_h = \{x \in \mathcal{A} : x^* = x\}$ (Hermitian elements of $\mathcal{A}$), $\mathcal{A}_{ah} = \{x \in \mathcal{A} : x^* = -x\}$ (anti-Hermitian elements of $\mathcal{A}$); recall that $\mathcal{A}_h \oplus \mathcal{A}_{ah} = \mathcal{A}$. $G_{\mathcal{A}}$ is the group of invertible elements of $\mathcal{A}$, G^+ is the space of positive elements of $G_{\mathcal{A}}$, $\mathcal{U}_{\mathcal{A}}$ is the group of unitary elements of $\mathcal{A}$.

If $\mathcal{A}$ is represented in a Hilbert space $(\mathcal{L}, \langle\ ,\ \rangle)$, then G^+ can be thought of as a set of inner product on $\mathcal{L}$ which are equivalent to $\langle\ ,\ \rangle$: if $a \in G^+$,

$$\langle \xi, \eta \rangle_a := \langle a\xi, \eta \rangle,\ \ \xi, \eta \in \mathcal{L}.$$

Each $a \in G^+$ defines also an involution ^{*_a} in $\mathcal{A}$: for $x \in \mathcal{A}$, $x^{*_a} = a^{-1}xa$. If $\mathcal{A}$ is represented in $\mathcal{L}$, then ^{*_a} is the adjoint with respect to $\langle\ ,\rangle_a$.

Note that for $a \in G^+$, if we denote

$$\mathcal{A}_h^a := \{x \in \mathcal{A} : x^{*_a} = x\} \quad \text{and} \quad \mathcal{A}_{ah}^a := \{x \in \mathcal{A} : x^{*_a} = -x\},$$

then it holds that $\mathcal{A} = \mathcal{A}_h^a \oplus \mathcal{A}_{ah}^a$ (i.e., the decomposition as a-Hermitian and a-anti-Hermitian elements).

Since $G_{\mathcal{A}}$ is an open subset of $\mathcal{A}$, its manifold structure is trivial, and for each $g \in G_{\mathcal{A}}$, the tangent space $(TG_{\mathcal{A}})_g$ identifies with $\mathcal{A}$. Similarly, G^+ is open in $\mathcal{A}_h$, and thus $(TG^+)_a \simeq \mathcal{A}_h$ for each $a \in G^+$.

The group $G_{\mathcal{A}}$ acts on G^+ by means of

$$L : G_{\mathcal{A}} \times G^+ \to G^+\ ,\ \ L_g a = (g^*)^{-1} a g^{-1},$$

for $g \in G_{\mathcal{A}}$ and $a \in G^+$. The action is transitive: for each pair $a, b \in G^+$ it holds that $L_g a = b$ for $g = b^{-1/2}a^{1/2}$.

Along this paper, for the left action of a group G on a manifold M, we use both notations $L_g m$ and $g \cdot m$, for $g \in G$ and $m \in M$.

The *isotropy subgroup* of $a \in G^+$ for the action L is $\mathcal{I}_a = \{g \in G_{\mathcal{A}} : (g^*)^{-1}ag^{-1} = a\}$. Note that $\mathcal{I}_1 = \mathcal{U}_{\mathcal{A}}$. Again, if $\mathcal{A}$ is represented in $\mathcal{L}$, then $\mathcal{I}_a = \{g \in G_{\mathcal{A}} : g : (\mathcal{L}, \langle\ ,\ \rangle_a) \to (\mathcal{L}, \langle\ ,\ \rangle_a)$ is unitary$\}$. For any $a \in G^+$, the map $\mathbf{p}_a : G_{\mathcal{A}} \to G^+$, $\mathbf{p}_a(g) = L_g a = (g^*)^{-1}ag^{-1}$ is a submersion with a global section $\mathbf{s}_a : G^+ \to G_{\mathcal{A}}$, $\mathbf{s}_a(b) = b^{-1/2}a^{1/2}$. As a consequence, we get a diffeomorphism

$$G_{\mathcal{A}}/\mathcal{U}_{\mathcal{A}} \longleftrightarrow G^+,\ g\mathcal{U}_{\mathcal{A}} \mapsto (g^*)^{-1}g^{-1} = (gg^*)^{-1},$$

whose inverse is $b \mapsto b^{-1/2}\mathcal{U}_{\mathcal{A}}$.

Notice that the tangent map of $\mathbf{p}_a$ at $1 \in G_{\mathcal{A}}$ is $(T\mathbf{p}_a)_1 : \mathcal{A} \to \mathcal{A}_h$, $T(\mathbf{p}_a)_1 x = -(ax + x^*a)$. Its kernel is the Banach-Lie algebra of $\mathcal{I}_a$: $\mathfrak{i}_a = N((T\mathbf{p}_a)_1) = (T\mathcal{I}_a)_1$. If $\mathcal{A}$ is represented in $\mathcal{L}$, then $(T\mathcal{I}_a)_1$ consists of the elements of $\mathcal{A}$ which are anti-Hermitian operators in $(\mathcal{L}, \langle\ ,\ \rangle_a)$. Therefore, a natural complement for $(T\mathcal{I}_a)_1$ in $(TG_{\mathcal{A}})_1 = \mathcal{A}$ is

$$\mathbf{H}_a = \{x \in \mathcal{A} : ax = x^*a\},$$

i.e., the a-Hermitian elements of $\mathcal{A}$. It is a relevant fact that the restriction $(T\mathbf{p}_a)_1|_{\mathbf{H}_a} : \mathbf{H}_a \to \mathcal{A}_h$ is a linear isomorphism, whose inverse is $\mathbf{K}_a : \mathcal{A}_h \to \mathbf{H}_a$, $\mathbf{K}_a(z) = -\frac{1}{2}a^{-1}z$. It is easy to verify that $\mathbf{H}_a$ has the following properties:

1. for $g \in \mathcal{I}_a$ and $y \in \mathbf{H}_a$, it holds that $gyg^{-1} \in \mathbf{H}_a$;
2. for $x \in \mathfrak{i}_a = (T\mathcal{I}_a)_1$ and $y \in \mathbf{H}_a$, it holds that $[x, y] = xy - yx \in \mathbf{H}_a$;
3. the map $a \mapsto \mathbf{H}_a$ is smooth.

This shows that the distribution of subspaces $a \mapsto \mathbf{H}_a$ defines a *reductive structure* in G^+. Usually, $\mathbf{H}_a$ is called the *horizontal space* of $\mathcal{A} = (TG_{\mathcal{A}})_a$ at $a \in G^+$.

The map $a \mapsto \mathbf{K}_a$ will be called the 1-*form* of the reductive homogeneous spaces G^+.

As with any reductive homogenous space, for any smooth curve $a = a_t : [0, 1] \to G^+$ there exists a unique *horizontal lifting* of a, i.e., a smooth curve $g = g_t : [0, 1] \to G$ such that $\mathbf{p}_{a_0} g = a$ and $\dot{g}_t := \frac{dg}{dt} \in \mathbf{H}_{g_t}$ for all $t \in [0, 1]$. The horizontal lifting g is the unique solution of the linear differential equation

$$\begin{cases} \dot{g} = \mathbf{K}_a(\dot{a})g \\ g_0 = 1, \end{cases}$$

which we call the *transport equation* for G^+. The solution g provides the parallel transport of vectors along the curve a. Explicitly, the transport equation for G^+ is

$$\begin{cases} \dot{g} = -\frac{1}{2}a^{-1}\dot{a}g \\ g_0 = 1. \end{cases}$$

The solution g is used to define (via parallel transport) the *covariant derivative* of a vector field $X = X_t$ along $a = a_t$:

$$\frac{DX}{dt} := (g_t^*)^{-1}\left(\frac{d}{dt}(TL_{g_t})_{a_t}X_t\right)g_t^{-1} = \dot{X} - \frac{1}{2}(Xa^{-1}\dot{a} + \dot{a}a^{-1}X).$$

The vector field is *parallel* if $\dfrac{DX}{dt} = 0$. A curve $a = a_t$ is called a *geodesic* of this linear connection if $\dot a$ is parallel, i.e.,

$$\ddot a = \dot a a^{-1} \dot a. \tag{1}$$

There are two classical problems associated to the equation (1), namely:

1. the *initial values* problem

$$\begin{cases} \ddot a = \dot a a^{-1} \dot a \\ a_0 = b \\ \dot a_0 = x, \quad \text{for } b \in G^+, x \in (TG^+)_b \end{cases}$$

and

2. the *boundary values* problem

$$\begin{cases} \ddot a = \dot a a^{-1} \dot a \\ a_0 = b \\ a_1 = c, \quad \text{for } b, c \in G^+ \end{cases}$$

Both have a unique solution. For the initial values problem the solution is

$$a_t = e^{-\frac{t}{2}x} = e^{t\mathbf{K}_1(x)} \text{ if } b = 1,$$

or

$$a_t = b^{1/2} e^{-\frac{t}{2} b^{-1/2} x b^{-1/2}} b^{1/2} \tag{2}$$

for $b \in G^+$ and $x \in \mathcal{A}_h = (TG^+)_1$ arbitrary. The solution for the boundary valued problem is

$$a_t = b^{1/2}(b^{-1/2} c b^{-1/2})^t b^{1/2}. \tag{3}$$

Note that the velocity vector at $t = 0$ of this geodesic is $x = -2\log(b^{-1/2} c b^{-1/2})$

The *exponential map* of G^+ is therefore given by

$$\exp_a : (TG^+)_a \simeq \mathcal{A}_h \to G^+ \ , \ \ \exp_a(x) = a_1 = a^{1/2} e^{a^{-1/2} x a^{-1/2}} a^{1/2}, \tag{4}$$

which is a global diffeomorphism with inverse

$$\log_a : G^+ \to \mathcal{A}_h \ , \ \ \log_a b = a^{1/2} \log(a^{-1/2} b a^{-1/2}) a^{1/2}. \tag{5}$$

There is a natural metric in G^+ which comes from a Finsler structure, i.e., a continuous distribution of norms $a \mapsto \| \ \|_a$ on $(TG^+)_a$. Define $\|X\|_a := \|a^{-1/2} X a^{-1/2}\|$. It is easy to show that $\|X\|_a = \|g^* X g\|_{g^* a g}$. Put $d(a,b) = \inf \text{length}(\gamma)$, for γ a smooth curve joining a and b, where $\text{length}(\gamma) = \int_0^1 \|\dot\gamma(t)\|_{\gamma(t)} dt$. Therefore d is invariant under the action of $G_{\mathcal{A}}$, which means that G acts isometrically on G^+.

Theorem 2.1.

1. *If $b, c \in G^+$ and $a_t = b^{1/2}(b^{-1/2} c b^{-1/2})^t b^{1/2}$ is the unique geodesic joining b and c, then $a = a_t$ is shortest among all smooth curves joining b and c in G^+. Then*

$$d(b,c) = \text{length}(a) = \| \log(b^{-1/2} c b^{-1/2}) \|.$$

2. *If $a = a_t, b = b_t$ are geodesics in G^+, then $f(t) = d(a_t, b_t)$ is a convex function in $\mathbb{R}$*

Remark 2.2. Metric spaces with these properties are called *non positive curvature spaces* in the sense of Alexandroff [18], [7], [9]. Note that the geodesics are defined for all $t \in \mathbb{R}$, and that he distance d is $G_{\mathcal{A}}$-invariant.

3 Polar decomposition of reflections

If T is a bounded linear operator on a Hilbert space, there exists a unique partial isometry V such that $T = V|T| = |T^*|V$, with $|T| = (T^*T)^{1/2}$ such that $N(V) = N(T)$ (N denotes the nullspace). In an abstract C*-algebra $\mathcal{A}$, a similar result holds, except that the partial isometry may not belong to $\mathcal{A}$, but to $\mathcal{A}''$, the enveloping von Neumann algebra of $\mathcal{A}$. However, if T is invertible, then $V = T|T|^{-1} \in \mathcal{A}$. This is called the *polar decomposition* of T.

Proposition 3.1. *If $a \in G_{\mathcal{A}}$ then $\sigma_a := a|a|^{-1} = |a^*|^{-1}a \in \mathcal{U}_{\mathcal{A}}$.*

For further use, we remark that if $a^2 = 1$, then $\sigma_a = \sigma_a^{-1} = \sigma_a^*$ and $\sigma_a|a| = |a|^{-1}\sigma_a$.

4 A space of idempotents related to a symmetry

From now on, we shall deal with the C*-algebra $M_2 = M_2(\mathcal{A}) = \mathcal{A}^{2\times 2}$, which can be regarded as the algebra of all $\mathcal{A}$-linear adjointable operators acting in the right C* $\mathcal{A}$-module $\mathcal{A}^2$. The $\mathcal{A}$-valued inner product of this module is given by

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^*\mathbf{y} = x_1^* y_1 + x_2^* y_2,$$

for $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$. The adjoint of a matrix $a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in M_2$ is given by

$$a^* = \begin{pmatrix} a_{11}^* & a_{21}^* \\ a_{12}^* & a_{22}^* \end{pmatrix}.$$

If $a \in M_2$ and $\mathcal{S}$ is a submodule of $\mathcal{A}^2$, we write $a \geq 0$ in $\mathcal{S}$ if $\langle a\mathbf{x}, \mathbf{x} \rangle \geq 0$ for all $\mathbf{x} \in \mathcal{S}$.

We define

$$\mathcal{Q} = \{q \in M_2 : q^2 = q\} \tag{6}$$

and

$$\mathcal{P} = \{p \in \mathcal{Q} : p^* = p\}. \tag{7}$$

Notice that the map $q \mapsto 2q - 1$ establishes natural bijections

$$\mathcal{Q} \to \{a \in M_2 : a^2 = 1\}.$$

For later use, we recall that $\mathcal{Q}$ is a smooth submanifold of M_2 with tangent spaces given by

$$(T\mathcal{Q})_q = \{x \in M_2 : xq + qx = x\},$$

for $q \in \mathcal{Q}$. Also $\mathcal{P}$ is a smooth submanifold of $\mathcal{Q}$, and for $p \in \mathcal{P}$,

$$(T\mathcal{P})_p = \{x \in M_2 : xp + px = x \text{ and } x^* = x\}.$$

Proposition 4.1. *Given $q \in \mathcal{Q}$ and a symmetry σ, the following conditions are equivalent:*

1. *$\sigma(2q - 1) \in G_2^+$.*
2. *There exists $\lambda \in G_2^+$ such that $2q - 1 = \lambda^2\sigma$.*

3. There exists $\lambda \in G_2^+$ such that $q = \lambda(\frac{1}{2}(1+\sigma))\lambda^{-1}$ and $\sigma\lambda\sigma = \lambda^{-1}$.

4. $\sigma \geq 0$ in $R(q)$, $\sigma \leq 0$ in $N(q)$ and $q = \sigma q^* \sigma$.

This result gives several characterizations of the set

$$\mathcal{Q}_\sigma = \{q \in \mathcal{Q} : \sigma(2q-1) \in G_2^+\}, \tag{8}$$

which plays a central role in this exposition. Notice also the immersion

$$\mathcal{Q}_\sigma \hookrightarrow G_{M_2}^+, \quad \text{defined by} \quad q \mapsto (2q-1)\sigma. \tag{9}$$

The fiber $\mathcal{Q}_\sigma$ is a smooth submanifold of $\mathcal{Q}$, and for $q \in \mathcal{Q}_\sigma$

$$(T\mathcal{Q}_\sigma)_q = \{x \in M_2 : xq + qx = x \text{ and } \sigma x^* = x\sigma\}.$$

5 The unitary group of a symmetry

Every symmetry $\sigma \in M_2$ defines an $\mathcal{A}$-valued sesquilinear form θ_σ in $\mathcal{A}^2$:

$$\theta_\sigma : \mathcal{A}^2 \times \mathcal{A}^2 \to \mathcal{A}\ , \quad \theta_\sigma(\mathbf{x}, \mathbf{y}) = \langle \sigma\mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^* \sigma \mathbf{y}. \tag{10}$$

Every $a \in M_2$ admits a θ_σ-adjoint $a^{\sharp_\sigma} \in M_2$,

$$\theta_\sigma(a\mathbf{x}, \mathbf{y}) = \theta_\sigma(\mathbf{x}, a^{\sharp_\sigma}\mathbf{y}), \quad \text{i.e.,} \quad a^{\sharp_\sigma} = \sigma a^* \sigma. \tag{11}$$

The *unitary group* $\mathcal{U}_\sigma$ of θ_σ is

$$\mathcal{U}_\sigma = \{a \in G_{M_2} : \theta_\sigma(a\mathbf{x}, a\mathbf{y}) = \theta_\sigma(\mathbf{x}, \mathbf{y}) \text{ for all } \mathbf{x}, \mathbf{y} \in \mathcal{A}^2\} = \{a \in G_{M_2} : \sigma a^* \sigma = a^{-1}\}. \tag{12}$$

In this paper we shall study the group $\mathcal{U}_\sigma$ and some relevant homogeneous spaces, only for the special symmetries

$$\sigma = \rho := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \sigma = \rho' := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

In the remaining of this section we focus in the case ρ. Explicitly,

$$\theta_\rho : \mathcal{A}^2 \times \mathcal{A}^2 \to \mathcal{A}\ , \theta_\rho(\mathbf{x}, \mathbf{y}) = x_1^* y_1 - x_2^* y_2 = \mathbf{x}^* \rho \mathbf{y}.$$

By (11), for $a \in M_2$ it holds that $a^{\sharp_\rho} = \rho a^* \rho = \begin{pmatrix} a_{11}^* & -a_{21}^* \\ -a_{12}^* & a_{22}^* \end{pmatrix}$. Therefore, $a \in \mathcal{U}_\rho$ if and only if $a \in G_{M_2}$ and satisfies $\rho a^* \rho = a^{-1}$; explicitly

$$\begin{cases} a_{11}^* a_{11} - a_{21}^* a_{21} = 1 \\ a_{11}^* a_{12} - a_{21}^* a_{22} = 0 \\ a_{12}^* a_{11} - a_{22}^* a_{21} = 0 \\ a_{22}^* a_{22} - a_{12}^* a_{12} = 1 \end{cases} \tag{13}$$

Remark 5.1. It can be shown that $\mathcal{U}_\rho$ is a Banach-Lie subgroup of G_{M_2} ([1]). Its Banach-Lie algebra is given by

$$\mathfrak{U}_\rho = \{x \in M_2 : \rho x + x^* \rho = 0\} = \{\begin{pmatrix} b & x^* \\ x & c \end{pmatrix} : b, c \in \mathcal{A}_{ah}, x \in \mathcal{A}\},$$

where $\mathcal{A}_{ah} = \{x \in \mathcal{A} : x^* + x = 0\}$.

Observe also that the Banach-Lie algebra of the subgroup $\mathcal{U}_\rho \cap \mathcal{U}_{M_2}$ is

$$(T(\mathcal{U}_\rho \cap \mathcal{U}_{M_2}))_1 = \{\begin{pmatrix} b & 0 \\ 0 & c \end{pmatrix} : b, c \in \mathcal{A}_{ah}\}.$$

Using (13) it follows that the subgroup $\mathcal{U}_\rho$ is closed for the adjoint operation and that $a_{11}, a_{22} \in G_{\mathcal{A}}$.

Proposition 5.2. *The map*

$$\mathcal{Q}_\rho \to \mathcal{U}_\rho^+, \ q \mapsto ((2q-1)\rho)^{1/2}$$

is a diffeomorphism with inverse $\lambda \mapsto \lambda p \lambda^{-1}$.

Remark 5.3. Notice that the diffeomorphism above has the explicit matrix form

$$q \mapsto \begin{pmatrix} (1 + q_{11}^{-1/2} q_{21}^* q_{21} q_{11}^{-1/2})^{1/2} & q_{11}^{-1/2} q_{21}^* \\ q_{21} q_{11}^{-1/2} & (1 + q_{21} q_{11}^{-1} q_{21}^*)^{1/2} \end{pmatrix}. \tag{14}$$

The elements of $\mathcal{U}_\rho$ are known in different areas of mathematics as Möbius, linear fractional [29], [24], [31] or symplectic transformations [28], [16], [35].

6 $\mathcal{Q}_\rho$ as a homogeneous space of $\mathcal{U}_\rho$

In this section we prove that $\mathcal{Q}_\rho$ and $\mathcal{D}$ are diffeomorphic as reductive homogeneous spaces of $\mathcal{U}_\rho$, by means of an $\mathcal{U}_\rho$-equivariant diffeomorphism.

The set

$$\mathcal{K}_\rho := \{\mathbf{x} \in \mathcal{A}^2 : \mathbf{x}^* \rho \mathbf{x} = 1, \ x_1 \in G_{\mathcal{A}}\} \tag{15}$$

plays a role which is similar to that of the punctured plane in the stereographic projection. Notice that $\mathcal{K}_\rho$ consists of the elements of the unit sphere of θ_ρ whose first coordinate is invertible.

In order to identify the spaces $\mathcal{Q}_\rho$ and $\mathcal{D}$ we shall use projective methods. In fact, we present both spaces as subsets of the "projective line" associated with $\mathcal{A}^2$. For instance, $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathcal{K}_\rho$ will serve to indicate the element $z \in \mathcal{D}$, $z = x_2 x_1^{-1}$ by means of its "homogeneous coordinates". Analogously, a point $q \in \mathcal{Q}_\rho$ is represented by $\mathbf{x} \in \mathcal{K}_\rho$ as the θ_ρ-orthogonal projection over the $\mathcal{A}$-submodule $[\mathbf{x}]$ spanned by $\mathbf{x}$.

We collect in the next proposition several useful facts which relate $\mathcal{K}_\rho$ with the group $\mathcal{U}_\rho$.

Proposition 6.1.

1. *If* $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathcal{K}_\rho$, *then* $\|x_2 x_1^{-1}\| < 1$.

2. *If $a \in \mathcal{U}_\rho$ and $\mathbf{x} \in \mathcal{K}_\rho$, then $a\mathbf{x} \in \mathcal{K}_\rho$.*

3. *If $\mathbf{x} \in \mathcal{K}_\rho$ then there exists $\mathbf{y} \in \mathcal{A}^2$ such that $\mathbf{x}^*\rho\mathbf{y} = 0$ and $\mathbf{y}^*\rho\mathbf{y} = -1$; in particular $a = \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} \in \mathcal{U}_\rho$.*

4. *The action of $\mathcal{U}_\rho$ on $\mathcal{K}_\rho$ by left multiplication (see 2. above) is transitive: $\mathcal{K}_\rho = \mathcal{U}_\rho\mathbf{x}$ for any $\mathbf{x} \in \mathcal{K}_\rho$.*

The next theorem is a useful characterization of $\mathcal{Q}_\rho$ by means of elements of $\mathcal{K}_\rho$; it will be frequently used in the sequel. First we need to establish a few facts.

Definition 6.2. *If $\mathbf{x} \in \mathcal{K}_\rho$, denote*

$$p_{\mathbf{x}} = \mathbf{x}\mathbf{x}^*\rho \in M_2. \tag{16}$$

Lemma 6.3. *For $\mathbf{x} \in \mathcal{K}_\rho$ it holds that $p_{\mathbf{x}} \in \mathcal{Q}$ and $R(p_{\mathbf{x}}) = [\mathbf{x}] := \{\mathbf{x}a : a \in \mathcal{A}\}$. In particular, this latter space is a closed and complemented submodule of $\mathcal{A}^2$. For $\mathbf{x}, \mathbf{x}' \in \mathcal{K}_\rho$, $p_{\mathbf{x}} = p_{\mathbf{x}'}$ if and only if $[\mathbf{x}] = [\mathbf{x}']$, if and only if there exists $u \in \mathcal{U}_\mathcal{A}$ such that $\mathbf{x}' = \mathbf{x}u$.*

Theorem 6.4. *$\mathcal{Q}_\rho$ coincides with the set $\{p_{\mathbf{x}} : \mathbf{x} \in \mathcal{K}_\rho\}$.*

Theorem 6.5.

1. *The map $\varphi_\rho : \mathcal{K}_\rho \to \mathcal{Q}_\rho \quad \varphi_\rho(\mathbf{x}) = p_{\mathbf{x}}$ is a smooth principal fibration, with group $\mathcal{U}_\mathcal{A}$ and with a global cross section: $\sigma_\rho : \mathcal{Q}_\rho \to \mathcal{K}_\rho$, $\sigma_\rho(q) = \lambda_q\mathbf{e}_1 = \begin{pmatrix} q_{11}^{1/2} \\ q_{21}q_{11}^{-1/2} \end{pmatrix}$.*

2. *The map $\Phi : \mathcal{K}_\rho \to \mathcal{Q}_\rho \times \mathcal{U}_\mathcal{A}$, $\Phi(\mathbf{x}) = (p_{\mathbf{x}}, \mathbf{x}^*\rho\sigma_\rho(p_{\mathbf{x}}))$ is a diffeomorphism with inverse $\Phi^{-1}(q, u) = \sigma_\rho(q)u^*$.*

Remark 6.6. One should notice that, with the current assumptions and notations, $\mathbf{x}^*\rho\sigma_\rho(p_{\mathbf{x}}) = x_1^*(x_1x_1^*)^{-1/2}$

The *tautological bundle* of $\mathcal{Q}_\rho$ is defined by

$$\{(q, \mathbf{x}) \in \mathcal{Q}_\rho \times \mathcal{A}^2 : \mathbf{x} \in R(q)\}$$

with the projection $(q, \mathbf{x}) \mapsto q$. We get the diagram

$$\begin{array}{ccl} \mathcal{K}_\rho & \hookrightarrow & \{(a, \mathbf{x}) :\in \mathcal{Q}_\rho \times \mathcal{A}^2 : \mathbf{x} \in R(q)\} \\ & \searrow & \downarrow \\ & & \mathcal{Q}_\rho \end{array} \tag{17}$$

where the horizontal arrow is given by $\mathbf{x} \mapsto (p_{\mathbf{x}}, \mathbf{x})$ ($\mathbf{x} \in \mathcal{K}_\rho$). A corollary of Theorem 6.4 is that $\varphi_\rho : \mathcal{K}_\rho \to \mathcal{Q}_\rho$ is the principal bundle with group $\mathcal{U}_\mathcal{A}$ associated to the tautological bundle. We shall pursue this matter later (Section 14).

Proposition 6.7.

1. *$\mathcal{U}_\rho$ acts transitively on $\mathcal{Q}_\rho$ by similarity: $\mathcal{U}_\rho \times \mathcal{Q}_\rho \to \mathcal{Q}_\rho$, $(a, q) \mapsto aqa^{-1}$.*

2. *For every $q_0 \in \mathcal{Q}_\rho$, the map $\pi_{q_0} : \mathcal{U}_\rho \to \mathcal{Q}_\rho$, $\pi_{q_0}(a) = aq_0a^{-1}$ is smooth and onto. Moreover, π_{q_0} admits a global cross section.*

Corollary 6.8. *For $q \in \mathcal{Q}_\rho$, the isotropy group H_q of q, $H_q = \{a \in \mathcal{U}_\rho : aqa^{-1} = q\}$ is a closed Lie-Banach subgroup of $\mathcal{U}_\rho$ and the bijection $\mathcal{U}_\rho/H_q \leftrightarrow \mathcal{Q}_\rho$ is a diffeomorphism. In other terms, Q_ρ is a reductive homogeneous space of $\mathcal{U}_\rho$.*

The proof is straightforward. The cross section of the above proposition induces the inverse diffeomorphism.

We note now that $\mathcal{Q}_\rho$ is a reductive homogeneous space of $\mathcal{U}_\rho$. We proved before (Remark 5.1) that the Lie algebra of $\mathcal{U}_\rho$ is

$$\mathfrak{u}_\rho = \{x \in M_2 : \rho x^* \rho = -x\} = \{\begin{pmatrix} b & c^* \\ c & d \end{pmatrix} : b, c, d \in \mathcal{A},\ b + b^* = 0,\ d + d^* = 0\}.$$

The isotropy group of p in $\mathcal{U}_\rho$ is

$$\mathcal{I}_p = \{a \in \mathcal{U}_\rho : ap = pa\} = \{\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} : u, v \in \mathcal{U}_\mathcal{A}\}.$$

Then its Lie algebra is

$$\mathfrak{i}_p = \{\begin{pmatrix} b & 0 \\ 0 & d \end{pmatrix} : b + b^* = d + d^* = 0\},$$

and a natural supplement for $\mathfrak{i}_p$ in $\mathfrak{u}_\rho$ is

$$\mathbf{H}_p = \{\begin{pmatrix} 0 & c^* \\ c & 0 \end{pmatrix} : c \in \mathcal{A}\}. \tag{18}$$

It is fairly obvious that $a\mathbf{H}_p a^{-1} \subset \mathbf{H}_p$ for every $a \in \mathcal{I}_p$.

It follows then that the unique geodesic δ in $\mathcal{Q}_\rho$ with $\delta(0) = p$ and $\dot{\delta}(0) = X \in \mathbf{H}_p$ is

$$\delta(t) = e^{tX} \cdot p = e^{\frac{t}{2}X} p\ e^{-\frac{t}{2}X}. \tag{19}$$

Recall from (9) the immersion

$$\Phi : \mathcal{Q}_\rho \to G^+_{M_2}\ , \quad q :\mapsto ((2q - 1)\rho)^{1/2},$$

which satisfies that $im\ \Phi = \mathcal{U}^+_\rho$.

We shall prove later that this immersion preserves the relevant geometrical structures. In particular, it translates geodesics of $\mathcal{Q}_\rho$ into geodesics of $G^+_{M_2}$. Conversely, every geodesic of $G^+_{M_2}$ contained in $\mathcal{U}^+_\rho$ is the image of a geodesic in $\mathcal{Q}_\rho$. Using this translation we are able to compute in an explicit way all geodesics of $\mathcal{Q}_\rho$.

If $q_0, q_1 \in \mathcal{Q}_\rho$, then

$$\mu_0 = ((2q_0 - 1)\rho)^{1/2}\ , \quad \mu_1 = ((2q_1 - 1)\rho)^{1/2} \tag{20}$$

belong to $G^+_{M_2}$ and, therefore, they are joined by a unique geodesic μ in $G^+_{M_2}$. Explicitly, the curve

$$\mu : [0,1] \to G^+_{M_2} \ , \quad \mu(t) = \mu_0^{1/2} \left(\mu_0^{-1/2} \mu_1 \mu_0^{-1/2}\right)^t \mu_0^{1/2}$$

is the unique geodesic in $G^+_{M_2}$ such that $\mu(0) = \mu_0$ and $\mu(1) = \mu_1$ (recall (3) in Section 2). By definition, $\mu_0, \mu_1 \in im\ \Phi = \mathcal{U}^+_\rho$; it is an easy exercise to show that $\mu(t) \in \mathcal{U}^+_\rho$ for all $t \in [0,1]$ (for all $t \in \mathbb{R}$, in fact). From the polar decompositions

$$(2q_0 - 1) = (2q_0 - 1)\rho\rho = \mu_0^2 \rho, \ (2q_1 - 1) = (2q_1 - 1)\rho\rho = \mu_1^2 \rho$$

and the fact (Proposition 4.1) that $q_0 = \mu_0 p \mu_0^{-1}$ and $q_1 = \mu_1 p \mu_1^{-1}$ we see that

$$q(t) = \mu(t) p \mu^{-1}(t) \tag{21}$$

is the geodesic in $\mathcal{Q}_\rho$ joining q_0 and q_1.

Remark 6.9. It is apparent that $\mathcal{U}^+_\rho$ is also a reductive homogeneous space of $\mathcal{U}_\rho$. We do not pursue its study here, since its geometry is that of $\mathcal{Q}_\rho$.

7 $\mathcal{D}$ as a homogeneous space of $\mathcal{U}_\rho$

Our next step is to show that the Poincaré disk

$$\mathcal{D} = \{z \in \mathcal{A} : z^* z < 1\} = \{z \in \mathcal{A} : \|z\| < 1\}$$

is a reductive homogeneous space of $\mathcal{U}_\rho$, equivariantly diffeomorphic to $\mathcal{Q}_\rho$.

Recall from Proposition 6.1 that $x_2 x_1^{-1} \in \mathcal{D}$ for every $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathcal{K}_\rho$.

Proposition 7.1. *The map*

$$\varphi_D : \mathcal{K}_\rho \to \mathcal{D} \ , \quad \varphi_D(\mathbf{x}) = x_2 x_1^{-1}$$

is a smooth submersion with a smooth global cross section. Namely,

$$\mathbf{x}_z := \begin{pmatrix} 1 \\ z \end{pmatrix} (1 - z^* z)^{-1/2} \in \mathcal{K}_\rho,$$

and $\varphi_D(\mathbf{x}_z) = z$*; thus* $\sigma_D(z) = \mathbf{x}_z$ *defines a smooth global cross section for* φ_D*.*

Notice that $\sigma_D(\varphi_D(\mathbf{x})) = \mathbf{x}_{x_2 x_1^{-1}} = \begin{pmatrix} 1 \\ x_2 x_1^{-1} \end{pmatrix} (x_1 x_1^*)^{1/2}$.

We have the following proposition:

Proposition 7.2. *If* $\mathbf{x} \in \mathcal{K}_\rho$ *then* $\mathbf{x}^* \rho \sigma_D(x_2 x_1^{-1}) \in \mathcal{U}_\mathcal{A}$*, and the map*

$$\Phi_D : \mathcal{K}_\rho \to \mathcal{D} \times \mathcal{U}_\mathcal{A}, \ \Phi_D(\mathbf{x}) = (x_2 x_1^{-1}, \mathbf{x}^* \rho \sigma_D(x_2 x_1^{-1}))$$

is a diffeomorphism with inverse

$$\Phi_D^{-1}(z, u) = \sigma_D(z) u^* = \begin{pmatrix} 1 \\ z \end{pmatrix} (1 - z^* z)^{-1/2} u^*.$$

Definition 7.3. *Define the action of $\mathcal{U}_\rho$ on $\mathcal{D}$ as follows: for $a \in \mathcal{U}_\rho$ and $z \in \mathcal{D}$, put*

$$a \cdot z := \varphi_D(a\sigma_D(z)) = \varphi_D(a \begin{pmatrix} 1 \\ z \end{pmatrix} (1 - z^*z)^{-1/2}) = (a_{21} + a_{22}z)(a_{11} + a_{12}z)^{-1}.$$

Proposition 7.4. *The action of $\mathcal{U}_\rho$ on $\mathcal{D}$ is transitive.*

With a similar argument as in Proposition 6.1.4, it easy to show that

$$g_z = \begin{pmatrix} (1 - z^*z)^{-1/2} & (1 - z^*z)^{-1/2}z^* \\ z(1 - z^*z)^{-1/2} & (1 - zz^*)^{-1/2} \end{pmatrix} \in \mathcal{U}_\rho \tag{22}$$

and $g_z \cdot 0 = z$.

For each $z_0 \in \mathcal{D}$, consider

$$\pi_{z_0} : \mathcal{U}_\rho \to \mathcal{D} \ , \quad \pi_{z_0}(a) = a \cdot z_0.$$

Theorem 7.5. *$\mathcal{D}$ is a reductive homogeneous space of $\mathcal{U}_\rho$.*

Consider $z_0 = 0$. The isotropy subgroup $\mathcal{I}_0 = \{a \in \mathcal{U}_\rho : a \cdot 0 = 0\}$ is

$$\mathcal{I}_0 = \{\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} : u, v \in \mathcal{U}_\mathcal{A}\}.$$

The Lie algebra of $\mathcal{I}_0$ is

$$\mathfrak{i}_0 = \{\begin{pmatrix} b & 0 \\ 0 & d \end{pmatrix} : b^* = -b, \ d^* = -d\}$$

and a supplement for $\mathfrak{i}_0$ in $\mathfrak{u}_\rho$ is

$$\mathbf{H}_0 = \{\begin{pmatrix} 0 & c^* \\ c & 0 \end{pmatrix} : c \in \mathcal{A}\}$$

which is a reductive complement.

As in the case of $\mathcal{Q}_\rho$, the geodesics of the affine connection of $\mathcal{D}$ starting at 0 are, exactly, those of the form $\delta(t) = e^{tX} \cdot 0$, with $X \in \mathbf{H}_0$; the geodesics starting at $z \in \mathcal{D}$ are $g_z \cdot (\delta(t))$.

The next result identifies $\mathcal{Q}_\rho$ and $\mathcal{D}$ as reductive homogeneous spaces of $\mathcal{U}_\rho$.

Theorem 7.6. *There exists a diffeomorphism $\alpha_D : \mathcal{Q}_\rho \to \mathcal{D}$ which makes the following diagram commutative*

$$\begin{array}{ccc} & \mathcal{K}_\rho & \\ {}^{\varphi_\rho}\swarrow & & \searrow^{\varphi_D} \\ \mathcal{Q}_\rho & \xrightarrow{\ \alpha_D\ } & \mathcal{D}, \end{array} \tag{23}$$

All maps in the diagram are $\mathcal{U}_\rho$-equivariant, i.e., for every $\mathbf{x} \in \mathcal{K}_\rho$, $a \in \mathcal{U}_\rho$ and $q \in \mathcal{Q}_\rho$ it holds that

$$\varphi_\rho(a\mathbf{x}) = ap_\mathbf{x}a^{-1} \ , \quad \varphi_D(a\mathbf{x}) = a \cdot (x_2x_1^{-1}) \quad \textit{and} \quad \alpha_D(ap_\mathbf{x}a^{-1}) = p_{a\mathbf{x}}.$$

We continue here the study of the geodesics of $\mathcal{D}$.

It is easy to see that for any $z \in \mathcal{D}$, the unique geodesic δ of $\mathcal{D}$ such that $\delta(0) = 0$ and $\delta(1) = z$ is given by

$$\delta(t) = e^{t\begin{pmatrix} 0 & \alpha^* \\ \alpha & 0 \end{pmatrix}} \cdot 0 \ , \quad \text{where} \ \ \alpha = z \sum_{k=0}^{\infty} \frac{1}{2k+1}(z^*z)^k. \tag{24}$$

Note here that $\sum_{k=0}^{\infty} \frac{t^{2k+1}}{2k+1}$ is the Taylor series of the function $f(t) = \frac{1}{2}\log(\frac{1+t}{1-t})$, in the interval $(-1, 1)$. We shall compute below the explicit form of δ in $\mathcal{D}$, joining 0 and $z \in \mathcal{D}$ at time $t = 1$:

Lemma 7.7. *Given $z \in \mathcal{D}$, the unique geodesic δ of $\mathcal{D}$ with $\delta(0) = 0$ and $\delta(1) = z$ is given by*

$$\delta(t) = \omega \tanh(t|\alpha|), \tag{25}$$

where α is given in (24) *above, and ω is the partial isometry in the polar decomposition of α: $\alpha = \omega|\alpha|$, performed in $\mathcal{A}^{**}$.*

Notice that $\omega \in \mathcal{A}^{**}$ need not belong to $\mathcal{A}$. However, $\delta(t) \in \mathcal{A}$ for all t. Concerning partial isometries in C*-algebras, see [21].

Let us relate the polar decompositions of z and α.

Proposition 7.8. *If $z \in \mathcal{D}$ and α as in* (24)*, then*

$$\alpha = \frac{1}{2}\omega(\log(1 + |z|) - \log(1 - |z|)), \quad \text{and} \quad z = \omega|z|,$$

i.e., the partial isometry $\omega \in \mathcal{A}^{**}$ is the same for α and z.

Corollary 7.9. *The exponential map $\exp_0$ of $\mathcal{D}$ at 0, and its inverse $\log_0$ can be written explicitly as follows: if $z = \omega|z| \in \mathcal{D}$*

$$\log_0 : \mathcal{D} \to (T\mathcal{D})_0 \ , \ \log_0(z) = \frac{1}{2}\omega \log\left((1 + |z|)(1 - |z|)^{-1}\right).$$

If $\alpha = \omega|\alpha| \in \mathcal{A} \simeq (T\mathcal{D})_0$, then

$$\exp_0 : (T\mathcal{D})_0 \to \mathcal{D} \ , \ \exp_0(\alpha) = \omega \tanh(|\alpha|).$$

In particular, if $\alpha = \log_0(z)$ (or, equivalently, $z = \exp_0(\alpha)$), then z and α have the same partial isometry in the polar decomposition. Also,

$$|\exp_0(\alpha)| = \tanh(|\alpha|) \quad \text{and} \quad |\log_0(z)| = \frac{1}{2}\log\left((1 + |z|)(1 - |z|)^{-1}\right). \tag{26}$$

8 $\mathcal{H}$ as a reductive homogeneous space

Consider now the symmetry

$$\rho' := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{27}$$

the $\mathcal{A}$-valued form $\theta_{\rho'}$

$$\theta_{\rho'} : \mathcal{A}^2 \times \mathcal{A}^2 \to \mathcal{A},\ \theta_{\rho'}(\mathbf{x}, \mathbf{y}) = \mathbf{x}^* \rho' \mathbf{y} = -i(x_1^* y_2 - x_2^* y_1) = 2 \text{ Im } (x_1^* y_2), \tag{28}$$

its unitary group

$$\mathcal{U}_{\rho'} = \{a \in G_{M_2} : \rho' a^* \rho' = a^{-1}\}, \tag{29}$$

and the hyperboloid

$$\mathcal{K}_{\rho'} = \{\mathbf{x} \in \mathcal{A}^2 : \mathbf{x}^* \rho' \mathbf{x} = 1, x_1 \in G_{\mathcal{A}}\}. \tag{30}$$

Consider also the unitary matrix in M_2

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}. \tag{31}$$

Proposition 8.1. *The matrix U has the following properties,*

1. *it intertwines ρ and ρ': $\rho' = U \rho U^*$;*
2. *it conjugates the unitary groups: $\mathcal{U}_{\rho'} = U \mathcal{U}_\rho U^*$;*
3. *it maps one hyperboloid onto the other: $\mathcal{K}_{\rho'} = U \mathcal{K}_\rho$.*

Proposition 8.2.

1. *If $\mathbf{x} \in \mathcal{K}_{\rho'}$ then $x_2 x_1^{-1} \in \mathcal{H}$.*
2. *The map*

$$\varphi_H : \mathcal{K}_{\rho'} \to \mathcal{H} \ , \quad \varphi_H(\mathbf{x}) = x_2 x_1^{-1} \tag{32}$$

is a submersion with a global cross section

$$\sigma_H : \mathcal{H} \to \mathcal{K}_{\rho'} \ , \quad \sigma_H(h) = \begin{pmatrix} 1 \\ h \end{pmatrix} (2 \ Im \ h)^{-1/2}. \tag{33}$$

As in the case of ρ, we define

$$\mathcal{Q}_{\rho'} = \{q \in \mathcal{Q} : (2q - 1)\rho' \in G^+_{M_2}\} \tag{34}$$

and we get analogous results and properties. In particular, if $\mathbf{x} \in \mathcal{K}_{\rho'}$ and $p_{\mathbf{x}} = \mathbf{x}\mathbf{x}^* \rho'$, then $p_{\mathbf{x}} \in \mathcal{Q}_{\rho'}$ and these idempotents exhaust $\mathcal{Q}_{\rho'}$:

$$\mathcal{Q}_{\rho'} = \{\mathbf{x}\mathbf{x}^* \rho' : \mathbf{x} \in \mathcal{K}_{\rho'}\}.$$

We get an analogous commutative diagram

$$\begin{array}{ccc} & \mathcal{K}_{\rho'} & \\ {\scriptstyle \varphi_{\rho'}} \swarrow & & \searrow {\scriptstyle \varphi_H} \\ \mathcal{Q}_{\rho'} & \xrightarrow{\ \alpha_H\ } & \mathcal{H}, \end{array} \tag{35}$$

where $\varphi_{\rho'}(\mathbf{x}) = \mathbf{x}\mathbf{x}^*\rho'$ and $\alpha_H(p_{\mathbf{x}}) = x_2 x_1^{-1}$ is a well defined diffeomorphism.

The group $\mathcal{U}_{\rho'}$ acts on $\mathcal{H}$ (as $\mathcal{U}_\rho$ acts on $\mathcal{D}$) as follows. First, note that, if $a \in \mathcal{U}_{\rho'}$ and $h \in \mathcal{H}$ then

$$\mathbf{x}_h = \begin{pmatrix} 1 \\ h \end{pmatrix} (2 \text{ Im } h)^{-1/2} \in \mathcal{K}_{\rho'} \tag{36}$$

and also $a\mathbf{x}_h \in \mathcal{K}_{\rho'}$ by Proposition 8.1.

Definition 8.3. *For $a \in \mathcal{U}_{\rho'}$ and $h \in \mathcal{H}$ put*

$$a \cdot h = \varphi_H(a\mathbf{x}_h) = (a\mathbf{x}_h)_2 (a\mathbf{x}_h)_1^{-1} = (a_{21} + a_{22}h)(a_{11} + a_{12}h)^{-1}.$$

It is a routine calculation to prove that this action is well defined and that all maps in the diagram (35) are $\mathcal{U}_{\rho'}$-equivariant. The action of $\mathcal{U}_{\rho'}$ on $\mathcal{H}$ is transitive because the unitary U implements an isomorphism between diagrams (23) and (35). However, in the next section we shall prove this result without using the unitary U.

As in the case of $\mathcal{D}$, $\mathcal{H}$ is a reductive homogeneous space of $\mathcal{U}_{\rho'}$ and we get that $\mathcal{H}$ and $\mathcal{Q}_{\rho'}$ are isomorphic (as reductive homogeneous spaces). Combining diagrams (23) and (35) by means of U, we get a bigger diagram

$$\begin{array}{ccccccccc} & & \mathcal{K}_\rho & & \stackrel{U}{\longrightarrow} & & \mathcal{K}_{\rho'} & & \\ \varphi_\rho & \swarrow & & \searrow & \varphi_D \quad & \varphi_{\rho'} & \swarrow & \searrow & \varphi_H \ . \\ \mathcal{D} & & \stackrel{\longrightarrow}{\alpha_D^{-1}} & & \mathcal{Q}_\rho \quad \stackrel{\longrightarrow}{Ad(U)} & \mathcal{Q}_{\rho'} & & \stackrel{\longrightarrow}{\alpha_H} & \mathcal{H} \end{array} \tag{37}$$

Here $Ad(U) : \mathcal{Q}_\rho \to \mathcal{Q}_{\rho'}$ denotes ths isomomorphism $Ad(U)q = UqU^*$.

Remark 8.4. The composition $\alpha_H \circ Ad(U) \circ \alpha_D^{-1}$ provides the (linear fractional) diffeomorphism

$$\Gamma : \mathcal{D} \to \mathcal{H} \ , \quad \Gamma(z) = i(1+z)(1-z)^{-1}$$

with inverse

$$\Gamma^{-1} : \mathcal{H} \to \mathcal{D} \ , \quad \Gamma^{-1}(h) = (1+ih)(1-ih)^{-1}.$$

9 The Borel subgroup of $\mathcal{U}_{\rho'}$

We have proven several results about the relations between the groups $\mathcal{U}_\rho$ and the manifolds $\mathcal{K}_\rho$, $\mathcal{Q}_\rho$ and $\mathcal{D}$. The unitary matrix U allowed us to translate these relations to the analogous spaces in the case of ρ'. We have chosen to proceed in this order (first ρ, then ρ') because many of the properties reviewed had easier and more natural proofs in the former realm. In this section we introduce a subgroup of $\mathcal{U}_{\rho'}$ which is easier to understand, and, more important, it admits a clear geometric interpretation.

Definition 9.1. *The Borel subgroup of* $\mathcal{U}_{\rho'}$ *is*

$$\mathcal{B}' := \{a = \begin{pmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{pmatrix} \in \mathcal{U}_{\rho'}\}.$$

Proposition 9.2. *The subgroup* $\mathcal{B}'$ *coincides with*

$$\{a = \begin{pmatrix} g & 0 \\ (g^*)^{-1}x & (g^*)^{-1} \end{pmatrix} : g \in G_{\mathcal{A}}, x \in \mathcal{A}_h\}.$$

Remark 9.3. Observe that, for $a = \begin{pmatrix} g & 0 \\ (g^*)^{-1}x & (g^*)^{-1} \end{pmatrix} \in \mathcal{B}'$ and $h \in \mathcal{H}$, it holds that

$$a \cdot h = (a\begin{pmatrix} 1 \\ h \end{pmatrix})_2 (a\begin{pmatrix} 1 \\ h \end{pmatrix})_1^{-1} = (g^*)^{-1}(x+h)g^{-1} = L_g(a+h).$$

That is, the action of $\mathcal{B}'$ on h is the affine translation by $x^* = x$ (which remains in $\mathcal{H}$: Im $(x+h)=$ Im $(h) > 0$), followed by the action L_g of $G_{\mathcal{A}}$ extended to $\mathcal{H}$: $L_g k = (g^*)^{-1} k g^{-1}$.

Significantly, there is no loss of geometric information if we restrict the action to the subgroup $\mathcal{B}'$:

Proposition 9.4. $\mathcal{B}'$ *acts transitively on* $\mathcal{H}$.

Proposition 9.5. *The action of* $\mathcal{B}'$ *on* $\mathcal{K}_{\rho'}$ *is free. So in particular,* $\mathcal{B}'$ *is diffeomorphic to* $\mathcal{K}_{\rho'}$.

Specifically, we get the diffeomorphism

$$\mathcal{K}_{\rho'} \to \mathcal{B}' \quad \mathbf{x} \mapsto \begin{pmatrix} \sqrt{2}x_1 & 0 \\ \frac{1}{\sqrt{2}}(x_1^*)^{-1}(x_1^* x_2 + x_2^* x_1) & \frac{1}{\sqrt{2}}(x_1^*)^{-1} \end{pmatrix} \tag{38}$$

with inverse

$$\mathcal{B}' \to \mathcal{K}_{\rho'}\ ,\ \begin{pmatrix} g & 0 \\ (g^*)^{-1}x & (g^*)^{-1} \end{pmatrix} \mapsto \begin{pmatrix} g \\ (g^*)^{-1}(x+i) \end{pmatrix}.$$

Remark 9.6. The diffeomorphism $\mathcal{K}_{\rho'} \simeq \mathcal{B}'$ proved above defines a group structure on $\mathcal{K}_{\rho'}$. On the other hand, as in (17), we have the diagram

$$\begin{array}{ccl} \mathcal{K}_{\rho'} & \hookrightarrow & \{(a, \mathbf{x}) :\in \mathcal{Q}_{\rho'} \times \mathcal{A}^2 : \mathbf{x} \in R(q)\} \\ & \searrow & \downarrow \\ & & \mathcal{Q}_{\rho'}, \end{array} \tag{39}$$

which proves that the map $\mathcal{K}_{\rho'} \to \mathcal{Q}_{\rho'}$, $\mathbf{x} \mapsto p_{\mathbf{x}}$ is a principal fiber bundle with structure group $\mathcal{U}_{\mathcal{A}}$, associated to the tautological bundle on $\mathcal{Q}_{\rho'}$, cf. [6]. This configuration is similar to that of the classical *Hopf fibration*

$$\begin{array}{cccl} \mathbf{S^1} & \hookrightarrow & \mathbf{S^3} & \subset \mathbb{C}^2 \\ & & \downarrow & \\ & & \mathbf{S^2} & = \mathbb{CP}^1, \end{array}$$

where $\mathbf{S^3} \to \mathbf{S^2}$ is the principal $\mathbf{S^1}$-bundle associated to the tautological bundle on $\mathbb{CP}^1$. Moreover, $\mathbf{S^3}$ can be identified with the group Spin (3), which is isomorphic to the group of unit quaternions.

In order to understand the structure of $\mathcal{U}_{\rho'}$, it will be also useful to consider the subgroup $\mathcal{T} \subset \mathcal{U}_{\rho'}$,

$$\mathcal{T} := \{ \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix} : \tau \in \mathcal{A}_h \}. \tag{40}$$

$\mathcal{B}'$ and $\mathcal{T}$ locally decompose $\mathcal{U}_{\rho'}$, in the following sense:

Proposition 9.7. *$\mathcal{B}'$ and $\mathcal{T}$ are Banach-Lie subgroups of $\mathcal{U}_{\rho'}$, and complemented submanifolds of M_2. They generate an open and closed subgroup of $\mathcal{U}_{\rho'}$, which contains the connected component of the identity.*

Remark 9.8. If $\mathcal{A}$ is a von Neumann algebra, then $\mathcal{U}_{\rho'}$ is connected. Since $\mathcal{U}^+_{\rho'}$ is clearly connected (in fact, contractible), one needs to show that the unitary part $\mathbf{D}_{\rho'} = \mathcal{U}_{\rho'} \cap \mathcal{U}_{M_2}$ of $\mathcal{U}_{\rho'}$ is connected. Note that $\mathbf{D}_{\rho'} = \mathcal{U}_{M_2} \cap \{J\}'$. Since J is anti-Hermitian, it follows that $\{J\}' \subset M_2$ is a von Neumann algebra, and therefore $\mathcal{U}_{M_2} \cap \{J\}'$ is the unitary group of a von Neumann algebra, thus connected.

Theorem 9.9. *The unitary part $\mathbf{D}_{\rho'}$ of $\mathcal{U}_{\rho'}$ is isomorphic to $\mathcal{U}_\mathcal{A} \times \mathcal{U}_\mathcal{A}$. The group $\Pi_0(\mathcal{U}_{\rho'})$ of connected components of $\mathcal{U}_{\rho'}$, is isomorphic to $\Pi_0(\mathcal{U}_\mathcal{A}) \times \Pi_0(\mathcal{U}_\mathcal{A})$.*

Corollary 9.10. *$\mathcal{B}'$ and $\mathcal{T}$ generate $\mathcal{U}_{\rho'}$ if and only if $\mathcal{U}_\mathcal{A}$ is connected.*

Remark 9.11. Note that if $c \in \mathcal{T}$, then $c^* \in \mathcal{B}'$. In particular, $\mathcal{B}'$ and $(\mathcal{B}')^* = \{b^* : b \in \mathcal{B}'\}$ generate a subgroup $\mathcal{B}' \vee (\mathcal{B}')^*$ of $\mathcal{U}_{\rho'}$ which contains the connected component of the identity.

The matrix $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is an element of $\mathcal{U}_{\rho'}$. Notice that $\mathcal{B}'$ and J generate the subgroup $\mathcal{B}' \vee J = \mathcal{B}' \vee (\mathcal{B}')^*$. Indeed, if $b = \begin{pmatrix} g & 0 \\ (g^*)^{-1}x & (g^*)^{-1} \end{pmatrix} \in \mathcal{B}'$ $(x^* = x)$, then

$$JbJ^{-1} = \begin{pmatrix} (g^*)^{-1} & -(g^*)^{-1} \\ 0 & g \end{pmatrix},$$

which is the adjoint of $\begin{pmatrix} g^{-1} & 0 \\ -xg^{-1} & g^* \end{pmatrix} = \begin{pmatrix} h & 0 \\ (h^*)^{-1}y & (h^*)^{-1} \end{pmatrix}$, an element of $\mathcal{B}'$; here $h = g^{-1}$ and $y = -h^*xh \in \mathcal{A}_h$.

Observe that J acts as a symmetry of $\mathcal{H}$ in the sense that it is an involutive isometry of the homogeneous reductive space $\mathcal{H}$ and inverts geodesics through $i \in \mathcal{H}$. Also we have a similar symmetry with respect to any point in $\mathcal{H}$ (by means of the action of the group $\mathcal{U}_{\rho'}$). For this reason we can consider $\mathcal{H}$ as a symmetric space as in the classical setting [20].

10 The covariant derivative in $\mathcal{H}$

In this section we compute explicitly the covariant derivative induced by the reductive structure (see for instance [22]). As in the case of $\mathcal{U}_\rho$, the Banach-Lie algebra $\mathfrak{u}_{\rho'}$ of $\mathcal{U}_{\rho'}$ decomposes as

$$\mathfrak{u}_{\rho'} = \mathfrak{u}_{0,\rho'} \oplus \mathfrak{u}_{1,\rho'} = \{\beta \in M_2(\mathcal{A})_{as} : \beta J = J\beta\} \oplus \{\gamma \in M_2(\mathcal{A})_s : \gamma J = -J\gamma\},$$

where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ as above. Then, elements $X \in \mathfrak{u}_{\rho'}$ are of the form $X = X_0 + X_h$,

$$X = \begin{pmatrix} x_{11} & x_{12} \\ -x_{12} & -x_{11} \end{pmatrix} + \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix}$$

with $x_{11}^* = -x_{11}$, and all other entries selfadjoint. The left hand subspace $\mathfrak{u}_{0,\rho'}$ is the Banach-Lie algebra of the isotropy group of the action at the element $i \in \mathcal{H}$. The right hand subspace $\mathfrak{u}_{1,\rho'}$ is the *horizontal* space at this point.

Theorem 10.1. *The* 1*-form of the reductive connection at* $i \in \mathcal{H}$ *is*

$$\varsigma_i(\zeta) = \frac{1}{2}\begin{pmatrix} -\Upsilon & \chi \\ \chi & \Upsilon \end{pmatrix}, \text{ if } \zeta = \chi + i\Upsilon.$$

The covariant derivative is given by

$$\frac{D\zeta}{dt} = \zeta' - Re(x'y_0^{-1}\Upsilon + y'y_0^{-1}\chi) + iRe(x'y_0^{-1}\chi - y'y_0^{-1}\Upsilon). \tag{41}$$

Therefore, the geodesics ζ of $\mathcal{H}$, which start in $h_0 = x_0 + iy_0$, are the solutions of $\frac{D\dot\zeta}{dt} = 0$, which amounts to

$$\begin{cases} \ddot\chi = Re\ (x'y_0^{-1}\dot\Upsilon + y'y_0^{-1}\dot\chi) \\ \ddot\Upsilon = Re\ (x'y_0^{-1}\dot\chi - y'y_0^{-1}\dot\Upsilon) \end{cases}.$$

11 Finsler metrics

By "Finsler metric" or "Finsler structure", we mean a continuous distribution of norms in the tangent spaces as in Section 2 (cf. [32], Section 12).

In this section we define Finsler structures in $\mathcal{Q}_\rho$ and $\mathcal{D}$, which are invariant with respect to the action of $\mathcal{U}_\rho$. The case of $\mathcal{Q}_{\rho'}$ and $\mathcal{H}$ as homogeneus metric spaces of $\mathcal{U}_{\rho'}$ can be dealt similarly.

11.1 Finsler metric in $\mathcal{Q}_\rho$

The space $\mathcal{Q}_\rho$ has a Finsler metric, which is invariant under the action of $\mathcal{U}_\rho$. It is defined as follows:

1. If $\mathbf{X} \in (T\mathcal{Q}_\rho)_p$, define $\|\mathbf{X}\|_p = \|\mathbf{X}\|$.
2. If $\mathbf{X} \in (T\mathcal{Q}_\rho)_q$, then $a^{-1}\mathbf{X}a \in (T\mathcal{Q}_\rho)_p$ for every $a \in \mathcal{U}_\rho$ such that $a \cdot p = q$. We define $\|\mathbf{X}\|_q = \|a^{-1}\mathbf{X}a\|$.

Note that $\|a^{-1}\mathbf{X}a\| = \|b^{-1}\mathbf{X}b\|$ if $a, b \in \mathcal{U}_\rho$ verify that $a \cdot p = b \cdot p = q$. Indeed, in this case $b = aw$ for $w \in \mathcal{I}_p$; in particular, as we saw earlier, w is a unitary element in M_2. Thus

$$\|b^{-1}\mathbf{X}b\| = \|w^* a^{-1}\mathbf{X}aw\| = \|a^{-1}\mathbf{X}a\|.$$

Thus the Finsler metric below is well defined:

Definition 11.1. *For* $\mathbf{X} \in (T\mathcal{Q}_\rho)_q$,

$$\|\mathbf{X}\|_q = \|a^{-1}\mathbf{X}a\||,$$

for any $a \in \mathcal{U}_\rho$ *such that* $a \cdot p = q$

This metric is invariant under the action of $\mathcal{U}_\rho$:

$$\|c\mathbf{X}c^{-1}\|_{cqc^{-1}} = \|\mathbf{X}\|_q \ , \quad \text{for all } q \in \mathcal{Q}_\rho,\ \mathbf{X} \in (T\mathcal{Q}_\rho)_q \text{ and } c \in \mathcal{U}_\rho. \tag{42}$$

11.2 Finsler metric in $\mathcal{D}$

We introduce a Finsler metric on $\mathcal{D}$, which will be shown to be invariant under the action of $\mathcal{U}_\rho$. In other words, for each $a \in \mathcal{U}_\rho$, if $L_a : \mathcal{D} \to \mathcal{D}$ denotes the translation $L_a z = a \cdot z$, then $(TL_a)_z : (T\mathcal{D})_z \to (T\mathcal{D})_{a\cdot z}$ will be an isometry.

Definition 11.2.

1. *For* $x \in (T\mathcal{D})_0 \simeq \mathcal{A}$ *we put*
$$\|x\|_0 := \|x\|, \tag{43}$$
the usual norm of $\mathcal{A}$.

2. *If* $x \in (T\mathcal{D})_z$ *where* $z = a \cdot 0$, *then* $(TL_{a^{-1}})_z x \in (T\mathcal{D})_0$ *and we put*
$$\|x\|_z := \|(TL_{a^{-1}})_z x\|_0 = \|(TL_{a^{-1}})_z x\|. \tag{44}$$

Next, we show that the diffeomorphism α_D in the diagram (23) is isometric.

Proposition 11.3. *The diffeomorphism* $\alpha_D : \mathcal{Q}_\rho \to \mathcal{D}$, $\alpha_D(q) = q_{21}q_{11}^{-1}$ *is isometric, i.e.*

$$\|(T\alpha_D)_q x\|_{\alpha_D(q)} = \|x\|_q,$$

for every $x \in (T\mathcal{Q}_\rho)_q$.

Remark 11.4. There is a similar result for the isomorphism $\mathcal{Q}_{\rho'} \to \mathcal{H}$.

12 Immersions in $G^+_{M_2}$

We have now all geometric tools necessary to profit from our knowledge of the geometry of $\mathcal{Q}_\rho$ and $\mathcal{Q}_{\rho'}$. First, we need the following definition:

Definition 12.1. *If M (resp. M') is a homogeneous space of the Lie-Banach group G (resp. G'), a morphism from (G, M) to (G', M') is a smooth map*

$$G \times M \to G' \times M' \ , \quad (g, m) \mapsto (\varphi(g), \Phi(m))$$

such that $\varphi : G \to G'$ is a smooth group homomorphism, $\Phi : M \to M'$ is a smooth map and

$$\Phi(g \cdot m) = \varphi(g) \cdot \Phi(m) \tag{45}$$

for all $g \in g$, $m \in M$.

Examples 12.2.

1. $\mathcal{U}_\rho \times \mathcal{D} \to \mathcal{U}_\rho \times \mathcal{Q}_\rho$, $\varphi(a) = a$, $\Phi(z) = p_{\mathbf{x}_z}$, where as before, $\mathbf{x}_z = \begin{pmatrix} 1 \\ z \end{pmatrix} (1 - z^* z)^{-1/2}$. In this case it is an isomorphism, i.e., the inverse map is also an morphism.
2. $\mathcal{U}_\rho \times \mathcal{Q}_\rho \to G_{M_2} \times G^+_{M_2}$, $\varphi(a) = (a^*)^{-1}$ (i.e., φ is the inclusion $\mathcal{U}_\rho \hookrightarrow G_{M_2}$), $\Phi(q) = (2q - 1)\rho$ ($q \in \mathcal{Q}_\rho$). Indeed, it holds that $\Phi(aqa^{-1}) = a(2q - 1)a^*$.

   Analogous examples of morphisms are provided by the following versions of 1. and 2.:
3. $\mathcal{U}_{\rho'} \times \mathcal{H} \to \mathcal{U}_{\rho'} \times \mathcal{Q}_{\rho'}$, $\varphi(a) = a$, $\Phi(h) = p_{\mathbf{x}_h}$, where $\mathbf{x}_h = \begin{pmatrix} 1 \\ h \end{pmatrix} (\text{Im } h)^{-1/2}$, as in (36).
4. $\mathcal{U}_{\rho'} \times \mathcal{Q}_{\rho'} \to G_{M_2} \times G^+_{M_2}$, $\varphi(a) = (a^*)^{-1}$, $\Phi'(q') = (2q' - 1)\rho'$. Here also *im* $\Phi' = \mathcal{U}^+_{\rho'} = \mathcal{U}_{\rho'} \cap G^+_{M_2}$.

It is easy to check that the immersions $\Phi : \mathcal{Q}_\rho \to G^+_{M_2}$ and $\Phi' : \mathcal{Q}_{\rho'} \to G^+_{M_2}$ preserve the Finsler metrics.

Therefore, we get:

Theorem 12.3. *The spaces $\mathcal{D}$, $\mathcal{H}$, $\mathcal{Q}_\rho$ and $\mathcal{Q}_{\rho'}$ share with $G^+_{M_2}$ the following properties:*

1. *any two points can be joined by a unique geodesic;*
2. *every geodesic is defined for all $t \in \mathbb{R}$;*
3. *the geodesic joining two points m_1, m_2 is shortest among all smooth curves joining them in the manifold;*
4. *if γ, δ are two geodesics, then $f(t) = d(\gamma(t), \delta(t))$ is a convex function.*

All these spaces are non positive curvature spaces in the sense of Alexandroff.

Example 12.4. Let us compute the distance $d_D(0,z)$ between 0 and z in $\mathcal{D}$. Recall from (26) the explicit form of the modulus of inverse of the exponential map (based at $0 \in \mathcal{D}$):

$$|\log_0(z)| = \frac{1}{2}\log\left((1+|z|)(1-|z|)^{-1}\right).$$

Then

$$d(0,z) = \|\alpha\| = \|\log_0(z)\| = \|\log_0(z)\| = \frac{1}{2}\|\log\left((1+|z|)(1-|z|)^{-1}\right)\|.$$

The function $f(t) = \log(1+t) - \log(1-t)$ is strictly increasing in $[0,1)$, with $f(0) = 0$. Thus (using that $\||z|\| = \|z\|$),

$$\|\log(1+|z|) - \log(1-|z|)\| = \max\{|f(t)| : t \in \sigma(|z|)\} = f(\|z\|).$$

Then,

$$d(0,z) = \frac{1}{2}\log(\frac{1+\|z\|}{1-\|z\|}).$$

Recall, for $z \in \mathcal{D}$, the element $g_z \in \mathcal{U}_\rho$ such that $g_z \cdot 0 = z$ given in (22),

$$g_z = \begin{pmatrix} (1-z^*z)^{-1/2} & (1-z^*z)^{-1/2}z^* \\ z(1-z^*z)^{-1/2} & (1-zz^*)^{-1/2} \end{pmatrix}.$$

Using the fact that the action of $\mathcal{B}$ on $\mathcal{D}$ is isometric, and the above construction of g_z, for arbitrary $z_1, z_2 \in \mathcal{D}$, we can compute $d(z_1,z_2)$ as follows:

$$d(z_1,z_2) = d(0, g_{z_1}^{-1} \cdot z_2) = \frac{1}{2}\log(\frac{1+\|g_{z_1}^{-1} \cdot z_2\|}{1-\|g_{z_1}^{-1} \cdot z_2\|}). \tag{46}$$

In order to compute g_z^{-1}, recall that elements g in $\mathcal{U}_\rho$ are characterized by the relation $\rho g^* \rho = g^{-1}$. Then

$$g_z^{-1} = \begin{pmatrix} (1-z^*z)^{-1/2} & -(1-z^*z)^{-1/2}z^* \\ -z(1-z^*z)^{-1/2} & (1-zz^*)^{-1/2} \end{pmatrix}.$$

Then, after straightforward computations,

$$g_{z_1}^{-1} \cdot z_2 = (1-z_1z_1^*)^{-1/2}(z_2 - z_1^*)(1-z_1^*z_2)^{-1}(1-z_1^*z_1)^{1/2}.$$

Remark 12.5. In the scalar case $\mathcal{A} = \mathbb{C}$, one has

$$d(0,z) = \frac{1}{2}\log\left(\frac{1+|z|}{1-|z|}\right),$$

which is ($\frac{1}{2}$ times) the Poincaré distance in the open unit disk $\mathbb{D}$.

13 A lifting form

We remarked before that the range of $q = p_{\mathbf{x}} \in \mathcal{Q}_\rho$ has basis $\{\mathbf{x}\}$ ($\mathbf{x} \in \mathcal{K}_\rho$), as right $\mathcal{A}$-submodule of $\mathcal{A}^2$. We complete this observation by showing that also $N(q)$ has a basis, and that together with $\mathbf{x}$ they form a θ_ρ-orthonormal basis of $\mathcal{A}^2$.

Proposition 13.1. *Let* $\mathbf{x} \in \mathcal{K}_\rho$. *There exist unique elements* $\mathbf{y} \in R(p_\mathbf{x})$ *and* $\mathbf{z} \in N(p_\mathbf{x})$, $\mathbf{y}, \mathbf{z} \in \mathcal{K}_\rho$, *with* y_1, z_1 *positive and invertible. Namely*

$$\mathbf{y} = \mathbf{x} x_1^* (x_1 x_1^*)^{-1/2} \quad \textit{and } \mathbf{z} = \begin{pmatrix} x_1 x_1^* (1 + x_2 x_2^*)^{-1/2} \\ (1 + x_2 x_2^*)^{-1/2} \end{pmatrix}.$$

The unitary group $\mathcal{U}_\mathcal{A}$ acts on each $\mathbf{x}$ in $\mathcal{K}_\rho$. We get the tangent map of this action

$$\iota_\mathbf{x} : \mathcal{A}_{ah} = (T\mathcal{U}_\mathcal{A})_1 \to (T\mathcal{K}_\rho)_\mathbf{x} \tag{47}$$

which is injective and splits: one has the decomposition $(T\mathcal{K}_\rho)_\mathbf{x} = N(p_\mathbf{x}) \oplus im\ \iota_\mathbf{x}$.

Remark 13.2. $N(p_\mathbf{x})$ is the unique singly generated $\mathcal{A}$-submodule of $(T\mathcal{K}_\rho)_\mathbf{x}$.

This short section deals with several properties of the tangent map at $\mathbf{x} \in \mathcal{K}_\rho$ of $\varphi_\rho : \mathcal{K}_\rho \to \mathcal{Q}_\rho$, $\varphi_\rho(\mathbf{x}) = p_\mathbf{x}$. Recall the tangent spaces

$$(T\mathcal{K}_\rho)_\mathbf{x} = \{\mathbf{z} \in \mathcal{A}^2 : \text{Re } \mathbf{x}^* \rho \mathbf{z} = 0\},\ \mathbf{x} \in \mathcal{K}_\rho,$$

and

$$(T\mathcal{Q}_\rho)_q = \{a \in M_2 : aq + qa = a, \rho a = a^* \rho\},\ q \in \mathcal{Q}_\rho.$$

It is easy to verify that $(T\varphi_\rho)_\mathbf{x} : (T\mathcal{K}_\rho)_\mathbf{x} \to (T\mathcal{Q}_\rho)_{p_\mathbf{x}}$ is given by

$$(T\varphi_\rho)_\mathbf{x} \mathbf{z} = \mathbf{z}\mathbf{x}^* \rho + \mathbf{x}\mathbf{z}^* \rho,\ \mathbf{z} \in (T\mathcal{K}_\rho)_\mathbf{x}.$$

Notice that if $\mathbf{x} \in \mathcal{K}_\rho$ and $a \in (T\mathcal{Q}_\rho)_{p_\mathbf{x}}$ it holds that $a\mathbf{x} \in N(p_\mathbf{x})$. Indeed, since $ap_\mathbf{x} + p_\mathbf{x} a = a$, it holds that $ap_\mathbf{x}\mathbf{x} + p_\mathbf{x} a\mathbf{x} = a\mathbf{x}$. Since $p_\mathbf{x}\mathbf{x} = \mathbf{x}$, this implies that $p_\mathbf{x} a\mathbf{x} = 0$.

Notice also that $N(p_\mathbf{x}) \subset (T\mathcal{K}_\rho)_\mathbf{x}$: if $\mathbf{z} \in \mathcal{A}^2$ and $\mathbf{x}\mathbf{x}^*\rho\mathbf{z} = 0$, then, since $x_1 \in G_\mathcal{A}$, it must be $\mathbf{x}^*\rho\mathbf{z} = 0$, a fortiori Re $\mathbf{x}^*\rho\mathbf{z} = 0$.

Proposition 13.3. *For every* $\mathbf{x} \in \mathcal{K}_\rho$ *the* $\mathcal{A}$*-linear map*

$$\kappa_\mathbf{x} : (T\mathcal{Q}_\rho)_{p_\mathbf{x}} \to (T\mathcal{K}_\rho)_\mathbf{x}\ ,\quad \kappa_\mathbf{x} a = a\mathbf{x}$$

satisfies

1. $((T\varphi_\rho)_\mathbf{x} \circ \kappa_\mathbf{x})a = a$, *for all* $a \in (T\mathcal{Q}_\rho)_{p_\mathbf{x}}$*;*
2. $\kappa_\mathbf{x} \circ (T\varphi_\rho)_\mathbf{x} = 1 - p_\mathbf{x}$*;*
3. $im\ \kappa_\mathbf{x} = N(p_\mathbf{x})$*.*

Definition 13.4. *We call* $\kappa_\mathbf{x}$ *the* lifting form *of* $\mathcal{Q}_\rho$*.*

Corollary 13.5. *For* $\mathbf{x} \in \mathcal{K}_\rho$*, there is the decomposition* $(T\mathcal{K}_\rho)_\mathbf{x} = N(p_\mathbf{x}) \oplus N((T\varphi_\rho)_\mathbf{x})$*.*

For further reference, we write

$$\mathbf{H}_\mathbf{x} = N(p_\mathbf{x}), \tag{48}$$

and call it the *horizontal space* over $\mathbf{x}$.

Corollary 13.6. *For* $\mathbf{x} \in \mathcal{K}_\rho$*, the restriction*

$$(T\varphi_\rho)_\mathbf{x}\big|_{N(p_\mathbf{x})} : N(p_\mathbf{x}) \to (T\mathcal{Q}_\rho)_{p_\mathbf{x}}$$

is a linear isomorphism with inverse $\kappa_\mathbf{x}$*.*

Remark 13.7. If $a \in (T\mathcal{Q}_\rho)_{p_\mathbf{x}}$ and $\mathbf{y} \in \mathcal{K}_\rho$ is such that $p_\mathbf{y} = p_\mathbf{x}$, then $\mathbf{y} = \mathbf{x}u$ for some $u \in \mathcal{U}_\mathcal{A}$ (see Lemma 6.3). Then $\kappa_\mathbf{y}(a) = a\mathbf{y} = a\mathbf{x}u$. Therefore, we can think of every $a \in (T\mathcal{Q}_\rho)_{p_\mathbf{x}}$ as the set of pairs $(\mathbf{x}u, \mathbf{z}u)$ with $\mathbf{z} \in N(p_\mathbf{x})$ and $u \in \mathcal{U}_\mathcal{A}$. We write $(\mathbf{x}, \mathbf{z}) \sim (\mathbf{x}u, \mathbf{z}u)$.

14 Kähler structures

14.1 The coefficient bundle

Let us describe a setting, which will allow us to present a useful approach to the three main vector bundles which we shall use to study the geometry of $\mathcal{Q}_\rho$. They are all based on the principal bundle $\varphi_\rho : \mathcal{K}_\rho \to \mathcal{Q}_\rho$.

Let $\eta : F \to \mathcal{Q}_\rho$ be a fiber bundle, consider the commutative diagram

$$\begin{array}{ccc} F_{\mathcal{K}_\rho} & \xrightarrow{\tilde{\varphi}_\rho} & F \\ \eta^* \downarrow & & \downarrow \eta \\ \mathcal{K}_\rho & \xrightarrow{\varphi_\rho} & \mathcal{Q}_\rho \end{array}, \tag{49}$$

where $F_{\mathcal{K}_\rho} = \{(f, \mathbf{x}) : f \in F,\ \mathbf{x} \in \mathcal{K}_\rho,\ \eta(f) = \varphi_\rho(\mathbf{x})\}$. This diagram describes η^* as the bundle over $\mathcal{K}_\rho$ induced by φ_ρ. Note that $F_{\mathcal{K}_\rho}$ carries a natural action from $\mathcal{U}_\mathcal{A}$, which lifts the action of $\mathcal{U}_\mathcal{A}$ over $\mathcal{K}_\rho$, and which presents the bundle $\eta : F \to \mathcal{Q}_\rho$ as the space of orbits of $\eta^* : F_{\mathcal{K}_\rho} \to \mathcal{K}_\rho$ under this action.

We can argue that an element $f \in F$ over $q \in \mathcal{Q}_\rho$ is represented by an element $\tilde{f} \in F_{\mathcal{K}_\rho}$ in the basis $\mathbf{x} \in \mathcal{K}_\rho$ ($\varphi_\rho(\mathbf{x}) = q$, f lies over the same $\mathbf{x}$). More precisely, $(\tilde{f}, \mathbf{x})$ and $(\tilde{f}', \mathbf{x}')$ represent the same element if and only if there exists $u \in \mathcal{U}_\mathcal{A}$ such that $\tilde{f}' = \tilde{f}u$ and $\mathbf{x}' = \mathbf{x}u$.

Definition 14.1. *Given a bundle $\eta : F \to \mathcal{Q}_\rho$, we call a $\mathcal{K}_\rho$-presentation of η a bundle $\zeta : Z \to \mathcal{K}_\rho$, with an action of $\mathcal{U}_\mathcal{A}$ over T which lifts the action of $\mathcal{U}_\mathcal{A}$ over $\mathcal{K}_\rho$, and a bundle isomorphism $\Phi : Z \to F_{\mathcal{K}_\rho}$ preserving the action of $\mathcal{U}_\mathcal{A}$,*

$$\begin{array}{ccc} Z & \xrightarrow{\Phi} & F_{\mathcal{K}_\rho} \\ \zeta \searrow & & \swarrow \eta \\ & \mathcal{K}_\rho & \end{array}.$$

Let us give three examples of $\mathcal{K}_\rho$-presentations which will be relevant in this paper.

14.2 The *tautological bundle* over $\mathcal{Q}_\rho$

Consider

$$\xi : \mathcal{E} \to \mathcal{Q}_\rho,$$

where $\mathcal{E} = \{(q, \mathbf{x}) \in M_2(\mathcal{A}) \times \mathcal{A}^2 : q \in \mathcal{Q}_\rho, \mathbf{x} \in R(q)\}$ and $\xi(q, \mathbf{x}) = q$. Over each idempotent q we put the vectors in the range of q.

Proposition 14.2. *ξ is a locally trivial vector bundle.*

Then we have the induced bundle over $\mathcal{K}_\rho$, and the diagram (49) in this case:

$$\begin{array}{ccc} \mathcal{E}_{\mathcal{K}_\rho} & \xrightarrow{\tilde{\varphi}_\rho} & \mathcal{E} \\ \xi^* \downarrow & & \downarrow \xi \\ \mathcal{K}_\rho & \xrightarrow{\varphi_\rho} & \mathcal{Q}_\rho \end{array},$$

Consider the product bundle $\mathcal{K}_\rho \times \mathcal{A} \to \mathcal{A}$ and the isomorphism $\beta : \mathcal{K}_\rho \times \mathcal{A} \to \mathcal{E}$, $\beta(\mathbf{x}, a) = (p_\mathbf{x}, \mathbf{x}a)$. The action of $\mathcal{U}_\mathcal{A}$ on $\mathcal{K}_\rho \times \mathcal{A}$ is given by $(\mathbf{x}, a) \cdot u = (\mathbf{x}u, u^* a)$. The isomorphism β gives a $\mathcal{K}_\rho$-presentation for $\mathcal{E}$. Thus, an element $(q, \mathbf{y})$ of $\mathcal{E}$ is represented by a pair $(\mathbf{x}, a)$, and two pairs $(\mathbf{x}, a)$, $(\mathbf{x}', a')$ represent the same element of $\mathcal{E}$ if and only if there exists $u \in \mathcal{U}_\mathcal{A}$ such that $\mathbf{x}' = \mathbf{x}u$ and $a' = u^* a$.

14.3 The *coefficient* bundle

The fiber bundle defined now will be a central feature of our exposition. It will provide a setting for the most relevant notions studied in this paper.

We shall describe a bundle over $\mathcal{Q}_\rho$, whose fiber over q is the space of right $\mathcal{A}$-module endomorphisms of $R(q)$, which we shall call the *coefficient bundle*

$$\gamma : \mathcal{C} \to \mathcal{Q}_\rho.$$

Formally, $\mathcal{C} = \{(q, \varphi) \in \mathcal{Q}_\rho \times \mathcal{L}_\mathcal{A}(R(q)) : q \in \mathcal{Q}_\rho\}$, where $\mathcal{L}_\mathcal{A}(R(q))$ denotes the space of right $\mathcal{A}$-module endomorphisms of the module $R(q)$, and the map $\gamma : \mathcal{C} \to \mathcal{Q}_\rho$ is $\gamma(q, \varphi) = q$. Given $\mathbf{x} \in \mathcal{K}_\rho$ with $p_\mathbf{x} = q$, an endomorphism φ of $R(q)$ gives $\varphi(\mathbf{x}) = \mathbf{x}a$. We say that $\mathbf{x}$ is a *basis* for $R(q)$, and that $a \in \mathcal{A}$ is the *matrix* of φ in the basis $\mathbf{x}$. If we change $\mathbf{x}$ for $\mathbf{y} = \mathbf{x}u \in R(q)$ for $u \in \mathcal{U}_\mathcal{A}$, and denote by b the matrix of φ at $\mathbf{y}$, $\varphi(\mathbf{y}) = \mathbf{y}b$, then

$$\mathbf{x}ub = \mathbf{y}b = \varphi(\mathbf{y}) = \varphi(\mathbf{x})u = \mathbf{x}au, \tag{50}$$

i.e., $b = u^*au$. In other words, the endomorphisms can be identified with the pairs $(\mathbf{x}, a) \in \mathcal{K}_\rho \times \mathcal{A}$, subject to the equivalence relation

$$(\mathbf{x}, a) \sim (\mathbf{x}u, u^*au)\ , \quad u \in \mathcal{U}_\mathcal{A}. \tag{51}$$

Notation 14.3. We denote by $\Big[(\mathbf{x}, a)\Big]$ the endomorphism of the module $R(p_\mathbf{x})$ which has matrix a in the basis $\mathbf{x}$.

Remark 14.4. Observe that $\mathcal{C}$ is a fiber bundle whose fibers are C*-algebras. If, additionally, $\mathcal{A}$ is commutative then $\mathcal{C}$ is the product bundle $\mathcal{Q}_\rho \times \mathcal{A}$.

Proposition 14.5. *The space $\mathcal{C}$ is a C^∞ differentiable manifold and the map $\gamma : \mathcal{C} \to \mathcal{Q}_\rho$ is a C^∞ trivial fiber bundle.*

As in the previous cases, the coefficient bundle $\gamma : \mathcal{C} \to \mathcal{Q}_\rho$ induces the bundle $\gamma^* : \mathcal{C}_{\mathcal{K}_\rho} \to \mathcal{K}_\rho$ and the commutative diagram (49) in this setting becomes

$$\begin{array}{ccc} \mathcal{C}_{\mathcal{K}_\rho} & \stackrel{\tilde{\varphi}_\rho}{\longrightarrow} & \mathcal{C} \\ \gamma^* \downarrow & & \downarrow \gamma \\ \mathcal{K}_\rho & \stackrel{\varphi_\rho}{\longrightarrow} & \mathcal{Q}_\rho \end{array} .$$

The fiber $(\mathcal{C}_{\mathcal{K}_\rho})_\mathbf{x}$ over $\mathbf{x} \in \mathcal{K}_\rho$ consists of the endomorphisms of $R(p_\mathbf{x})$. Therefore, one has the representation $\Phi : \mathcal{K}_\rho \times \mathcal{A} \to \mathcal{C}_{\mathcal{K}_\rho}$, given by $\Phi(\mathbf{x}, a) = (\mathbf{x}, [(\mathbf{x}, a)])$.

14.4 The connection in the coefficient bundle

Since $\mathcal{C}_q$ consists of endomorphisms ξ_q, $q \in \mathcal{Q}_\rho$, the standard connection on $\mathcal{C}$ is given by the *Leibnitz formula.* Let $\mathbf{X} \in T\mathcal{Q}_\rho$ and $\mathbf{x} = \mathbf{x}(t)$ adjusted to $\mathbf{X}$. Let σ and φ be cross sections of ξ and $\mathcal{C}$, respectively. We define the covariant derivative $D_\mathbf{X}\varphi$ by the rule

$$(D_\mathbf{X}\varphi)\sigma := D_\mathbf{X}(\varphi\sigma) - \varphi(D_\mathbf{X}\sigma). \tag{52}$$

Straightforward computations prove

$$(D_\mathbf{X}\varphi)\sigma = \mathbf{x}\left(\dot{\lambda} + [\theta_\rho(\mathbf{x}, \dot{\mathbf{x}}), \lambda]\right) a = \mathbf{x}\left(\dot{\lambda} + [\mathbf{x}^*\rho\mathbf{x}, \lambda]\right) a. \tag{53}$$

14.5 The tangent bundle $T\mathcal{Q}_\rho$

For any $\mathbf{x} \in \mathcal{K}_\rho$, the nullspace $N(p_\mathbf{x})$ is a subspace of $(T\mathcal{K}_\rho)_\mathbf{x}$, which we called the *horizontal space* $\mathbf{H}_\mathbf{x}$. The tangent map of the fibration $\varphi_\rho : \mathcal{K}_\rho \to \mathcal{Q}_\rho$

$$(T\varphi_\rho)_\mathbf{x} : (T\mathcal{K}_\rho)_\mathbf{x} \to (T\mathcal{Q}_\rho)_{p_\mathbf{x}}$$

is bijective when restricted to $\mathbf{H}_\mathbf{x}$. Then we have the bundle $\nu : \mathbf{H} \to \mathcal{K}_\rho$, where $\mathbf{H} = \{(\mathbf{z}, \mathbf{x}) : \mathbf{x} \in \mathcal{K}_\rho,\ \mathbf{z} \in \mathbf{H}_\mathbf{x}\}$, and the map is given by $\nu(\mathbf{z}, \mathbf{x}) = \mathbf{x}$. Thus, we have the $\mathcal{K}_\rho$-presentation

$$\begin{array}{ccc} \mathbf{H} & \stackrel{\Phi}{\longrightarrow} & (T\mathcal{Q}_\rho)_{\mathcal{K}_\rho} \\ \nu \searrow & & \swarrow \tau^* \\ & \mathcal{K}_\rho & \end{array} ,$$

where $\tau : T\mathcal{Q}_\rho \to \mathcal{Q}_\rho$ is the tangent bundle and $\tau^* : (T\mathcal{Q}_\rho)_{\mathcal{K}_\rho} \to \mathcal{K}_\rho$ is the induced bundle. The isomorphism Φ is given by the differential of φ_ρ restricted to $\mathbf{H}$, namely

$$\Phi(\mathbf{x}, \mathbf{z}) = (\mathbf{x}, (d\varphi_\rho)_\mathbf{x}(\mathbf{z}))\ , \text{ for } \mathbf{x} \in \mathcal{K}_\rho\ ,\ \mathbf{z} \in \mathbf{H}_\mathbf{x}. \tag{54}$$

Then a tangent vector to $\mathcal{Q}_\rho$ at $p_\mathbf{x}$ is represented by a pair $(\mathbf{x}, \mathbf{z})$, $\mathbf{x} \in \mathcal{K}_\rho$, $\mathbf{z} \in \mathbf{H}_\mathbf{x}$; $(\mathbf{x}, \mathbf{z})$ and $(\mathbf{x}u, \mathbf{z}')$ represent the same vector if and only if $\mathbf{z}' = \mathbf{z}u$ $(u \in \mathcal{U}_\mathcal{A})$.

Remark 14.6. We may consider the *tautological* 1-form with values in the tangent bundle $T\mathcal{Q}_\rho$ that associates to a tangent vector $\mathbf{X}$ to $\mathcal{Q}_\rho$, the same vector $\mathbf{X}$ as an element of $T\mathcal{Q}_\rho$. The lifting form given in Definition 13.4 is just an expression of the values of this tautological form in terms of the principal bundle $\mathcal{K}_\rho$.

14.5.1 $T\mathcal{Q}_\rho$ as a $\mathcal{C}$-module

Recall from (52) that the fiber $\mathcal{C}_q$ can be presented as the set of pairs $(\mathbf{x}, a) \in \mathcal{K}_\rho \times \mathcal{A}$ modulo the equivalence $(\mathbf{x}, a) \sim (\mathbf{x}u, u^*au)$ for $u \in \mathcal{U}_\mathcal{A}$. Since each submodule $R(q)$ has a basis consisting of one element, then $\mathcal{L}_\mathcal{A}(R(q))$ is in one to one correspondence with $\mathcal{A}$: having chosen a basis $\mathbf{x} \in \mathcal{K}_\rho$ for $R(q)$, the map $\mathcal{A} \to \mathcal{C}_q$, $a \mapsto (\mathbf{x}, a)$ is a $*$-isomorphism. Thus, each fiber $\mathcal{C}_q$ has the structure of a C*-algebra, namely $\mathcal{A}$. Clearly the structure does not depend on the choice of the basis; another basis $\mathbf{x}u$ provides the same structure, because $Ad(u)$ is a $*$-automorphism of $\mathcal{A}$.

On the other hand, we saw before that the tangent bundle has a similar presentation. Each $\mathbf{X} \in (T\mathcal{Q}_\rho)_q$ is given by the set of pairs $(\mathbf{x}, \mathbf{v}) \in \mathcal{K}_\rho \times \mathcal{A}^2$, modulo the equivalence $(\mathbf{x}, \mathbf{v}) \sim (\mathbf{x}u, \mathbf{v}u)$ for $u \in \mathcal{U}_\mathcal{A}$ and $\mathbf{v} = \mathbf{X}\mathbf{x} = \kappa_\mathbf{x}(\mathbf{X})$. Accordingly, we denote $\mathbf{X} \in (T\mathcal{Q}_\rho)_q$ as $[(\mathbf{x}, \mathbf{v})]$.

Definition 14.7. *For* $\mathbf{X} \in (T\mathcal{Q}_\rho)_q$ *and* $\varphi \in \mathcal{L}_\mathcal{A}(R(q))$ *we define the product*

$$\mathbf{X} \cdot \varphi := [(\mathbf{x}, \mathbf{v}a)] \in (T\mathcal{Q}_\rho)_q$$

if $\mathbf{x} \in \mathcal{K}_\rho$, $p_\mathbf{x} = q$, $\varphi = [(\mathbf{x}, a)]$ *and* $\mathbf{X} = [(\mathbf{x}, \mathbf{v})]$.

Observe that this operation is well defined: if we change the basis to $\mathbf{x}u$, then φ and $\mathbf{X}$ are represented by $(\mathbf{x}u, u^*au)$ and $(\mathbf{x}u, \mathbf{v}u)$ respectively, and the product in the new referential is $(\mathbf{x}u, \mathbf{v}uu^*au) = (\mathbf{x}u, \mathbf{v}au)$ which represents the same tangent vector $\mathbf{X} \cdot \varphi$ as $(\mathbf{x}, \mathbf{v}a)$.

Thus, $T\mathcal{Q}_\rho$ can be regarded as a bundle of right modules over $\mathcal{C}$, which is a bundle of C*-algebras (isomorphic to $\mathcal{A}$).

Remark 14.8. *The tautological bundle over $\mathcal{K}_\rho$.*

Let us consider the pullback $\mathcal{E}' = \varphi_\rho^* \mathcal{E}$ of $\mathcal{E}$ by φ_ρ,

$$\xi' : \mathcal{E}' = \{(\mathbf{x}, \mathbf{v}) \in \mathcal{K}_\rho \times \mathcal{A}^2 : p_\mathbf{x} \mathbf{v} = \mathbf{v}\} \to \mathcal{K}_\rho \ , \quad \xi'(\mathbf{x}, \mathbf{v}) = \mathbf{x}.$$

Pick $\mathbf{x} \in \mathcal{K}_\rho$ and let $\mathcal{E}'_\mathbf{x}$ be the fiber of $\mathcal{X}'$ over $\mathbf{x}$. Then the map

$$\mathbf{a} \to \mathcal{E}'_\mathbf{x} \ , \quad a \mapsto \mathbf{x}a$$

is an isomorphism of right $\mathcal{A}$-modules, with inverse $\mathbf{v} \mapsto \theta_\rho(\mathbf{x}, \mathbf{v})$. It can be regarded as a chart for $\mathcal{E}'_\mathbf{x}$ associated to $\mathbf{x}$. Let $\mathbf{y} \in \mathcal{E}_{p_\mathbf{x}}$ (i.e., $p_\mathbf{x} = p_\mathbf{y} = q$); then there exists a unitary $u \in \mathcal{U}_\mathcal{A}$ such that $\mathbf{y} = \mathbf{x}u$. The isomorphism between $\mathcal{A}$ and $\mathcal{E}'_\mathbf{y}$ is $\mathcal{A} \ni b \mapsto \mathbf{y}a$. The coordinates a and b of a given $\mathbf{v} \in \mathcal{E}_q$ are related by $\mathbf{x}ub = \mathbf{y}b = \mathbf{v} = \mathbf{x}a$, and thus, $a = ub$. Then we can regard the vectors $\mathbf{v} \in \mathcal{E}_q$ as pairs $(\mathbf{x}, a) \in \mathcal{K}_\rho \times \mathcal{A}$, with the following equivalence relation:

$$(\mathbf{x}, a) \sim (\mathbf{x}u, u^* a) \ , \quad u \in \mathcal{U}_\mathcal{A}.$$

Remark 14.9. *The connection in the tautological bundle.*

The covariant derivative in the tautological bundle $\xi : \mathcal{E} \to \mathcal{Q}_\rho$ is given by

$$D_\mathbf{X} \sigma = q(\mathbf{X} \bullet \sigma),$$

where σ is a cross section for ξ and $\mathbf{X} \bullet \sigma$ is the directional derivative of σ, considering σ as a function with values in $\mathcal{A}^2$. Let us write this explicitly, in referential terms: let $q(t)$ be a smooth curve in $\mathcal{Q}_\rho$, with $\dot{q} = \mathbf{X}$, and let $\mathbf{x}(t)$ be a smooth lifting of $q(t)$ in $\mathcal{K}_\rho$, i.e, $p_{\mathbf{x}(t)} = q(t)$. Then $\sigma(t) = \mathbf{x}(t)a(t)$ where $a(t) \in \mathcal{A}$ is a smooth curve. We have

$$D_\mathbf{X} \sigma = \frac{d}{dt} q(\mathbf{x}(t)a(t)) = q(\dot{\mathbf{x}}(t)a(t) + \mathbf{x}(t)\dot{a}(t)) = \mathbf{x}\left(\dot{a}(t) + \theta_\rho(\mathbf{x}(t), \dot{\mathbf{x}}(t))a(t)\right).$$

This formula can be read as follows: the covariant derivative has two terms in the fibers of ξ_q, $\mathbf{x}a$ and the component of $\dot{\mathbf{x}}$ in $R(q)$ multiplied by a. Note also that since $q\mathbf{x} = \mathbf{x}$, $\dot{\mathbf{x}}$ decomposes as $q\mathbf{x}$ in $R(q)$ plus $\mathbf{X}\mathbf{x}$ in $N(q)$.

15 The complex structure of $\mathcal{Q}_\rho$

Notice that, for any $q \in \mathcal{Q}_\rho$, the tangent space $(T\mathcal{Q}_\rho)_q$ is a *real* vector space. In this section, we shall define a generalized complex structure on $\mathcal{Q}_\rho$. This means a smooth map $q \mapsto \mathcal{J}_q$, where

$$\mathcal{J}_q : (T\mathcal{Q}_\rho)_q \to (T\mathcal{Q}_\rho)_q \tag{55}$$

is a bounded linear map such that $\mathcal{J}_q^2 = -1_{(T\mathcal{Q}_\rho)_q}$, with an integrability property. For general ideas about complex structures in finite dimensional manifolds, we refer the reader to [22].

Recall, from Theorem 6.5, the map $\varphi_\rho : \mathcal{K}_\rho \to \mathcal{Q}_\rho$, $\varphi_\rho(\mathbf{x}) = \mathbf{x}\mathbf{x}^*\rho$ and its tangent map $(T\varphi_\rho)_\mathbf{x}\mathbf{Y} = (\mathbf{Y}\mathbf{x}^* + \mathbf{x}\mathbf{Y}^*)\rho$.

Fix $q \in \mathcal{Q}_\rho$ and $\mathbf{x} \in \mathcal{K}_\rho$ such that $q = p_\mathbf{x}$. Then, for all $\mathbf{Z} \in (T\mathcal{Q}_\rho)_q$, it holds that

$$(T\varphi_\rho)_\mathbf{x} \ i\mathbf{Z}\mathbf{x} = i \ (\mathbf{Z}\mathbf{x}\mathbf{x}^* - \mathbf{x}\mathbf{x}^*\mathbf{Z}^*)\rho = i \ (\mathbf{Z}q - q\mathbf{Z}) = i \ \mathbf{Z}(2q-1),$$

because $q = p_\mathbf{x} = \mathbf{x}\mathbf{x}^*\rho$, $\mathbf{Z}^*\rho = \rho\mathbf{Z}$ and $\mathbf{Z}q + q\mathbf{Z} = \mathbf{Z}$. This shows that

$$\mathcal{J}_q \mathbf{Z} := (T\varphi_\rho)_\mathbf{x} \ i \ \mathbf{Z}_\mathbf{x} = i \ \mathbf{Z}(2q-1)$$

is a well defined map $\mathcal{J}_q : (T\mathcal{Q}_\rho)_q \to (T\mathcal{Q}_\rho)_q$. The proof that $\mathcal{J}_q(\mathcal{J}_q\mathbf{Z}) = -\mathbf{Z}$ is a simple computation.

Remark 15.1. In the particular case when $q = p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, and $\mathbf{x} = \mathbf{e}_1$, we have, in matrix form, $\mathbf{Z} = \begin{pmatrix} 0 & Z_1 \\ -Z_1^* & 0 \end{pmatrix}$, and $\mathcal{J}_p\mathbf{Z} = i\mathbf{Z}\rho = \begin{pmatrix} 0 & -iZ_1 \\ -iZ_1^* & 0 \end{pmatrix} = -i \begin{pmatrix} 0 & Z_1 \\ Z_1^* & 0 \end{pmatrix}$.

In order to see that this complex structure is integrable we proceed as follows. First, we note that it is invariant under the action of the group $\mathcal{U}_\rho$. Next, we identify $\mathcal{Q}_\rho$ with the disk $\mathcal{D} = \{a \in \mathcal{A} : \|a\| < 1\} \subset \mathcal{A}$. The disk $\mathcal{D}$ has a natural complex structure, as an open subset of the Banach space $\mathcal{A}$, which is invariant for the action of $\mathcal{U}_\rho$ (the identification $\mathcal{Q}_\rho \simeq \mathcal{D}$ is $\mathcal{U}_\rho$-equivariant). In order to prove that our complex structure is *integrable*, it suffices to show that these structures coincide at the point $q = p$, which corresponds to $0 \in \mathcal{D}$ in the identification $\mathcal{Q}_\rho \simeq \mathcal{D}$:

Lemma 15.2. *The complex multiplication defined in $\mathcal{Q}_\rho$ corresponds, under the identification $\mathcal{Q}_\rho \simeq \mathcal{D}$, to the usual multiplication by the imaginary constant i.*

In [6] the reader will find a similar treatment of complex structures in a C*-algebra context.

16 The Hilbertian product in $\mathcal{Q}_\rho$

We define now on $\mathcal{Q}_\rho$ a generalization of the notion of Hermitian structure. It consists of a Hilbertian product on each $(T\mathcal{Q}_\rho)_q$ with values in $\mathcal{C}_q$, which is invariant under the action of $\mathcal{U}_\rho$.

Definition 16.1. *Given $\mathbf{X}, \mathbf{Y} \in (T\mathcal{Q}_\rho)_q$, we put*

$$\langle \mathbf{X}, \mathbf{Y} \rangle_q = \text{ the endomorphism of } R(q) \text{ given by the pair } (\mathbf{x}, -\theta_\rho(\kappa_{\mathbf{x}}(\mathbf{X}), \kappa_{\mathbf{x}}(\mathbf{Y})))$$

$$= \Big[(\mathbf{x}, -\theta_\rho(\kappa_{\mathbf{x}}(\mathbf{X}), \kappa_{\mathbf{x}}(\mathbf{Y})))\Big],$$

where $\mathbf{x} \in \mathcal{K}_\rho$ is such that $p_{\mathbf{x}} = q$, and the brackets denote the equivalence class as in (51).

Note that (regarding $\mathbf{X}$ and $\mathbf{Y}$ as matrices in $M_2(\mathcal{A})$)

$$-\theta_\rho(\kappa_{\mathbf{x}}(\mathbf{X}), \kappa_{\mathbf{x}}(\mathbf{Y})) = -(\mathbf{X}\mathbf{x})^*\rho(\mathbf{Y}\mathbf{x}) = -\mathbf{x}^*\mathbf{X}^*\rho\mathbf{Y}\mathbf{x}.$$

Summarizing,

$$\langle \mathbf{X}, \mathbf{Y} \rangle_q = \Big[(\mathbf{x}, -\mathbf{x}^*\mathbf{X}^*\rho\mathbf{X}\mathbf{x})\Big]. \tag{56}$$

Remark 16.2. Some remarks are in order.

1. The minus sign is needed so that the form is positive. Indeed, recall that θ_ρ is negative in $N(q)$ and $\mathbf{X}\mathbf{x} \in N(q)$,
$$\langle \mathbf{X}, \mathbf{X} \rangle_q = [(\mathbf{x}, -\theta_\rho(\mathbf{X}\mathbf{x}, \mathbf{X}\mathbf{x}))].$$
Recall that the C*-algebra structure of the fibers $\mathcal{C}_q$ of $\mathcal{C}$ is that of $\mathcal{A}$: an endomorphism φ of $\mathcal{L}_{\mathcal{A}}(R(q) \simeq \mathcal{A}$ is positive if and only if any of its matrices is a positive element of $\mathcal{A}$. Recall from the definition of $\mathcal{Q}_\rho$, that the projection q decomposes θ_ρ, i.e., θ_ρ is positive in $R(q)$ and negative in $N(q)$. On the other hand, as seen above, $\mathbf{X}\mathbf{x} \in N(q)$.

2. If one changes $\mathbf{x}$ for $\mathbf{x}u$ for some $u \in \mathcal{U}_\mathcal{A}$, then

$$\theta_\rho(\kappa_{\mathbf{x}u}(\mathbf{X}u), \kappa_{\mathbf{x}u}(\mathbf{Y}u)) = u^* \theta_\rho(\kappa_\mathbf{x}(\mathbf{X}), \kappa_\mathbf{x}(\mathbf{Y}))u,$$

   i.e. $\langle \mathbf{X}, \mathbf{Y} \rangle_q$ is indeed an element of $\mathcal{C}_q$.

3. This Hilbertian product is linear with respect to the product defined in 14.7: if $q = p_\mathbf{x} \in \mathcal{Q}_\rho$, $\mathbf{X}, \mathbf{Y} \in (T\mathcal{Q}_\rho)_q$ and $\varphi \in \mathcal{L}_\mathcal{A}(R(q))$, then

$$\langle \mathbf{X}, \mathbf{Y} \cdot \varphi \rangle_q = \langle \mathbf{X}, \mathbf{Y} \rangle_q \cdot \varphi.$$

   Indeed, if we use the basis $\mathbf{x} \in \mathcal{K}_\rho$ for $R(q)$, then $\mathbf{Y} \cdot \varphi$ is represented by the pair $(\mathbf{x}, \mathbf{Y}\mathbf{x}a)$, and

$$\theta_\rho(\kappa_\mathbf{x}(\mathbf{X}), \kappa_\mathbf{x}(\mathbf{Y} \cdot \varphi)) = (\mathbf{X}\mathbf{x})^* \rho \mathbf{Y}\mathbf{x}a,$$

   which is the element of $\mathcal{A}$ that one uses to define $\langle \mathbf{X}, \mathbf{Y} \rangle_q \cdot \varphi$ (in terms of the basis $\mathbf{x}$).

4. Recall from the end of Section 13, that if $\dot{v}, \dot{w} \in (T\mathcal{H})_\zeta$ for some $\zeta \in \mathcal{H}$, then

$$\langle \dot{v}, \dot{w} \rangle_\zeta = -\frac{1}{4}(y^{-1/2}\dot{v}y^{-1/2})^*(y^{-1/2}\dot{w}y^{-1/2}),$$

   which is the Hilbertian product, in its version for $\mathcal{H}$.

Theorem 16.3. *The complex structure on $\mathcal{Q}_\rho$ is compatible with the Hilbertian product in the following sense:*

$$\langle \mathcal{J}_q \mathbf{X}, \mathbf{Y} \rangle_q = -\langle \mathbf{X}, \mathcal{J}_q \mathbf{Y} \rangle_q,$$

for $q = p_\mathbf{x} \in \mathcal{Q}_\rho$ and $\mathbf{X}, \mathbf{Y} \in (T\mathcal{Q}_\rho)_q$.

Remark 16.4. We saw earlier that the tangent space $(T\mathcal{Q}_\rho)_q$ is a right module over the corresponding fiber $\mathcal{C}_q$ of the coefficient bundle. The scalar field $\mathbb{C}$ lies naturally inside $\mathcal{C}_q$ as

$$\mathbb{C} \ni z \longleftrightarrow \Big[(\mathbf{x}, zI)\Big] \in \mathcal{C}_q,$$

because $uzIu^* = zI$ for all $u \in \mathcal{U}_\mathcal{A}$. Thus $(T\mathcal{Q}_\rho)_q$ is a $\mathbb{C}$-vector space. It is clear that the complex structure in $T\mathcal{Q}_\rho$ which we defined above (in terms of the identification Φ of (54)) coincides with the one induced by the operator $\mathcal{J}$.

Remark 16.5. One might wish to extend the argument above. Namely, to use the identification $\mathcal{Q}_\rho \simeq \mathcal{D} \subset \mathcal{A}$ in order to endow the tangent bundle $T\mathcal{Q}_\rho$ with a right action of the algebra $\mathcal{A}$. However, this action depends on the immersion (at the tangent level) and is not intrinsic. It works in the case of $\mathbb{C}$, as it works also for the center of $\mathcal{A}$: endomorphisms with matrices in the center of $\mathcal{A}$ have the same matrix for any basis of the given submodule.

17 The invariant Finsler structure

We proved in Section 11 that $\mathcal{Q}_\rho$ has a Finsler structure, which is invariant under $\mathcal{U}_\rho$. If $q \in \mathcal{Q}_\rho$ and $\mathbf{X} \in (T\mathcal{Q}_\rho)_q$, define the norm

$$\|X\|_q = \||2q - I|^{-1/2}\mathbf{X}|2q - I|^{1/2}\|.$$

We recall that it is precisely this structure, which when translated to the disk $\mathcal{D}$, gives the Poincaré metric of the disk (Section 11). In this section we shall see that the Hilbertian product, induces also in a natural way the same Finsler structure.

Let $\varphi \in \mathcal{L}_{\mathcal{A}}(R(q))$. Define

$$|\varphi| := \sup_{0 \neq \mathbf{y} \in R(q)} \frac{\|\theta_\rho(\varphi(\mathbf{y}), \varphi(\mathbf{y}))\|^{1/2}}{\|\theta_\rho(\mathbf{y}, \mathbf{y})\|^{1/2}}. \tag{57}$$

Alternatively, θ_ρ is a C*-Hilbert module (positively) inner product in $R(q)$. Thus, the above formula is just the usual way to compute the norm of an endomorphism, when the module $R(q)$ is endowed with the C*-module norm.

In the presence of a basis $\mathbf{x} \in \mathcal{K}_\rho$ of $R(q)$ and a matrix $a \in \mathcal{A}$ for φ, the norm of the endomorphism is the norm of the matrix.

Lemma 17.1. *If $\varphi = [(\mathbf{x}, a)]$, then $|\varphi| = \|a\|$.*

Theorem 17.2. *Let $q \in \mathcal{Q}_\rho$ and $\mathbf{X} \in (T\mathcal{Q}_\rho)_q$. Then*

$$|\langle \mathbf{X}, \mathbf{X} \rangle_q| = \|\mathbf{X}\|_q^2.$$

18 The symplectic form ω

References for symplectic geometry are [33], [10], [15].

The differential forms that we consider here may have at least three types of coefficients, with respect to the scalar multiplying of the tangent vectors and also the values of the forms. The concept of form varies substantially according to the type the coefficients:

1. coefficients in $\mathcal{C}$;
2. if $\mathcal{A}$ has a center valued trace, i.e., there exists a tracial linear map $\tau : \mathcal{A} \to \mathcal{B}$, with $\mathcal{B}$ a subalgebra of the center of $\mathcal{A}$, then the coefficients are the elements of $\mathcal{B}$;
3. if $\mathcal{A}$ has a scalar trace $\tau : \mathcal{A} \to \mathbb{C}$, then the coefficients are the complex numbers.

If $\mathbf{X}, \mathbf{Y} \in (T\mathcal{Q}_\rho)_q$, the decomposition in selfadjoint and anti-selfadjoint parts

$$\langle \mathbf{X}, \mathbf{Y} \rangle_q = (\mathbf{X}, \mathbf{Y})_q + i\omega(\mathbf{X}, \mathbf{Y})_q$$

provides a generalized *Riemannian* form

$$(\mathbf{X}, \mathbf{Y})_q = Re\langle \mathbf{X}, \mathbf{Y} \rangle_q$$

and a generalized symplectic form

$$\omega(\mathbf{X}, \mathbf{Y})_q = Im\langle \mathbf{X}, \mathbf{Y} \rangle_q.$$

In this paper we are interested in ω. Notice that ω is sesqui-linear and $\omega(\mathbf{Y}, \mathbf{X}) = -\omega(\mathbf{X}, \mathbf{Y})$.

We compute now the curvature of the canonical connection, which turns out to be a scalar multiple of the symplectic form ω.

To simplify the computation, we make the assumption that given $\mathbf{X}, \mathbf{Y} \in (T\mathcal{Q}_\rho)_q$, we can construct a smooth map $q(t,s) \in \mathcal{Q}_\rho$ such that

$$q(0,0) = q, \ \frac{\partial}{\partial t} q|_{(0,0)} = \mathbf{X} \quad \text{and} \quad \frac{\partial}{\partial s} q|_{(0,0)} = \mathbf{Y},$$

and this map $q(t,s)$ is lifted to a map $\mathbf{x}(t,s) \in \mathcal{K}_\rho$. We assume also the existence of a cross section σ defined on a neighbourhood of q. $\sigma = \mathbf{x}a$, where both $\mathbf{x}$ and a are functions of (t,s). To abbreviate, we shall write with a dot $^{\cdot}$ the derivatives with respect to t, and with a tilde $'$ the derivatives with respect to s. Then, differentiating

$$D_{\mathbf{X}}\sigma = \mathbf{x}(\dot{a} + \theta_\rho(\mathbf{x}, \dot{\mathbf{x}})a)$$

with respect to s, we get

$$\mathbf{x}\left(\dot{a}' + \theta_\rho(\mathbf{x}', \dot{\mathbf{x}})a + \theta_\rho(\mathbf{x}, \dot{\mathbf{x}}')a + \theta_\rho(\mathbf{x}, \dot{\mathbf{x}})a' + \theta_\rho(\mathbf{x}, \mathbf{x}')(\dot{a} + \theta_\rho(\mathbf{x}, \dot{\mathbf{x}})a)\right) = D_{\mathbf{Y}} D_{\mathbf{X}} \sigma.$$

Interchanging $'$ with $^{\cdot}$ we get

$$\mathbf{x}\left(\dot{a'} + \theta_\rho(\dot{\mathbf{x}}, \mathbf{x}')a + \theta_\rho(\mathbf{x}, \dot{\mathbf{x'}})a + \theta_\rho(\mathbf{x}, \mathbf{x}')\dot{a} + \theta_\rho(\mathbf{x}, \dot{\mathbf{x}})(a' + \theta_\rho(\mathbf{x}, \mathbf{x}')a)\right) = D_{\mathbf{X}} D_{\mathbf{Y}} \sigma.$$

This should be specialized at $(t,s) = (0,0)$. Clearly, the lifting of $q(t,s)$ can be done in order that $\mathbf{x}'$ and $\dot{\mathbf{x}}$ are *horizontal* at $(0,0)$ (i.e., that they belong to $N(q)$). Then we get

$$R(\mathbf{X}, \mathbf{Y})\sigma = D_{\mathbf{X}} D_{\mathbf{Y}} \sigma - D_{\mathbf{Y}} D_{\mathbf{X}} \sigma = \mathbf{x}(\theta_\rho(\dot{\mathbf{x}}, \mathbf{x}') - \theta_\rho(\mathbf{x}', \dot{\mathbf{x}}))a.$$

Since $\dot{\mathbf{x}}$ and $\mathbf{x}'$ are horizontal, we have that $\dot{\mathbf{x}} = \kappa_{\mathbf{x}}(\mathbf{X}) = \mathbf{X}\mathbf{x}$ and $\mathbf{x}' = \mathbf{Y}\mathbf{x}$. Then

$$R(\mathbf{X}, \mathbf{Y})\sigma = \Big[(\mathbf{x}, \mathbf{x}(\theta_\rho(\mathbf{X}\mathbf{x}, \mathbf{Y}\mathbf{x}) - \theta_\rho(\mathbf{Y}\mathbf{x}, \mathbf{X}\mathbf{x}))a)\Big] = \Big[(\mathbf{x}, \mathbf{x}\theta_\rho(\mathbf{x}, [\mathbf{X}, \mathbf{Y}]\mathbf{x})a)\Big]. \tag{58}$$

Here we use that $\mathbf{X}$ and $\mathbf{Y}$, being tangent vectors of $\mathcal{Q}_\rho$, are θ_ρ-symmetric, i.e., $\mathbf{X}^*\rho = \rho\mathbf{X}$.

Note that $R(\mathbf{X}, \mathbf{Y})$ is an endomorphism of $\mathcal{E}_q$. Recall the Hilbertian $\mathcal{C}$-valued product in $\mathcal{Q}_\rho$

$$\langle \mathbf{X}, \mathbf{Y} \rangle_q = -\Big[(\mathbf{x}, \theta_\rho(\mathbf{X}\mathbf{x}, \mathbf{Y}\mathbf{x})\Big]$$

for $\mathbf{x} \in \mathcal{K}_\rho$ such that $p_{\mathbf{x}} = q$, which is an element of $\mathcal{C}_q$, i.e., an endomorphism of $\mathcal{E}_q$. Its imaginary part is

$$Im\langle \mathbf{X}, \mathbf{Y} \rangle_q = -\frac{i}{2}\Big[(\mathbf{x}, -\theta_\rho(\mathbf{X}\mathbf{x}, \mathbf{Y}\mathbf{x}) + \theta_\rho(\mathbf{X}\mathbf{x}, \mathbf{Y}\mathbf{x})^*)\Big] = -\frac{i}{2}\Big[(\mathbf{x}, \theta_\rho(\mathbf{Y}\mathbf{x}, \mathbf{X}\mathbf{x}) - \theta_\rho(\mathbf{X}\mathbf{x}, \mathbf{Y}\mathbf{x}))\Big]$$

$$= \frac{i}{2}\Big[(\mathbf{x}, \theta_\rho(\mathbf{x}, [\mathbf{X}, \mathbf{Y}]\mathbf{x})\Big].$$

Thus, we have proved the following result, which is a kind of prequantization of $\mathcal{Q}_\rho$ [34]:

Theorem 18.1. *The curvature of the tautological bundle $\mathcal{E}$ and the Hilbertian product in $\mathcal{Q}_\rho$ are related by the following formula*

$$\frac{i}{2}\ R(\mathbf{X}, \mathbf{Y})_q = Im\langle \mathbf{X}, \mathbf{Y} \rangle_q = \omega(\mathbf{X}, \mathbf{Y})_q.$$

19 The moment map

Consider a symplectic manifold M, and a Lie group G (with Lie algebra $\mathfrak{G}$) which acts on M by symplectomorphisms. In this situation a *moment map* is an equivariant function $\mu : M \to \mathfrak{G}^*$, the dual of $\mathfrak{G}$. Equivalently, μ can be given as a map $M \times \mathfrak{G} \to \mathbb{R}$. We shall adopt this latter point of view, and in our case the scalars will be replaced by the coefficient bundle $\mathcal{C}$.

Recall from Remark 5.1 the Banach-Lie algebra $\mathfrak{U}_\rho$ of the Banach-Lie group $\mathcal{U}_\rho$, $\mathfrak{U}_\rho = \mathfrak{U}_{0,\rho} \oplus \mathfrak{U}_{1,\rho}$, where

$$\mathfrak{U}_{0,\rho} = \{\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \in M_2(\mathcal{A}) : a_1, a_2 \in \mathcal{A}_{ah}\} \text{ and } \mathfrak{U}_{1,\rho} = \{\begin{pmatrix} 0 & a^* \\ a & 0 \end{pmatrix} \in M_2(\mathcal{A}) : a \in \mathcal{A}\}. \quad (59)$$

If $a \in \mathfrak{U}_\rho$ and $q \in \mathcal{Q}_\rho$, we put

$$f_a(q) = \frac{1}{2i}\Big[(\mathbf{x}, \theta_\rho(\mathbf{x}, a\mathbf{x}))\Big] = \frac{1}{2i}\Big[(\mathbf{x}, \mathbf{x}^*\rho a\mathbf{x})\Big], \quad (60)$$

for $\mathbf{x} \in \mathcal{K}_\rho$ such that $p_\mathbf{x} = q$. Again, a simple computation shows that if one chooses $\mathbf{x}u$ instead, the element $\theta_\rho(\mathbf{x}, a\mathbf{x})$ varies accordingly: $\theta_\rho(\mathbf{x}u, a\mathbf{x}u) = \langle \rho\mathbf{x}u, a\mathbf{x}u\rangle = u^*\langle \rho\mathbf{x}, a\mathbf{x}\rangle u = u^*\theta_\rho(\mathbf{x}, a\mathbf{x})u$. Thus, $f_a(q) \in \mathcal{C}_q$. We call $f_a : \mathcal{Q}_\rho \to \mathcal{C}$ is *the moment map* of the generalized symplectic manifold $\mathcal{Q}_\rho$.

Proposition 19.1. *The moment map is equivariant with respect to the action of $\mathcal{U}_\rho$: if $c \in \mathcal{U}_\rho$, $q \in \mathcal{Q}_\rho$ and $a \in \mathfrak{U}_\rho$,*

$$f_{cac^{-1}}(cqc^{-1}) = f_a(q).$$

Let us compute the covariant derivative of f_a. We use a horizontal lifting $\mathbf{x}(t) \in \mathcal{K}_\rho$ of the curve $q(t)$ in the direction of $\mathbf{X}$ at q (i.e., $q(0) = q$, $\dot q(0) = \mathbf{X}$ and $\dot{\mathbf{x}}(t) \in N(q(t))$). Then

$$D_\mathbf{X} f_a = \Big[(\mathbf{x}, \frac{1}{2i}(\theta_\rho(\dot{\mathbf{x}}, a\mathbf{x}) + \theta_\rho(\mathbf{x}, a\dot{\mathbf{x}})))\Big] = \Big[(\mathbf{x},\ Im\ \theta_\rho(\dot{\mathbf{x}}, a\mathbf{x}))\Big],$$

where we use that a is θ_ρ-anti-symmetric. Since $\dot{\mathbf{x}}$ is horizontal, it holds that $\dot{\mathbf{x}} = \mathbf{X}\mathbf{x}$. Then

$$D_\mathbf{X} f_a = \Big[(\mathbf{x},\ Im\ \theta_\rho(\mathbf{X}\mathbf{x}, a\mathbf{x})\Big]. \quad (61)$$

On the other hand, when computing the curvature of the canonical connection of $\mathcal{E}$, we proved that for any $\mathbf{X}, \mathbf{Y} \in (T\mathcal{Q}_\rho)_q$

$$Im\ \langle \mathbf{X}, \mathbf{Y}\rangle_q = \Big[(\mathbf{x}, \frac{1}{2i}(\theta_\rho(\mathbf{X}\mathbf{x}, \mathbf{Y}\mathbf{x}) - \theta_\rho(\mathbf{X}\mathbf{x}, \mathbf{Y}\mathbf{x})^*))\Big] = -\Big[(\mathbf{x}, Im\ \theta_\rho(\mathbf{X}\mathbf{x}, \mathbf{Y}\mathbf{x}))\Big].$$

Now we consider for each $a \in \mathfrak{U}_\rho$ the vector field in $\mathcal{Q}_\rho$, $\mathbf{X}_a(q) = (d\pi_q)_1(a) = [a, q]$, where $\pi_q : \mathcal{U}_\rho \to \mathcal{Q}_\rho$ denotes the action map: $\pi_q(c) = cqc^{-1}$. Applying the formula above with $\mathbf{X} = \mathbf{X}_a$, we get

$$\omega(\mathbf{X}_a, \mathbf{Y})_q = Im\ \langle \mathbf{X}_a, \mathbf{Y}\rangle_q = \Big[(\mathbf{x}, -Im\ \theta_\rho(\mathbf{X}_a\mathbf{x}, \mathbf{Y}\mathbf{x}))\Big].$$

Observe that $\mathbf{X}_a(q)\mathbf{x} = [a, q]\mathbf{x} = aq\mathbf{x} - qa\mathbf{x} = a\mathbf{x} - qa\mathbf{x}$ (recall that $q\mathbf{x} = p_\mathbf{x}\mathbf{x} = \mathbf{x}$). Note also that $\theta_\rho(qa\mathbf{x}, \mathbf{Y}\mathbf{x}) = 0$ because $\mathbf{Y}\mathbf{x} \in N(q)$. Thus $\theta_\rho(\mathbf{X}_a\mathbf{x}, \mathbf{Y}\mathbf{x}) = \theta_\rho(a\mathbf{x}, \mathbf{Y}\mathbf{x})$. Therefore

$$\omega(\mathbf{X}_a, \mathbf{Y})_q = \Big[(\mathbf{x}, -Im\ \theta_\rho(a\mathbf{x}, \mathbf{Y}\mathbf{x}))\Big].$$

We proved before that $D_{\mathbf{Y}} f_a = \Big[\mathbf{x},\ Im\ \theta_\rho(\mathbf{Y}\mathbf{x}, a\mathbf{x}))\Big]$. Thus, we have shown that

$$D_{\mathbf{Y}} f_a = \omega(\mathbf{X}_a, \mathbf{Y}).$$

Suppose now that f is a section of $\mathcal{C}$ over some open subset $\mathcal{W}$ of $\mathcal{Q}_\rho$. Consider the covariant derivative Df as a 1-form on $\mathcal{W}$ with values in $\mathcal{C}$: $Df(\mathbf{X}) = D_{\mathbf{X}} f$.

Definition 19.2. *We say that a field $\mathbf{X}_f$ is the* symplectic gradient *of f if*

$$D_{\mathbf{X}} f = \omega(\mathbf{X}, \mathbf{X}_f).$$

We are mainly interested in the case of functions of the form f_a, for $a \in \mathfrak{U}_\rho$,

$$f_a(q) = \frac{1}{2i}\Big[(\mathbf{x}, \theta_\rho(\mathbf{x}, a\mathbf{x}))\Big] = \frac{1}{2i}\Big[(\mathbf{x}, \mathbf{x}^*\rho a\mathbf{x})\Big],$$

where $q \in \mathcal{Q}_\rho$ and $\mathbf{x} \in \mathcal{K}_\rho$ satisfies $p_{\mathbf{x}} = q$. Recall that the Lie algebra $\mathfrak{U}_\rho$ consists of all matrices $a = \begin{pmatrix} a_{11} & \alpha \\ \alpha^* & a_{22} \end{pmatrix}$, where $a_{ii}^* = -a_{ii}$. Recall also the identity $D_{\mathbf{X}} f_a = \omega(\mathbf{X}, \mathbf{X}_a)$. This says that $\mathbf{X}_a$ is the symplectic gradient of f_a.

Recall also that $\mathbf{X}_a(q) = [a, q]$. Now given $a, b \in \mathfrak{U}_\rho$, we want to find the value of $\omega(\mathbf{X}_a, \mathbf{X}_b)$. at $q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, and correspondingly $\mathbf{x} = \mathbf{e}_1 \in \mathcal{K}_\rho$. Since

$$X_a = [a, q] = \begin{pmatrix} 0 & -\alpha \\ \alpha^* & 0 \end{pmatrix}, \quad X_b = [b, q] = \begin{pmatrix} 0 & -\beta \\ \beta^* & 0 \end{pmatrix}$$

for some $\alpha, \beta \in \mathcal{A}$, then $\langle \mathbf{X}_a, \mathbf{X}_b \rangle_q = \theta_\rho\Big(\begin{pmatrix} 0 \\ \alpha^* \end{pmatrix}, \begin{pmatrix} 0 \\ \beta^* \end{pmatrix}\Big) = -\alpha\beta^*$, and

$$\omega(\mathbf{X}_a, \mathbf{X}_b)_q = Im\ \langle \mathbf{X}_a, \mathbf{X}_b \rangle_q = \frac{1}{2i}(\beta\alpha^* - \alpha\beta^*).$$

We want to establish the relationship between $\omega(\mathbf{X}_a, \mathbf{X}_b)$ and $f_{[a,b]}$. To this effect, observe first that

$$\theta_\rho(\mathbf{e}_1, [a, b]\mathbf{e}_1) = -\theta_\rho(a\mathbf{e}_1, b\mathbf{e}_1) + \theta_\rho(b\mathbf{e}_1, a\mathbf{e}_1) = \alpha\beta^* - \beta\alpha^* + a_{11}b_{11} - b_{11}a_{11}.$$

Then $\omega(\mathbf{X}_a, \mathbf{X}_b) = \frac{1}{2i}(\beta\alpha^* - \alpha\beta^*)$ and $f_{[a,b]} = \frac{1}{2i}(\alpha\beta^* - \beta\alpha^* + a_{11}b_{11} - b_{11}a_{11})$. So

$$\omega(\mathbf{X}_a, \mathbf{X}_b) = -f_{[a,b]} + \frac{1}{2i}(a_{11}b_{11} - b_{11}a_{11}).$$

Finally, $f_a(q) = \frac{1}{2i}\theta_\rho(\mathbf{e}_1, a\mathbf{e}_1) = \frac{1}{2i}a_{11}$, and

$$\omega(\mathbf{X}_a, \mathbf{X}_b) = -f_{[a,b]} + 2i[f_a, f_b]. \tag{62}$$

This equality holds at $q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, but a simple argument shows that (62) holds at every $q \in \mathcal{Q}_\rho$. Indeed, each of the terms involved is a function in $q \in \mathcal{Q}_\rho$ with $a, b \in \mathfrak{U}_\rho$ fixed. If we change q by cqc^{-1} for $c \in \mathcal{U}_\rho$, the values change by an inner automorphism. Finally, since $\mathcal{U}_\rho$ acts transitively on $\mathcal{Q}_\rho$, our claim is proven.

Given two sections f, g with their respective symplectic gradients $\mathbf{X}_f$ and $\mathbf{X}_g$, the *Poisson bracket* $\{f, g\}$ is defined as

$$\{f, g\} = \omega(\mathbf{X}_f, \mathbf{X}_g).$$

Then, we get

$$\{f_a, f_b\} = -f_{[a,b]} + 2i[f_a, f_b].$$

The term $2i[f_a, f_b]$ occurs because of the non-commutativity of the C*-algebras $\mathcal{C}_q$, where the moment map takes its values.

Summarizing the facts of the last sections:

Theorem 19.3. *Consider the manifold $\mathcal{Q}_\rho$, with the tautological bundle $\mathcal{E}$ and the coefficient bundle $\mathcal{C}$. Then*

1. *There exist invariant connections in $\mathcal{E}$ and $\mathcal{C}$, linked by Leibnitz' rule (see (52)).*
2. *There exists a Hilbertian product in $T\mathcal{Q}_\rho$, with values in $\mathcal{C}$. This product is compatible with the right $\mathcal{C}$-module structure of $T\mathcal{Q}_\rho$.*
3. *The imaginary part ω of the Hilbertian product in $T\mathcal{Q}_\rho$ is the curvature of the tautological connection of $\mathcal{E}$. In this sense, the symplectic form ω is exact.*
4. *The map f_a ($a \in \mathfrak{U}_\rho$) is a moment map: the field $\mathcal{X}_a$ is the symplectic gradient of the function (cross section for $\mathcal{C}$) f_a. Here gradients are computed using the covariant derivative.*

20 The Liouville 1-form

We begin this section with the explicit computation of the lifting form introduced in Section 13, for the case of $\mathcal{H}$.

20.1 The lifting form of $\mathcal{H}$

Pick $\mathbf{x} \in \mathcal{K}_{\rho'}$, which means that $2\ Im\ x_1^* x_2 = 1$. Note the fact that $x_1 \in G_\mathcal{A}$ implies that also $x_2 \in G_\mathcal{A}$. As in the model $\mathcal{D}$, every $\mathbf{x} \in \mathcal{K}_{\rho'}$ gives rise to a projection (its representative in $\mathcal{Q}_{\rho'}$): $p_\mathbf{x} = \mathbf{x}\mathbf{x}^* \rho_H$. Consider its complement

$$1 - p_\mathbf{x} = \begin{pmatrix} 1 - ix_1x_2^* & ix_1x_1^* \\ -ix_2x_1^* & 1 + ix_2x_1^* \end{pmatrix}.$$

Note the second column $\begin{pmatrix} ix_1x_1^* \\ 1 + ix_2x_1^* \end{pmatrix} = \begin{pmatrix} ix_1 \\ (x_1^*)^{-1} + ix_2 \end{pmatrix} x_1^*$, which means that the vector

$$\mathbf{x}_\perp := \begin{pmatrix} ix_1 \\ (x_1^*)^{-1} + ix_2 \end{pmatrix}. \tag{63}$$

generates the nullspace of $p_{\mathbf{x}}$. Note that $\theta_{\rho'}(\mathbf{x}_\perp, \mathbf{x}_\perp) = -1$. The pair $\{\mathbf{x}, \mathbf{x}_\perp\}$ forms a $\theta_{\rho'}$-orthogonal basis for $\mathcal{A}^2$. Any element $\mathbf{z} \in A^2$ is written

$$\mathbf{z} = \mathbf{x}\theta_{\rho'}(\mathbf{x}, \mathbf{z}) - \mathbf{x}_\perp \theta_{\rho'}(\mathbf{x}_\perp, \mathbf{z}).$$

Note also that $\mathbf{x}_\perp = \mathbf{x}i + \mathbf{e}_2(x_1^*)^{-1}$, which gives $\mathbf{x}_\perp = \mathbf{x} - \mathbf{e}_2 i(x_1^*)^{-1}$.

The map $\mathcal{K}_{\rho'} \to \mathcal{H}$, $\mathbf{x} \mapsto \zeta = x_2 x_1^{-1}$ is the $\mathcal{H}$ valued version of the projection map $\mathbf{x} \to p_{\mathbf{x}}$. Its tangent map at $\mathbf{x}$, is $\mathbf{v} \mapsto \dot{v} := v_2 x_1^{-1} - x_2 x_1^{-1} v_1 x_1^{-1}$. Suppose now that $\mathbf{v} \in N(p_{\mathbf{x}})$: $\mathbf{v} = \begin{pmatrix} x_1 \\ x_2 - i(x_1^*)^{-1} \end{pmatrix} \lambda$, so that

$$\dot{v} = (x_2 - i(x_1^*)^{-1})\lambda x_1^{-1} - x_2 x_1^{-1} x_1 \lambda x_1^{-1} = (x_2\lambda - i(x_1^*)^{-1}\lambda - x_2\lambda)x_1^{-1} = -i(x_1^*)^{-1}\lambda x_1^{-1},$$

and thus $\lambda = i\ x_1^* \dot{v} x_1$. Then, given $\mathbf{x}$ in $k_{\rho'}$ which projects over ζ in $\mathcal{H}$, and $\dot{v} \in (T\mathcal{H})_\zeta$, we have the (lifting) form, as in Section 13

$$\kappa_{\mathbf{x}}^H(\dot{v}) = \begin{pmatrix} x_1 \\ x_2 - i\ x_1(x_1^*)^{-1} \end{pmatrix} i x_1^* \dot{v} x_1 = \begin{pmatrix} x_1 x_1^* \\ x_2 x_1^* - i \end{pmatrix} i\ \dot{v} x_1.$$

Or equivalently

$$\kappa_{\mathbf{x}}^H(\dot{v}) = (\mathbf{x}\ i - \mathbf{e}_2 i\ (x_1^*)^{-1}) i\ x_1^* \dot{v} x_1 = (-\mathbf{x} x_1^* + \mathbf{e}_2)\dot{v} x_1.$$

There is a natural global cross section for the map $\mathcal{K}_{\rho'} \ni \mathbf{x} \mapsto \zeta \in \mathcal{H}$, namely $\zeta = x + iy \mapsto \begin{pmatrix} 1 \\ \zeta \end{pmatrix} (2y)^{-1/2}$. Then we have

$$\kappa_{\mathbf{x}}^H(\dot{v}) = \begin{pmatrix} (2y)^{-1} \\ \zeta(2y)^{-1} - i \end{pmatrix} i\ \dot{v}(2y)^{-1/2}. \tag{64}$$

We want to compute the following product in $\mathcal{H}$.

Consider the expressions (64) for two tangent vectors $\dot{v}, \dot{w}$ in $(T\mathcal{H})_\zeta$:

$$\mathbf{v} = \begin{pmatrix} (2y)^{-1} \\ \zeta(2y)^{-1} - i \end{pmatrix} i\ \dot{v}(2y)^{-1/2}\ , \quad \mathbf{w} = \begin{pmatrix} (2y)^{-1} \\ \zeta(2y)^{-1} - i \end{pmatrix} i\ \dot{w}(2y)^{-1/2}.$$

Then we have

$$\theta_{\rho'}(\mathbf{v}, \mathbf{w}) = \langle \dot{v}, \dot{w} \rangle = (2y)^{-1/2} \dot{v}^* \theta_{\rho'} \Big(\begin{pmatrix} (2y)^{-1} \\ \zeta(2y)^{-1} - 1 \end{pmatrix}, \begin{pmatrix} (2y)^{-1} \\ \zeta(2y)^{-1} - 1 \end{pmatrix} \Big) \dot{w}(2y)^{-1/2}$$

$$= -(2y)^{-1/2}\dot{v}^*(2y)^{-1}\dot{w}(2y)^{-1/2} = -\frac{1}{4}(y^{-1/2}\dot{v}y^{-1/2})^*(y^{-1/2}\dot{w}y^{-1/2}).$$

Suppose now that the algebra has a trace τ onto a commutative subalgebra $\mathcal{B} \subset \mathcal{A}$. If we take the trace of the above expression, we get

$$\tau\langle \dot{v}, \dot{w} \rangle = -\tau((2y)^{-1}\dot{v}^*(2y)^{-1}\dot{w}). \tag{65}$$

Here $y^{-1/2}\dot{v}y^{-1/2}$ is the translation to the point $y^{-1/2}\zeta y^{-1/2}$ of the vector $\dot{v}$. Note that $y^{-1/2}\zeta y^{-1/2} = y^{-1/2} x y^{-1/2} + i$.

Therefore $\langle \dot{v}, \dot{w} \rangle_\zeta = \langle \dot{v}_0, \dot{w}_0 \rangle_{\zeta_0}$, where $\zeta_0 = y^{-1/2}xy^{-1/2} + i$. When the imaginary part equals 1, the inner product has the simpler expression $\langle \dot{v}_0, \dot{w}_0 \rangle_{\zeta_0} = -\frac{1}{4}\dot{v}_0^* \dot{w}_0$. Or, explicitely, if $\dot{v}_0 = \dot{x} + i\ \dot{y}$ and $\dot{w}_0 = \dot{\xi} + i\ \dot{\eta}$, then $\dot{v}_0^* \dot{w}_0 = (\dot{x}\dot{\xi} + \dot{y}\dot{\eta}) + i(\dot{x}\dot{\eta} - \dot{y}\dot{\xi})$, and

$$\tau \langle \dot{v}_0, \dot{w}_0 \rangle_{\zeta_0} = -\frac{1}{4}\{\tau(\dot{x}\dot{\xi} + \dot{y}\dot{\eta}) + i\ \tau(\dot{x}\dot{\eta} - \dot{y}\dot{\xi})\}.$$

Both traces in the above expression are selfadjoint elements of $\mathcal{B}$ (or real numbers if the trace is numerical). The imaginary part of $\tau\langle \dot{v}_0, \dot{w}_0 \rangle_{\zeta_0}$ is essentially the (trace of the) symplectic form ω. This expression corresponds strictly to $dp \wedge dq$ in the classical setting, where p varies in the imaginary part and q in the real part. This shows that $\mathcal{H} = TG^+$ as a true dynamical system.

Let us write down the real and imaginary part of $\tau\langle\ ,\ \rangle$ at any point $\zeta \in \mathcal{H}$ (non necessarily with $Im\ \zeta = 1$). With the current notation, we get

$$\tau \langle \dot{v}, \dot{w} \rangle_\zeta = -\frac{1}{4}\{\tau(\dot{x}\dot{\xi} + \dot{y}\dot{\eta}) + i\ \tau(\dot{x}\dot{\eta} - \dot{y}\dot{\xi})\}, \tag{66}$$

where $\dot{x} + i\ \dot{y} = y^{-1/2}\dot{v}y^{-1/2}$ and $\dot{\xi} + i\ \dot{\eta} = y^{-1/2}\dot{w}y^{-1/2}$.

Let us compute now the 1-*form of Liouville* which induces the symplectic form ω of $\mathcal{H}$. Recall that it receives different names: tautological 1-form, canonical 1-form, Poincaré 1-form, symplectic potential. The 1-form of Liouville is a form on the cotangent bundle of the manifold. Here it shall be presented as a 1-form in the *tangent* bundle $TG^+ \simeq \mathcal{H}$, identifying the tangent bundle TG^+ with the co-tangent bundle T^*G^+. To perform this identification, we shall use the trace τ, and the action of $G_\mathcal{A}$ on G^+. Given a vector $\dot{v} \in (T\mathcal{H})_\zeta$, for $\zeta = x + i\ y \in \mathcal{H}$, the Liouville form α maps ζ into an element of $\mathcal{B}$. We must project $\dot{v}$ from its tangency point ζ to the point $y \in G^+$, having in mind that $\zeta = x + i\ y$ represents the tangent x at the point y. Finally, we evaluate x in $\dot{v}$. To perform this task, we translate to $y = 1$, and compute

$$\alpha_y(\dot{v}) = \tau(y^{-1/2}xy^{-1/2}y^{-1/2}\dot{v}y^{-1/2}) = \tau(y^{-1}xy^{-1}\dot{v}). \tag{67}$$

This is the Liouville 1-form.

Now we differentiate this 1-form in $\mathcal{H}$, i.e., we compute $d\alpha(\dot{v}, v') = \dot{v}\alpha(v') - v'\alpha(\dot{v}) - \alpha([\dot{v}, v'])$. Consider a function in the variables s, t whose derivatives produce the fields $\dot{v}$, v' when differentiated with respect to t and s, respectively. We must compute

$$\frac{\partial}{\partial t}\alpha_\zeta(v') - \frac{\partial}{\partial s}\alpha_\zeta(\dot{v}),$$

because there is no need to substract $\alpha([\dot{v}, v'])$, since it is trivial. Then

$$\frac{\partial}{\partial t}\alpha_\zeta(v') = \tau(-y^{-1}\dot{v}y^{-1}xy^{-1}v' + y^{-1}\dot{x}y^{-1}y' - y^{-1}xy^{-1}\dot{y}y^{-1}y' + y^{-1}xy^{-1}\dot{y}')$$

and

$$\frac{\partial}{\partial s}\alpha_\zeta(\dot{v}) = \tau(-y^{-1}y'y^{-1}xy^{-1}\dot{y} + y^{-1}x'y^{-1}\dot{y} - y^{-1}xy^{-1}y'y^{-1}\dot{y} + y^{-1}xy^{-1}\dot{y}').$$

Therefore

$$(d\alpha)_\zeta(\dot{v}, v') = \frac{\partial}{\partial t}\alpha_\zeta(v') - \frac{\partial}{\partial s}\alpha_\zeta(\dot{v}) = \tau(y^{-1}\dot{x}y^{-1}y' - y^{-1}x'y^{-1}\dot{y}).$$

Now, comparing this last expression with the imaginary part of the trace of the Hilbertian product form, and adapting the notation of both computations, we get

Theorem 20.1. *The symplectic form* ω *is* exact*:*

$$(d\alpha)_\zeta(\dot v, \dot w) = \omega(\dot v, \dot w).$$

Remark 20.2. Since G^+ can be regarded as a submanifold of $\mathcal{H}$:

$$G^+ \ni y \mapsto iy \in \mathcal{H},$$

then we can compute the Hilbertian product of $\mathcal{H}$ on this submanifold. This coincides with the Riemannian structure of G^+. If $x_1, x_2 \in (TG^+)_y$, we have that $\zeta_1 = x_1 + i\ y$ and $\zeta_2 = x_2 + i\ y$ belong to $\mathcal{H}$. Define

$$[x_1, x_2]_y = \tau((y^{-1/2}x_1y^{-1/2})(y^{-1/2}x_2y^{-1/2})) = \tau(y^{-1}x_1y^{-1}x_2).$$

It is a $\mathcal{B}$-valued, bilinear (it takes selfadjoint values) and positive semidefinite form. Note also that this form is invariant under the action of G on G^+. In fact, $g \cdot y = (g^{-1})^* y g^{-1}$, for $g \in G$ and $y \in G^+$. The action defines a linear isomorphism in $\mathcal{A}$, so that the same formula gives the induced action in the tangent spaces of G^+: $g \cdot x = (g^{-1})^* x g$, if $x \in (TG^+)_y$. A straightforward computation shows that

$$[g \cdot x_1, g \cdot x_2]_{g \cdot y} = [x_1, x_2]_y.$$

If $\mathcal{A} = M_n(\mathbb{C})$, $\mathcal{B} = \mathbb{C}$ and τ is the usual trace for $n \times n$ matrices, this inner product is the usual Riemannian metric for the homogeneous space of positive definite $n \times n$ matrices.

On the other hand, if we embed $G^+ \hookrightarrow \mathcal{H}$, by means of $y \mapsto i\ y$, i.e., we regard G^+ as the imaginary positive axis of $\mathcal{H}$, then for $x_1, x_2 \in (TG^+)_y$ it holds

$$\langle i\ x_1, i\ x_2 \rangle_{i\ y} = -\frac{1}{4}(y^{-1/2} i\ x_1 y^{-1/2})^* (y^{-1/2} i\ x_2 y^{-1/2}) = -\frac{1}{4}(y^{-1/2}x_1y^{-1/2})(y^{-1/2}x_2y^{-1/2}),$$

and, thus,

$$\tau\langle i\ x_1, i\ x_2 \rangle_{i\ y} = -\frac{1}{4}[x_1, x_2]_y.$$

This means that the trace of the Hilbertian product, restricted to the positive imaginary axis in $\mathcal{H}$, gives, essentially, the Riemannian metric of the space G^+.

A geometric case of an algebra with a central trace is the following. Consider a complex vector bundle $E \to B$ with compact base space B, endowed with a Riemannian metric $\langle e, e' \rangle_b$, $b \in B$, $e, e' \in E_b$ (the fiber of E over b). Consider the fiber bundle $\mathrm{End}(E) \to B$ of endomorphisms of the vector bundle E, and let $\mathcal{A}$ be the algebra $\Gamma(\mathrm{End}(E))$ of the continuous global cross sections of $\mathrm{End}(E)$. Since each E_b is a (finite dimensional) Hilbert space, $\mathrm{End}(E_b)$ is a C*-algebra. The space $\Gamma(\mathrm{End}(E))$ of cross sections has therefore the norm $\|\varphi\| = \sup_{b \in B} \|\varphi_b\|$, where $\varphi_b : E_b \to E_b$ and $\|\varphi_b\|$ is the usual norm of linear operators. With this norm, $\mathcal{A}$ is a C*-algebra. The center $Z(\mathcal{A})$ of this algebra is the space of scalar sections λ in $\mathrm{End}(E)$ (homotetic in each fiber). The central trace is given by $\mathbf{tr} : \mathcal{A} \to Z(\mathcal{A})$, $\mathbf{tr}(\sigma)_b = Tr(\sigma_b)$, $b \in B$, with Tr the usual trace of E_b. More specifically, B could be a compact manifold, and E the complexification of its tangent bundle, with an Hermitian metric. This case is interesting due to the equivalence, as homogeneous spaces, of the disk $\mathcal{D}$ and the Poincaré halfspace $\mathcal{H}$ of the algebra $\mathcal{A}$. This homogeneous space can be thought as the tangent bundle TG^+ of the space G^+ of positive and invertible elements of $\mathcal{A}$. In this context, an element of $\mathcal{H}$ is a pair (X, a) with $a \in G^+$ and $X \in (TG^+)_a$. The element $a \in G^+$ represents a Riemannian metric in B, and a possible tangent vector X (a selfadjoint element of $\mathcal{A}$) could be the Ricci curvature of the metric a. In this manner, the geometry of $\mathcal{H}$ is linked to the deformation of the pairs (*Riemannian metric, Ricci curvature*), viewed as elements of TG^+.

21 The scalar case

In this section we shall consider the classical case when $\mathcal{A} = \mathbb{C}$. We shall check that the geometry induced by $\mathcal{U}_\rho$ on $\mathcal{D}$ is the classical hyperbolic geometry of the Poincaré disk. In the scalar case, or more generally, when $\mathcal{A}$ is commutative, the coefficient bundle $\mathcal{C}$ consists of $\mathcal{A}$ in each fiber: the Hilbertian product, the moment map, and so forth, take values in $\mathcal{A}$. Indeed, if φ is an endomorphism of $R(q)$, with bases $\mathbf{x}$ and $\mathbf{x}u$, then the corresponding matrices of φ in these bases are

$$a \ \text{ and } uau^* = a,$$

respectively. That is, the coefficient bundle $\mathcal{C}$ ends up being $\mathcal{A}$.

In particular, in the case $\mathcal{A} = \mathbb{C}$, one obtains a complex structure for the unit disk $\mathcal{D}$. The goal of this section is to show that this structure is the classical complex structure of the Poincaré disk.

We first recall the isomorphism

$$\Phi_{\mathcal{D}} : \mathcal{D} \to \mathcal{Q}_\rho.$$

In the scalar case we get

$$\mathcal{D} \ni z \mapsto p_z = \frac{1}{1-|z|^2} \begin{pmatrix} 1 & -\bar{z} \\ z & -|z|^2 \end{pmatrix}.$$

The tangent spaces $(T\mathcal{D})_x$ identify with $\mathbb{C}$. Given $a \in \mathbb{C}$, regarded as a tangent vector in $T(\mathcal{D})_z$, let us denote by $\mathbf{X}_a = (d\Phi_{\mathcal{D}})_z(a)$ the corresponding tangent vector in $T(\mathcal{Q}_\rho)_{p_z}$. Clearly, one gets

$$\mathbf{X}_a = \frac{1}{(1-|z|^2)^2} \begin{pmatrix} \bar{a}z + a\bar{z} & -\bar{a} - a\bar{z}^2 \\ a + \bar{a}z^2 & -\bar{a}z - a\bar{z} \end{pmatrix}.$$

21.1 The complex inner product

For the module $R(p_z)$, we chose the basis $\mathbf{x}_z = \frac{1}{(1-|z|^2)^{1/2}} \begin{pmatrix} 1 \\ z \end{pmatrix}$. Thus, the lifting $\kappa_{\mathbf{x}_z}(\mathbf{X}_a)$ is given by

$$\mathbf{X}_a \mathbf{x}_z = \frac{1}{(1-|z|^2)^{5/2}} \begin{pmatrix} \bar{a}z + a\bar{z} & -\bar{a} - a\bar{z}^2 \\ a + \bar{a}z^2 & -\bar{a}z - a\bar{z} \end{pmatrix} \begin{pmatrix} 1 \\ z \end{pmatrix} = \frac{1}{(1-|z|^2)^{3/2}} \begin{pmatrix} a\bar{z} \\ a \end{pmatrix}.$$

Therefore, if $z \in \mathcal{D}$ and $a, b \in \mathbb{C}$ $(= (T\mathcal{D})_z)$

$$\langle a, b \rangle_z = -\theta_\rho(\mathbf{X}_a \mathbf{x}_z, \mathbf{X}_b \mathbf{x}_z) = -\frac{1}{(1-|z|^2)^{3/2}} \theta_\rho\left(\begin{pmatrix} a\bar{z} \\ a \end{pmatrix}, \begin{pmatrix} b\bar{z} \\ b \end{pmatrix} \right) = \frac{\bar{a}b}{(1-|z|^2)^2}, \tag{68}$$

which is the classical complex inner product in the Poincaré disk.

21.2 The linear connection

Let $\mathbf{X}$ be a tangent field defined on a neighbourhood of q in $\mathcal{Q}_\rho$, and $\mathbf{Y} \in (T\mathcal{Q}_\rho)_q$. Let $q(t)$ be a smooth curve adapted to $\mathbf{Y}$: $q(0) = q$ and $\dot{q}(0) = \mathbf{Y}$. Then the covariant derivative in $\mathcal{Q}_\rho$ is given by [12]

$$\nabla_{\mathbf{Y}}\mathbf{X}_q = \frac{d}{dt}\mathbf{X}_{q(t)}|_{t=0} + [\mathbf{X}_q, [\mathbf{Y}, q]].$$

Let now $a = a(z)$ be a $\mathbb{C}$-valued smooth map defined on a neighbourhood of $z_0 \in \mathcal{D}$, regarded as a tangent vector field in $\mathcal{D}$, and $b \in \mathbb{C}$ a tangent vector at z_0. To compute $\nabla_b a_{z_0}$, we have to compute $\nabla_{\mathbf{X}_b}\mathbf{X}_a$ at $z = z_0$, and identify this tangent vector as a matrix $\mathbf{X}_c$ for certain $c \in \mathbb{C}$ (at z!), and then $c = \nabla_b a_{z_0}$. In order to simplify this computation, we shall consider the case $z_0 = 0$. In fact, due to the invariance of the linear connection under the action of $\mathcal{U}_\rho$, this will suffice to identify the covariant derivative. In this case ($z_0 = 0$), we have

$$\mathbf{X}_a(z) = \frac{1}{(1-|z|^2)^2}\begin{pmatrix} \bar{a}(z)z + a(z)\bar{z} & -\bar{a}(z) - a(z)\bar{z}^2 \\ a(z) + \bar{a}(z)z^2 & -\bar{a}(z)z - a(z)\bar{z} \end{pmatrix}, \ \mathbf{X}_b = \begin{pmatrix} 0 & -\bar{b} \\ b & 0 \end{pmatrix}.$$

Applying the formula above and choosing a smooth $z(t) \in \mathcal{D}$ such that $z(0) = 0$ and $\dot{z}(0) = b$ (for instance $z(t) = tb$), after straightforward computations one obtains

$$\nabla_{X_a}(X_b)_{E_1} = \begin{pmatrix} 0 & -\frac{\partial}{\partial b}\bar{a}(0) \\ \frac{\partial}{\partial b}a(0) & 0 \end{pmatrix} = X_{\frac{\partial}{\partial b}a(0)} \quad \text{at } z = 0.$$

Thus,

$$\nabla_b a_0 = \frac{\partial}{\partial b}a(0). \tag{69}$$

This coincides with the Levi-Civita connection of the classical metric of the Poincaré disk (at the origin).

21.3 The moment map

Note that the Lie algebra $\mathfrak{U}_\rho$ is given in this case by all matrices of the form

$$a = \begin{pmatrix} i\alpha & \omega \\ \bar{\omega} & i\beta \end{pmatrix} = i\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} + \begin{pmatrix} 0 & \omega \\ \bar{\omega} & 0 \end{pmatrix}, \ \alpha, \beta \in \mathbb{R}, \ \omega \in \mathbb{C}.$$

Recall that if $q = p_{\mathbf{x}} \in \mathcal{Q}_\rho$ and $a \in \mathfrak{U}_\rho$, then the moment map is given by

$$f_a(q)) = \frac{1}{2i}\theta_\rho(\mathbf{x}, a\mathbf{x}).$$

If $z \in \mathcal{D}$ and $q = p_z$, we choose as above the basis $\dfrac{1}{(1-|z|^2)^{1/2}}\begin{pmatrix} 1 \\ z \end{pmatrix}$, and a is given as above, then

$$f_a(z)) = f_a(q) = \frac{1}{1-|z|^2}\left\{\frac{1}{2}\theta_\rho(\begin{pmatrix} 1 \\ z \end{pmatrix}, \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}\begin{pmatrix} 1 \\ z \end{pmatrix}) + \frac{1}{2i}\theta_\rho(\begin{pmatrix} 1 \\ z \end{pmatrix}, \begin{pmatrix} 0 & \omega \\ \bar{\omega} & 0 \end{pmatrix}\begin{pmatrix} 1 \\ z \end{pmatrix})\right\}$$

$$= \frac{1}{1-|z|^2}\left(\frac{1}{2}(\alpha - \beta|z|^2) + \frac{1}{2i}(\omega z - \bar{\omega}\bar{z})\right). \tag{70}$$

21.4 Commutative C*-algebras

If $\mathcal{A}$ is commutative (i.e., $\mathcal{A} = C(\Omega, \mathbb{C})$ for some compact Hausdorf space Ω), the coefficient bundle also reduces to $\mathcal{A}$, as remarked at the beginning of this section. Moreover, it is clear that the computations done in this section can be carried over exactly in the same way. Thus, one gets a near classical situation, in which the Hilbertian product, the metric and the moment map take values in $\mathcal{A}$, and the formulas look the same as in the scalar case, replacing complex numbers by continuous functions.

22 Valuations

In this section we introduce valuation maps. Once a valuation map onto a commutative C*-algebra is chosen, the non commutative Kähler structure becomes a classical Kähler structure: measurements take values in a fixed scalar field, instead of being elements of the coefficient bundle $\mathcal{C}$. Most important, valuation maps will allow us to examine the convexity properties of the moment map.

In what follows, $\mathcal{F}$ denotes a commutative C*-algebra.

Definition 22.1. *A* valuation *ν in $\mathcal{Q}_\rho$ is a differentiable map $\nu : \mathcal{C} \to \mathcal{F}$ with the following properties:*

1. *ν is positive in the following sense: for any $q \in \mathcal{Q}_\rho$, $\nu|_{\mathcal{C}_q} : \mathcal{C}_q \to \mathcal{F}$ is a positive linear map between C*-algebras. In particular, this implies that $\nu|_{\mathcal{C}_q}$ is bounded.*
2. *ν is tracial: for any $q \in \mathcal{Q}_\rho$ and $a, b \in \mathcal{C}_q$, $\nu(ab) = \nu(ba)$.*

   Additionally, we say that ν is *faithful* if
3. *for any $q \in \mathcal{Q}_\rho$ and $a \in \mathcal{C}_q$, $\nu(a^*a) = 0$ implies $a = 0$.*

Let us introduce the following examples, which show that the existence of ν is not an unlikely event.

Examples 22.2.

1. If the base algebra $\mathcal{A}$ admits a trace τ with values in a commutative subalgebra $\mathcal{F} \subset \mathcal{A}$, then naturally a valuation ν is defined: $\nu(\varphi) = \tau(a)$, where a is the matrix of φ in $\mathbf{x}$, for any $q \in \mathcal{C}_q$.
2. $\mathcal{A}$ need not admit a trace. Suppose that there exists a $*$-homomorphism onto a commutative algebra $\pi : \mathcal{A} \to \mathcal{F}$ (here $\mathcal{F}$ need not be a subalgebra of $\mathcal{A}$). In this case, $\pi(ab) = \pi(a)\pi(b) = \pi(b)\pi(a) = \pi(ba)$. One such example is the Toeplitz C*-algebra $\mathcal{T}(C(\mathbb{T})) = \{T_f : f \in C(\mathbb{T})\}$, where T_f denotes the Toeplitz operator with symbol f. $\mathcal{T}(C(\mathbb{T}))$ does not admit a trace, but it has a $*$-homomorphism onto a commutative algebra, namely
$$\pi : \mathcal{T}(C(\mathbb{T})) \to \mathcal{T}(C(\mathbb{T}))/\mathcal{K}(L^2(\mathbb{T})) \simeq C(\mathbb{T}).$$
   Another example of this sort is the algebra $\mathcal{D}+\mathcal{K} = \{D+K : D \text{ diagonal and } K \text{ compact}\}$ in $\mathcal{B}(\ell^2)$. In this case, there is a homomorphism
$$\pi : \mathcal{D} + \mathcal{K} \to \mathcal{D} + \mathcal{K}/\mathcal{K}(\ell^2) \simeq \ell^\infty/c_0.$$

3. A third sort of example is obtained if $\mathcal{A}$ has a positive tracial map onto a commutative algebra.

Let us fix a valuation $\nu : \mathcal{C} \to \mathcal{F}$.

The structures that we have defined in $\mathcal{Q}_\rho$, with values in $\mathcal{C}$, have valuations which transform them in structures with values in $\mathcal{F}$. The main feature is the $\mathcal{C}$-valued Hilbertian product: if $q = p_{\mathbf{x}}$,

$$\langle \mathbf{X}, \mathbf{Y} \rangle_q = -\theta_\rho(\kappa_{\mathbf{x}}(\mathbf{X}), \kappa_{\mathbf{y}}(\mathbf{Y})) = -\theta_\rho(\mathbf{X}\mathbf{x}, \mathbf{Y}\mathbf{y}) \in \mathcal{C}_q,$$

and applying ν

$$\langle \mathbf{X}, \mathbf{Y} \rangle_q^\nu := -\nu\theta_\rho(\mathbf{X}, \mathbf{x}, \mathbf{Y}\mathbf{y}). \tag{71}$$

This $\mathcal{F}$-valued inner product defines a Hermitian structure in $\mathcal{Q}_\rho$, where the scalar "field" is $\mathcal{F}$. Note that if ν is not faithful, then $\langle \ , \ \rangle^\nu$ is positive semi-definite.

Next, we consider the connection and curvatures, and the symplectic form. We saw in Theorem 18.1 that the canonical connection in the tautological bundle $\xi \to \mathcal{Q}_\rho$ has curvature equal to ($-2i$-times) the imaginary part of the Hilbertian product (both terms in this assertion, considered as 2-forms in $\mathcal{Q}_\rho$ with values in $\mathcal{C}$).

Let us consider now the following: pick a field of bases $\mathbf{x} \in \mathcal{K}_\rho$ defined on an open set and a cross section σ of ξ on this open set. We can write

$$D_{\mathbf{X}}\sigma = \mathbf{x}(\mathbf{X} \cdot a + \alpha(\mathbf{X})a)$$

where $\sigma = \mathbf{x}a$, a is an $\mathcal{A}$-valued function, $\mathbf{X} \cdot a$ is the directional derivative of a in the direction $\mathbf{X}$, and $\alpha(\mathbf{X})$ is an $\mathcal{A}$-valued 1-form in $\mathcal{Q}_\rho$ (what we called the 1-form of $\mathcal{Q}_\rho$ in the basis $\mathbf{x}$) . If we compute

$$D_{\mathbf{X}}D_{\mathbf{Y}}\sigma - D_{\mathbf{Y}}D_{\mathbf{X}}\sigma - D_{[\mathbf{X},\mathbf{Y}]}\sigma$$

for fields $\mathbf{X}, \mathbf{Y}$ in $(T\mathcal{Q}_\rho)_q$, we obtain the following expression for the curvature $R(\mathbf{X}, \mathbf{Y})\sigma$:

$$R(\mathbf{X}, \mathbf{Y})\sigma = \mathbf{x}((d\alpha)(\mathbf{X}, \mathbf{Y})a + [\alpha(\mathbf{X}), \alpha(\mathbf{Y})]a),$$

where $[\alpha(\mathbf{X}), \alpha(\mathbf{Y})] = \alpha(\mathbf{X})\alpha(\mathbf{Y}) - \alpha(\mathbf{Y})\alpha(\mathbf{X})$. On the other hand, we saw in Theorem 18.1 that

$$-\frac{1}{2i}R(\mathbf{X}, \mathbf{Y})_q = Im\ \langle \mathbf{X}, \mathbf{Y} \rangle_q,$$

where both terms are $\mathcal{C}$-valued, and $\mathbf{X}, \mathbf{Y} \in (T\mathcal{Q}_\rho)_q$. Applying the valuation ν we get

$$\nu(R(\mathbf{X}, \mathbf{Y})_q) = \nu(d\alpha(\mathbf{X}, \mathbf{Y}) + [\alpha(\mathbf{X}), \alpha(\mathbf{Y})]) = \nu d\alpha(\mathbf{X}, \mathbf{Y}).$$

Note that $\nu(\mathbf{X} \cdot \alpha(\mathbf{Y})) = \mathbf{X} \cdot d\alpha(\mathbf{Y})$, because ν is linear and bounded (it commutes with the derivatives). Hence

$$\nu\ d\alpha(\mathbf{X}, \mathbf{Y}) = \nu(\mathbf{X} \cdot \alpha(\mathbf{Y}) - \mathbf{Y}\alpha(\mathbf{X}) - \alpha([\mathbf{X}, \mathbf{Y}] = \mathbf{X} \cdot \nu\alpha(\mathbf{Y}) - \mathbf{Y} \cdot \nu\alpha(\mathbf{X}) - \nu\alpha([\mathbf{X}, \mathbf{Y}]).$$

In other words: $\nu\ d\alpha(\mathbf{X}, \mathbf{Y}) = d\ \nu\alpha(\mathbf{X}, \mathbf{Y})$. Therefore

$$Im\ \langle(\mathbf{X}, \mathbf{Y}\rangle_q^\nu = \nu\ Im\langle(\mathbf{X}, \mathbf{Y}\rangle_q = -\frac{i}{2i}\nu R(\mathbf{X}, \mathbf{Y})_q = -\frac{1}{2i}\nu\ d\alpha(\mathbf{X}, \mathbf{Y}) = -\frac{1}{2i}d\ \nu\alpha(\mathbf{X}, \mathbf{Y}).$$

Thus, recalling that $\omega(\mathbf{X},\mathbf{Y}) = Im\ \langle \mathbf{X},\mathbf{Y}\rangle$, we have that the alternate $\mathcal{F}$-valued 2-form $\nu\omega$ over $\mathcal{Q}_\rho$ satisfies the equality

$$\nu\omega(\mathbf{X},\mathbf{Y}) = -\frac{1}{2i} d\ \nu\alpha(\mathbf{X},\mathbf{Y}). \tag{72}$$

We shall call $\nu\omega$ the *valuated symplectic form* over $\mathcal{Q}_\rho$, with values in $\mathcal{F}$. Then we have that the valuated symplectic form $\nu\omega$ is *exact*. Indeed, there exist global bases $\mathbf{x}$ defined in the whole $\mathcal{Q}_\rho$.

Recall the moment map defined in Section 19: for $a \in \mathfrak{U}_\rho$, $f_a : \mathcal{Q}_\rho \to \mathcal{C}$, is defined by $f_a(q) = \frac{1}{2i}\Big[(\mathbf{x}, \mathbf{x}^*\rho a\mathbf{x})\Big]$. Recall also the equality $D_\mathbf{X} f_a = \omega(\mathbf{X},\mathbf{X}_a)$. Let us apply the valuation to the moment map

$$f_a^\nu = \nu f_a : \mathcal{Q}_\rho \to \mathcal{F}.$$

Note that

$$\mathbf{X}\cdot f_a^\nu = X \cdot \nu f_a = \nu D_\mathbf{X}\cdot f_a,$$

Therefore $\mathbf{X}\cdot f_a^\nu = \nu\omega(\mathbf{X},\mathbf{X}_a)$. This means that f^ν is a moment map. Let us understand now the map f^ν as a map from $\mathcal{Q}_\rho$ to the dual of $\mathfrak{U}_\rho$. Here "dual" means the space of bounded linear "functionals" with values in $\mathcal{F}$. To do this, we shall consider the following tracial functional :

$$\tau : M_2(\mathcal{A}) \to \mathcal{F}\ ,\quad \tau\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix} = \frac{1}{2}\nu(a_{11}+a_{22}).$$

Apparently, τ is positive, $\tau(I) = 1$ and verifies $\tau(ab) = \tau(ba)$. Then, if $q = p_\mathbf{x}$, we have

$$f_a^\nu(q) = \nu\theta_\rho(\mathbf{x}, a\mathbf{x}) = \nu(\mathbf{x}^*\rho a\mathbf{x}) = \tau(\mathbf{x}^*\rho a\mathbf{x}),$$

where the last equality uses the fact that, on elements a of $\mathcal{A}$ (regarded as scalar matrices $\begin{pmatrix} a & 0 \\ 0 & a\end{pmatrix}$), ν coincides with τ. Then, using the trace property of τ, $\tau(\mathbf{x}^*\rho a\mathbf{x}) = \tau(\mathbf{x}\mathbf{x}^*\rho a) = \tau(qa)$, i.e.,

$$f_a^\nu(q) = \tau(qa). \tag{73}$$

This formula allows one to clearly identify (by means of ν) the moment map as a map from $\mathcal{Q}_\rho$ with values in the tangent space of the Lie algebra $\mathfrak{U}_\rho$ of the group $\mathcal{U}_\rho$: to $q \in \mathcal{Q}_\rho$ corresponds the $\mathcal{F}$-valued linear functional $\tau(q\ \cdot)$, with density matrix q.

Remark 22.3. The dual space considered is subordinated to the valuation ν. It consists of bounded linear functionals defined on $\mathfrak{U}_\rho$, with values in $\mathcal{F}$. If, additionally, the algebra of scalars $\mathcal{F}$ in which one chooses to take measurements is a subalgebra of $\mathcal{A}$, then these functionals $\tau(q\ \cdot)$ are also $\mathcal{F}$-linear.

Next we shall discuss a property of the moment map, which mimicks the theorem of [23], [4], [19] and [8] on compact symplectic manifolds acted by a torus. We shall consider therefore a subgroup of the full group $\mathcal{U}_\rho$ acting in $\mathcal{Q}_\rho$, namely the diagonal group

$$\mathbf{D}_\rho = \{\begin{pmatrix} u_1 & 0 \\ 0 & u_2\end{pmatrix} : u_i \in \mathcal{U}_\mathcal{A}\} \subset \mathcal{U}_\rho,$$

which will play the role of a torus. Note that $\mathbf{D}_\rho = \mathcal{U}_\rho \cap \mathcal{U}_2(\mathcal{A})$, i.e., the unitary matrices in $M_2(\mathcal{A})$ which preserve the form θ_ρ.

In classical symplectic geometry, the restriction of the action to the subgroup induces a restricted moment map, which, when regarded as a map from the manifold to the dual of the Lie algebra of the acting group, consists in composing the moment of the full group with the projection of the Lie algebra of the full group onto the Lie algebra of the subgroup.

In our case, we shall restrict the action to $\mathbf{D}_\rho$. Note that the Lie algebra of $\mathbf{D}_\rho$ is the subalgebra $\mathfrak{U}_{0,\rho}$ given in (59)

$$\mathfrak{U}_{0,\rho} = \{\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} : a_i^* = -a_i\}.$$

The projection is given by

$$\mathfrak{U}_\rho \to \mathfrak{U}_{0,\rho} \ , \quad \begin{pmatrix} a_1 & b \\ b^* & a_2 \end{pmatrix} \mapsto \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}.$$

The following is an Atiyah-Guillemin-Sternberg's type of convexity result.

Theorem 22.4. *The image of the moment map f^ν of the restricted action of the group $\mathbf{D}_\rho$,*

$$\mathcal{Q}_\rho \ni q \overset{f^\nu}{\longmapsto} \tau(q\ \cdot) \in \mathcal{L}(\mathfrak{U}_{0,\rho}, \mathcal{F}),$$

regarded as a subset of the space $\mathcal{L}(\mathfrak{U}_{0,\rho}, \mathcal{F})$ of linear functionals $\mathfrak{U}_{0,\rho} \to \mathcal{F}$, is a convex set.

23 The hyperbolic part of $\mathcal{A}\mathbb{P}_1$

We say that an element $\mathbf{x} \in \mathcal{A}^2$ is *regular* if $\mathbf{x} \in Gl_2(\mathcal{A})\mathbf{e_1}$, i.e., if $\mathbf{x}$ is the first column of a matrix in $Gl_2(\mathcal{A})$. A *regular rank one submodule* of $\mathcal{A}^2$ is a right $\mathcal{A}$-module $\ell \subset \mathcal{A}^2$ of the form $\ell = [\mathbf{x}] := \{\mathbf{x}a; a \in \mathcal{A}\}$ for some regular $\mathbf{x} \in \mathcal{A}^2$.

Definition 23.1. *The* projective line *of $\mathcal{A}$ is the set $\mathcal{A}\mathbb{P}_1$ of all regular rank one submodules of $\mathcal{A}^2$.*

We define the *cross ratio* $CR(\ell_1, \ell_2, \ell_3, \ell_4)$ of four submodules ℓ_1, ℓ_2, ℓ_3, ℓ_4 in $\mathcal{A}\mathbb{P}_1$, following ideas by M. I. Zelikin [36].

Definition 23.2. *We denote by $CR(\ell_1, \ell_2, \ell_3, \ell_4)$ the (possibly empty) set of module homomorphisms $\varphi : \ell_3 \to \ell_3$ of the form $\varphi = \psi\eta$, where the homomorphisms $\eta : \ell_3 \to \ell_2$, $\psi : \ell_2 \to \ell_3$ satisfy $x - \psi(x) \in \ell_4$ and $y - \eta(y) \in \ell_1$, for all $x \in \ell_2$, $y \in \ell_3$.*

This set may contain more than one element. In the next section, we discuss existence and uniqueness of elements in $CR(\ell_1, \ell_2, \ell_3, \ell_4)$.

Consider the subset $\mathcal{A}\mathbb{P}_1^\rho$ of $\mathcal{A}\mathbb{P}_1$, which we call the *hyperbolic part* of $\mathcal{A}\mathbb{P}_1$:

$$\mathcal{A}\mathbb{P}_1^\rho = \{[\mathbf{x}] : \mathbf{x} \in \mathcal{K}_\rho\}.$$

Note that the elements of $\mathcal{K}_\rho$ are regular because they are the first columns of matrices in $\mathcal{U}_\rho$. It is easy to see that $\mathcal{A}\mathbb{P}_1^\rho$ is properly contained in $\mathcal{A}\mathbb{P}_1$.

Proposition 23.3. *The group $\mathcal{U}_\rho$ acts transitively on $\mathcal{A}\mathbb{P}_1^\rho$.*

There exist many natural ways of endowing $\mathcal{A}\mathbb{P}_1$ with a differentiable structure. Here we choose to define the differentiable structure only on $\mathcal{A}\mathbb{P}_1^\rho$. We have shown that $\mathcal{Q}_\rho = \{p_{\mathbf{x}} : \mathbf{x} \in \mathcal{K}_\rho\}$, where $p_{\mathbf{x}}$ is the unique θ_ρ-orthogonal projection with range $[\mathbf{x}]$.

Proposition 23.4. *The mapping* $\mathbf{R} : \mathcal{Q}_\rho \to \mathcal{A}\mathbb{P}_1^\rho$ *defined by* $\mathbf{R}(q) = R(q)$ *is a bijection, with* $\mathbf{R}^{-1}([\mathbf{x}]) = p_{\mathbf{x}}$*. Morever,* $\mathbf{R}(aqa^{-1}) = a \cdot \mathbf{R}(q)$*.*

Proof. Straightforward computation. □

The natural bijection $\mathbf{R}$ allows one to transfer the differentiable structure of $\mathcal{Q}_\rho$ to $\mathcal{A}\mathbb{P}_1^\rho$. For instance, the tangent space at $[\mathbf{x}] \in \mathcal{A}\mathbb{P}_1^\rho$, for $\mathbf{x} \in \mathcal{K}_\rho$, is identified as

$$(T\mathcal{A}\mathbb{P}_1^\rho)_{[\mathbf{x}]} = \{X \in M_2(\mathcal{A}) : X^\sharp = X, p_{\mathbf{x}}X = X(1 - p_{\mathbf{x}})\}$$

$$\simeq \{[\mathbf{y}] : \mathbf{y} \in \mathcal{A}^2, \theta_\rho(\mathbf{x}, \mathbf{y}) = 0, \theta_\rho(\mathbf{y}, \mathbf{y}) = 1\} = N(p_{\mathbf{x}}).$$

The isomorphism implementing $\simeq$ above is the map $X \mapsto X\mathbf{x}$, from $(T\mathcal{A}\mathbb{P}_1^\rho)_{[\mathbf{x}]}$ onto $N(p_{\mathbf{x}})$.

If $\ell \in \mathcal{A}\mathbb{P}_1^\rho$, denote by

$$\ell^{\perp_\rho} = \{\mathbf{y} \in \mathcal{A}^2 : \theta_\rho(\mathbf{y}, \mathbf{x}) = 0 \text{ for all } \mathbf{x} \in \ell\}.$$

Note that $(T\mathcal{A}\mathbb{P}_1^\rho)_{[\mathbf{x}]} \simeq [\mathbf{x}]^{\perp_\rho}$.

24 The invariant metric in $\mathcal{A}\mathbb{P}_1^\rho$

Given $\ell \in \mathcal{A}\mathbb{P}_1^\rho$ and $V \in (T\mathcal{A}\mathbb{P}_1^\rho)_\ell$, fix a generator $\mathbf{x}_0 \in \mathcal{K}_\rho$ for ℓ, i.e., $[\mathbf{x}_0] = \ell$ and $\theta_\rho(\mathbf{x}_0, \mathbf{x}_0) = 1$. Recall that $\mathbf{x}_0$ is determined up to a unitary element of $\mathcal{A}$: if $\mathbf{x}_0'$ is another such generator, then there exists $u \in \mathcal{U}_\mathcal{A}$ such that $\mathbf{x}_0' = \mathbf{x}_0 u$. Having fixed a generator for ℓ, as we saw above, $(T\mathcal{A}\mathbb{P}_1^\rho)_\ell$ identifies with $\ell^{\perp_\rho}$, and to the tangent vector V corresponds an element $\mathbf{v} \in \ell^{\perp_\rho}$. We define

$$|V|_\ell := \|\theta_\rho(\mathbf{v}, \mathbf{v})\|^{1/2}. \tag{74}$$

Note that $|\ \ |_\ell$ does not depend on the choice of the generator. If we choose $\mathbf{x}_0' = \mathbf{x}_0 u$ instead, the tangent vector V is represented by $\mathbf{v}' = \mathbf{v}u \in \ell^{\perp_\rho}$, and therefore

$$\|\theta_\rho(\mathbf{v}', \mathbf{v}')\| = \|\theta_\rho(\mathbf{v}u, \mathbf{v}u)\| = \|u^*\theta_\rho(\mathbf{v}, \mathbf{v})u\| = \|\theta_\rho(\mathbf{v}, \mathbf{v})\|.$$

Next, recall from Lemma that θ_ρ is negative definite (non degenerate), and therefore the expression (74) above defines a proper norm in $(T\mathcal{A}\mathbb{P}_1^\rho)_\ell$.

Remark 24.1. If $V \in (T\mathcal{A}\mathbb{P}_1^\rho)_{[\mathbf{x}_0]}$, V is represented by some $\mathbf{v} \in [\mathbf{x}_0]^{\perp_\rho}$; since $\mathbf{y}_0 = \begin{pmatrix} (x_1^*)^{-1}x_2^* \\ 1 \end{pmatrix}$, for $\mathbf{x}_0 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, is a generator of $[\mathbf{x}_0]^{\perp_\rho}$, there exists $a \in \mathcal{A}$ such that $\mathbf{v} = \mathbf{y}_0 a$. Then

$$|V|_{[\mathbf{x}_0]} = \|\theta_\rho(\mathbf{y}_0 a, \mathbf{y}_0 a)\|^{1/2} = \|a^*(1 - x_2x_1^{-1}(x_2x_1^{-1})^*a\|^{1/2} = \|(1 - |(x_2x_1^{-1})^*|^2)a\|.$$

Note also that the norm of $\mathbf{v} = \mathbf{y}_0 a$ in $\mathcal{A}^2$ is

$$\|\mathbf{y}_0 a\| = \|\langle \mathbf{y}_0 a, \mathbf{y}_0 a\rangle\|^{1/2} = \|(1 + |(x_2x_1^{-1})^*|^2)a\|.$$

Proposition 24.2. *For any $\ell \in \mathcal{A}\mathbb{P}_1^\rho$, the norm $|\ \ |_\ell$ of $(T\mathcal{A}\mathbb{P}_1^\rho)_\ell$ is complete.*

The distribution $\mathcal{A}\mathbb{P}_1^\rho \ni \ell \mapsto |\ \ |_\ell$ is clearly continuous. Thus, $\mathcal{A}\mathbb{P}_1^\rho$ is endowed with a Finsler metric.

The following result is tautological, but of the utmost importance for our discussion:

Theorem 24.3. *The Finsler metric defined in (74) is invariant under the action of $\mathcal{U}_\rho$.*

24.1 Invariant metric in $\mathcal{A}\mathbb{P}_1^\rho$

Lemma 24.4. *The differential of the map $\mathcal{A}\mathbb{P}_1^\rho \to \mathcal{D}$, $[\mathbf{x}] \mapsto x_2 x_1^{-1}$, at $[\mathbf{e_1}]$ is the tangent map*

$$\left[\begin{pmatrix} 0 \\ x \end{pmatrix}\right] \longmapsto x \ , \quad x \in \mathcal{A}.$$

Theorem 24.5. *The identification $[\mathbf{x}] \longleftrightarrow x_2 x_1^{-1}$ between $\mathcal{A}\mathbb{P}_1^\rho$ and $\mathcal{D}$, is isometric.*

25 Limit points of geodesics

One of our concerns in computing the geodesic δ, and the above results on the polar decomposition, is to establish the following result:

Theorem 25.1. *For $z \in \mathcal{D}$, let δ be the unique geodesic of $\mathcal{D}$ such that $\delta(0) = 0$ and $\delta(1) = z$. Put $z = \omega|z|$ the polar decomposition (i.e., $\omega \in \mathcal{A}^{**}$); then*

$$\mathrm{SOT} - \lim_{t\to\infty} \delta(t) = \omega \quad \textit{and} \quad \mathrm{SOT} - \lim_{t\to-\infty} \delta(t) = -\omega.$$

This geometric role of the partial isometry ω in the polar decomposition of $z \in \mathcal{D}$ (or, more generally, of every $z \in \mathcal{A} \setminus \{0\}$) has not been noticed before, to the authors' knowledge.

In order to compute the limit points of arbitrary geodesics, it will be useful to extend the action of $\mathcal{U}_\rho$ to the strong operator border

$$\partial\mathcal{D} := \{a \in \mathcal{A}^{**} : \|a\| = 1\}$$

of $\mathcal{D}$, i.e., to define $g \cdot a$ for $a \in \mathcal{A}^{**}$ with $\|a\| = 1$.

Lemma 25.2. *If $g \in \mathcal{U}_\rho$ and $a \in \partial\mathcal{D}$, then $g_{11} + g_{12}a$ is invertible in $\mathcal{A}^{**}$.*

Proposition 25.3. *If $a \in \partial\mathcal{D}$, and $g \in \mathcal{U}_\rho$, then*

$$g \cdot a := (g_{21} + g_{22}a)(g_{11} + g_{12}a)^{-1} \in \partial\mathcal{D},$$

defines a left action of $\mathcal{U}_\rho$ on $\partial\mathcal{D}$.

Using this result, we can compute the limit points of arbitrary geodesics. Since the action of $\mathcal{U}_\rho$ is transitive, given $z_1, z_2 \in \mathcal{D}$, there exists $g \in \mathcal{U}_\rho$ such that $g \cdot z_1 = 0$.

Corollary 25.4. *Let $z_0, z_1 \in \mathcal{D}$ and $g \in \mathcal{U}_\rho$ such that $g \cdot z_0 = 0$. Let δ be the unique geodesic of $\mathcal{D}$ such that $\delta(0) = z_0$ and $\delta(1) = z_1$. Denote by $\dot{\delta}_0$ the initial velocity of δ. Then*

$$\mathrm{SOT} - \lim_{t\to+\infty} \delta(t) = g \cdot \omega_0 \ \textit{and}\ \mathrm{SOT} - \lim_{t\to-\infty} \delta(t) = g \cdot (-\omega_0),$$

where $\omega_0 \in \mathcal{A}^{**}$ is the partial isometry in the polar decomposition of $\dot{\delta}_0$: $\dot{\delta}_0 = \omega_0|\dot{\delta}_0|$.

Remark 25.5. In order to identify these limit points in $\mathcal{D}$, following the notation of the above Corollary, note that if $g = |g^*|v = \begin{pmatrix} (1+b^*b)^{1/2} & b^* \\ b & (1+bb^*)^{1/2} \end{pmatrix} \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix}$ (the reversed polar decomposition), then $v \cdot \omega_0 = v_2 \omega_0 v_1^*$ is a partial isometry. Therefore, the limit points of geodesics are elements in $\partial\mathcal{D}$ of the form

$$(b + (1+bb^*)^{1/2}\omega)((1+b^*b)^{1/2} + b^*\omega)^{-1} \text{ and } (b - (1+bb^*)^{1/2}\omega)((1+b^*b)^{1/2} - b^*\omega)^{-1},$$

where $b \in \mathcal{A}$ is arbitrary and $\omega \in \mathcal{A}^{**}$ is a partial isometry.

Note that not every partial isometry in $\mathcal{A}^{**}$ occurs in the polar decomposition of an element in $\mathcal{D}$. For instance, if $\mathcal{A} = C([0,1])$ (continuous functions in the unit interval), the polar decomposition of $f \in \mathcal{A}$ is $f = w|f|$, where $w \in L^\infty(0,1)$ is given by $w(t) = \begin{cases} f(t)/|f(t)| \text{ if } f(t) \neq 0 \\ 0 \text{ if } f(t) = 0 \end{cases}$. An arbitrary partial isometry in $L^\infty(0,1)$ is a measurable function whose values are zero or complex numbers of modulus 1. The set of zeros of such a function is an arbitrary measurable set, whereas the set of zeros of partial isometries which occur in the polar decomposition of a continuous function, are closed subsets of $[0,1]$.

Another way to study the limit points of geodesics, is by using the Borel subgroup $\mathcal{B} \subset \mathcal{U}_\rho$. This is the conjugate group of $\mathcal{B}' \subset \mathcal{U}_{\rho'}$ defined before (Definition 9.1), by means of the unitary matrix U. Indeed, since the action of this group is transitive in $\mathcal{D}$, any limit point of a geodesic is either of the form $g \cdot v$ or $g \cdot (-v)$, for $g \in \mathcal{B}$. Consider the following example:

Example 25.6. Suppose that $\mathcal{A}$ is a von Neumann algebra, and let $p \neq 0$ be a projection in $\mathcal{A}$. Write $b = \begin{pmatrix} \frac{g+\hat{g}}{2} - \hat{g}x & \frac{g-\hat{g}}{2} - \hat{g}x \\ \frac{g-\hat{g}}{2} + \hat{g}x & \frac{g+\hat{g}}{2} + \hat{g}x \end{pmatrix}$, which is the general form of an element in $\mathcal{B}$ (here we put $\hat{g} = (g^*)^{-1}$). Let us compute $b \cdot p$. After straightforward computations,

$$b \cdot p = (g(1+p) + \hat{g}\,(-1+p+2x(1+p)))\,(g(1+p) + \hat{g}\,(1-p-2x(1+p)))^{-1}.$$

Note that $1+p$ is invertible and that $(1-p)(1+p)^{-1} = 1-p$. Then

$$b \cdot p = \left(1 + \hat{g}\,(p-1+2x(1+p))\,(1+p)^{-1}g^{-1}\right)\left(1 + \hat{g}\,(1-p-2x(1+p))\,(1+p)^{-1}g^{-1}\right)^{-1} =$$

$$= \left(1 + \hat{g}(p-1+2x)g^{-1}\right)\left(1 + \hat{g}(1-p-2x)g^{-1}\right)^{-1}.$$

Denote $\alpha = \hat{g}(p-1)g^{-1}$ and $\beta = 2\hat{g}xg^{-1}$. Observe that α is a non-invertible selfadjoint element, $\alpha \leq 0$ and its range is proper and closed; β is an arbitrary anti-Hermitien element. Then

$$b \cdot p = (1+\alpha+\beta)\,(1-(\alpha+\beta))^{-1}.$$

Note that $1-(\alpha+\beta)$ is invertible because $Re(1-(\alpha+\beta)) = 1-\alpha \geq 1$. If one picks $p = 1$, then $\alpha = 0$ and

$$b \cdot 1 = (1+\beta)(1-\beta)^{-1},$$

which is a unitary operator such that -1 does not belong to its spectrum. In particular, this shows that the action of $\mathcal{B}$ ceases to be transitive in $\partial\mathcal{D}$.

Our next result shows a necessary condition for an element $a \in \mathcal{A}^{**}$ with $\|a\| = 1$ to be a limit point of a geodesic of $\mathcal{D}$.

Proposition 25.7. *If $a \in \mathcal{A}^{**}$ is the limit point at $+\infty$ of a geodesic in $\mathcal{D}$, then $1 - a^*a = hqh$, where $h \in G^+$ and $q \in \mathcal{A}^{**}$ is a projection. In particular, not every element of norm 1 in $\mathcal{A}^{**}$ is the limit point of a geodesic: such elements satisfy that the defect element $1 - a^*a$ has closed range.*

Remark 25.8. The characterization of the partial isometries which appear in the polar decompositions of all limit points $a \in \mathcal{A}^{**}$ is an interesting open problem.

26 Operator cross ratio in the hyperbolic part of the projective line

Here we state the main result of this part of the monograph, relating the metric of $\mathcal{A}\mathbb{P}_1^\rho$ introduced in Section 24, with the so called operator cross ratio, as defined in the Grassmann manifold of a Hilbert space by M.I. Zelikin [36]. We shall apply these ideas to the rank one submodules in $\mathcal{A}\mathbb{P}_1^\rho$. To this effect, the isometry between $\mathcal{A}\mathbb{P}_1^\rho$ and the disk $\mathcal{D}$ will be important.

Consider $\ell = \left[\begin{pmatrix} 1 \\ z \end{pmatrix}\right] \in \mathcal{A}\mathbb{P}_1^\rho$, for $z \in \mathcal{D}$. Let $z = \omega|z|$ be the polar decomposition. Let δ be the geodesic of $\mathcal{A}\mathbb{P}_1^\rho$ such that $\delta(0) = \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]$ and $\delta(1) = \left[\begin{pmatrix} 1 \\ z \end{pmatrix}\right]$. Equivalently, regarded in $\mathcal{D}$: $\delta(0) = 0$ and $\delta(1) = z$. As seen in Section 25,

$$\text{SOT} - \lim_{t\to+\infty} \delta(t) = \omega \text{ and } \text{SOT} - \lim_{t\to-\infty} \delta(t) = -\omega.$$

Four points are determined: $-\omega, 0, z, \omega$, or better, four rank one submodules

$$\ell_{-\infty} := \left[\begin{pmatrix} 1 \\ -\omega \end{pmatrix}\right], \ell_0 := \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right], \ell = \left[\begin{pmatrix} 1 \\ z \end{pmatrix}\right], \ell_{+\infty} := \left[\begin{pmatrix} 1 \\ \omega \end{pmatrix}\right],$$

where the limit lines lie in $\partial\mathcal{A}\mathbb{P}_1^\rho$.

In Section 23 we defined the operator cross ratio of four elements in $\mathcal{A}\mathbb{P}_1$, as a (possibly empty) set of module endomorphisms, following ideas of Zelikin [36]. Here we compute the operator cross ratio $CR(\ell_{-\infty}, \ell_0, \ell, \ell_{+\infty})$, proving that it is non empty, and that there exists a natural ℓ-endomorphism to choose from this set.

Recall that elements of $CR(\ell_{-\infty}, \ell_0, \ell, \ell_{+\infty})$ are (module) endomorphisms of ℓ, defined as the composition of *the projection from ℓ to ℓ_0 parallel to $\ell_{-\infty}$, followed by the projection from ℓ_0 to ℓ parallel to $\ell_{+\infty}$*.

In coordinates, by choosing generators in the respective submodules

$$\begin{pmatrix} 1 \\ z \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix}\lambda, \begin{pmatrix} 1-\lambda \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -\omega \end{pmatrix}\mu.$$

Then $1 - \lambda = \mu$ and $z = -\omega\mu$. Then $\omega|z| = -\omega\mu$. If z is invertible (and then ω is unitary) this implies $\mu = -|z|$, otherwise this is just one possible solution. Non uniqueness of solutions of these equations reflect the geometric fact that the modules ℓ_0 and ℓ_∞ may not be in direct sum. Explicitly, all solutions of these equations are of the form

$$\lambda = 1 + |z| - \Omega \ , \quad \mu = -|z| + \Omega,$$

where $\Omega \in \mathcal{A}^{**}$ is such that $\omega\Omega = 0$. In particular, $|z|\Omega = |z|\omega^*\omega\Omega = 0$. We choose the solution with $\Omega = 0$. Note that $\lambda = 1 + |z|$, and therefore the first projection in the above composition is given by

$$\begin{pmatrix} 1 \\ z \end{pmatrix} \mapsto \begin{pmatrix} 1+|z| \\ 0 \end{pmatrix}.$$

Next

$$\begin{pmatrix} 1+|z| \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ z \end{pmatrix}\gamma \ , \ \begin{pmatrix} 1+|z|-\gamma \\ -z\gamma \end{pmatrix} = \begin{pmatrix} 1 \\ \omega \end{pmatrix}\epsilon.$$

So that $1+|z|-\gamma = \epsilon$ and $-z\gamma = \omega\epsilon$, and then (the unique solution if ω is unitary, or a possible solution that we choose, otherwise) $1 + |z| - \gamma = -|z|\omega$, i.e.,

$$\gamma = (1 - |z|)^{-1}.$$

Other solutions of the above equation are of the form

$$\gamma = (1 - |z|)^{-1} + (1 - |z|)^{-1}\Omega',$$

where $\Omega' \in \mathcal{A}^{**}$ is such that $\omega\Omega' = 0$. In general, the possible endomorphisms $\ell \to \ell$ are given (in these coordinates) by

$$\begin{pmatrix} 1 \\ z \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ z \end{pmatrix}(1+|z|+\Omega)(1-|z|)^{-1}(1+\Omega') = (1+|z|)(1-|z|)^{-1} + \Omega' + \Omega(1-|z|)^{-1} + \Omega\Omega',$$

where we use that $\omega\Omega = \omega\Omega' = 0$, and thus $(1 \pm |z|)^{\pm 1}\Omega = \Omega$ (and the same for Ω').

As noted, if z is invertible, there is a unique solution with $\Omega = \Omega' = 0$. Our choice of cross ratio, picking $\Omega = \Omega' = 0$ in any case, is justified below.

Remark 26.1. The following fact is known (see for instance [21]). Let $a_n \in \mathcal{A}$ with $\|a_n\| \le 1$. If $a_n \to a$ strongly, then $|a_n| \to |a|$ strongly.

Remark 26.2. Dixmier and Marechal [17] proved that the set on invertible elements of a von Neumann algebra is strong operator dense in the algebra. The argument in [17] proceeds as follows. Let $a = u|a|$ be the polar decomposition of a ($u \in \mathcal{A}^{**}$). First, the algebra $\mathcal{A}^{**}$ is factored in its finite and properly infinite parts. In the finite part u can be chosen unitary. In the properly infinite part, one readily sees that it suffices to consider the cases in which u is an isometry or a co-isometry. In the case that u is an isometry, Dixmier and Maréchal prove that u is the strong limit of unitaries u_n. If u is a co-isometry, they show that there exist invertible elements g_n which converge strongly to u, with norms $\|g_n\| = 1$ (this is clear in the proof, though it is not stated in their result). Summarizing, if $\mathcal{A}$ is a von Neumann algebra, and $a \in \mathcal{A}$, there exist $g_n \in G$ with $\|g_n\| \le \|a\|$ such that

$$\text{SOT} - \lim_{n\to\infty} g_n = a.$$

Using this fact, it is clear that, if $\mathcal{A}$ is a von Neumann algebra, and $z \in \mathcal{D}$, then there exist $z_n \in G$ with $\|z_n\| \le \|z\|$, such that $z_n \to z$ strongly.

Proposition 26.3. *Let $z_n, z \in \mathcal{D}$ such that $z_n \to z$ strongly and $\|z_n\| \le \|z\|$. Then*

$$(1+|z_n|)(1-|z_n|)^{-1} \to (1+|z|)(1-|z|)^{-1} \textit{ strongly.}$$

Definition 26.4. *Let $z \in \mathcal{D}$. We define $cr(0,z) \in CR(\ell_{-\infty}, \ell_0, \ell_z, \ell_\infty)$, for $\ell_z = \left[\begin{pmatrix} 1 \\ z \end{pmatrix}\right]$, as the endomorphism*

$$cr(0,z) : \ell_z \to \ell_z \ , \quad cr(0,z)(\begin{pmatrix} 1 \\ z \end{pmatrix} a) = \begin{pmatrix} 1 \\ z \end{pmatrix} (1+|z|)(1-|z|)^{-1}a,$$

for $a \in \mathcal{A}$.

We use the action of $\mathcal{U}_\rho$ to extend this definition to any pair $z_0 \neq z_1 \in \mathcal{D}$.

Definition 26.5. *Let $z_0, z_1 \in \mathcal{D}$, $z_0 \neq z_1$. Pick $g \in \mathcal{U}_\rho$ such that $g \cdot 0 = z_0$ and denote $z = g^{-1} \cdot z_1$. We define*

$$cr(z_0, z_1) = g \ cr(0,z) g^{-1}.$$

Before checking that the definition does not depend on the choice of g, we remark the following. Let δ be the unique geodesic of $\mathcal{D}$ such that $\delta(0) = z_0$ and $\delta(1) = z_1$, and let

$$z_{-\infty} = \mathrm{SOT} - \lim_{t \to -\infty} \delta(t) \ \text{ and } z_{+\infty} = \mathrm{SOT} - \lim_{t \to +\infty} \delta(t).$$

Then $cr(z_0, z_1) \in CR(\ell_{z_{-\infty}}, \ell_{z_0}, \ell_{z_1}, \ell_{z_{+\infty}})$, because g is a module homomorphism which maps ℓ_z onto ℓ_{z_1}. Indeed, if $\mathbf{x} = \begin{pmatrix} 1 \\ z \end{pmatrix} a \in \ell_z$, then clearly

$$g\mathbf{x} = \begin{pmatrix} 1 \\ g \cdot z \end{pmatrix} (g_{11} + g_{12} z) a = \begin{pmatrix} 1 \\ z_1 \end{pmatrix} (g_{11} + g_{12} z) a \in \ell_{z_1}.$$

Let us check that $cr(z_1, z_2)$ is well defined, i.e., that it does not depend on the choice of g.

To prove this, recall that if $k \in \mathcal{U}_\rho$ satisfies $k \cdot 0 = 0$, then $k = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix}$, with $u_1, u_2 \in \mathcal{U}_\mathcal{A}$.

Proposition 26.6. *With the above notations, the endomorphism*

$$cr(z_0, z_1) \in CR(\ell_{z_{-\infty}}, \ell_{z_0}, \ell_{z_1}, \ell_{z_{+\infty}})$$

does not depend on the choice of g. Namely, if $h \in \mathcal{U}_\rho$ satisfies $h \cdot 0 = z_0$, and $z' = h^{-1} \cdot z_1$, then

$$h \ cr(0, z') h^{-1} = g \ cr(0,z) g^{-1}.$$

Suppose that $\mathcal{A}$ is a von Neumann algebra. If $z \in \mathcal{D}$ is non invertible, there exist $z_n \in \mathcal{D}$ which are invertible such that $z_n \to z$ strongly and $\|z_n\| \leq \|z\|$. Then $cr(0, z_n), cr(0,z)$ are endomorphisms of different submodules. In order to compare them, we can regard them as $\mathcal{A}$-module morphisms of $\mathcal{A}^2$, embedding each module in $\mathcal{A}^2$ using the θ_ρ-orthogonal projections $p_{\ell_{z_n}}, p_{\ell_z}$ onto the submodules ℓ_{z_n}, ℓ_z, respectively. For $z' \in \mathcal{D}$,

$$p_{\ell_{z'}}(\mathbf{x}) = (1 - |z'|^2)^{-1/2} \theta_\rho(\begin{pmatrix} 1 \\ z' \end{pmatrix}, \mathbf{x}) \begin{pmatrix} 1 \\ z' \end{pmatrix} (1 - |z'|^2)^{-1/2}.$$

Proposition 26.7. *Let $\mathcal{A}$ be a von Neumann algebra.*

1. *If $z \in \mathcal{D} \cap G_\mathcal{A}$, the set $CR(\ell_{-\infty}, \ell_0, \ell_z, \ell_\infty)$ consists of a single element $cr(0,z)$.*

2. If $z \in \mathcal{D}$ is non invertible, there exist $z_n \in \mathcal{D}$ which are invertible such that $z_n \to z$ strongly and $\|z_n\| \leq \|z\|$. Then, with the above notations,

$$cr(0, z_n)p_{\ell_{z_n}}(\mathbf{x}) \to cr(0, z)p_{\ell_z}(\mathbf{x})$$

strongly in $\mathcal{A}^2$.

Observe that in the von Neumann algebra case $cr(0, z)$ is then uniquely defined when z is invertible, and is defined by strong continuity in the remaining cases.

Remark 26.8. As a corollary we get that, even if the set $CR(\ell_1, \ell_2, \ell_3, \ell_4)$ may be empty for general $\ell_1, \ell_2, \ell_3, \ell_4$, the particular set $CR(\ell_{-\infty}, \ell_0, \ell_z, \ell_\infty)$ is not, and $cr(0, z)$ is a distinguished element of this set.

As a first approximation of the deep relationship between the cross ratio and the metric in $\mathcal{A}\mathbb{P}_1^\rho$, we can state the following result. It can be regarded of the scalar version of our main result, here we state the equality between the norm of the endomorphism $cr(0, z)$ and the distance between 0 and z in the Poincaré disk $\mathcal{D}$.

Theorem 26.9. *Let $z \in \mathcal{D}$, then*

$$\frac{1}{2}\|cr(0, z)\|_{\mathcal{B}(\ell_z)} = d(0, z),$$

where $\| \quad \|_{\mathcal{B}(\ell_z)}$ denotes the norm of operators acting in $\ell_z \subset \mathcal{A}^2$.

27 The cross ratio, the logarithm and the exponential

We have just defined a Hilbertian $\mathcal{C}$-valued structure in $\mathcal{Q}_\rho \simeq \mathcal{A}\mathbb{P}_1^\rho$, or, equivalently, in $\mathcal{D}$. In particular, the product

$$\langle \log_0(z), \log_0(z) \rangle_0$$

takes values in the set of endomorphisms of $\ell_0 = \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]$, where $\log_0$ is defined in Corollary 7.9. It is a positive module endomorphism (given by multiplying the generator $\mathbf{e_1}$ by a positive element of a). Thus, it has a unique positive square root $\langle \log_0(z), \log_0(z) \rangle_0^{1/2}$, which we denote by $mod_0(\log_0(z))$ of $\log_0(z)$. Explicitly, in the generator $\mathbf{e_1}$, $mod_0(\log_0(z))$ consists in multiplying the generator by $\log\big((1 + |z|)(1 - |z|)^{-1}\big)$.

On the other hand, we saw that, for $z \in \mathcal{D}$, the endomorphism of ℓ_z denoted by $cr(0, z)$, is given by the same coefficient $\log\big((1 + |z|)(1 - |z|)^{-1}\big)$, which multiplies the generator $\begin{pmatrix} 1 \\ z \end{pmatrix}$ of ℓ_z.

We shall translate the endomorphism $cr(0, z)$ from ℓ_z to ℓ_0 by means of the parallel transport of $\mathcal{A}\mathbb{P}_1^\rho$, along the geodesic δ, with $\delta(0) = \ell_0$ and $\delta(1) = \ell_z$ (i.e., the same former δ, which under the identification $\mathcal{D} \simeq \mathcal{A}\mathbb{P}_1^\rho$ joins $\delta(0) = 0$ and $\delta(1) = z$ in $\mathcal{D}$: $\delta(t) = \omega \tanh(t|\alpha|)$).

The parallel transport of elements of $\mathcal{D}$ (or $\mathcal{A}\mathbb{P}_1^\rho$) along the geodesic $\delta(t) = e^{t\begin{pmatrix} 0 & \alpha^* \\ \alpha & 0 \end{pmatrix}} \cdot 0$, where α is, as in (24)

$$\alpha = z \sum_{k=0}^{\infty} \frac{1}{2k+1}(z^* z)^k,$$

is given by the left action of the invertible matrix
$e^{t\begin{pmatrix} 0 & \alpha^* \\ \alpha & 0 \end{pmatrix}} : \ell_0 \to \ell_{\delta(t)}$. The endomorphism $cr(0,z)$ of ℓ_z is transported to ℓ_0 as

$$cr(0,z)_0 := e^{-\begin{pmatrix} 0 & \alpha^* \\ \alpha & 0 \end{pmatrix}} cr(0,z) e^{\begin{pmatrix} 0 & \alpha^* \\ \alpha & 0 \end{pmatrix}} : \ell_0 \to \ell_0.$$

Our main result (for the origin) is the following:

Theorem 27.1. *With the current notation, if $z \in \mathcal{D}$ (or $\ell_z \in \mathcal{A}\mathbb{P}_1^\rho$),*

$$e^{mod_0(\log_0(z))} = cr(0,z)_0 \tag{75}$$

or, equivalently,

$$mod_0(\log_0(z)) = \log(cr(0,z)_0), \tag{76}$$

where the exponential in the first equality is the usual exponential of $\mathcal{A}$, log *in the second equality is the usual logarithm of G^+, and each endomorphism of ℓ_0 is identified with its coefficient in the basis $\mathbf{e_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.*

As in Definition 26.5, let $z_0 \neq z_1 \in \mathcal{D}$ ($\ell_{z_0} \neq \ell_{z_1} \in \mathcal{A}\mathbb{P}_1^\rho$). Pick $g \in \mathcal{U}_\rho$ such that $g \cdot 0 = z_0$, and denote by $z = g^{-1} \cdot z_1$ as before. Let δ be the geodesic such that $\delta(0) = 0$ and $\delta(1) = z_1$. Then $\delta_{z_0,z_1} = g \cdot \delta$ is the geodesic which joins z_0 and z_1 at $t = 0$ and $t = 1$, respectively. Recall that $cr(z_0,z_1) = gcr(z_0,z_1)g^{-1}$. Likewise, we put

$$\log_{z_0}(z_1) := g \log_0(z) g^{-1}, \quad \text{and} \quad mod_{z_0}(z_1) = \langle \log_{z_0}(z_1), \log_{z_0}(z_1) \rangle_{z_0}^{1/2},$$

where $\langle \varphi, \psi \rangle_{z_0} = g\langle g\varphi g^{-1}, g\psi g^{-1} \rangle_0 g^{-1}$, and $\log_{z_0}$ is the inverse of the exponential $exp_{z_0} : (T\mathcal{D})_{z_0} \to \mathcal{D}$. It is not difficult to verify that these definitions do not depend on the choice of g.

Finally, let us denote by $cr(z_0,z_1)_{z_0}$ the parallel transport of $cr(z_0,z_1)$ from ℓ_{z_1} to ℓ_{z_0} along the geodesic δ_{z_0,z_1} (obtained by conjugation as in the case of the origin, by the value at $t = 1$ of the one parameter group in $\mathcal{U}_\rho$ which determines δ_{z_0,z_1}). The $\mathcal{U}_\rho$-covariance of the data involved enables us to prove the following:

Theorem 27.2. *With the current notations,*

$$mod_{z_0}(\log_{z_0}(z_1)) = \log\left(cr(z_0,z_1)_{z_0}\right).$$

In particular, $\|\log_{z_0}(z_1)\|_{z_0} = \|\log(cr(z_0,z_1)_{z_0})\|$.

Remark 27.3. If we think of $mod_{z_0}(\log_{z_0}(z_1))$ as an *distance operator* from z_0 to z_1, then the identity in the above theorem shows a projective way of computing this operator distance.

28 An example

Suppose that the algebra $\mathcal{A}$ has a trace $\mathbf{tr}$ onto a central subalgebra, that is, there exists a C*-subalgebra $\mathcal{B} \subset Z(\mathcal{A})$ of the center of $\mathcal{A}$ and a conditional expectation $\mathbf{tr} : \mathcal{A} \to \mathcal{B}$ satisfying $\mathbf{tr}(xy) = \mathbf{tr}(yx)$ for all $x, y \in \mathcal{A}$. This happens, for instance, if $\mathcal{A}$ is a finite von Neumann algebra.

We can define a Hilbertian $\mathcal{B}$-valued inner product, by means of

$$\langle X, Y \rangle_{\mathbf{tr},q} = -\mathbf{tr}(\theta_\rho(X\mathbf{x}, Y\mathbf{y})).$$

Indeed, since $\mathbf{tr}$ is tracial, the value of $-\mathbf{tr}(\theta_\rho(X\mathbf{x}, Y\mathbf{y}))$ is independent of the choice of $\mathbf{x} \in \mathcal{K}_\rho$ satisfying $q = p_{\mathbf{x}}$. On the other hand, $cr(z_0, z_1)$ is an element of Γ_{z_0}, which has matrix a in a unital base $\mathbf{x} \in R(q)$, as explained before. We put $cr(z_0, z_1)_{\mathbf{tr}}$, for $\mathbf{tr}(a)$. Clearly, $cr(z_0, z_1)_{\mathbf{tr}}$ does not depend on the basis $\mathbf{x}$. With these notations, the formula in Corollary 27.2, can be written

$$\langle \log_{z_0} z_1, \log_{z_0} z_1 \rangle_{\mathbf{tr}}^{1/2} = \log cr(z_0, z_1)_{\mathbf{tr}}, \tag{77}$$

which is an identity involving elements in $\mathcal{B}$.

More specifically, if $\mathcal{A}$ is commutative, we can choose $\mathbf{tr}$ the identity $\mathcal{A} = \mathcal{B}$, and we have

$$|\log_{z_0} z_1| = \log cr(z_0, z_1), \tag{78}$$

as elements in $\mathcal{A}$.

Declarations.

The authors state that there is no conflict of interest. No datasets were generated or analyzed during the current study.

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Esteban Andruchow
Instituto de Ciencias, Universidad Nacional de Gral. Sarmiento,
J.M. Gutierrez 1150, (1613) Los Polvorines, Argentina
and Instituto Argentino de Matemática, 'Alberto P. Calderón', CONICET,
Saavedra 15 3er. piso, (1083) Buenos Aires, Argentina.
e-mail: eandruch@ungs.edu.ar

Gustavo Corach
Instituto Argentino de Matemática, 'Alberto P. Calderón', CONICET,

Saavedra 15 3er. piso, (1083) Buenos Aires, Argentina.
e-mail: gcorach@gmail.com

Lázaro Recht
Instituto Argentino de Matemática, 'Alberto P. Calderón', CONICET,
Saavedra 15 3er. piso, (1083) Buenos Aires, Argentina.
e-mail: lrecht@gmail.com