Moment of a subspace and joint numerical range

Abel Klobouk[a] and Alejandro Varela[b]

[a]Departamento de Ciencias Básicas, Universidad Nacional Luján, Luján, Argentina;
[b]Instituto de Ciencias, Universidad Nacional de General Sarmiento, Los Polvorines, Argentina

Abstract
For a given complex finite dimensional subspace S of $\mathbb{C}^n$ and a fixed basis, we study the compact and convex subset of $(\mathbb{R}_{\geqslant 0})^n$ that we call the moment of S

$$\begin{aligned} m_S =& \text{convex hull}\left(\{|s|^2 \in \mathbb{R}^n_{\geqslant 0} : s \in S \wedge \|s\| = 1\}\right) \\ \simeq& \{\text{Diag}(Y) \in M_n^h(\mathbb{C}) : Y \geqslant 0, \text{tr}(Y) = 1, P_S Y P_S = Y\} \end{aligned}$$

where $|s|^2 = (|s_1|^2, |s_2|^2, \ldots, |s_n|^2)$. This set is relevant in the determination of minimal hermitian matrices ($M \in M_n^h$ such that $\|M + D\| \leqslant D$ for every diagonal D and $\|\ \|$ the spectral norm). We describe extremal points and curves of m_S in terms of principal vectors that minimize the angle between S and the coordinate axes. We also relate m_S to the joint numerical range W of n rank one $n \times n$ matrices constructed with the orthogonal projection P_S and the fixed basis used. This connection provides a new approach to the description of m_S and to minimal matrices. As a consequence the intersection of two of these joint numerical ranges allow the construction or detection of a minimal matrix, a fact that is easier to corroborate than the equivalent condition for moments. It is also proved that m_S is a semi-algebraic set equal to the intersection of the mentioned W with a hyperplane and whose generated positive cone coincides with that of W.

KEYWORDS
moment of a subspace; extremal points; minimal hermitian matrix; diagonal matrix; best approximation; convexity; joint numerical range

AMS CLASSIFICATION
65F35, 15B57, 15A60, 52A20.

1. Introduction

Given a subspace S of $\mathbb{C}^n$, our main interest in the study of the set

$$m_S = \text{conv}\left(\{|s|^2 \in \mathbb{R}^n_{\geqslant 0} : s \in S \wedge \|s\| = 1\}\right)$$

where $|s|^2 = \left(|s_1|^2, |s_2|^2, \ldots, |s_n|^2\right)$ with s_j, $j = 1, \ldots, n$ are the coordinates of s in a fixed basis $E = \{e_1, e_2, \ldots, e_n\}$ (see Proposition 3.2 for other equivalent definitions of m_S). Its importance lies on its fundamental relation with what we call minimal hermitian matrices $M \in M_n^(\mathbb{C})$, that satisfy $\|M\| \leqslant \|M + D\|$, with D a diagonal matrix and $\|M\| = \sup_{\|x\|=1} \|Mx\|$. Note that the consideration of diagonal matrices D

CONTACT A. Varela. Email: avarela@ungs.edu.ar

requires the choice of a fixed orthonormal basis E of $\mathbb{C}^n$. The relation between minimal matrices and m_S is the following

$$M \text{ is minimal } \Leftrightarrow M = \|M\| \left(P_{S_{\|M\|}} - P_{S_{-\|M\|}}\right) + R \text{ and } m_{S_{\|M\|}} \cap m_{S_{-\|M\|}} \neq \varnothing \quad (1)$$

where P_S denotes de orthogonal projection onto S and $\|R\| < \|M\|$, $\mathrm{Im}(R) \perp S_{\|M\|}, S_{-\|M\|}$, with S_λ denoting the eigenspaces of M corresponding to the eigenvalue λ (see the more detailed Remark 2). These matrices, in turn, allow the concrete description of some metric geodesics in flag manifolds as studied in [1,2].

We describe many extremal points of m_S (including curves of them) that can be easily described in terms of some principal vectors of S or the matrix of the orthogonal projection P_S in the E basis. The projections of the mentioned curves of m_S in certain 2 dimensional coordinate planes are parts of ellipses centred at the origin that can be easily obtained.

We also study a very close relationship between the moment m_S and a joint numerical range of some chosen hermitian matrices of rank one $P_S E_i P_S = (P_S e_i) \cdot (P_S e_i)^* = (P_S e_i) \otimes (P_S e_i)$, for $i = 1, \ldots, n$ (see Theorem 6.3). This allows a translation of the condition of being a minimal matrix (1) to a property of intersection of the two corresponding joint numerical ranges (see Theorem 6.7). Recent results using algebraic geometry allow a description of a subset $T^\sim$ whose convex hull gives these joint numerical ranges (see [3]). This is a generalization of a Theorem of Kippenhahn where it is proved that the convex hull of certain algebraic curve equals the classic numerical range of a matrix $W(A)$.

In what follows we list the main contents of each section of this work. In Section 2 we state most of the notation used. Section 3 includes previous results and different characterizations related to m_S. We also introduce there what could be considered the centre of m_S (see (5)) and some of its properties.

In Section 4 we introduce the principal standard vectors $\{v^j\}_{j=1}^n$ of a subspace S related to a fixed basis E. These are the ones that minimize the angle between S and the coordinate axis of a fixed basis $E = \{e_1, e_2, \ldots, e_n\}$. We describe some of its properties, particularly the ones related to the elements $|v^j|^2 \in m_S$, $j = 1, \ldots, n$.

Section 5 is dedicated to the presentation and description of certain particular curves of extreme points of m_S that join elements $|v^j|^2 \in m_S$ of what we call the principal standard vectors $\{v^j\}_{j=1}^n$.

In Section 6 we show the close relation between m_S and the joint numerical range

$$\begin{aligned} W_{S,E} &= W(P_S E_1 P_S, \ldots, P_S E_n P_S) \\ &= \left\{(\mathrm{tr}(P_S E_1 P_S \rho, \ldots, \mathrm{tr}(P_S E_1 P_S \rho) \in \mathbb{R}^n_{\geqslant} 0 : \rho \geqslant 0, \mathrm{tr}(\rho) = 1\right\}, \end{aligned}$$

where $E_i = e_i \otimes e_i$ is the orthogonal projection onto the subspace generated by $e_i \in E$, $i = 1, \ldots n$. Here E is the fixed standard basis we are considering. This allows restating many results of m_S and minimal matrices in terms of these joint numerical ranges and obtain some general properties about m_S. Among them it is proved that m_S is a semi-algebraic set (Remark 17 c)), it is a subset of $W_{S,E}$ included in a hyperplane that generates the same cone than $W_{S,E}$ (Theorem 6.3 and Proposition 6.5) and the known algorithms used to generate $W_{S,E}$ can be used to approximate m_S. It is also shown that the map $P_S \mapsto m_S$ satisfies $\mathrm{dist}_H(m_S, m_V) \leqslant (2\sqrt{n} + 1)\|P_S - P_V\|$ (Proposition 6.5) where dist_H is the Hausdorff distance. Moreover, for a hermitian matrix $M = \|M\|(P_{\mathrm{Eig}_{\|M\|}} - P_{\mathrm{Eig}_{-\|M\|}}) + R$ (with $\mathrm{Eig}_{\|M\|}$, $\mathrm{Eig}_{-\|M\|}$ the eigenspaces of $\pm\|M\|$, $R \in$

$M_n^h(\mathbb{C})$, $R(Eig_{\pm\|M\|}) = 0$), holds that $W_{\mathrm{Eig}_{\|M\|},E} \cap W_{\mathrm{Eig}_{-\|M\|},E} \neq \{0\}$ if and only if M is minimal.

2. Preliminaries and notation

Let $\mathbb{C}^n$ be the finite n-dimensional Hilbert space of vectors of complex numbers with its usual inner product denoted with $\langle\ ,\ \rangle$ and norm $\|\ \|$. We denote with $E = \{e_1, e_2, \ldots, e_n\}$ a fixed orthonormal ordered basis of $\mathbb{C}^n$ which we will call the standard basis. The vectors $e_i \in E$, $i = 1, \ldots, n$ will be called the standard vectors. Our adoption of a fixed basis is required since we are going to work with diagonal matrices that may not remain as such when a different basis is used.

Given a vector $x \in \mathbb{C}^n$ we will denote with x_1, x_2, $\ldots$,x_n its standard coordinates (in the E basis) with the only exception of $\{e_1, e_2, \ldots, e_n\}$ where the subscripts denote the different standard vectors.

We will denote with $|x|^2 = (|x_1|^2, \ldots, |x_n|^2) \in \mathbb{R}^n_{\geqslant 0}$ the vector of the squared modulus of its E coordinates $x_i = \langle x, e_i\rangle$, for $i = 1, \ldots, n$. The vector $|x|^2$ has been described in previous works with $x \circ \bar{x}$ using the entry-wise Schur (or Hadamard) product $\circ$ where $\bar{x}$ denotes the vector whose coordinates are those of x conjugated. We state here some of these notation for further references.

Notation 1. We will use the following notations:

a) $E = \{e_1, e_2, \ldots, e_n\}$ will denote a fixed orthonormal ordered basis in $\mathbb{C}^n$, considered the standard basis,
b) given $x \in \mathbb{C}^n$, the x_j will be the j^{th} coordinate of x in the standard basis, that is, $x_j = \langle x, e_j\rangle$,
c) given a vector $x \in \mathbb{C}^n$ we denote with $|x|$ the vector of the modules of its coordinates $|x| = (|x_1|, \ldots, |x_n|)$ and similarly $|x|^2 = (|x_1|^2, \ldots, |x_n|^2)$.

With $M_n(\mathbb{C})$ we denote the set of $n \times n$ matrices with coefficients in $\mathbb{C}$, and if $A \in M_n(\mathbb{C})$ then A^* is its adjoint matrix (conjugate transpose of A). We write $A \in M_n^h(\mathbb{C})$ if A is a matrix such that $A = A^*$, and we will call it hermitian. Given a square matrix A, then $\mathrm{Diag}(A)$ is the diagonal square matrix with the same diagonal than A written using a fixed basis of $\mathbb{C}^n$. Sometimes we will identify the matrix $\mathrm{Diag}(A)$ with the vector whose entries are obtained from the corresponding diagonal.

The rank one orthogonal projections onto the subspace generated by an $x \in \mathbb{C}^n$, with $\|x\| = 1$ will be denoted with $x \otimes x$ ($(x \otimes x)(v) = \langle v, x\rangle x$, for all $v \in \mathbb{C}^n$). Its corresponding matrix in the E basis can be written by $x \cdot x^*$ where $\cdot$ is the matrix product and x is identified with the $n \times 1$ column matrix of coordinates of x in the E basis ($x_{i,1} = \langle x, e_i\rangle$).

If $T \subset \mathbb{C}^n$ or $\mathbb{R}^n$, we will write $\mathrm{conv}(T)$ to denote the convex closure or convex hull of T. Equivalently the set of convex combinations $t\,a + (1-t)b$ for $t \in [0,1]$, for all $a, b \in T$.

3. Moment of a subspace

The term "moment" in the following definition of the set m_S of a subspace S of $\dim(S) = r$ is motivated by its relation to certain moment map defined in the symplectic manifold $(\mathbb{C}^n)^r$ (see Section 4 in [4] for details).

Definition 3.1. Let S a subspace of $\mathbb{C}^n$, $\{0\} \subsetneq S \subsetneq \mathbb{C}^n$ with $\dim(S) = r$. The moment of S related to a fixed orthonormal ordered basis $E = \{e_1, e_2, \ldots, e_n\}$ of $\mathbb{C}^n$ or just the **moment set of** S is the subset of $(\mathbb{R}_{\geqslant 0})^n$ defined by

$$m_{S,E} = \operatorname{conv}\left(\{|v|^2 \in \mathbb{R}^n_{\geqslant 0} : v \in S \wedge \|v\| = 1\}\right). \tag{2}$$

If the basis E is fixed or clear from the context we will use the shorter notation m_S instead of $m_{S,E}$.

Remark 1. The set m_S is a compact and convex set of $\mathbb{R}^n$, but note that if the convex closure (conv) is not used in (3.1) then m_S would not be necessarily convex (see Remark 4 in [5]).

There are several equivalent statements that we can consider to define the set m_S some of which do not require to take the convex hull. See for example Proposition 3.2.

Remark 2. Our motivation to study the set m_S relies on the following property. Let $\mathbb{V}$ and $\mathbb{W}$ be two non trivial orthogonal subspaces of $\mathbb{C}^n$, and a hermitian matrix $R \in M_n^h(\mathbb{C})$ such that $RP_{\mathbb{V}} = RP_{\mathbb{W}} = 0$ and $\|R\| < \lambda$. Then, if $m_{\mathbb{V}} \cap m_{\mathbb{W}} \neq \varnothing$, follows that the matrix $M = \lambda(P_{\mathbb{V}} - P_{\mathbb{W}}) + R$ is a minimal hermitian matrix in the sense that

$$\|M\| = \operatorname{dist}(M, \operatorname{Diag}_n(\mathbb{R})) = \operatorname{dist}(M, \operatorname{Diag}_n(\mathbb{C}))$$

for $\operatorname{Diag}_n(X)$ the diagonal matrices with coefficients in X. Moreover, every minimal matrix can be written in this way (see Corollary 3 of [5] and Theorem 3 of [4] to obtain a detailed study and other equivalent statements).

Remark 3. We denote the positive cone generated by a set $X \subset \mathbb{R}^n$ as

$$\operatorname{cone}(X) = \{t\, x \in \mathbb{R}^n : t \in \mathbb{R}_{\geqslant 0} \wedge x \in X\}. \tag{3}$$

Note that we can consider the positive cone generated by m_S in $\mathbb{R}^n_{\geqslant 0}$ instead of just m_S in order to obtain the sufficient condition mentioned in the previous Remark 2. Nevertheless, we choose the present Definition 3.1 of m_S in order to work with bounded sets.

The following are equivalent characterizations of the moment m_S. The statements (1), (2), (3), can be obtained from characterizations made in [5], (4) is Lemma 6.2 and (5) is Theorem 6.3 of section 6.

Proposition 3.2. *The following are equivalent conditions to define m_S, the moment of S related to a basis $E = \{e_1, \ldots, e_n\}$, with dim$(S) = r$.*

(1) Let $\operatorname{Diag}(X) \in M_n(\mathbb{C})$ *be the diagonal matrix with the same diagonal than $X \in M_n(\mathbb{C})$ when its written using the basis E, then*

$$m_S \simeq \{\operatorname{Diag}(Y) \in M_n^h(\mathbb{C}) : Y \geqslant 0, tr(Y) = 1, Im(Y) \subset S\},$$

where we denote with $\simeq$ the usual identification of vectors of $\mathbb{C}^n$ with diagonal matrices of $\mathbb{C}^{n\times n}$.

(2)

$$m_S = \operatorname{conv}\left\{|v|^2 : v \in S \wedge \|v\| = 1\right\}$$

(see the meaning of $|v|^2$ in Notation 1)

(3)

$$m_S = \bigcup_{\{s^i\}_{i=1}^r o.n.\ set\ in\ S} \operatorname{conv}\left(\{|s^i|^2\}_{i=1}^r\right).$$

(4)

$$m_S = \{(tr(E_1Y), tr(E_2Y), \ldots, tr(E_nY)) \in \mathbb{R}^n_{\geqslant 0} : Y \in \mathcal{D}_S\}$$

where $E_i = e_i \cdot (e_i)^t$ *and* $\mathcal{D}_S = \{Y \in M_n^h(\mathbb{C}) : Y \geqslant 0, tr(Y) = 1, P_S Y = Y\}$.

(5)

$$m_S = W(P_SE_1P_S, \ldots, P_SE_nP_S) \cap \left\{x \in \mathbb{R}^n_{\geqslant 0} : \sum_{i=1}^n x_i = 1\right\},$$

where P_S *is the orthogonal projection onto* S, $E_i = e_i \cdot (e_i)^*$ *and* W *is the joint numerical range defined in 52.*

Remark 4. Note that in particular the condition (3) implies that the Carathéodory number of $\left\{|v|^2 : v \in S \wedge \|v\| = 1\right\}$ is less or equal than $r = \dim(S)$ (see (2)) .

Example 3.3. Let $U_d \in M_n(\mathbb{C})$ be a unitary such that its matrix with respect to the E basis is diagonal, S a subspace of $\mathbb{C}^n$ and $U_d(S) = \{U_d(s) : s \in S\}$, the subspace that is the image of S under U_d. Then it is trivial that $m_{S,E} = m_{U_d(S),E}$. Nevertheless, if $\mathbb{V}, \mathbb{W}$ are subspaces of $\mathbb{C}^n$ such that $m_{\mathbb{V}} = m_{\mathbb{W}}$ it is not true that there exists a diagonal unitary U_d (relative to the base E) such that $U_d(\mathbb{V}) = \mathbb{W}$. For instance, if we consider the subspaces $\mathbb{V} = \operatorname{span}\{(1, 1, 0), (0, 1, 1)\}$ and $\mathbb{W} = \operatorname{span}\{(-1, e^{i\pi/4}, 0), (0, e^{i\pi/3}, e^{i\pi/6})\}$ it can be proved that $m_{\mathbb{V}} = m_{\mathbb{W}}$ but there is not any diagonal unitary U_d such that $U_d(\mathbb{V}) = \mathbb{W}$. Their corresponding orthogonal projections written in the E basis are

$$P_{\mathbb{V}} = \begin{pmatrix} 2/3 & 1/3 & -1/3 \\ 1/3 & 2/3 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{pmatrix} \text{ and } P_{\mathbb{W}} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3}e^{-\frac{i\pi}{4}} & \frac{1}{3}e^{-\frac{i\pi}{12}} \\ -\frac{1}{3}e^{\frac{i\pi}{4}} & \frac{2}{3} & \frac{1}{3}e^{\frac{i\pi}{6}} \\ \frac{1}{3}e^{\frac{i\pi}{12}} & \frac{1}{3}e^{-\frac{i\pi}{6}} & \frac{2}{3} \end{pmatrix}.$$

Remark 5. Let $S \subset \mathbb{C}^n$ be a subspace of dimension r and E a fixed basis of $\mathbb{C}^n$. If $\{s^1, s^2, \ldots, s^r\}$ and $= \{w^1, w^2, \ldots, w^r\}$ are two orthonormal basis of S, then

$$\sum_{i=1}^r |s^i|^2 = \sum_{i=1}^r |w^i|^2. \tag{4}$$

The proof of this fact follows after considering that $P_S = \sum_{i=1}^r s^i \otimes s^i = \sum_{i=1}^r w^i \otimes w^i$ and therefore their diagonal matrices coincide $\operatorname{diag}(P_S) = \operatorname{diag}\left(\sum_{i=1}^r s^i \otimes s^i\right) = \sum_{i=1}^r |s^i|^2 = \operatorname{diag}\left(\sum_{i=1}^r w^i \otimes w^i\right) = \sum_{i=1}^r |w^i|^2$.

The element

$$c(m_{S,E}) = \frac{1}{r}\sum_{i=1}^r |s^i|^2 = \frac{1}{\dim(S)} \operatorname{diag}(P_S) \tag{5}$$

for (any) orthogonal basis $\{s^1, s^2, \ldots, s^r\}$ of S fulfils some interesting symmetric properties in the moment set m_S.

Proposition 3.4. *Let S be a non trivial subspace of $\mathbb{C}^n$, E a fixed basis of $\mathbb{C}^n$ and $c(m_S) = c(m_{S,E})$ defined as in (5). Then $c(m_S)$ satisfies the following properties.*

(a) $c(m_S) \in m_S$.

(b) *$c(m_S)$ coincides with the barycentre or centroid of the simplex generated by $\{|w^1|^2, |w^2|^2, \ldots, |w^r|^2\} \subset \mathbb{R}^n_{\geqslant 0}$ obtained from any orthonormal basis $\{w^1, w^2, \ldots, w^r\}$ of S.*

(c) *Let S and V be subspaces of $\mathbb{C}^n$, with one of them not trivial, $S \perp V$, with $dim(S) = r$ and $dim(V) = k$. Then*

$$c\left(m_{S \oplus V}\right) = \frac{1}{r+k}(r\, c(m_S) + k\, c(m_V)). \tag{6}$$

This can be generalized to any number of mutually orthogonal subspaces.

(d) $c(m_{\mathbb{C}^n}) = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$.

(e) *If $dim(S) = r < n$, then $c(m_{S^\perp}) = \frac{1}{n-r}((1,1,\ldots,1) - r\, c(m_S))$.*

(f) *If $dim(S) = r < n$ then the i^{th} coordinate of $c(m_S)$ satisfies $c(m_S)_i \leqslant \frac{1}{r}$ for every $i = 1, \ldots, n$.*

(g) *Given a subspace $D \subset S$, with $dim(D) = d < dim(S) = r$, then $c(m_{S \ominus D}) = c(m_{S \cap D^\perp}) = \frac{1}{r-d}\left(r\, c(m_S) - d\, c(m_D)\right)$.*

(h) *Let S and V be two subspaces of $\mathbb{C}^n$ with dimensions r and k respectively and $D = S \cap V$ of dimension d such that $\left(S \cap D^\perp\right) \perp \left(V \cap D^\perp\right)$ holds. Then $c(m_{S+V}) = \frac{1}{r+k-d}(r\, c(m_S) + k\, c(m_V) - d\, c(m_D))$.*

Proof. (a) This follows after considering the definition of $c(m_S)$ (5) and the statement (3) in Proposition 3.2.

(b) The centroid or barycentre of the simplex of r vectors $\{w^i\}_{i=1}^r \subset \mathbb{R}^n$ is $\frac{1}{r}\sum_{i=1}^r w^i$. Then consider (4) and (5).

(c) This property can be proved using an orthonormal basis of $S \oplus V$ of elements of S and V and then (5).

(d) and (e) are apparent.

(f) Using the previous (c), (d) and (e) follows that $c(m_{\mathbb{C}^n})_i = 1/n = \frac{r\, c(m_S)_i + (n-r)\, c(m_{S^\perp})_i}{n}$, which implies that $r\, c(m_S)_i = 1 - (n-r)\, c(m_{S^\perp})_i \leqslant 1$.

(g) Consider a completion of an orthonormal basis of D to one of S and use (5).

(h) This equation can be proved using similar techniques of basis completions and (f).

□

Remark 6. Note the similarity of the equation (6) with the one used to calculate the geometric centroid or barycentre of m disjoint sets A_j with $j = 1, \ldots, m$ using $c(\cup_{j=1}^m A_j) = \frac{\sum_{j=1}^m c(A_j)\mu(A_j)}{\sum_{j=1}^m \mu(A_j)}$ where μ is the corresponding measure.

Proposition 3.5. *Let S and W be two subspaces of $\mathbb{C}^n$ such that $S \perp W$ and such that there exist two corresponding orthonormal basis $\{s^h\}_{h=1}^r$ and $\{w^j\}_{j=1}^k$ that satisfy*

$$\sum_{h=1}^r t_h |s^h|^2 = \sum_{j=1}^k u_j |w^j|^2 \tag{7}$$

for $t_h, u_j \geqslant 0$ and $\sum_{h=1}^r t_h = \sum_{j=1}^k u_j = 1$. That is, (S,W) form a support (see Definition 1 and theorems 2 and 3 in [4]).
Then

$$\left(\sum_{h=1}^r t_h|s^h|^2\right)_m = \left(\sum_{j=1}^k u_j|w^j|^2\right)_m \leqslant \frac{1}{2}, \quad \textit{for every } m = 1, \ldots, n. \tag{8}$$

Proof. Suppose there is m_0 such that $\left(\sum_{h=1}^r t_h|s^h|^2\right)_{m_0} > \frac{1}{2}$. Then define the vectors $x = \sum_{h=1}^r \sqrt{t_h}\ e^{-i\arg(s^h_{m_0})}\ s^h$ and $y = \sum_{j=1}^k \sqrt{u_j}\ e^{-i\arg(w^j_{m_0})}\ w^j$. Observe that $x \in V$ and $\|x\| = \langle x, x\rangle^{1/2} = (\sum_{h=1}^r t_h)^{1/2} = 1$. Similarly, follows that $y \in W$, $\|y\| = 1$ and $x \perp y$.
Observe that the m_0 coordinate of x satisfies $|x_{m_0}|^2 = \left(\sum_{h=1}^r \sqrt{t_h}\ e^{-i\arg(s^h_{m_0})}\ s^h\right)^2_{m_0} = \left(\sum_{h=1}^r \sqrt{t_h}\ |s^h_{m_0}|\right)^2 \geqslant \sum_{h=1}^r t_h|s^h_{m_0}|^2 > \frac{1}{2}$, and in the case of y also $|y_{m_0}|^2 > \frac{1}{2}$.
Now consider the dimension two subspace $C = \text{gen}\{x, y\}$ where $\{x, y\}$ is an orthonormal basis of C and $c(m_C) = \frac{1}{2}(|x|^2 + |y|^2)$ (see Proposition 3.4 (c)). Then $c(m_C)_{m_0} = \frac{|x_{m_0}|^2 + |y_{m_0}|^2}{2} > \frac{1/2+1/2}{2} = \frac{1}{2}$ which contradicts Proposition 3.4 (f). □

Remark 7. Note that the condition (7) is equivalent to $m_S \cap m_W \neq \varnothing$ (see (3) in Proposition 3.2). Then (8) implies that $m_S \cap m_W \subset [0, 1/2]^n$ must always hold. This information is relevant since the supports (S, W) allow the construction of minimal matrices (see Theorem 3, [4]).

4. Generic subspaces and its principal standard vectors

Definition 4.1. We call a subspace S of $\mathbb{C}^n$ a **generic subspace** with respect to the basis $E = \{e_i\}_{i=1}^n$ if for every $j = 1, \ldots, n$ there is $x \in S$ such that $x_j = \langle x, e_i\rangle \neq 0$.
This condition is equivalent to any of the following statements:

- S is not included in a $(n-1)$-dimensional coordinate space $C_j = \{z \in \mathbb{C}^n : z_j = 0\} = (\text{span}\{e_j\})^\perp$ for $j = 1, \ldots, n$,
- or $\left(P_S(e_j)\right)_j \neq 0$ for all j (otherwise $0 = \langle P_S(e_j), e_j\rangle = \langle P_S(e_j), P_S(e_j)\rangle = \|P_S(e_j)\|^2$ which implies $e_j \perp S$).

Remark 8. Observe that if S is not a generic subspace of $\mathbb{C}^n$ we can find $m < n$ such that a natural immersion of S in $\mathbb{C}^m$ becomes a generic subspace of $\mathbb{C}^m$.

Definition 4.2. Principal standard vectors. If $P_S(e_j) = P_S e_j \neq 0$ we will denote with

$$v^j = \frac{P_S e_j}{\|P_S e_j\|} \in S, j = 1, \ldots, n \tag{9}$$

the unique principal vectors related to the standard basis $E = \{e_i\}_{i=1}^n$ that satisfy $(v^j)_j = v^j_j > 0$ ($v^j_j = \langle \frac{P_S e_j}{\|P_S e_j\|}, e_j\rangle = \|P_S e_j\|$) and minimize the angle between the subspace S and the one dimensional coordinate axes $\text{span}(\{e_j\})$, that is

$$\langle v^j, e_j\rangle = \max_{s \in S, \|s\|=1} |\langle s, e_j\rangle|.$$

Note that the existence of these principal standard vectors for all j requires that S is a generic subspace (where $P_S e_j \neq 0$ for all j, see Definition 4.1).

To prove the uniqueness of v^j suppose there exists $w \in S$ ($w \neq v^j$) with $\|w\| = 1$ and such that

$$\langle w, e_j \rangle = \max_{s \in S, \|s\|=1} |\langle s, e_j \rangle|.$$

Then $w_j = v^j_j > 0$ and $v^j + w \neq 0$. If we suppose that $\|v^j + w\| = 2$ then it can be proved that $|\langle v^j, w \rangle| \geqslant 1$ and therefore $v^j = \lambda w$ with $|\lambda| = 1$, but since $v^j_j = w_j > 0$ follows that $\lambda = 1$ and then $v^j = w$. Now suppose $\|v^j + w\| < 2$, and define $x = \frac{v^j + w}{\|v^j + w\|} \in S$, then

$$x_j = \langle x, e_j \rangle = \frac{1/2(v^j_j + w_j)}{1/2\|v^j + w\|} > 1/2(v^j_j + w_j) = v^j_j, \tag{10}$$

a contradiction.

Remark 9. We state here some properties of the principal standard vectors of a generic subspace S defined in (9) that follow after direct computations.

a) For every $j = 1, \ldots, n$, $v^j_j = \|P_S(e_j)\| > 0$.
b) If we also denote with P_S the corresponding $n \times n$ matrix in standard coordinates, then its j$^{\text{th}}$-column satisfies $\operatorname{col}_j(P_S) = P_S e_j = v^j_j v^j$ and its j,k entry $(P_S)_{j,k} = v^j_j v^j_k$. In particular $(P_S)_{j,j} = \left(v^j_j\right)^2$.
c) Since the matrix P_S is hermitian then $v^j_j v^j_k = \overline{v^k_k v^k_j} = v^k_k \overline{v^k_j}$ and then $\arg(v^j_k) = -\arg(v^k_j)$.
d) Directly from the previous item follows that if $P_S e_j$ and $P_S e_k$ are not null, then

$$\frac{v^j_k}{v^k_k} = \frac{\overline{v^k_j}}{v^j_j} \text{ and } \frac{\overline{v^j_k}}{v^k_k} = \frac{v^k_j}{v^j_j}. \tag{11}$$

e) Observe that $0 = v^j_k = \langle v^j, e_k \rangle = \langle v^j, P_S e_k \rangle = \|P_S(e_k)\| \langle v^j, v^k \rangle \Leftrightarrow v^j \perp v^k$ or $P_S(e_k) = 0$ or $P_S(e_j) = 0$. Therefore, if S is a generic subspace

$$v^j_k = 0 \Leftrightarrow v^j \perp v^k. \tag{12}$$

Lemma 4.3. *Let S be a generic subspace of $\mathbb{C}^n$ and $w \in S$, with $\|w\| = 1$. Then the following properties hold*

(1) $|v^j_j| = v^j_j \geqslant |w_j|$ *for all* $j = 1, \cdots, n$.
(2) Moreover,

$$v^j_j = |w_j| \ \Leftrightarrow w = e^{i \arg(w_j)} v^j \ , \tag{13}$$

and in particular

(3)

$$v_j^j = |v_j^k| \Leftrightarrow v^k = e^{i\arg(v_j^k)}v^j \Leftrightarrow v^j = e^{i\arg(v_k^j)}v^k \Leftrightarrow |v_i^j| = |v_i^k|, \forall i = 1, \ldots n. \quad (14)$$

(4) As a consequence

$$\{v^j, v^k\} \text{ is linearly independent } \Leftrightarrow v_j^j \neq |v_j^k| \ (v_j^j > |v_j^k|) \Leftrightarrow v_k^k \neq |v_k^j| \ (v_k^k > |v_k^j|). \quad (15)$$

Proof. The first statement is apparent from the definition of v^j. For the second statement, observe that $v_j^j = \|P_S e_j\| > 0$ (see Remark 9). Then w, with $\|w\| = 1$ is a multiple of v^j if and only if $w = e^{i\arg(w_j)}v^j$. Then

$$\begin{aligned} w = e^{i\arg(w_j)}v^j \Leftrightarrow |\langle w, v^j\rangle| = 1 \Leftrightarrow \left|\langle w, \frac{P_S e_j}{\|P_S e_j\|}\rangle\right| = 1 \Leftrightarrow \left|\langle P_S w, \frac{e_j}{v_j^j}\rangle\right| = 1 \Leftrightarrow \\ \Leftrightarrow \frac{|\langle w, e_j\rangle|}{v_j^j} = 1 \Leftrightarrow |w_j| = v_j^j \end{aligned} \quad (16)$$

The statement (14) follows after replacing w from (13) with v^k and applying some of the properties listed in Remark 9. □

Remark 10. Given S a generic subspace and $x \in S$ with $\|x\| = 1$, then its j^{th}-coordinate can be calculated as

$$x_j = \|P_S e_j\|\langle x \ , \ v^j\rangle = v_j^j\langle x \ , \ v^j\rangle \quad (17)$$

This follows since $x_j = \langle x \ , \ e_j\rangle = \langle P_S x \ , \ e_j\rangle = \langle x \ , \ P_S e_j\rangle = \langle x \ , \ \|P_S e_j\| v^j\rangle = \|P_S e_j\|\langle x \ , \ v^j\rangle$. Therefore, $x_j = 0$ if and only if $\langle x \ , \ v^j\rangle = 0$ or $\|P_S e_j\| = 0$, but this second condition cannot happen in a generic space.

Proposition 4.4. *Given a subspace S of $\mathbb{C}^n$ and e_j a member of the standard basis E such that $P_S e_j$ is not null (for example if S is a generic subspace), then $|v^j|^2 = (|v_1^j|^2, |v_2^j|^2, \ldots, |v_n^j|^2)$ is an extreme point in m_S.*

Moreover, if $|v^j|^2$ is a convex combination of $|y|^2$ and $|z|^2$ with $y, z \in S$, then y and z must be multiples of v^j.

Proof. Suppose that there are two vectors $y, z \in S$, with $\|y\| = \|z\| = 1$ such that $|v^j|^2 = t|y|^2 + (1-t)|z|^2$ with $0 \leqslant t \leqslant 1$. Then in particular $|v_j^j|^2 = t|y_j|^2 + (1-t)|z_j|^2$ which implies that $|v_j^j| = v_j^j = |y_j^j| = |z_j^j|$. Then using (13) in Lemma 4.3 we obtain that both y and z must be multiples (by a complex with modulus one) of v^j. This implies that $|v^j|^2 = |y|^2 = |z|^2$. □

5. Curves of extreme points in m_S

Definition 5.1. Let S be a generic subspace and v^j and v^k (defined in 9) be linearly independent. We define a curve $\overset{j \frown k}{v} : [0, \pi/2] \to S$ that starts in v^j and passes through

$e^{i\arg(v^j_k)}v^k$:

$$\overset{j\curvearrowright k}{v}(t) = \cos(t)v^j + \sin(t)\; e^{i\arg(v^j_k)}\frac{(v^k - \langle v^k, v^j\rangle v^j)}{\|v^k - \langle v^k, v^j\rangle v^j\|},\; t \in [0, \pi/2]. \tag{18}$$

Note that $\|\overset{j\curvearrowright k}{v}(t)\| = 1$ and that using basic properties of v^j and v^k (see Remark 9) the j and k coordinates of $\overset{j\curvearrowright k}{v}(t)$ (that we denote with $\overset{j\curvearrowright k}{v_j}(t)$ and $\overset{j\curvearrowright k}{v_k}(t)$) can be computed

$$\overset{j\curvearrowright k}{v_j}(t) = \cos(t)v^j_j\;, \quad \text{and} \quad \overset{j\curvearrowright k}{v_k}(t) = \cos(t)v^j_k + \sin(t)e^{i\arg(v^j_k)}\sqrt{(v^k_k)^2 - |v^j_k|^2}. \tag{19}$$

Remark 11. The curve vectors $\overset{j\curvearrowright k}{v}(t)$ can be obtained by projections of particular linear combinations of the e_j and e_k standard vectors. Namely,

$$\overset{j\curvearrowright k}{v}(t) = P_S\left(\overset{j\curvearrowright k}{\beta}(t)\right),\;\; \text{for}\;\; \overset{j\curvearrowright k}{\beta}(t) = \cos(t)\frac{e_j}{\|P_S(e_j)\|} + \sin(t)\; e^{i\arg(v^j_k)}\frac{\left(\frac{e_k}{\|P_S(e_k)\|} - \langle v^k, v^j\rangle\frac{e_j}{\|P_S(e_j)\|}\right)}{\|v^k - \langle v^k, v^j\rangle v^j\|}. \tag{20}$$

This implies that for each $t \in [0, \pi/2]$, the vector $\overset{j\curvearrowright k}{v}(t)$ attains the minimal angle between S and the one dimensional subspace spanned by $\overset{j\curvearrowright k}{e}(t) = \frac{\overset{j\curvearrowright k}{\beta}(t)}{\|\overset{j\curvearrowright k}{\beta}(t)\|}$

$$\langle\overset{j\curvearrowright k}{v}(t), \overset{j\curvearrowright k}{e}(t)\rangle = \max_{s\in S, \|s\|=1} |\langle s, \overset{j\curvearrowright k}{e}(t)\rangle|.$$

Note that $\overset{j\curvearrowright k}{e}(t)$ is included in $\text{span}\{e_j, e^{i\arg(v^j_k)}\; e_k\} = \text{span}\{e_j, e_k\}$ for all $t \in [0, \pi/2]$.

Moreover, it can be computed that $\langle\overset{j\curvearrowright k}{v}(t), \overset{j\curvearrowright k}{e}(t)\rangle = \left\|P_S\left(\overset{j\curvearrowright k}{e}(t)\right)\right\| = \frac{1}{\|\overset{j\curvearrowright k}{\beta}(t)\|} > 0$ (if zero then v^j and v^k should be linearly dependent), $\overset{j\curvearrowright k}{v}(t) = \frac{P_S\left(\overset{j\curvearrowright k}{e}(t)\right)}{\left\|P_S\left(\overset{j\curvearrowright k}{e}(t)\right)\right\|}$, and $\overset{j\curvearrowright k}{v}(t)$ is unique among the unit vectors $s \in S$ that attain this maximum with the property that $\langle s, \overset{j\curvearrowright k}{e}(t)\rangle > 0$ (this can be proved as done in (10)).

Following the same procedures as those used in Lemma 4.3 for v^j, similar results can be obtained for $\overset{j\curvearrowright k}{v}(t)$ as stated in the next lemma.

Lemma 5.2. *Let S be a generic subspace of $\mathbb{C}^n$ and $w \in S$, with $\|w\| = 1$. Then the following properties hold*

(1) $|\langle\overset{j\curvearrowright k}{v}(t), \overset{j\curvearrowright k}{e}(t)\rangle| = \langle\overset{j\curvearrowright k}{v}(t), \overset{j\curvearrowright k}{e}(t)\rangle \geqslant |\langle w, \overset{j\curvearrowright k}{e}(t)\rangle|$ *for all* $t \in [0, \pi/2]$.

(2) Moreover,

$$\langle\overset{j\curvearrowright k}{v}(t), \overset{j\curvearrowright k}{e}(t)\rangle = |\langle w, \overset{j\curvearrowright k}{e}(t)\rangle| \;\Leftrightarrow w = e^{i\arg(\langle w, \overset{j\curvearrowright k}{e}(t)\rangle)}\; \overset{j\curvearrowright k}{v}(t)\;, \tag{21}$$

(3) *and in particular*

$$
\begin{aligned}
\langle \overset{j\curvearrowright k}{v}(t), \overset{j\curvearrowright k}{e}(t)\rangle = |\langle \overset{j\curvearrowright k}{v}(s_0), \overset{j\curvearrowright k}{e}(t)\rangle| \text{ , for } s_0 \in [0,\pi/2] \Leftrightarrow \\
\Leftrightarrow \overset{j\curvearrowright k}{v}(s_0) = e^{i\arg(\langle \overset{j\curvearrowright k}{v}(s_0), \overset{j\curvearrowright k}{e}(t)\rangle)} \overset{j\curvearrowright k}{v}(t) \\
\Leftrightarrow |\langle \overset{j\curvearrowright k}{v}(t), \overset{j\curvearrowright k}{e}(u)\rangle| = |\langle \overset{j\curvearrowright k}{v}(s_0), \overset{j\curvearrowright k}{e}(u)\rangle| \text{ , } \forall u \in [0,\pi/2].
\end{aligned} \tag{22}
$$

(4) *As a consequence*

$$
\left\{\overset{j\curvearrowright k}{v}(t), \overset{j\curvearrowright k}{v}(s)\right\} \text{ is linearly independent } \Leftrightarrow \langle \overset{j\curvearrowright k}{v}(t), \overset{j\curvearrowright k}{e}(t)\rangle \neq |\langle \overset{j\curvearrowright k}{v}(s), \overset{j\curvearrowright k}{e}(t)\rangle|. \tag{23}
$$

Proposition 5.3. *Let S be a generic subspace of $\mathbb{C}^n$ and v^j and v^k from* (9)*. Consider $\overset{j\curvearrowright k}{v} : [0, \frac{\pi}{2}] \to Im(\overset{j\curvearrowright k}{v}) \subset S$ the curve defined in* (18)*, then:*

(1) *the map $\overset{j\curvearrowright k}{v}$ is bijective,*
(2) *for all $t \in [0, \frac{\pi}{2}]$, holds that*

$$
\langle \overset{j\curvearrowright k}{v}(t), v^j\rangle \in \mathbb{R}_{\geqslant 0} \qquad \text{and} \quad \langle \overset{j\curvearrowright k}{v}(t), e^{i\arg\left(v^j_k\right)} v^k\rangle \in \mathbb{R}_{\geqslant 0} \tag{24}
$$

(3) $\overset{j\curvearrowright k}{v}(0) = v^j$ *and* $\overset{j\curvearrowright k}{v}\left(\arccos\left(|v^j_k|/v^k_k\right)\right) = e^{i\arg(v^j_k)} v^k$.

Proof. (1) To prove the bijectivity of $\overset{j\curvearrowright k}{v}$ it is enough to observe that the j-coordinate of the curve is $\overset{j\curvearrowright k}{v_j}(t) = \cos(t) v^j_j$ (see (19)) which is a strictly decreasing real function from $[0, \frac{\pi}{2}]$ onto $[0, v^j_j]$.

(2) Fix $t \in [0, \frac{\pi}{2}]$ and recall that $\overset{j\curvearrowright k}{v}(t)$ has norm 1 (see (18)). Moreover, the fact that S is a generic subspace implies that $v^j_j, v^k_k \in \mathbb{R}_{>0}$, $e^{i\arg(v^j_k)} v^k_j \in \mathbb{R}_{\geqslant} 0$ and $\arg(v^j_k) = \arg(e^{i\arg(v^j_k)} v^k_k)$.

On the other hand

$$
\arg(\langle v^j, e^{i\arg(v^j_k)} v^k\rangle) = \arg(e^{-i\arg(v^j_k)}\langle v^j, v^k\rangle) = \arg\left(\|P_S e_k\| \langle v^j, e_k\rangle\right) - \arg(v^j_k) = 0
$$

and consequently

$$
v^j_j,\ e^{i\arg(v^j_k)} v^k_j,\ e^{i\arg\left(v^j_k\right)} \frac{\left(v^k_j - \langle v^k, v^j\rangle v^j_j\right)}{\|v^k - \langle v^k, v^j\rangle v^j\|} \in \mathbb{R}_{\geqslant 0} \text{ , hence } \overset{j\curvearrowright k}{v_j}(t) \in \mathbb{R}_{\geqslant 0}. \tag{25}
$$

Similarly

$$
\arg(v^j_k) = \arg(e^{i\arg(v^j_k)} v^k_k) = \arg\left(e^{i\arg\left(v^j_k\right)} \frac{\left(v^k_k - \langle v^k, v^j\rangle v^j_k\right)}{\|v^k - \langle v^k, v^j\rangle v^j\|}\right) = \arg\left(\overset{j\curvearrowright k}{v_k}(t)\right) \tag{26}
$$

The previous equalities (25) and (26) and Remark 10 imply that

$$\langle \overset{j\curvearrowright k}{v}(t), v^j \rangle = \overset{j\curvearrowright k}{v_j}(t)/v_j^j \in \mathbb{R}_{\geqslant 0}$$

and

$$\langle \overset{j\curvearrowright k}{v}(t), e^{i\arg(v_k^j)}v^k \rangle = e^{-i\arg(v_k^j)}\langle \overset{j\curvearrowright k}{v}(t), v^k \rangle \overset{(26)}{=} e^{-i\arg\left(\overset{j\curvearrowright k}{v_k}(t)\right)} \overset{j\curvearrowright k}{v_k}(t)/v_k^k \in \mathbb{R}_{\geqslant 0}.$$

(3) It is trivial that $\overset{j\curvearrowright k}{v}(0) = v^j$. Now if v^j, v^k are linearly independent, then $0 < \|v^k - \langle v_k, v^j\rangle v^j\| = \sqrt{1 - \frac{|v_j^k|^2}{(v_j^j)^2}} = \frac{\sqrt{(v_j^j)^2 - |v_j^k|^2}}{v_j^j} < 1$, and therefore there exists $t_0 \in (0, \pi/2)$ such that

$$\sin(t_0) = \|v^k - \langle v_k, v^j\rangle v^j\| = \frac{\sqrt{(v_j^j)^2 - |v_j^k|^2}}{v_j^j}.$$

Using that $\left(\frac{\sqrt{(v_j^j)^2 - |v_j^k|^2}}{v_j^j}\right)^2 + \left(|v_j^k|/v_j^j\right)^2 = 1$, then $t_0 = \arccos(|v_j^k|/v_j^j) = \arccos(|v_k^j|/v_k^k) \in (0, \pi/2)$ since $|v_j^k|/v_j^j = |v_k^j|/v_k^k < 1$. Otherwise, if $|v_j^k|/v_j^j = 1$, v^j and v^k must be linearly dependent (see (15) in Lemma 4.3).
Evaluating the sine and cosine in that t_0 we obtain:

$$\begin{aligned}
\overset{j\curvearrowright k}{v}(t_0) &= \cos(t_0)v^j + \sin(t_0)\, e^{i\arg(v_k^j)} \frac{\left(v^k - \langle v^k, v^j\rangle v^j\right)}{\|v^k - \langle v^k, v^j\rangle v^j\|} \\
&= \left(|v_k^j|/v_k^k\right) v^j + \|v^k - \langle v^k, v^j\rangle v^j\|\, e^{i\arg(v_k^j)} \frac{\left(v^k - \langle v^k, v^j\rangle v^j\right)}{\|v^k - \langle v^k, v^j\rangle v^j\|} \qquad (27)\\
&= \left(|v_k^j|/v_k^k\right) v^j + e^{i\arg(v_k^j)}\left(v^k - \langle v^k, v^j\rangle v^j\right)
\end{aligned}$$

And using that $e^{i\arg(v_k^j)}\langle v^k, v^j\rangle = e^{i\arg(v_k^j)} v_j^k/v_j^j = |v_j^k|/v_j^j$ we obtain

$$\begin{aligned}
\overset{j\curvearrowright k}{v}(t_0) &= \left(|v_k^j|/v_k^k\right) v^j + e^{i\arg(v_k^j)}\left(v^k - \langle v^k, v^j\rangle v^j\right) = \left(|v_k^j|/v_k^k\right) v^j + e^{i\arg(v_k^j)} v^k - (|v_j^k|/v_j^j) v^j \\
&= e^{i\arg(v_k^j)} v^k.
\end{aligned} \qquad (28)$$

□

Proposition 5.4. *Consider the vectors $\overset{j\curvearrowright k}{v}(t)$ from Definition 5.1, $x \in S$ with $\|x\| = 1$ and $\overline{e_j e_k}$ the segment between e_j and e_k projected to the j and k coordinates (in $\mathbb{R}^2$). Then there exists $t_x \in [0, \pi/2]$ such that*

$$|x_j| \leqslant \left|\overset{j\curvearrowright k}{v_j}(t_x)\right| \quad , \quad |x_k| \leqslant \left|\overset{j\curvearrowright k}{v_k}(t_x)\right| \quad \textit{and} \qquad (29)$$

$$\operatorname{dist}\left(\left(\left|\overset{j\curvearrowright k}{v_j}(t_x)\right|^2, \left|\overset{j\curvearrowright k}{v_k}(t_x)\right|^2\right), \overline{e_j e_k}\right) \leqslant \operatorname{dist}\left((|x_j|^2, |x_k|^2), \overline{e_j e_k}\right). \qquad (30)$$

Proof. In order to alleviate the notation let us write

$$\widetilde{w}^{jk} = e^{i\arg\left(v^j_k\right)} \frac{\left(v^k - \langle v^k, v^j\rangle v^j\right)}{\|v^k - \langle v^k, v^j\rangle v^j\|}. \tag{31}$$

Recall that $\overset{j\frown k}{v_j}(t) = \cos(t)v^j + \sin(t)\widetilde{w}^{jk}$ with $v^j \perp \widetilde{w}^{jk}$ (see (18)). Then we can write $x \in S$ as a linear combination of v^j, $\widetilde{w}^{jk}$ and a vector $y \in S$ with $\|y\| = 1$, orthogonal to the subspace spanned by the other two vectors

$$x = a\ v^j + b\ \widetilde{w}^{jk} + c\ y\ ,\ \text{with } a, b, c \in \mathbb{C}. \tag{32}$$

We will consider two cases: $c = 0$ and $c \neq 0$.

- Case $c = 0$

  Recalling the j^{th} and k^{th} standard coordinates of v^j and $\widetilde{w}^{jk}$ as in (19) we can write (suppose $j < k$):

$$x = a\, v^j + b\, \widetilde{w}^{jk} = \left(\ldots, av^j_j, \ldots, av^j_k + be^{\mathrm{i}\arg(v^j_k)}\sqrt{(v^k_k)^2 - |v^j_k|^2}, \ldots\right).$$

  The orthonormality of $\{v^j, \widetilde{w}^{jk}\}$ and $\|x\| = 1$ implies that $|a|^2 + |b|^2 = 1$. Now define $\alpha = \arccos(|a|) \in [0, \pi/2]$ and consider

$$\overset{j\frown k}{v}(\alpha) = \cos(\alpha)\, v^j + \sin(\alpha)\, \widetilde{w}^{jk}.$$

  Then

$$|x_j| = |a|\, |v^j_j| = \left|\overset{j\frown k}{v_j}(\alpha)\right| \tag{33}$$

  and

$$\begin{aligned} |x_k| &= \left|a\, v^j_k + b\, e^{i\arg(v^j_k)}\sqrt{(v^k_k)^2 - |v^j_k|^2}\right| \\ &\leqslant |a|\, |v^j_k| + |b| \left|\sqrt{(v^k_k)^2 - |v^j_k|^2}\right| = \cos(\alpha)|v^j_k| + \sin(\alpha)\sqrt{(v^k_k)^2 - |v^j_k|^2} \\ &= \left|\overset{j\frown k}{v_k}(\alpha)\right| \end{aligned} \tag{34}$$

  where in the last equality we have used that v^j and $\widetilde{w}^{jk}$ have the same argument in its k^{th} coordinate (see (19)). Therefore, $|x_k| \leqslant \left|\overset{j\frown k}{v_k}(\alpha)\right|$ which together with (33) proves (29) in this case. Then follows that

$$\operatorname{dist}\left(\left(\left|\overset{j\frown k}{v_j}(t_x)\right|^2, \left|\overset{j\frown k}{v_k}(t_x)\right|^2\right), \overline{e_j e_k}\right) \leqslant \operatorname{dist}\left((|x_1|^2, |x_2|^2), \overline{e_j e_k}\right).$$

- Case $c \neq 0$ (and $y \neq 0$)

  Using Remark 10 and the fact that $y \perp \operatorname{span}\{v^j, \widetilde{w}^{jk}\} = \operatorname{span}\{v^j, v^k\}$ it can be proved that $y_j = \|P_S e_j\| \langle y\ ,\ v^j\rangle = 0$ and similarly $y_k = 0$. Then, $x_j = a\, v^j_j + b\, \widetilde{w}^{jk}_j$ and $x_k = a\, v^j_k + b\, \widetilde{w}^{jk}_k$.

If $x_j = 0$ and $x_k = 0$ it is enough to take $t_x = 0$ ($\overset{j\curvearrowright k}{v}(t_x) = v^j$).
Consider now the case when $x_j \neq 0$ or $x_k \neq 0$ and the define the vector $\widehat{x} \in S$ as

$$\widehat{x} = \frac{P_{\mathrm{span}\{v^j,\widetilde{w}^{jk}\}}(x)}{\|P_{\mathrm{span}\{v^j,\widetilde{w}^{jk}\}}(x)\|} = \frac{av^j + b\widetilde{w}^{jk}}{\|av^j + b\widetilde{w}^{jk}\|}.$$

Since $\|av^j + b\widetilde{w}^{jk}\| < \|x\| = 1$ (because $y \perp v^j$, $y \perp \widetilde{w}^{jk}$) and the j^{th} and k^{th} coordinates of $av^j + b\widetilde{w}^{jk}$ and x coincide ($y_j = y_k = 0$), we have that

$$|\widehat{x}_j| \geqslant |x_j| \quad \text{and} \quad |\widehat{x}_k| \geqslant |x_k|. \tag{35}$$

Since $\widehat{x}$ is included in the case already considered when $c = 0$, there exists $t_{\widehat{x}}$ that satisfies (29), (see (33) and (34)). That is, $|\widehat{x}_j| = |\overset{j\curvearrowright k}{v_j}(t_{\widehat{x}})|$ and $|\widehat{x}_k| \leqslant |\overset{j\curvearrowright k}{v_k}(t_{\widehat{x}})|$. Then using (35) we obtain

$$|x_j| \leqslant |\overset{j\curvearrowright k}{v_j}(t_{\widehat{x}})| \quad , \quad |x_k| \leqslant |\overset{j\curvearrowright k}{v_k}(t_{\widehat{x}})|$$

which in turn proves

$$d\left((|\overset{j\curvearrowright k}{v_j}(t_{\widehat{x}})|^2, |\overset{j\curvearrowright k}{v_k}(t_{\widehat{x}})|^2), \overline{e_j e_k}\right) \leqslant d\left((|x_1|^2, |x_2|^2), \overline{e_j e_k}\right).$$

□

Remark 12. There are many different $x \in S$ with $\|x\| = 1$ that satisfy (29) for the same value of t_x (see for example Figure 1).

Theorem 5.5. *Consider the curve* $\overset{j\curvearrowright k}{v} : [0, \pi/2] \to \mathbb{R}^n_{\geqslant 0}$ *from Definition 5.1 and* $x \in S$, *with* $\|x\| = 1$. *Then there exists a unique* $t_x \in [0, \frac{\pi}{2}]$ *such that*

$$|x_j| = |\overset{j\curvearrowright k}{v_j}(t_x)| \quad \textit{and} \quad |x_k| \leqslant |\overset{j\curvearrowright k}{v_k}(t_x)|. \tag{36}$$

Moreover, if $x = a\,v^j + b\,\widetilde{w}^{jk} + c\,y$ *(with* $\widetilde{w}^{jk}$ *as in* (31) *and* $y \perp v^j, \widetilde{w}^{jk}$*), then* $t_x = \arccos(|a|)$.

Proof. Consider first the the existence of t_x.
We continue using the notation $\widetilde{w}^{jk}$ as in (31). As in 32, we write

$$x = a\,v^j + b\,\widetilde{w}^{jk} + c\,y\ , \text{ with } a, b, c \in \mathbb{C} \text{ and } y \perp v^j, \widetilde{w}^{jk}.$$

Now we consider different cases. If $c = 0$ we can choose $t_x = \arccos(|a|)$ as in (33) and (34) in the proof of the previous Proposition 5.4 to obtain the equality and inequality required.
Now consider the case $c \neq 0$ and some sub-cases

- If $a = 0$ then it must be $x_j = 0\ v^j_j + b\ \widetilde{w}^{jk}_j + c\ 0 = 0$ where we have used that $\widetilde{w}^{jk}_j = 0$ (a direct computation) and $y_j = 0$ (see Case $c \neq 0$ in Proposition 5.4). Then we can choose $t_x = \frac{\pi}{2}$, and obtain $|\overset{j\curvearrowright k}{v_j}(\pi/2)| = 0 = |x_j|$.

For the k^{th}-coordinate, $y_k = 0$ and (19) imply
$|x_k| = \left|0\, v_k^j + b\, e^{i\arg(v_k^j)}\sqrt{(v_k^k)^2 - |v_k^j|^2}\right| = \left|b\sqrt{(v_k^k)^2 - |v_k^j|^2}\right| \leqslant |1\, \widetilde{w}_k^{jk}| = |\overset{j\frown k}{v_k}(\tfrac{\pi}{2})|$

- Consider now $a \neq 0$. If we choose $t_x = \arccos(|a|)$, then using again that $\widetilde{w}_j^{jk} = 0$ and $y_j = 0$ we obtain

$$|x_j| = |a\, v_j^j| = \cos(t_x) v_j^j = |\overset{j\frown k}{v_j}(t_x)|.$$

Moreover, since $|a|^2 + |b|^2 + |c|^2 = 1$,

$$\begin{aligned} |x_k| &= \left|a\, v_k^j + b\, e^{i\arg(v_k^j)}\sqrt{(v_k^k)^2 - |v_k^j|^2}\right| \\ &\leqslant \left|a\, v_k^j + \sqrt{1-|a|^2}\, e^{i\arg(v_k^j)}\ \sqrt{(v_k^k)^2 - |v_k^j|^2}\right| = |\overset{j\frown k}{v_k}(t_x)| \end{aligned}$$

which ends the proof of the existence of t_x.
The uniqueness of $t_x \in [0, \pi/2]$ can be proved using (as in the proof of (1) in Proposition 5.3) the bijectivity of the map $t \mapsto \overset{j\frown k}{v_j}(t) = \cos(t) v_j^j$ for $t \in [0, \pi/2]$.

□

Remark 13. The maximality of v^i in its i-coordinate v_i^i (see Lemma 4.3) implies that if $x \in S$ with $\|x\| = 1$ then $|x_j| \in [0, v_j^j]$ and $|x_k| \in [0, v_k^k]$. Then, the projection of m_S to the j and k coordinates is included in the rectangle $[0, (v_j^j)^2] \times [0, (v_k^k)^2]$. In the following results we will show more precise boundaries of m_S.

Theorem 5.6. *Let $S \subset \mathbb{C}^n$ be a generic subspace, $\{v^j, v^k\}$ two linearly independent principal standard vectors, m_S the moment of S as in Definitions 4.2 and 3.1 respectively, and $\gamma_{j,k} : [0, \pi/2] :\to m_S \subset \mathbb{R}^n_{\geqslant 0}$ the curve defined by*

$$\gamma_{j,k}(t) = \left|\overset{j\frown k}{v}(t)\right|^2 = \left(\left|\overset{j\frown k}{v_1}(t)\right|^2, \ldots, \left|\overset{j\frown k}{v_n}(t)\right|^2\right) \tag{37}$$

with $\overset{j\frown k}{v}(t)$ as in Definition 5.1.
Then

1. *the projection of the j^{th} and k^{th} coordinates of $\sqrt{\gamma_{j,k}}$ to $\mathbb{R}^2$ given by $t \mapsto \left(|\overset{j\frown k}{v_j}(t)|, |\overset{j\frown k}{v_k}(t)|\right)$ is part of an ellipse centred at the origin,*
2. *if v^j and v^k are not orthogonal, the points $\left(|\overset{j\frown k}{v_j}(t)|^2, |\overset{j\frown k}{v_k}(t)|^2\right)$ from the projected curve $\gamma_{j,k}$ to the plane spanned by e_j, e_k are all extreme points of the projection of m_S to the same plane,*
3. *in case $v^j \perp v^k$ then $\left(|\overset{j\frown k}{v_j}(t)|^2, |\overset{j\frown k}{v_k}(t)|^2\right)$ parametrizes a segment that is in the boundary of the projection of m_S to the plane spanned by e_j, e_k.*

Proof. Let $P_{j,k} : \mathbb{R}^n \to \mathbb{R}^2$ be the projection $P_{j,k}(x) = (x_j, x_k)$, then the curve

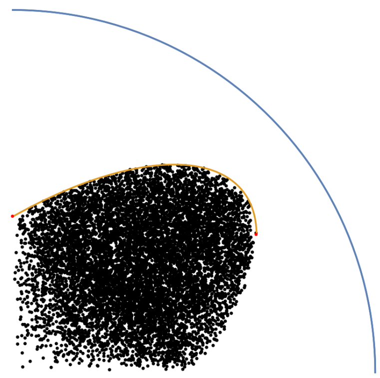

Figure 1. For $j = 1$ and $k = 2$ image of the curve $|\overset{j\frown k}{v}|$ projected on the j and k coordinates for a 3 dimensional subspace S of $\mathbb{C}^9$ (in orange). The blue curve is a quarter of the unit circle and the black dots are projected points of m_S (with the square roots of its entries) taken randomly.

obtained in the plane after the projection of $\sqrt{\gamma_{j,k}(t)}$ is (see (19))

$$\begin{aligned} P_{j,k}\left(\left|\overset{j\frown k}{v}(t)\right|\right) &= \left(\left|\cos(t)v_j^j\right|\ ,\ \left|\cos(t)v_k^j + \sin(t)e^{i\arg\left(v_k^j\right)}\sqrt{(v_k^k)^2 - |v_k^j|^2}\right|\right) \\ &= \left(\cos(t)v_j^j\ ,\ \cos(t)|v_k^j| + \sin(t)\sqrt{(v_k^k)^2 - |v_k^j|^2}\right) \\ &= \cos(t)\ \left(v_j^j, |v_k^j|\right) + \sin(t)\ \left(0, \sqrt{(v_k^k)^2 - |v_k^j|^2}\right) \end{aligned} \tag{38}$$

which is clearly part of an ellipse centred at the origin (recall that $t \in [0, \pi/2]$). It starts in $P_{j,k}(|v^j|)$, passes through $P_{j,k}(|v^k|)$ (see Proposition 5.3 part (3)) and ends in the $\operatorname{span}\{(0,1)\}$ axis. See for example Figure 1.

Now squaring the coordinates of (38) we obtain

$$\begin{aligned} P_{j,k}\left(\left|\overset{j\frown k}{v}(t)\right|\right)^2 &= \Big(\cos^2(t)(v_j^j)^2\ ,\ \cos^2(t)|v_k^j|^2 + \sin^2(t)\left((v_k^k)^2 - |v_k^j|^2\right) + \\ &\qquad + 2\cos(t)|v_k^j|\sin(t)\sqrt{(v_k^k)^2 - |v_k^j|^2}\Big) \\ &= \cos^2(t)\left((v_j^j)^2\ ,\ |v_k^j|^2\right) + \sin^2(t)\left(0, \left((v_k^k)^2 - |v_k^j|^2\right)\right) + \\ &\qquad + \left(0, 2\cos(t)|v_k^j|\sin(t)\sqrt{(v_k^k)^2 - |v_k^j|^2}\right). \end{aligned} \tag{39}$$

This is the parametrization of a segment that joins $\left((v_j^j)^2\ ,\ |v_k^j|^2\right)$ with $\left(0, \left((v_k^k)^2 - |v_k^j|^2\right)\right)$ plus a vertical vector with second coordinate $\geqslant 0$. Note that this second coordinate is zero only if $t = 0$, $t = \pi/2$, $v_k^j = 0$ or $v_k^k = |v_k^j|$. This last case is not possible because otherwise using (15) the vectors v^j and v^k would be linearly dependent and we are supposing in the hypothesis that they are not (see Definition

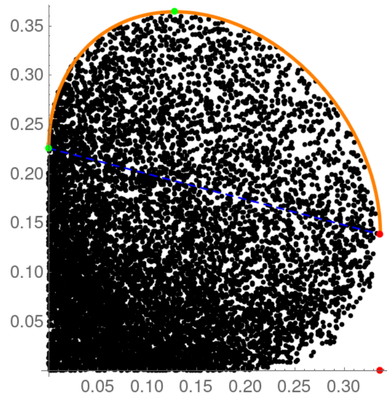

Figure 2. Curve $P_{j,k}\left(|{}^{j}\overrightarrow{v}^{k}(t)|^2\right)$ for $t \in [0, \pi/2]$ in orange, segment between $((v^j_j)^2, |v^j_k|^2)$ and $(0, (v^k_k)^2 - |v^j_k|^2)$ dashed in blue, red dots are $((v^j_j)^2, |v^j_k|^2)$ and $((v^j_j)^2, 0)$, and the green dots are $(0, (v^k_k)^2 - |v^j_k|^2)$ and $(|v^k_j|^2, (v^k_k)^2)$. The black dots are projected random points of m_S.

5.1). Then the curve in (39) is a segment only when $v^j_k = 0$, which is equivalent to

$$0 = v^j_k = \langle v^j, e_k\rangle = \|P_S(e_k)\|\langle v^j, v^k\rangle \Leftrightarrow v^j \perp v^k \tag{40}$$

(1) Case $v^j \not\perp v^k$ (equivalently $v^j_k \neq 0$)

The curve parametrized with $P_{j,k}\left(\left|{}^{j}\overrightarrow{v}^{k}(t)\right|\right)^2$ is the graph of a map $f : [0, (v^j_j)^2] \to \mathbb{R}_{>0}$ (see the proof of (1) in Proposition 5.3). An example of this situation can be seen in Figure 2. The fact that $2\cos(t)|v^j_k|\sin(t)\sqrt{(v^k_k)^2 - |v^j_k|^2}$ is strictly concave for $t \in (0, \pi/2)$ and that the graph of the map f was obtained adding this coordinate to the second coordinate of the mentioned segment implies the concavity of f. Then for $t \in (0, \pi/2)$, if there exists $x, y \in S$, with $\|x\| = \|y\| = 1$, and $0 \leqslant \lambda \leqslant 1$ such that

$$P_{j,k}\left(\left|{}^{j}\overrightarrow{v}^{k}(t)\right|\right)^2 = \lambda(|x_j|^2, |x_k|^2) + (1-\lambda)(|y_j|^2, |y_k|^2)$$

then there exist t_x and t_y such that (36) holds for x and y. But if any of the inequalities in the k^{th} coordinate is strict, then the concavity of the map f is contradicted. Therefore all must be equalities and then $P_{j,k}\left(\left|{}^{j}\overrightarrow{v}^{k}(t)\right|\right)^2$ is an extreme point in $P_{j,k}(m_S)$. This in turn implies that $|{}^{j}\overrightarrow{v}^{k}(t)|^2$ is in the boundary of m_S for $t \in (0, \pi/2)$.

(2) Case $v^j \perp v^k$

As we have seen in (40) this is equivalent to $v^j_k = 0$.

Then $P_{j,k}\left(\left|{}^{j}\overrightarrow{v}^{k}(t)\right|\right)^2 = \cos^2(t)\left((v^j_j)^2\ ,\ |v^j_k|^2\right) + \sin^2(t)\left(0, \left((v^k_k)^2 - |v^j_k|^2\right)\right)$ which is a segment in $P_{j,k}(m_S)$. If there exists $x, y \in S$, with $\|x\| = \|y\| = 1$, and $0 \leqslant \lambda \leqslant 1$ such that

$$P_{j,k}\left(\left|{}^{j}\overrightarrow{v}^{k}(t)\right|\right)^2 = \lambda(|x_j|^2, |x_k|^2) + (1-\lambda)(|y_j|^2, |y_k|^2).$$

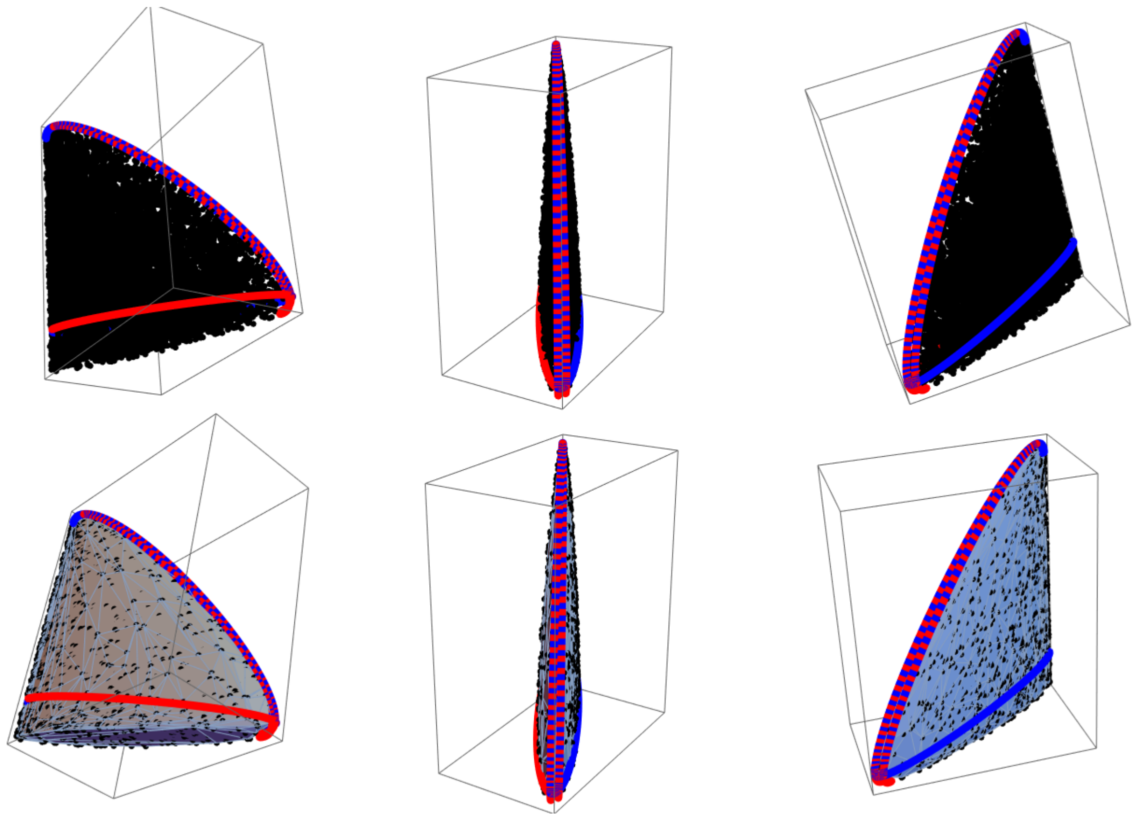

Figure 3. For a subspace $S \subset \mathbb{C}^7$ with $\dim(S) = 3$, this is an approximation (using random points above and its convex hull below) of the projection of m_S to $\mathbb{R}^3_{\geqslant 0}$ in 3 fixed coordinates and the plot of 6 curves $\overset{j\curvearrowright k}{v}$ (see (18) coinciding with the j,k coordinates being projected. Note the overlapping of $\left|\overset{j\curvearrowright k}{v}\right|^2$ and $\left|\overset{k\curvearrowright j}{v}\right|^2$ in some parts (see Theorem 5.8).

we can choose t_x, t_y as in the previous case such that (36) holds. Now if any of the two inequalities given by (36) is strict there must be t_0 such that $P_{j,k}\left(\left|\overset{j\curvearrowright k}{v}(t_0)\right|\right)^2$ is not in the segment, which is a contradiction. Then, all inequalities of (36) are equalities and therefore $P_{j,k}\left(\left|\overset{j\curvearrowright k}{v}(t)\right|\right)^2$ is a boundary point of $P_{j,k}(m_S)$. Then $\left|\overset{j\curvearrowright k}{v}(t)\right|$ is a boundary point of m_S for $t \in (0, \pi/2)$.

□

Theorem 5.7. *Let S be a generic subspace of $\mathbb{C}^n$, v^j, v^k linearly independent principal vectors such that $v^j \not\perp v^k$, m_S the moment of S and the curve $\overset{j\curvearrowright k}{v} : [0, \pi/2] \to S$ from Definition 5.1.*

Then the elements $|\overset{j\curvearrowright k}{v}(t)|^2$ are extremal points of m_S for every $t \in [0, \pi/2)$.

Moreover, if $s|w|^2 + (1-s)|z|^2 = |\overset{j\curvearrowright k}{v}(t_0)|^2$ with $0 < s < 1$, $\|w\| = \|z\| = 1$, $w, z \in S$, $t_0 \in [0, \pi/2)$, then w and z must be multiples of $\overset{j\curvearrowright k}{v}(t_0)$.

Proof. The case $t_0 = 0$ has been proved in Proposition 4.4.

Suppose there exist $w, z \in S$ with $\|w\| = \|z\| = 1$ such that

$$s|w|^2 + (1-s)|z|^2 = \left|\overset{j\curvearrowright k}{v}(t_0)\right|^2 \quad \text{with} \quad 0 < s < 1 \tag{41}$$

Then, since $v^j \not\perp v^k$ using (2) of Theorem 5.6 follows that $|z_j| = \left|\overset{j\curvearrowright k}{v_j}(t_0)\right|$ and $|z_k| = \left|\overset{j\curvearrowright k}{v_k}(t_0)\right|$.

As it was done in 32 under the notation 31, we can write z from (41) as a linear combination of v^j, $\widetilde{w}^{jk}$ and a vector $y \in S$ with $\|y\| = 1$, orthogonal to the subspace $\text{span}\{v^j, \widetilde{w}^{jk}\}$

$$z = a\, v^j + b\, \widetilde{w}^{jk} + c\, y\ , \text{ with } a, b, c \in \mathbb{C}.$$

Recall that as in the proof of Proposition 5.4 follows that $\widetilde{w}^{jk}_j = y_j = 0$ and $y_k = 0$, and then it must be $|a| = \cos(t_0)$. Now observe that

$$\begin{aligned}|z_k| &= \left|\cos(t_0)e^{i\,\arg(a)}\, v^j_k + b\,\widetilde{w}^{jk}_k\right| \leqslant \cos(t_0)\,|v^j_k| + |b|\,\left|\widetilde{w}^{jk}_k\right| \\ &\leqslant \cos(t_0)\,|v^j_k| + \sin(t_0)\,\left|\widetilde{w}^{jk}_k\right| = \left|\overset{j\curvearrowright k}{v_k}(t_0)\right|\end{aligned} \tag{42}$$

where in the last inequality we used that $|z|^2 = \cos^2(t_0) + |b|^2 + |c|^2 = 1$ which implies $|b| \leqslant \sin(t_0)$.

Since we are supposing (41) the extremes of the inequality obtained in (42) are equal and then the complex numbers $\cos(t_0)e^{i\,\arg(a)}\, v^j_k$ and $b\,\widetilde{w}^{jk}_k$ must be collinear. This last statement can only be true if $\arg(a) = \arg(b)$. Moreover, since the inequalities in (42) are equalities, follows that $|b| = \sin(t_0)$, and then $c = 0$, which implies that $a = e^{i\arg(a)}\cos(t_0)$ and $b = e^{i\arg(a)}\sin(t_0)$. Then

$$\begin{aligned}z &= a\,v^j + b\,\widetilde{w}^{jk} = e^{i\arg(a)}\left(\cos(t_0)v^j + \sin(t_0)\ e^{i\arg\left(v^j_k\right)}\frac{\left(v^k_k - \langle v^k, v^j\rangle v^j_k\right)}{\|v^k - \langle v^k, v^j\rangle v^j\|}\right) \\ &= e^{i\,\arg(a)}\overset{j\curvearrowright k}{v}(t_0)\end{aligned} \tag{43}$$

This proves in particular that $|z|^2 = |\overset{j\curvearrowright k}{v}(t_0)|^2$.

Following the same steps it can be proved that w is a multiple of $\overset{j\curvearrowright k}{v}(t_0)$, and that $|w|^2 = |\overset{j\curvearrowright k}{v}(t_0)|^2$, which implies that $|\overset{j\curvearrowright k}{v}(t_0)|^2$ is an extremal point in m_S. $\square$

Theorem 5.8. *Let S be a generic subspace, $1 \leqslant j, k \leqslant n$ with $j \neq k$, and $\overset{j\curvearrowright k}{v}$ the curve from Definition 5.1 but with domain in the interval $\left[0, \arccos\left(\left|v^k_j\right|/v^j_j\right)\right]$. Then*

$$Dom(\overset{j\curvearrowright k}{v}) = Dom(\overset{k\curvearrowright j}{v}) \quad \text{and} \quad Im(|\overset{j\curvearrowright k}{v}|^2) = Im(|\overset{k\curvearrowright j}{v}|^2)$$

(where Dom and Im denote the domain and image of the corresponding curves in $\mathbb{R}^n_{\geqslant 0}$), and for every $t \in \left[0, \arccos(|v^k_j|/v^j_j)\right]$ there exists a unique $s = \left(\arccos(|v^k_j|/v^j_j) - t\right) \in$

$\left[0, \arccos(|v_k^j|/v_k^k)\right]$ *such that*

$$\overset{j\curvearrowright k}{v}(t) = e^{i\arg(v_k^j)}\, \overset{k\curvearrowright j}{v}(s) \tag{44}$$

(see Figure 3).

Proof. The domains of $\overset{k\curvearrowright j}{v}$ and $\overset{j\curvearrowright k}{v}$ are equal since using (11) follows that $\frac{|v_k^j|}{v_k^k} = \frac{|v_j^k|}{v_j^j}$.

First observe that $\overset{j\curvearrowright k}{v}(0) = v^j = e^{\mathrm{i}\arg(v_k^j)}\, \overset{k\curvearrowright j}{v}\left(\arccos\left(|v_k^j|/v_k^k\right)\right)$ and that $\overset{j\curvearrowright k}{v}\left(\arccos\left(|v_j^k|/v_j^j\right)\right) = e^{\mathrm{i}\arg(v_k^j)}\, v^k = e^{\mathrm{i}\arg(v_k^j)}\, \overset{k\curvearrowright j}{v}(0)$.

Moreover, the following statements can be proved directly from the properties of v^j, v^k and $\overset{j\curvearrowright k}{v}$.

- $\arg(v_k^j) = \arg\left(\overline{v_j^k}\right)$
- $\overset{j\curvearrowright k}{v_j}(t) \in \mathbb{R}$, for all $t \in \left[0, \arccos\left(|v_j^k|/v_j^j\right)\right]$
- $\arg\left(\overset{j\curvearrowright k}{v_k}(t)\right) = \arg(v_k^j)$, for all $t \in \left[0, \arccos\left(|v_j^k|/v_j^j\right)\right]$
- $e^{\mathrm{i}\arg(v_k^j)}\, \overset{k\curvearrowright j}{v_j}(s) \in \mathbb{R}$, for all $s \in \left[0, \arccos\left(|v_k^j|/v_k^k\right)\right]$
- $\arg\left(\overset{k\curvearrowright j}{v_j}(s)\right) = -\arg(v_k^j)$, for all $s \in \left[0, \arccos\left(|v_k^j|/v_k^k\right)\right]$

Then for every t and s in the interval $\left[0, \arccos\left(\left|v_j^k\right|/v_j^j\right)\right] = \left[0, \arccos\left(\left|v_k^j\right|/v_k^k\right)\right]$ the following arguments coincide

$$\arg\left(\overset{j\curvearrowright k}{v_j}(t)\right) = \arg\left(e^{\mathrm{i}\arg(v_k^j)}\, \overset{k\curvearrowright j}{v_j}(s)\right) \quad \text{and} \quad \arg\left(\overset{j\curvearrowright k}{v_k}(t)\right) = \arg\left(e^{\mathrm{i}\arg(v_k^j)}\, \overset{k\curvearrowright j}{v_k}(s)\right). \tag{45}$$

The real map $\overset{j\curvearrowright k}{v_j}(t) = \cos(t) v_j^j$ is decreasing for $t \in \left[0, \arccos(|v_k^j|/v_j^j)\right]$ and using (19) we can write

$$\left|\overset{j\curvearrowright k}{v_k}(t)\right| = \left|\cos(t) v_k^j + \sin(t) e^{i\arg\left(v_k^j\right)} \sqrt{(v_k^k)^2 - |v_k^j|^2}\right| = \cos(t)\left|v_k^j\right| + \sin(t)\sqrt{(v_k^k)^2 - |v_k^j|^2}.$$

This map has only one critical point (maximum) in $[0, \frac{\pi}{2}]$ when $t = \arccos\left(|v_j^k|/v_j^j\right) = \arccos\left(|v_k^j|/v_k^k\right)$, and therefore is increasing in $\left[0, \arccos(|v_k^j|/v_k^k)\right]$.

Similarly $\overset{k\curvearrowright j}{v_k}(s)$ is decreasing in $\left[0, \arccos(|v_k^j|/v_k^k)\right]$, while $\left|\overset{k\curvearrowright j}{v_j}(s)\right|$ is increasing in such interval (see for example Figure 1).

Then if $0 \leqslant s_0 < s_1 \leqslant \arccos\left(|v_k^j|/v_k^k\right) = \arccos\left(|v_j^k|/v_j^j\right)$

$$\left|\overset{k\curvearrowright j}{v_j}(s_0)\right| < \left|\overset{k\curvearrowright j}{v_j}(s_1)\right| \quad \text{and} \quad \overset{k\curvearrowright j}{v_k}(s_0) > \overset{k\curvearrowright j}{v_k}(s_1) \quad \text{hold.} \tag{46}$$

Now given a fixed $s_0 \in [0, \arccos(|v_k^j|/v_k^k)]$ if we apply Proposition 5.4 to $x = \overset{k\curvearrowright j}{v}(s_0) \in$

S we can find $t_0 \in [0, \arccos(|v_j^k|/v_j^j)]$ such that

$$|\overset{k\curvearrowright j}{v}_j(s_0)| \leqslant |\overset{j\curvearrowright k}{v}_j(t_0)| \quad \text{and} \quad |\overset{k\curvearrowright j}{v}_k(s_0)| \leqslant |\overset{j\curvearrowright k}{v}_k(t_0)|. \tag{47}$$

Then applying again Proposition 5.4 for the vector $\overset{j\curvearrowright k}{v}(t_0) \in S$, we can find $s_1 \in [0, \arccos(|v_k^j|/v_k^k)]$ such that

$$|\overset{j\curvearrowright k}{v}_j(t_0)| \leqslant |\overset{k\curvearrowright j}{v}_j(s_1)| \quad \text{and} \quad |\overset{j\curvearrowright k}{v}_k(t_0)| \leqslant |\overset{k\curvearrowright j}{v}_k(s_1)| \tag{48}$$

Now considering (46), equations 47 and 48 imply that $s_0 = s_1$, and then from 45 we obtain that

$$e^{\mathrm{i}\arg(v_k^j)}\, \overset{k\curvearrowright j}{v}_j(s_0) = \overset{j\curvearrowright k}{v}_j(t_0) \quad \text{and} \quad e^{\mathrm{i}\arg(v_k^j)}\, \overset{k\curvearrowright j}{v}_k(s_0) = \overset{j\curvearrowright k}{v}_k(t_0).$$

The vectors $e^{\mathrm{i}\arg(v_k^j)}\overset{k\curvearrowright j}{v}(s_0)$ and $\overset{j\curvearrowright k}{v}(t_0)$ are equal in their j^{th} and k^{th} coordinates in a dimension 2 subspace spanned by the vectors v^j and $v^k \in S$. Then, for each $z \in \mathrm{gen}\{v^j, v^k\}$, if we consider the linear system $\alpha v^j + \beta e^{i\arg(v_k^j)} v^k = \begin{pmatrix} v^j \\ e^{i\arg(v_k^j)} v^k \end{pmatrix}^t \cdot \left(\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right) = z$, we know that it has a unique solution $\left(\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right) \in \mathbb{C}^2$ since v^j and $e^{i\arg(v_k^j)} v^k$ are linearly independent. Considering only the j^{th} and k^{th} coordinates of the linear system we can conclude that if $\det \begin{pmatrix} v_j^j & v_j^k \\ v_k^j & v_k^k \end{pmatrix} \neq 0$, then $e^{\mathrm{i}\arg(v_k^j)}\overset{k\curvearrowright j}{v}(s_0) = \overset{j\curvearrowright k}{v}(t_0)$. But the case $\det \begin{pmatrix} v_j^j & v_j^k \\ v_k^j & v_k^k \end{pmatrix} = 0$ is never possible. Otherwise $v_j^j v_k^k - v_k^j v_j^k = v_j^j v_k^k - |v_k^j|\,|v_j^k| = 0$ (recall that $\arg(v_k^j) = -\arg(v_j^k)$). Then using Lemma 4.3 we obtain that $v_j^j = |v_j^k|$ and $v_k^k = |v_k^j|$, and then that $v^j = e^{i\arg(v_k^j)} v^k$, a contradiction. Therefore for every $s_0 \in [0, \arccos(|v_k^j|/v_k^k)]$ there exists a unique $t_0 \in [0, \arccos(|v_j^k|/v_j^j)]$ such that

$$e^{\mathrm{i}\arg(v_k^j)}\overset{k\curvearrowright j}{v}(s_0) = \overset{j\curvearrowright k}{v}(t_0). \tag{49}$$

To prove the formula (44) recall that the domains of $\overset{k\curvearrowright j}{v}$ and $\overset{j\curvearrowright k}{v}$ are equal. Now take $t \in [0, \arccos(|v_j^k|/v_j^j)]$ and define $s = \arccos\left(|v_j^k|/v_j^j\right) - t$. Then

$$\begin{aligned} \cos(t) &= \cos(\arccos(|v_j^k|/v_j^j) - s) = \cos(\arccos(|v_j^k|/v_j^j))\cos(s) + \sin(\arccos(|v_j^k|/v_j^j))\sin(s) \\ &= \left(|v_j^k|/v_j^j\right)\cos(s) + \sqrt{1 - \left(|v_j^k|/v_j^j\right)^2}\,\sin(s) = \frac{|v_j^k|\cos(s) + \sqrt{(v_j^j)^2 - |v_j^k|^2}\,\sin(s)}{v_j^j} \end{aligned} \tag{50}$$

Therefore

$$\begin{aligned} \overset{j\curvearrowright k}{v}_j(t) &= \cos(t) v_j^j = |v_j^k|\cos(s) + \sqrt{(v_j^j)^2 - |v_j^k|^2}\,\sin(s) \\ &= \left| v_j^k\,\cos(s) + \sqrt{(v_j^j)^2 - |v_j^k|^2}\, e^{\mathrm{i}\arg(v_j^k)}\sin(s) \right| = e^{\mathrm{i}\arg(v_k^j)}\overset{k\curvearrowright j}{v}_j(s) \end{aligned} \tag{51}$$

The proof for the k^{th} coordinate is similar.

As was seen previously in the proof of (49) this equality in the j and k coordinates is enough to prove the formula (44). □

6. Relation of m_S with the joint algebraic numerical range

We will consider here the relation of the moment $m_S = m_{S,E}$ (see Definition 3.1) of a subspace S related to a fixed orthonormal basis E with the *joint numerical range*[3, 6,7]). The joint numerical range of m hermitian matrices $A_1, A_2, \ldots, A_m \in M_n^h(\mathbb{C})$, sometimes called the joint algebraic numerical range [8], is defined by

$$W(A_1, A_2, \ldots, A_m) = \{(\operatorname{tr}(A_1\rho), \operatorname{tr}(A_2\rho), \ldots, \operatorname{tr}(A_m\rho)) \in \mathbb{R}^n : \rho \in \mathcal{D}\} \tag{52}$$

with $\mathcal{D} = \{\rho \in M_n^h(\mathbb{C}) : \rho \geqslant 0, \operatorname{tr}(\rho) = 1\}$ the set of density matrices of $M_n(\mathbb{C})$ and tr the usual trace (sum of diagonal entries). This set is the convex hull of the also called joint numerical range in the literature (that we will denote classic joint numerical range)

$$\begin{aligned} W_{\text{class}}(A_1, A_2, \ldots, A_m) &= \\ &= \{(\langle A_1x, x\rangle, \langle A_2x, x\rangle, \ldots, \langle A_mx, x\rangle) \in \mathbb{R}^n : x \in \mathbb{C}^n, \|x\| = 1\} \end{aligned} \tag{53}$$

that is not necessarily convex in general. In the case W_{class} is convex then the equality $W = W_{\text{class}}$ holds.

More precisely, we will see that m_S can be described as a particular subset of a joint numerical range of some selected hermitian matrices (see Theorem 6.3).

Recall that one of the equivalent definitions of m_S seen in (1) of Proposition 3.2 is considering the following identification of the vectors of m_S with the diagonal entries of certain matrices

$$m_{S,E} \simeq \operatorname{Diag}\{Y \in M_n^h(\mathbb{C}) : Y \geqslant 0, \operatorname{tr}(Y) = 1, \operatorname{Im}(Y) \subset S\}.$$

Since these Y are hermitian matrices the condition $\operatorname{Im}(Y) \subset S$ is equivalent to $Y = P_SY = YP_S = P_SYP_S$. This allow us to rewrite the moment set m_S as

$$m_S \simeq \operatorname{Diag}\Big(\{Y \in M_n^h(\mathbb{C}) : Y \in \mathcal{D}_S\}\Big) \tag{54}$$

where $\mathcal{D}_S$

$$\begin{aligned} \mathcal{D}_S &= \Big\{Y \in M_n^h(\mathbb{C}) : Y \geqslant 0, \operatorname{tr}(Y) = 1, P_SY = Y (= YP_S)\Big\} \\ &= \{Y \in \mathcal{D} : Y = P_SY = YP_S = P_SYP_S\}\,. \end{aligned} \tag{55}$$

Lemma 6.1. *Consider $\mathcal{D}_S$ as in (55) and $0 \subsetneq S \subsetneq \mathbb{C}^n$ a subspace of $\mathbb{C}^n$. Then*

$$\mathcal{D}_S = \{\rho \in \mathcal{D} : tr(P_S\rho P_S) = 1\}\,.$$

Proof. Observe that

$$
\begin{aligned}
\rho \in \mathcal{D}_S &\Leftrightarrow \rho \geqslant 0, \mathrm{tr}(\rho) = 1, \rho = P_S\rho = \rho P_S = P_S\rho P_S \\
&\Leftrightarrow \rho \geqslant 0, \mathrm{tr}(\rho) = 1, \rho = \frac{1}{\mathrm{tr}(P_S\rho P_S)} P_S\rho P_S \\
&\Leftrightarrow \rho \geqslant 0, \mathrm{tr}(P_S\rho P_S) \neq 0, \rho = \frac{1}{\mathrm{tr}(P_S\rho P_S)} P_S\rho P_S
\end{aligned}
$$

and then

$$
\mathcal{D}_S = \left\{ \rho \in \mathcal{D} : \mathrm{tr}(P_S\rho P_S) \neq 0, \rho = \frac{1}{\mathrm{tr}(P_S\rho P_S)} P_S\rho P_S \right\} \tag{56}
$$

which implies that $\mathcal{D}_S \subset \{\rho \in \mathcal{D} : \mathrm{tr}(P_S\rho P_S) = 1\}$.

Now consider a $\rho \in \mathcal{D}$ with a spectral decomposition $\rho = \sum_{i=1}^k p_i \; x_i \cdot (x_i)^*$ with $\sum_{i=1}^k p_i = 1$, $p_i > 0$ and $x_i \in \mathbb{C}^n$, $\|x_i\| = 1$, $x_i \perp x_j$ (if $i \neq j$). Then the property $\mathrm{tr}(P_S\rho P_S) = 1$ implies that $1 = \sum_{i=1}^k p_i \; \mathrm{tr}\left(P_S x_i \cdot (x_i)^* P_S\right) = \sum_{i=1}^k p_i \; \mathrm{tr}\left(P_S x_i \cdot (P_S x_i)^*\right)$. Then it must be $\mathrm{tr}(P_S x_i \cdot (P_S x_i)^*) = 1$ for all $i = 1, \ldots, k$ which in turn implies that $\|P_S x_i\| = 1$ and therefore $x_i \in S$ and $P_S \, \rho = \rho = \rho^* = \rho \, P_S = P_S \, \rho \, P_S$. We have obtained that

$$
\{\rho \in \mathcal{D} : \mathrm{tr}(P_S\rho P_S) = 1\} \subset \mathcal{D}_S. \tag{57}
$$

which concludes the proof. □

Notation 2. Consider a fixed basis $E = \{e_1, e_2, \ldots, e_n\}$ of $\mathbb{C}^n$. The rank one orthogonal projections onto the subspace generated by a single $e_i \in E$ are described in the same E basis by the $n \times n$ matrices $E_i = e_i \cdot (e_i)^*$. Here $\cdot$ denotes the matrix product of the column vector e_i with the row vector $(e_i)^*$(conjugate transpose of e_i). In this case, the coordinates of the vector e_i are zeros with the exception of a 1 in its i^{th} coordinate and E_i is a $n \times n$ matrix of zeros and only a 1 in its i, i entry. In this case, the E_i projections are also denoted with $e_i \otimes e_i$.

Lemma 6.2. *If $\{0\} \subsetneq S \subsetneq \mathbb{C}^n$ is a subspace of $\mathbb{C}^n$ and $\mathcal{D}_S$ is as in* (55)*, then*

$$
m_S = \{(tr(E_1Y), tr(E_2Y), \ldots, tr(E_nY)) \in \mathbb{R}^n_{\geqslant 0} : Y \in \mathcal{D}_S\}
$$

where we denote with $E_i = e_i \otimes e_i$ (see Notation 2) and tr is the usual trace.

Proof. Let $Y \in \mathcal{D}_S$ and a generic element $\mathrm{Diag}(Y) \in m_S$ (see (54) and (55)) under the identification $\simeq$ between diagonals of positive definite matrices and vectors in $\mathbb{R}^n_{\geqslant 0}$. Then, if $E_i = e_i \otimes e_i$, the proof follows observing that

$$
\begin{aligned}
m_S \ni \mathrm{Diag}(Y) &\simeq (\mathrm{tr}(E_1YE_1), \mathrm{tr}(E_2YE_2), \ldots, \mathrm{tr}(E_nYE_n)) \\
&= (\mathrm{tr}(E_1Y), \mathrm{tr}(E_2Y), \ldots, \mathrm{tr}(E_nY))
\end{aligned}
$$

□

The next result shows the relation between m_S and a particular joint numerical range.

Theorem 6.3. *Let $E = \{e_i\}_{i=1}^n$ be a fixed orthonormal basis, m_S the moment of a subspace S with $\{0\} \subsetneq S \subsetneq \mathbb{C}^n$ related to E (see (3.1)), P_S the orthogonal projection onto S, $E_i = e_i \otimes e_i$ the rank one orthogonal projections onto the subspace generated by $e_i \in E$ (see details in Notation 2) and $W(A_1, A_2, \ldots, A_n)$ the joint numerical range of the hermitian matrices $A_1, A_2, \ldots, A_n \in M_n^h(\mathbb{C})$ (see (52)). Then*

$$m_S = W\left(P_S E_1 P_S, P_S E_2 P_S, \ldots, P_S E_n P_S\right) \bigcap \left\{x \in \mathbb{R}^n : \sum_i x_i = 1\right\}. \tag{58}$$

Proof. Suppose $x \in m_S$. Then using Lemma 6.2 we can find $Y \in \mathcal{D}_S$ such that $x = (\operatorname{tr}(E_1 Y), \operatorname{tr}(E_2 Y), \ldots, \operatorname{tr}(E_n Y))$. Then if we consider that $Y = Y P_S = P_S Y = Y P_S Y$ follows that

$$\begin{aligned} x &= (\operatorname{tr}(E_1 P_S Y P_S), \operatorname{tr}(E_2 P_S Y P_S), \ldots, \operatorname{tr}(E_1 P_S Y P_S)) \\ &= (\operatorname{tr}(P_S E_1 P_S Y), \operatorname{tr}(P_S E_2 P_S Y), \ldots, \operatorname{tr}(P_S E_1 P_S Y)). \end{aligned} \tag{59}$$

Then $Y \in \mathcal{D}_S \subset \mathcal{D}$ and (59) imply that $x \in W\left(P_S E_1 P_S, P_S E_2 P_S, \ldots, P_S E_1 P_S\right)$. Moreover, if $x = (x_1, x_2, \ldots, x_n)$ then

$$\sum_{i=1}^n x_i = \sum_{i=1}^n \operatorname{tr}(E_i Y) = \operatorname{tr}\left(\sum_{i=1}^n E_i Y\right) = \operatorname{tr}\left(Y \sum_{i=1}^n E_i\right) = \operatorname{tr}(Y I) = \operatorname{tr}(Y) = 1$$

which proves that m_S is included in $W\left(P_S E_1 P_S, P_S E_2 P_S, \ldots, P_S E_1 P_S\right)$ and its coordinates add to one.

Let us prove the other inclusion. If $x \in W\left(P_S E_1 P_S, P_S E_2 P_S, \ldots, P_S E_1 P_S\right)$ and its coordinates add to one, then there exists $\rho \in \mathcal{D}$ such that

$$\begin{aligned} x &= (\operatorname{tr}(P_S E_1 P_S \rho), \operatorname{tr}(P_S E_2 P_S \rho), \ldots, \operatorname{tr}(P_S E_n P_S \rho)) \\ &= (\operatorname{tr}(E_1 P_S \rho P_S), \operatorname{tr}(E_2 P_S \rho P_S), \ldots, \operatorname{tr}(E_n P_S \rho P_S)) \end{aligned} \tag{60}$$

and $1 = \sum_{i=1}^n \operatorname{tr}(E_i P_S \rho P_S) = \operatorname{tr}(P_S \rho P_S \sum_{i=1}^n E_i) = \operatorname{tr}(P_S \rho P_S)$. Then if we consider $Y = P_S \rho P_S$ it is apparent that $Y \in \mathcal{D}_S$ and therefore by Lemma 6.2 follows that $x = (\operatorname{tr}(E_1 Y), \operatorname{tr}(E_2 Y), \ldots, \operatorname{tr}(E_n Y)) \in m_S$. □

The following result is apparent after considering that $W_{\text{class}} = W$ if and only if W_{class} is convex.

Corollary 6.4. *Let us suppose that S satisfies the assumptions of the previous Theorem 6.3.*

Then if $W_{class}\left(P_S E_1 P_S, P_S E_2 P_S, \ldots, P_S E_n P_S\right)$ is convex,

$$m_S = W_{class}\left(P_S E_1 P_S, P_S E_2 P_S, \ldots, P_S E_n P_S\right) \bigcap \left\{x \in \mathbb{R}^n : \sum_i x_i = 1\right\}$$

and

$$cone\,(m_S) = cone\left(W_{class}\left(P_S E_1 P_S, P_S E_2 P_S, \ldots, P_S E_n P_S\right)\right).$$

Moreover, m_S and $W(P_SE_1P_S, P_SE_2P_S, \ldots, P_SE_nP_S)$ generate the same positive cone as showed in the following lemma (see (3) for a definition of positive cone).

Proposition 6.5. *Let $\{0\} \subsetneq S \subsetneq \mathbb{C}^n$ be a subspace of $\mathbb{C}^n$, then*

$$\textit{cone}\,(m_S) = \textit{cone}\,(W\,(P_SE_1P_S, P_SE_2P_S, \ldots, P_SE_nP_S))$$

where we use the definition of positive cone generated by a set given in (3).

Proof. The inclusion

$$\text{cone}\,(m_S) \subset \text{cone}\,(W\,(P_SE_1P_S, \ldots, P_SE_nP_S))$$

is obvious since using Theorem 6.3 follows that $m_S \subset W(P_SE_1P_S, \ldots, P_SE_nP_S)$.

Note that the null vector belongs to both cones. Suppose that $c \in \text{cone}\,(W\,(P_SE_1P_S, \ldots, P_SE_nP_S))$ and $c \neq (0, 0, \ldots, 0)$. Then there exists $t > 0$, $\rho \in \mathcal{D}$ such that

$$c = t\,(\text{tr}(P_SE_1P_S\rho), \text{tr}(P_SE_2P_S\rho), \ldots, \text{tr}(P_SE_nP_S\rho))$$

Now since $c \neq (0, 0, \ldots, 0)$ and $c_i = \text{tr}(P_SE_iP_S\rho) = \text{tr}(\rho^{1/2}P_SE_iP_S\rho^{1/2}) \geqslant 0$ then $\sum_{i=1}^n \text{tr}(P_SE_iP_S\rho) = \text{tr}\,(P_S\rho P_S \sum_{i=1}^n E_i) = \text{tr}(P_S\rho P_S) > 0$. Let us denote with $\tau = \text{tr}(P_S\rho P_S)$ and observe that

$$c = t\,\tau\,(\text{tr}\,(E_1(1/\tau)P_S\rho P_S)\,, \text{tr}\,(E_2(1/\tau)P_S\rho P_S)\,, \ldots, \text{tr}\,(E_n(1/\tau)P_S\rho P_S)) \qquad (61)$$

with $(1/\tau)P_S\rho P_S \geqslant 0$ and $\text{tr}\,((1/\tau)P_S\rho P_S) = 1$. This implies that $(1/\tau)P_S\rho P_S \in \mathcal{D}_S$ and therefore c is a positive multiple of an element of m_S (see (61) and Lemma 6.2). ☐

The following result gives a bound of how close are two moments sets of subspaces when their respective projections are close in norm.

Proposition 6.6. *Consider a pair of subspaces S and V of $\mathbb{C}^n$ and the notations of $\mathcal{D}_S$ as in* (55)*, m_S, E, E_i for $i = 1, \ldots, n$ and the joint numerical range W as in Theorem 6.3. Then, if $\|P_S - P_V\| < \frac{1}{2n}$,*

$$\begin{aligned} \text{dist}_H(m_S, m_V) &= \max\left\{\sup_{x\in m_S} \text{dist}(x, m_V),\ \sup_{z\in m_V} \text{dist}(z, m_S)\right\} \\ &\leqslant (2\sqrt{n} + 1)\|P_S - P_V\| \end{aligned} \qquad (62)$$

where dist_H denotes the Hausdorff distance of sets.

Proof. We will first prove that if $x \in m_S$ then $\text{dist}(x, m_V) < (2\sqrt{n} + 1)\|P_S - P_V\|$.

Using Lemma 6.2 if $x \in m_S$ then there exists $Y \in \mathcal{D}_S$ ($Y \geqslant 0$, $\text{tr}(Y) = 1$, $P_SY = Y$) such that $x = (\text{tr}(E_1Y), \ldots, \text{tr}(E_nY))$.

Now consider the vector $(\text{tr}(E_1P_VYP_V), \ldots, \text{tr}(E_nP_VYP_V))$ and observe that

$$\sum_{i=1}^n \text{tr}(E_iP_VYP_V) = \sum_{i=1}^n (P_VYP_V)_{i,i} = \text{tr}(P_VYP_V) \geqslant 0.$$

But that sum cannot be zero. Otherwise, note that $\mathrm{tr}(P_V Y P_V) = 0$ if and only if $P_V Y P_V = 0$ (recall that $Y \geqslant 0$). Then using the assumption that $\|P_S - P_V\| < \frac{1}{2n}$ follows that

$$\|P_S Y P_S - P_V Y P_V\| \leqslant \|P_S Y (P_S - P_V)\| + \|(P_S - P_V) Y P_V\| \leqslant 2\|P_S - P_V\| < \frac{1}{n}.$$

This would imply that $\|P_S Y P_S - 0\| = \|Y\| < 1/n$ which contradicts the assumptions that $Y \geqslant 0$ and $\mathrm{tr}(Y) = 1$.

Using again that $\mathrm{tr}(Y) = 1$

$$\begin{aligned}|\mathrm{tr}(P_V Y P_V) - \overbrace{\mathrm{tr}(P_S Y P_S)}^{1}| &\leqslant |\mathrm{tr}(P_V Y) - \mathrm{tr}(P_S Y)| \leqslant |\mathrm{tr}\left((P_V - P_S)Y\right)| \\ &\leqslant \|P_S - P_V\|\end{aligned} \tag{63}$$

which implies that $1 - \|P_S - P_V\| \leqslant \mathrm{tr}(P_V Y P_V) \leqslant 1 + \|P_S - P_V\|$ and

$$\begin{aligned}|\mathrm{tr}(E_i P_V Y P_V) - \mathrm{tr}(E_i P_S Y P_S)| &= |\mathrm{tr}\left(E_i (P_V Y P_V - P_S Y P_S)\right)| \\ &\leqslant \mathrm{tr}(E_i)\|P_V Y P_V - P_S Y P_S\| \leqslant 2\|P_S - P_V\|.\end{aligned} \tag{64}$$

It is clear that $\frac{1}{\mathrm{tr}(P_V Y P_V)} P_V Y P_V \in \mathcal{D}_{\mathbb{V}}$ and therefore the vector $z = \left(\frac{\mathrm{tr}(E_1 P_V Y P_V)}{\mathrm{tr}(P_V Y P_V)}, \ldots, \frac{\mathrm{tr}(E_n P_V Y P_V)}{\mathrm{tr}(P_V Y P_V)}\right)$ belongs to m_V (see Lemma 6.2 and Theorem 6.3). Then

$$\begin{aligned}\|x - z\| &\leqslant \|x - \mathrm{tr}(P_V Y P_V) z\| + \|\mathrm{tr}(P_V Y P_V) z - z\| \\ &\leqslant \left(\sum_{i=1}^{n} |\mathrm{tr}(E_i (P_S Y P_S - P_V Y P_V))|^2\right)^{1/2} + |\mathrm{tr}(P_V Y P_V) - 1|\,\|z\| \\ &\leqslant 2\sqrt{n}\|P_S - P_V\| + \|P_S - P_V\| = (2\sqrt{n} + 1)\|P_S - P_V\|\end{aligned}$$

where in the last inequality we used (64), (63) and the fact that $\|z\| \leqslant 1$ because its coordinates are positive and add to one.

Then we have proved that for every $x \in m_S$ there exists a $z \in m_V$ such that $\|x - z\| \leqslant (2\sqrt{n} + 1)\|P_S - P_V\|$. Therefore $\mathrm{dist}(x, m_V) \leqslant (2\sqrt{n} + 1)\|P_S - P_V\|$. We can argue in a similar way to prove that for every $z \in m_V$ holds $\mathrm{dist}(z, m_V) \leqslant (2\sqrt{n} + 1)\|P_S - P_V\|$. Then we have proved that

$$\mathrm{dist}_H(m_S, m_V) = \max\left\{\sup_{x \in m_S} \mathrm{dist}(x, m_V), \sup_{z \in m_V} \mathrm{dist}(z, m_S)\right\} \leqslant (2\sqrt{n} + 1)\|P_S - P_V\|$$

□

Remark 14. A reciprocal of the previous proposition does not hold. There exist orthogonal subspaces $S = \mathrm{gen}(x)$ and $V = \mathrm{gen}(\overline{x})$ generated by a vector $x \in \mathbb{C}^n$ and an orthogonal $\overline{x}$ obtained after conjugating the coordinates of x such that $m_S = \{|x|^2\} = m_V$ and $\|P_S - P_V\| = \sqrt{2}$.

The following are basic observations that describe elements of $W\left(P_S E_1 P_S, P_S E_2 P_S, \ldots, P_S E_n P_S\right)$. These can be obtained directly using properties of the trace, the projections and the principal standard vectors.

Remark 15. Let S be a subspace of $\mathbb{C}^n$, $\rho \in \mathcal{D}$, $E_i = e_i \otimes e_i$ for $i = 1, \ldots, n$, $\{v^i\}_{i=1}^n$ the principal standard vectors and denote with

$$\Delta(\rho) = (\operatorname{tr}(P_S E_1 P_S \rho), \operatorname{tr}(P_S E_2 P_S \rho), \ldots, \operatorname{tr}(P_S E_j P_S \rho), \ldots, \operatorname{tr}(P_S E_n P_S \rho))$$

the elements of $W(P_S E_1 P_S, P_S E_2 P_S, \ldots, P_S E_n P_S)$ corresponding to $\rho \in \mathcal{D}$. Then the following statements hold.

(1) If $\rho \in \mathcal{D}$, then

$$\Delta(\rho) = ((P_S \rho P_S)_{1,1}, (P_S \rho P_S)_{2,2}, \ldots, (P_S \rho P_S)_{n,n}).$$

(2) If $\rho \in \mathcal{D}_S$ (see (55)), then

$$\Delta(\rho) = (\rho_{1,1}, \rho_{2,2}, \ldots, \rho_{n,n}).$$

(3) If $x \in \mathbb{C}^n$ with $\|x\| = 1$, then $x \otimes x \in \mathcal{D}$ and

$$\Delta(x \otimes x) = \left(|(P_S x)_1|^2, |(P_S x)_2|^2, \ldots, |(P_S x)_n|^2\right)$$

(4) If $x^j \in \mathbb{C}^n$ with $\|x^j\| = 1$, for $j = 1, \ldots, k$, such that $\rho = \sum_{j=1}^k t_j (x^j \otimes x^j) \in \mathcal{D}$, then

$$\Delta(\rho) = \sum_{j=1}^k t_j \Delta(x^j \otimes x^j) = \sum_{j=1}^k t_j \left(|(P_S x^j)_1|^2, |(P_S x^j)_2|^2, \ldots, |(P_S x^j)_n|^2\right)$$

(5) If $x \in \mathbb{C}^n$ with $\|x\| = 1$, $x \otimes x \in \mathcal{D}$ and if $P_S = U I_r U^*$ is a spectral decomposition of P_S ($r = \dim(S)$, I_r a $n \times n$ matrix with 1 in the first r entries of its diagonal and zero elsewhere, and U a unitary whose columns are orthonormal eigenvectors of P_S, with the first r belonging to S), then

$$\begin{aligned}
\operatorname{tr}(I_r U^* E_i U I_r (x \otimes x)) &= \left|\left\langle (\overline{U_{1,i}}, \overline{U_{2,i}}, \ldots, \overline{U_{r,i}}), (x_1, x_2, \ldots, x_r) \right\rangle\right|^2 \\
&= |\langle \operatorname{row}_i(U^*), (x_1, x_2, \ldots, x_r, 0, \ldots, 0) \rangle|^2 \\
&= |\langle \operatorname{col}_i(U), (\overline{x_1}, \overline{x_2}, \ldots, \overline{x_r}, 0, \ldots, 0) \rangle|^2 \ , \text{for all } i = 1, \ldots, n.
\end{aligned}$$

Therefore, the vectors

$$\Delta(x \otimes x) = \left| U^* \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right|^2$$

are elements of $W(P_S E_1 P_S, \ldots, P_S E_n P_S)$ after using the unitary invariance of the joint numerical range. Moreover, the convex hull of these elements equals this joint numerical range.

(6) Consider $P_S = UI_rU^*$ as in the previous item, and$\rho = U^*\mu U \in \mathcal{D}$ for $\mu \in \mathcal{D}$ (any $\rho \in \mathcal{D}$ can be written in this format). Then

$$\begin{aligned}\Delta(\rho) &= (\langle I_r\mu I_r\,(Ue_1)\,I_r, I_r(Ue_1)I_r\rangle, \ldots, \ldots, \langle I_r\mu I_r\,(Ue_n)\,I_r, I_r(Ue_n)I_r\rangle)\\ &= \left(\left(\overline{U_{1,1}}, \ldots, \overline{U_{r,1}} \right) \cdot \begin{pmatrix} \mu_{1,1} & \cdots & \mu_{1,r} \\ \vdots & \ddots & \vdots \\ \mu_{r,1} & \cdots & \mu_{r,r} \end{pmatrix} \cdot \begin{pmatrix} U_{1,1} \\ \vdots \\ U_{r,1} \end{pmatrix}, \ldots, \right. \\ &\qquad \left. \ldots \left(\overline{U_{1,n}}, \ldots, \overline{U_{r,n}} \right) \cdot \begin{pmatrix} \mu_{1,1} & \cdots & \mu_{1,r} \\ \vdots & \ddots & \vdots \\ \mu_{r,1} & \cdots & \mu_{r,r} \end{pmatrix} \cdot \begin{pmatrix} U_{1,n} \\ \vdots \\ U_{r,n} \end{pmatrix} \right)\end{aligned}$$

(7) If e_i is a member of the standard basis E and we consider $e_i \otimes e_i \in \mathcal{D}$, then

$$\Delta(e_i \otimes e_i) = \left((v_1^1)^2 |v_i^1|^2, \ldots, (v_j^j)^2 |v_i^j|^2, \ldots, (v_n^n)^2 |v_i^n|^2 \right).$$

(8) If $s \in S$ with $\|s\| = 1$, then $s \otimes s \in \mathcal{D}_S \subset \mathcal{D}$ and

$$\Delta(s \otimes s) = \left(|s_1|^2, |s_2|^2, \ldots, |s_n|^2 \right).$$

(9) If $v^j \in S$ for $j = 1, \ldots, n$ is a principal standard vector (see Definition 4.2), then $v^j \otimes v^j \in \mathcal{D}_S$ and

$$\Delta(v^j \otimes v^j) = \left(|v_1^j|^2, |v_2^j|^2, \ldots, |v_n^j|^2 \right).$$

Remark 16. Let us consider now some properties related to the joint numerical range of these particular matrices. Here S is a $\dim(S) = r$ subspace of $\mathbb{C}^n$.

(1) Using basic properties of joint numerical ranges follows that if S is a generic subspace

$$W\left(P_S E_1 P_S, , \ldots, P_S E_n P_S\right) = \begin{pmatrix} (v_1^1)^2 & 0 & \cdots & 0 \\ 0 & (v_2^2)^2 & 0 & \vdots \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & (v_n^n)^2 \end{pmatrix} \cdot \Big(W\left(v^1 \otimes v^1, v^2 \otimes v^2, \ldots, v^n \otimes v^n\right) \Big)$$

which provides a way to write the first joint numerical range in terms of another involving only rank one projections related to the principal standard vectors of S (see 4.2).

(2) Denote with $W_{V,B} = W\left(P_V B_1 P_V, P_V B_2 P_V, \ldots, P_V B_n P_V\right)$, for V a subspace of $\mathbb{C}^n$, B an ordered basis of $\mathbb{C}^n$ and B_i the projection onto its i^{th} vector. If we consider a unitary matrix $U \in M_n(\mathbb{C})$, the subspace $U(S)$ and the basis $U \cdot E$ whose elements are the columns of U, then

$$\begin{aligned} W_{S,E} &= W\left(P_S E_1 P_S, , \ldots, P_S E_n P_S\right) \\ &= W\left(UP_SU^*UE_1U^*UP_SU^*, \ldots, UP_SU^*UE_nU^*UP_SU^*\right) \\ &= W_{U(S)\,,\,U\cdot E}. \end{aligned}$$

This follows using that $P_{U(S)} = UP_SU^*$ and the unitary invariance of the joint numerical range W.

(3) Consider a unitary matrix U such that $UP_SU^* = I_r$, with I_r the diagonal matrix with r ones at the beginning and zeroes afterwards The first r columns of U^*, that we denote $\{s^i\}_{i=1}^r$, form an orthonormal basis of S, and the $n-r$ other $o^1, \dots, o^{n-r}$ columns of U^* form an orthonormal basis of $S^\perp$. Then $U(S) = \text{gen}\,\{e_i\}_{i=1}^r$ and $U \cdot E = B = \{U \cdot e_i\}_{i=1}^n$. Then using the previous item

$$W_{S,E} = W_{\text{gen}\{e_i\}_{i=1}^r,\{U\cdot e_i\}_{i=1}^n} = W(I_rUe_1 \otimes I_rUe_1\,,\, I_rUe_2 \otimes I_rUe_2\,,\, \dots, I_rUe_n \otimes I_rUe_n)$$

which describes $W_{S,E}$ using only the first r coordinates of the vectors $\{Ue_i\}_{i=1}^n$.

The following results describe some relations between these joint numerical ranges and minimal hermitian matrices.

Theorem 6.7. *Under the previous assumptions and notations of Theorem 6.3, given $\mathbb{V}, \mathbb{W}$ non trivial subspaces of $\mathbb{C}^n$, the following statements are equivalent*

(1) $m_{\mathbb{V}} \cap m_{\mathbb{W}} \neq \varnothing$,
(2) $W(P_SE_1P_S, \dots, P_SE_nP_S) \cap W(P_{\mathbb{W}}E_1P_{\mathbb{W}}, \dots, P_{\mathbb{W}}E_nP_{\mathbb{W}}) \neq \{0\}$ *(there exists a non-zero vector in the intersection), and*
(3) The matrix $\lambda(P_{\mathbb{V}} - P_{\mathbb{W}}) + R$ *is a minimal hermitian matrix for* $R \in M_n^h(\mathbb{C})$ *such that* $RP_{\mathbb{V}} = RP_{\mathbb{W}} = 0$ *and* $\|R\| < \lambda$.

Proof. (1) $\Rightarrow$ (2). This is apparent, since $m_{\mathbb{V}} \subset W(P_{\mathbb{V}}E_1P_{\mathbb{V}}, \dots, P_{\mathbb{V}}E_nP_{\mathbb{V}})$ and $m_{\mathbb{W}} \subset W(P_{\mathbb{W}}E_1P_{\mathbb{W}}, \dots, P_{\mathbb{W}}E_nP_{\mathbb{W}})$ (see (58)) and these moments are not empty and do not have the null vector.

(2) $\Rightarrow$ (1). If assumption (2) holds then there exist $\rho, \mu \in \mathcal{D}$ such that

$$(\text{tr}(P_{\mathbb{V}}E_1P_{\mathbb{V}}\rho), \dots, \text{tr}(P_{\mathbb{V}}E_nP_{\mathbb{V}}\rho)) = (\text{tr}(P_{\mathbb{W}}E_1P_{\mathbb{W}}\mu), \dots, \text{tr}(P_{\mathbb{W}}E_nP_{\mathbb{W}}\mu)) \neq 0$$

and then

$$(\text{tr}(E_1P_{\mathbb{V}}\rho P_{\mathbb{V}}), \dots, \text{tr}(E_nP_{\mathbb{V}}\rho P_{\mathbb{V}})) = (\text{tr}(E_1P_{\mathbb{W}}\mu P_{\mathbb{W}}), \dots, \text{tr}(E_nP_{\mathbb{W}}\mu P_{\mathbb{W}}))\,. \tag{65}$$

Then since $P_{\mathbb{V}}\rho P_{\mathbb{V}} \geqslant 0$ and $P_{\mathbb{W}}\mu P_{\mathbb{W}} \geqslant 0$ and non zero

$$\text{tr}(P_{\mathbb{V}}\rho P_{\mathbb{V}}) = \sum_{i=1}^n \text{tr}(E_iP_{\mathbb{V}}\rho P_{\mathbb{V}}) = \sum_{i=1}^n \text{tr}(E_iP_{\mathbb{V}}\mu P_{\mathbb{V}}) = \text{tr}(P_{\mathbb{W}}\mu P_{\mathbb{W}}) \neq 0 \tag{66}$$

holds.

Now define

$$Y = \frac{1}{\text{tr}(P_{\mathbb{V}}\rho P_{\mathbb{V}})}P_{\mathbb{V}}\rho P_{\mathbb{V}} \quad \text{and} \quad X = \frac{1}{\text{tr}(P_{\mathbb{W}}\mu P_{\mathbb{W}})}P_{\mathbb{W}}\mu P_{\mathbb{W}}.$$

It is apparent that $Y \geqslant 0$, $P_{\mathbb{V}}Y = Y$, $\text{tr}(Y) = 1$ and therefore $Y \in \mathcal{D}_{\mathbb{V}}$. Similarly $X \in \mathcal{D}_{\mathbb{W}}$.

Then considering (65) and (66) follows that $\Phi(Y) = \Phi(X)$ which in turn implies that $m_{\mathbb{V}} \cap m_{\mathbb{W}} \neq \varnothing$.

(3) $\Leftrightarrow$ (1) This has been already proved elsewhere, see Remark 2 for details. □

Note that if $\mathbb{V}$ and $\mathbb{W}$ are orthogonal and satisfy any of the equivalent statements of the previous theorem then they form a support, the main object of study in [4] (see for example Theorem 3 in that work).

Corollary 6.8. *Let M be a hermitian matrix with $\|M\| = \lambda_{max} = -\lambda_{min}$, $\mathbb{V} = Eig_{\|M\|}$ and $\mathbb{W} = Eig_{-\|M\|}$ the eigenspaces of the eigenvalues $\pm\|M\|$. Then the following statements are equivalent*

(1) $(\mathbb{V}, \mathbb{W})$ *is a support*
(2) $m_{\mathbb{V}} \cap m_{\mathbb{W}} \neq \varnothing$
(3) M *is a minimal matrix*
(4) there exists a non-zero vector in the set $W(P_{\mathbb{V}}E_1P_{\mathbb{V}}, \ldots, P_{\mathbb{V}}E_nP_{\mathbb{V}}) \cap W(P_{\mathbb{W}}E_1P_{\mathbb{W}}, \ldots, P_{\mathbb{W}}E_nP_{\mathbb{W}})$.

Proof. The equivalence follows after considering the definition of a support (see [4]), Theorem 6.7 and the equivalences between (2) and (4) in Theorem 3 of [4]. □

Next we will write just W and W_{class} to denote the sets defined in (52) and (53) respectively.

The previous results in this section related to the joint numerical ranges $W(P_SE_1P_S, \ldots, P_SE_nP_S) \subset (\mathbb{R}_{\geqslant 0})^n$ show that they can be used to detect minimal hermitian matrices in a very similar way we used the moment set m_S. Nevertheless, the precise description of joint numerical ranges is not an easy task even in low dimensions.

For example, the classical joint numerical range W_{class} of (53) of any triple of hermitian $n \times n$ matrices is convex if $n > 2$. Here we are interested in n matrices of $n \times n$ then this property leave us only with the case of 3×3 matrices. The equivalent situation for $S \in \mathbb{C}^3$ where the convex hull in Definition 3.1 is not required was settled in [1].

In general, the convexity of W_{class} is an open problem for n-tuples of matrices when $n > 3$ (see [9,10]). Similarly, in the case of subspaces S with $\dim(S) > 3$ there exist non convex sets $\{|v|^2 \in \mathbb{R}^n_{\geqslant 0} : v \in S \wedge \|v\| = 1\}$ that even have not empty interior (see Remark 4 in [5] and [4]).

Since the set W_{class} may be significantly simpler than W, any positive result in these direction gives an easier characterization of m_S (using Corollary 6.4).

In [11] a detailed classification of the possible joint numerical ranges of 3 matrices of 3×3 is developed. In this case W is a three-dimensional oval in which every one dimensional face is a segment and every two dimensional face is a filled ellipse. A characterization of only ten configurations of these segments and ellipses are possible.

We include here some general and particular properties of joint numerical ranges found in the bibliography that provide information about m_S.

Remark 17. Let $\{0\} \subsetneq S \subsetneq \mathbb{C}^n$ be a subspace of $\mathbb{C}^n$, P_S its orthogonal projection and $E_i = e_i \cdot (e_i)^t = e_i \otimes e_i \in M_n^h(\mathbb{C})$, with e_i for $i = 1, \ldots, n$ the standard vectors of a fixed basis E as defined before.

Then, using the bibliography in joint numerical ranges, the following properties regarding W (see (52)), its classical version W_{class} (see (53)), $W_{S,E} = W(P_SE_1P_S, \ldots, P_SE_nP_S)$ and $m_{S,E}$ (see (3.1)) hold.

a) In [12] the authors presented an example where the analogous of the Kippenhahn boundary generating curve of the numerical range $W(A_1 + iA_2)$ for $A_1, A_2 \in M_n^h$ does not hold in the case of W of three 3×3 hermitian matrices. They consider an algebraic variety in the projective space CP^m and obtain a boundary generating hypersurface that, under some hypothesis, allow the detection of elements of W for any m and n (see Theorem 2.6 in that work).

b) A complete generalization of the so called boundary generating curve of Kippenhahn for joint numerical ranges of hermitian matrices of any size was developed in [3]. In that work it is proved that W is the convex hull of a semi-algebraic set $T^{\sim} \subset W$ that contains the exposed points of W and hence its extreme points. This subset $T^{\sim}$ is the euclidean closure of the union of the dual varieties of the regular points of the irreducible components of a hypersurface related to the zeroes of $\det(x_0 I + x_1 P_S E_1 P_S + \cdots + x_n P_S E_n P_S) \in \mathbb{R}[x_0, x_1, \ldots, x_n]$. Moreover, it is proved that $T^{\sim}$ contains the Zariski closure of the set of extreme points of W (see Remarks 1.3 4) and 4.15 of [3]). Nevertheless, $T^{\sim}$ is not necessarily contained in the boundary of W.

c) $W_{S,E}$ and $m_{S,E}$ are semi-algebraic sets of $\mathbb{R}^n$. This follows using Theorem 1.2 and Remark 1.3 3) of [3] and Theorem 6.3.

d) The study of the problem of finding all the unitary matrices U such that que $m_{U(S)} = m_S$ (see Example 3.3) or of the subspaces $\mathbb{V}, \mathbb{W}$ such that $m_{\mathbb{V}} = m_{\mathbb{W}}$ is closely related to the same problems in terms of the corresponding joint numerical ranges $W_{\mathbb{V},E}$ and $W_{\mathbb{W},E}$.

e) The work [13] describes which matrices can produce a given $W_{\text{class}}(A_1, \ldots, A_m)$ for $A_i \in M_n^h$, $i = 1, \ldots, m$. More precisely, under what conditions on m, n and/or the shape of $W_{\text{class}}(A_1, \ldots, A_m)$ the m-tuple $A_1, \ldots, A_m$ can be restored from it up to unitary similarity. The cases considered were $m = 2$, $n > 2$ and $n = m = 3$.

f) $W_{\text{class}}(A_1, A_2)$ is always convex and so it is $W_{\text{class}}(A_1, A_2, A_3)$ for $n \geqslant 3$ (see [9, 10]). Nevertheless, the convexity of W_{class} seems to be unanswered in the general theory of joint numerical ranges at least for big m.

g) There exist some standard numerical algorithms to generate or approximate the boundary of W. See for example the one detailed in page 6 of [15].

As seen in the item (1) of Remark 16, given a generic subspace S, the set $W_{S,E}$ can be easily described in terms of a joint numerical range of projections of rank one or as in and (3) of the same remark in terms of the coordinate subspace $\{e_i\}_{i=1}^r$ and matrices of rank one that are zero outside the $r \times r$ first block. The authors did not find any reference to studies of the joint numerical range of these particular n-tuple of $n \times n$ matrices.

References

[1] Andruchow, Esteban; Mata-Lorenzo, Luis E.; Mendoza, Alberto; Recht, Lázaro; Varela, Alejandro. Minimal matrices and the corresponding minimal curves on flag manifolds in low dimension. Linear Algebra Appl. 430 (2009), no. 8-9, 1906–1928.

[2] Durán, C.E., Mata-Lorenzo, L.E. and Recht, L., Metric geometry in homogeneous spaces of the unitary group of a C*-algebra: Part I–minimal curves, Adv. Math. 184 no. 2 (2004), 342-366.

[3] Plaumann, D., Sinn, R., Weis, S. Kippenhahn's Theorem for joint numerical ranges and quantum states. SIAM Journal on Applied Algebra and Geometry, vol. 5 no 1 (2021), p. 86–113.

[4] Mendoza, A., Recht, L., Varela, A. (2020). Supports for minimal hermitian matrices. Linear Algebra and its Applications, 584, 458–482.

[5] Andruchow, Esteban; Larotonda, Gabriel; Recht, Lázaro; Varela, Alejandro. A

characterization of minimal Hermitian matrices. Linear Algebra Appl., 436 (2012) 2366–2374.

[6] Bonsall, F. F. and Duncan, J. Numerical Ranges of Operators on Normed Spaces and of Elements of Normed Algebras, Cambridge University Press, London, 1971.

[7] Weis, S. On a theorem by Kippenhahn. Preprint, arXiv:1705.00935v1 [math.AG].

[8] Müller, V. The joint essential numerical range, compact perturbations, and the Olsen problem, Studia Mathematica, 197 (2010), 275–290.

[9] Gutkin, E., Jonckheere, E. A., Karow, M. Convexity of the joint numerical range: topological and differential geometric viewpoints, Linear Algebra and its Applications 376 (2004), 143-171.

[10] Li, C.-K. and Poon, Y.-T. Convexity of the joint numerical range, SIAM Journal on Matrix Analysis and Applications, 21 (2000), 668–678.

[11] Szymański, Konrad, Stephan Weis, and Karol Życzkowski. Classification of joint numerical ranges of three hermitian matrices of size three. Linear algebra and its applications, 545 (2018), 148–173.

[12] Chien, M. T. and Nakazato, H. Joint numerical range and its generating hypersurface. Linear algebra and its applications, 432 no 1 (2010), 173-179.

[13] Krupnik, N., and Spitkovsky, I.M. Sets of matrices with given joint numerical range. Linear algebra and its applications vol. 419, no 2-3 (2006), p. 569-585.

[14] Keeler, D. S., Rodman, L., Spitkovsky, I. M. The numerical range of 3×3 matrices. Linear algebra and its applications, 252 (1-3) (1997), 115-139.

[15] Szymański, K. Uncertainty relations and joint numerical ranges, MSc. thesis, preprint (2017) arXiv:1707.03464v1 [quant-ph].