

The set of partial isometries as a quotient Finsler space

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Abstract

A known general program, designed to endow the quotient space $\mathcal{U}_\mathcal{A}/\mathcal{U}_\mathcal{B}$ of the unitary groups $\mathcal{U}_\mathcal{A}$, $\mathcal{U}_\mathcal{B}$ of the C^* algebras $\mathcal{B} \subset \mathcal{A}$ with an invariant Finsler metric, is applied to obtain a metric for the space $\mathcal{I}(\mathcal{H})$ of partial isometries of a Hilbert space \mathcal{H} . $\mathcal{I}(\mathcal{H})$ is a quotient of the unitary group of $\mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H})$, where $\mathcal{B}(\mathcal{H})$ is the algebra of bounded linear operators in \mathcal{H} . Under this program, the solution of a linear best approximation problem leads to the computation of minimal geodesics in the quotient space. We find solutions of this best approximation problem, and study properties of the minimal geodesics obtained.

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1 Introduction

Let \mathcal{H} be a complex Hilbert space, a *partial isometry* V in \mathcal{H} is an operator which is an isometry $V : \mathcal{S}_i \rightarrow \mathcal{S}_f$ between two closed subspaces $\mathcal{S}_i, \mathcal{S}_f \subset \mathcal{H}$ (called *initial* and *final* subspaces of V , respectively), and is zero on \mathcal{S}_i^\perp . Algebraically, this is equivalent to $VV^*V = V$, and in this case V^*V and VV^* are the orthogonal projections onto the spaces \mathcal{S}_i and \mathcal{S}_f . Denote by

$$\mathcal{I}(\mathcal{H}) = \{\text{partial isometries in } \mathcal{H}\}.$$

The geometry of this set was thoroughly studied. Starting with Halmos and Mc Laughlin [14], who characterized the connected components. Later on, other papers appeared studying geometric or topological aspects of the set of partial isometries, for instance: [16], [18], [19], [1], [2], [4], [8], [9].

Perhaps the main feature of $\mathcal{I}(\mathcal{H})$ is the left action of the group $\mathcal{U}(\mathcal{H}) \times \mathcal{U}(\mathcal{H})$, where $\mathcal{U}(\mathcal{H})$ denotes the unitary group of \mathcal{H} :

$$(U, W) \cdot V = UVW^*, \quad (U, W) \in \mathcal{U}(\mathcal{H}) \times \mathcal{U}(\mathcal{H}), \quad V \in \mathcal{I}(\mathcal{H}). \quad (1)$$

The purpose of this note is to apply the program by C. Durán, L. Mata-Lorenzo and L. Recht [13], devised for the study of curves of minimal length in quotient spaces of the group of unitary

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elements in a C^* -algebra, to the space $\mathcal{I}(\mathcal{H})$. Indeed, $\mathcal{I}(\mathcal{H})$ is a quotient of the unitary group of the C^* -algebra $\mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H})$. The program in [13] proceeds (roughly) as follows. If M admits the transitive action of the unitary group $\mathcal{U}_{\mathcal{A}}$ of the C^* -algebra \mathcal{A} , then M can be regarded as a quotient $\mathcal{U}_{\mathcal{A}}/G$ for G a Banach-Lie subgroup of $\mathcal{U}_{\mathcal{A}}$. The program requires that G be the unitary group of a unital sub- C^* -algebra $\mathcal{B} \subset \mathcal{A}$ (though this requirement can be sometimes relaxed or bypassed, as it is done here). Therefore, the tangent spaces TM are naturally isomorphic to the quotient of the (real) Banach spaces $\mathcal{A}_h/\mathcal{B}_h$, where $\mathcal{B}_h \subset \mathcal{A}_h$ denote the sets of selfadjoint elements of \mathcal{B} and \mathcal{A} , respectively. Using this isomorphism, Durán, Mata-Lorenzo and Recht [13] endowed TM with quotient metric of $\mathcal{A}_h/\mathcal{B}_h$. The metric, by design, is invariant under the action of $\mathcal{U}_{\mathcal{A}}$ on M . Therefore, tangent vectors can be lifted to selfadjoint elements in \mathcal{A} : the norm of such a vector is given by the infimum of the norms (measured in \mathcal{A}) of all possible liftings. Their main result states that if $m \in M$ and $v \in (TM)_m$ are given, and one can find a lifting $x_0 \in \mathcal{A}_h$ of v , whose norm $\|x_0\|$ attains the infimum of all possible liftings of v , then the curve obtained as the uniparametric subgroup e^{itx_0} acting on m , which at $t = 0$ passes through m with velocity v , has minimal length for this metric, at least for time $|t| \leq \frac{\pi}{2\|x_0\|}$. Such liftings x_0 are called *minimal liftings*, their existence is not guaranteed, and even when they do exist, their characterization is an interesting problem, even in the case of finite dimensional algebras (i.e., matrix algebras): see for instance [5], [7], [15]. This problem is also related with non-commutative C^* -metrics and Leibniz seminorms [22]. An important background to the present work are the papers by E. Chiumiento [8], [9]. In these papers, quotient metrics and minimal liftings are studied in the orbits of partial isometries under the action of the so called restricted groups of unitaries (i.e., unitaries which are of the form $1 + K$, for K in an operator ideal).

Therefore, in dealing with particular examples, as is the case here, the focus is on the computation of such minimal liftings.

Another antecedent of this approach can be found in [3], where the space of isometries was studied, though not with the quotient norm considered here

The contents of this note are the following. In Section 2 we state the basic facts on the space $\mathcal{I}(\mathcal{H})$ and the action of $\mathcal{U}(\mathcal{H}) \times \mathcal{U}(\mathcal{H})$. In Section 3 we present an embedding of $\mathcal{I}(\mathcal{H})$ into the manifold of selfadjoint elements ϵ of $\mathcal{H} \times \mathcal{H}$ which satisfy $\epsilon^3 = \epsilon$, i.e., are operators of the form $\epsilon = E_+ - E_-$, with E_+, E_- mutually orthogonal projections. In Section 4 we recall from [13] the program of Durán, Mata-Lorenzo and Recht. Following these ideas, in Section 5 we introduce Finsler metrics in $\mathcal{I}(\mathcal{H})$ and in the space of the operators ϵ described in Section 3; with these metrics, the embedding of Section 3 is isometric. We show that the Finsler norm in $T\mathcal{I}(\mathcal{H})$ is equivalent to the usual operator norm in $\mathcal{B}(\mathcal{H})$. We also prove the main theorem of this note: that curves obtained by the method of [13] are not only minimal in $\mathcal{I}(\mathcal{H})$, but also minimal in the bigger manifold of the operators ϵ of Section 3. In Section 6 we consider the initial and final projections maps

$$\alpha(V) = V^*V, \quad \omega(V) = VV^*.$$

It is shown that if the set of projections is considered with its natural Finsler metric (see [11]), both maps are distance decreasing.

2 Preliminaries

Let us recall the basic facts of the space $\mathcal{I}(\mathcal{H})$ and the action (1):

Remark 2.1. Let $V, V_0, V_1, V_2 \in \mathcal{I}(\mathcal{H})$.

1. The connected components of $\mathcal{I}(\mathcal{H})$ are parametrized by three non negative integers $\leq +\infty$:

$$r(V) = \dim R(V), \quad n(V) = \dim N(V) \quad \text{and} \quad r^\perp(V) = \dim R(V)^\perp.$$

Namely, $V_1, V_2 \in \mathcal{I}(\mathcal{H})$ belong to the same connected component if and only if $r(V_1) = r(V_2)$, $n(V_1) = n(V_2)$ and $r^\perp(V_1) = r^\perp(V_2)$ (see [14]).

2. If $\|V_1 - V_2\| < 1$, then V_1 and V_2 lie in the same connected component of $\mathcal{I}(\mathcal{H})$ see [14].
3. These components coincide with the orbits of the action (1): V_1, V_2 lie in the same component if and only if there exist $U, W \in \mathcal{U}(\mathcal{H})$ such that $UV_1W^* = V_2$.
4. More recently, in [1] we considered the set of partial isometries of a C^* -algebra, as a homogeneous manifold. Each connected component / orbit, is a C^∞ complemented submanifold of the algebra. Back to the case when the algebra is $\mathcal{B}(\mathcal{H})$, if $\mathcal{I}(\mathcal{H})_{V_0}$ denotes the connected component of V_0 , then the map

$$\pi_{V_0} : \mathcal{U}(\mathcal{H}) \times \mathcal{U}(\mathcal{H}) \rightarrow \mathcal{I}(\mathcal{H})_{V_0}, \quad \pi_{V_0}(U, W) = UV_0W^* \quad (2)$$

is a C^∞ -submersion (see [1]). Note then that the whole space $\mathcal{I}(\mathcal{H})$ is a discrete union of complemented submanifolds (any two different components lying at distance of at least 1), and therefore is itself a complemented submanifold of $\mathcal{B}(\mathcal{H})$.

5. Given V_0 , the subgroup of elements in $\mathcal{U}(\mathcal{H}) \times \mathcal{U}(\mathcal{H})$ which fix the element V_0 , usually called the *isotropy subgroup* of V_0 , is given by

$$I_{V_0} = \{(G, H) \in \mathcal{U}(\mathcal{H}) \times \mathcal{U}(\mathcal{H}) : GV_0 = V_0H\}. \quad (3)$$

It is a C^∞ Banach-Lie group, whose Banach-Lie algebra is

$$\iota_{V_0} = \{(iX, iY) \in \mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H}) : X^* = X, Y^* = Y \text{ and } XV_0 = V_0Y\}. \quad (4)$$

$V \in \mathcal{I}(\mathcal{H})$ is called *balanced* if $n(V) = r^\perp(V)$. This type of partial isometries comprise full connected components of $\mathcal{I}(\mathcal{H})$. Note that $V \in \mathcal{I}(\mathcal{H})$ is balanced if and only if there exists an orthogonal projection P and unitaries U, W such that $V = UPW^*$. Indeed, if $V = UPW^*$, then V and P lie in the same connected component, and therefore, by Halmos-McLaughlin's characterization, $n(V) = n(P) = r^\perp(P) = r^\perp(V)$. The converse statement is clear.

Non-unitary isometries are examples of non balanced partial isometries.

The purpose of this note, is to introduce a natural invariant Finsler metric in $\mathcal{I}(\mathcal{H})$. That is, a metric $|\cdot|_V$ in each tangent space $(T\mathcal{I}(\mathcal{H}))_V$, which is invariant under the action of $\mathcal{U}(\mathcal{H}) \times \mathcal{U}(\mathcal{H})$: if $\mathcal{V} \in (T\mathcal{I}(\mathcal{H}))_V$ and $U, W \in \mathcal{U}(\mathcal{H})$, then

$$|\mathcal{V}|_V = |U\mathcal{V}W^*|_{UVW^*}.$$

Here, note that the action is linear: for fixed $U, W \in \mathcal{U}(\mathcal{H})$, the map $V \mapsto UVW^*$ is the restriction of a global bounded linear map $X \mapsto UXW^*$.

It will be useful to recall the form of the tangent spaces of $\mathcal{I}(\mathcal{H})$. The fact that the map (2) is a submersion, implies that its tangent maps are surjective. Then

$$(\mathrm{T}\pi_{V_0})_{(1,1)} : (\mathrm{T}\mathcal{U}(\mathcal{H}) \times \mathcal{U}(\mathcal{H}))_{(1,1)} = \{(iX, iY) : X^* = X, Y^* = Y\} \rightarrow (\mathrm{T}\mathcal{I}(\mathcal{H}))_{V_0},$$

$$(\mathrm{T}\pi_{V_0})_{(1,1)}((iX, iY)) = iXV_0 - iV_0Y$$

is surjective, and

$$(\mathrm{T}\mathcal{I}(\mathcal{H}))_{V_0} = \{iXV_0 - iV_0Y : X^* = X, Y^* = Y\}. \quad (5)$$

The metric that will be considered in $\mathrm{T}\mathcal{I}(\mathcal{H})$ is a *quotient metric*, using the homogeneous structure of $\mathcal{I}(\mathcal{H})$ (as a quotient of the group $\mathcal{U}(\mathcal{H}) \times \mathcal{U}(\mathcal{H})$), following the program outlined in the seminal paper by Durán, Mata-Lorenzo and Recht [13]. We shall describe it in Section 4.

Following this program, one can compute curves of minimal length (*metric geodesics*) of this Finsler metric, by finding *minimal liftings* of tangent vectors in $\mathrm{T}\mathcal{I}(\mathcal{H})$.

3 2×2 model for $\mathcal{I}(\mathcal{H})$

Given $V \in \mathcal{I}(\mathcal{H})$, consider $\epsilon_V \in \mathcal{B}(\mathcal{H} \times \mathcal{H})$ given by

$$\epsilon_V = \begin{pmatrix} 0 & V \\ V^* & 0 \end{pmatrix}.$$

Note that $\epsilon_V^* = \epsilon_V$,

$$\epsilon_V^2 = \begin{pmatrix} VV^* & 0 \\ 0 & V^*V \end{pmatrix},$$

where V^*V and VV^* are the initial and final projections of V , and that

$$\epsilon_V^3 = \begin{pmatrix} 0 & VV^*V \\ V^*VV^* & 0 \end{pmatrix} = \begin{pmatrix} 0 & V \\ V^* & 0 \end{pmatrix} = \epsilon_V$$

It follows that $\epsilon = \epsilon_V$ is a selfadjoint root of the polynomial $\mathbf{x}^3 - \mathbf{x}$, and therefore has a simple spectral decomposition of the form

$$\epsilon_V = 0 \cdot E_0 + 1 \cdot E_+ - 1 \cdot E_- = E_+ - E_-,$$

with

$$E_+ = \frac{1}{2}\{\epsilon_V^2 + \epsilon_V\}, \quad E_- = \frac{1}{2}\{\epsilon_V^2 - \epsilon_V\} \quad \text{and} \quad E_0 = 1 - \epsilon_V^2,$$

the mutually orthogonal spectral projections of ϵ_V .

Remark 3.1. Consider $\epsilon = \epsilon^*$ with $\epsilon^3 = \epsilon$. The unitary orbit of ϵ , under the inner action of the unitary group of $\mathcal{H} \times \mathcal{H}$,

$$\mathcal{O}_\epsilon = \{\mathbf{U} \epsilon \mathbf{U}^* : \mathbf{U} = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \in \mathcal{U}(\mathcal{H} \times \mathcal{H})\},$$

is a complemented C^∞ -submanifold of $\mathcal{B}(\mathcal{H} \times \mathcal{H})$, and a homogeneous space of the unitary group $\mathcal{U}(\mathcal{H} \times \mathcal{H})$ (see [10]). The isotropy group of ϵ is given by

$$I_\epsilon = \left\{ \mathbf{G} = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \in \mathcal{U}(\mathcal{H} \times \mathcal{H}) : \mathbf{G} \epsilon = \epsilon \mathbf{G} \right\}.$$

The Banach-Lie algebra, in the special case where $\epsilon = \epsilon_V$ for some $V \in \mathcal{I}(\mathcal{H})$, is

$$\iota_{\epsilon_V} = \left\{ i\mathbf{X} = i \begin{pmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{pmatrix} \in \mathcal{B}(\mathcal{H} \times \mathcal{H}) : X_{ii}^* = X_{ii} \text{ and } \begin{cases} X_{12}V^* = VX_{12}^*, \\ X_{12}^*V = V^*X_{12}, \\ X_{11}V = VX_{22} \end{cases} \right\}.$$

If we restrict the above inner action (on ϵ_V) to the diagonal subgroup

$$\Delta = \left\{ \begin{pmatrix} U & 0 \\ 0 & W \end{pmatrix} : U, W \in \mathcal{U}(\mathcal{H}) \right\} \subset \mathcal{U}(\mathcal{H} \times \mathcal{H}),$$

we obtain a copy of the connected component (orbit) of V :

Proposition 3.2. *Let $V \in \mathcal{I}(\mathcal{H})$, then*

$$\Delta \cdot \epsilon_V = \left\{ \begin{pmatrix} 0 & UVW^* \\ (UVW^*)^* & 0 \end{pmatrix} : U, W \in \mathcal{U}(\mathcal{H}) \right\} = \{ \epsilon_{UVW^*} : U, W \in \mathcal{U}(\mathcal{H}) \} \simeq \mathcal{I}_V.$$

Proof. It is a straightforward computation:

$$\begin{pmatrix} U & 0 \\ 0 & W \end{pmatrix} \begin{pmatrix} 0 & V \\ V^* & 0 \end{pmatrix} \begin{pmatrix} U^* & 0 \\ 0 & W^* \end{pmatrix} = \begin{pmatrix} 0 & UVW^* \\ WV^*U^* & 0 \end{pmatrix}$$

□

Note that

$$(T\Delta \cdot \epsilon_V)_{\epsilon_V} = \left\{ \begin{pmatrix} 0 & iXV - iVY \\ iYV^* - iV^*X & 0 \end{pmatrix} : X^* = X, Y^* = Y \right\}.$$

4 The program of Durán, Mata-Lorenzo and Recht

Let us briefly describe the context and main result of [13]. Let $\mathcal{B} \subset \mathcal{A}$ be unital C^* -algebras (with the same unit). Denote by $\mathcal{U}_{\mathcal{B}}$ and $\mathcal{U}_{\mathcal{A}}$ the unitary groups of \mathcal{B} and \mathcal{A} , respectively. In [13] a metric was introduced in the homogeneous (quotient) space $M = \mathcal{U}_{\mathcal{A}} / \mathcal{U}_{\mathcal{B}}$. If $u \in \mathcal{U}_{\mathcal{A}}$, let $[u] \in M$ be the class of u in the quotient space. The Banach-Lie algebras of $\mathcal{U}_{\mathcal{A}}$ and $\mathcal{U}_{\mathcal{B}}$ are, respectively

$$\mathfrak{u}_{\mathcal{A}} = \{ix : x^* = x \in \mathcal{A}\}, \quad \mathfrak{u}_{\mathcal{B}} = \{iy : y^* = y \in \mathcal{A}\}.$$

For $[u] \in M$, let

$$\pi_{[u]} : \mathcal{U}_{\mathcal{A}} \rightarrow M, \quad \pi_{[u]}(w) = [uw].$$

The tangent space of M at $[u]$ naturally identifies with the quotient of the Lie algebras of $\mathcal{U}_{\mathcal{A}}$ and $\mathcal{U}_{\mathcal{B}}$:

$$(TM)_{[u]} \simeq \mathfrak{u}_{\mathcal{A}} / \mathfrak{u}_{\mathcal{B}},$$

since $(T\pi_{[u]})_{[1]} : \mathfrak{u}_{\mathcal{A}} \rightarrow (TM)_{[u]}$ is an epimorphism with nullspace $\mathfrak{u}_{\mathcal{B}}$. In this tangent space they define the natural metric:

Definition 4.1. If $\mathbf{v} = ix + \mathbf{u}_{\mathcal{B}} \in (TM)_{[u]}$

$$|\mathbf{v}|_{[u]} := \inf\{\|x + y\| : y^* = y \in \mathcal{B}\}, \quad (6)$$

i.e., the usual metric in the quotient of (real) Banach spaces $\mathbf{u}_{\mathcal{A}} / \mathbf{u}_{\mathcal{B}}$.

Definition 4.2. Let $\mathbf{v} \in (TM)_{[u]}$. An element $x_0 = x_0^* \in \mathcal{A}$ is a minimal lifting of \mathbf{v} if

$$\mathbf{v} = ix_0 + \mathbf{u}_{\mathcal{B}}$$

and

$$\|x_0\| = \inf\{\|x_0 + y\| : y = y^* \in \mathcal{B}\} = |\mathbf{v}|_{[u]}.$$

That is, x_0 attains the norm of the class in the quotient norm.

In general, minimal liftings may not exist (see for instance the paper [7] for an interesting example). However, when they do exist, they provide curves of minimal length in M :

Theorem 4.3. (Durán, Mata-Lorenzo, Recht [13])

Let $[u] \in M$ and $\mathbf{v} \in (TM)_{[u]}$. Suppose that \mathbf{v} has a minimal lifting x_0 . Then the curve

$$\delta(t) = [e^{itx_0}u]$$

which satisfies the initial conditions

$$\delta(0) = [u] \quad \text{and} \quad \dot{\delta}(0) = \mathbf{v},$$

has minimal length along its path for $|t| \leq \frac{\pi}{2|\mathbf{v}|_{[u]}} = \frac{\pi}{2\|x_0\|}$.

Here, by *minimal length along its path* at the given interval of t , means that if $[t_0, t_1]$ is a subinterval of $[-\frac{\pi}{2|\mathbf{v}|_{[u]}}, \frac{\pi}{2|\mathbf{v}|_{[u]}}]$, and $\gamma(t)$ ($t \in I$) is an arbitrary smooth curve in M joining $\delta(t_0)$ and $\delta(t_1)$, then

$$\ell(\delta|_{[t_0, t_1]}) = \int_{t_0}^{t_1} |\dot{\delta}(t)|_{\delta(t)} dt \leq \int_I |\dot{\gamma}(t)|_{\gamma(t)} dt = \ell(\gamma).$$

Example 4.4. An example where minimal liftings exist at every tangent vector (at every point), occurs when both \mathcal{A} and \mathcal{B} are von Neumann algebras. For instance, if $V \in \mathcal{I}(\mathcal{H})$, then

$$M = \mathcal{O}_{\epsilon_V} \simeq \mathcal{U}(\mathcal{H} \times \mathcal{H}) / \mathcal{U}(\mathcal{H} \times \mathcal{H}) \cap \{\epsilon_V\}'$$

is such an example. Indeed, $\mathcal{U}(\mathcal{H} \times \mathcal{H}) \cap \{\epsilon_V\}'$ is the unitary group of the von Neumann algebra $\{\epsilon_V\}' \subset \mathcal{B}(\mathcal{H} \times \mathcal{H})$.

5 Finsler metric and minimal curves in $\mathcal{I}(\mathcal{H})$

In this section we show that minimal liftings of the homogeneous space

$$\mathcal{O}_{\epsilon_V} \simeq \mathcal{U}(\mathcal{H}) / \mathcal{U}(\mathcal{H}) \cap \{\epsilon_V\}'$$

induce in a simple manner minimal liftings in $\mathcal{I}(\mathcal{H})$, or more precisely, in the connected component $\mathcal{I}(\mathcal{H})_V \simeq \mathcal{U}(\mathcal{H}) \times \mathcal{U}(\mathcal{H}) / \mathbf{I}_V$ of V in $\mathcal{I}(\mathcal{H})$.

Note that I_V is not the unitary group of a selfadjoint algebra in a straightforward fashion. We shall use the 2×2 model for $\mathcal{I}(\mathcal{H})$, in order to be able to obtain minimal liftings, and as a byproduct, a stronger minimality result. Namely, that the metric geodesics obtained are not only (locally) minimal in $\mathcal{I}(\mathcal{H})$, but also in the ambient manifold \mathcal{O}_{ϵ_V} (regarding $\mathcal{I}(\mathcal{H})_V$ as a subset of \mathcal{O}_{ϵ_V} via the isometric embedding $\mathcal{I}(\mathcal{H})_V \hookrightarrow \mathcal{O}_{\epsilon_V}$, $V \mapsto \epsilon_V$).

Following the program in [13], we define the following Finsler metrics in \mathcal{O}_{ϵ_V} and in $\mathcal{I}(\mathcal{H})$. If $\epsilon^3 = \epsilon^* = \epsilon$, and $\mathcal{V} \in (T\mathcal{O}_\epsilon)_\epsilon$, we put

$$|\mathcal{V}|_\epsilon = \inf\{\|\mathbf{X}\| : (T\pi_\epsilon)_1(\mathbf{X}) = \mathcal{V}\} = \inf\{\|\mathbf{X} + \mathbf{Z}\| : \mathbf{Z} \in \iota_\epsilon\} \quad (7)$$

If $V \in \mathcal{I}(\mathcal{H})$ and $\mathcal{V} \in (T\mathcal{I}(\mathcal{H}))_V$, we put

$$|\mathcal{V}|_V = \inf\{\|(A, B)\| : A, B \in \mathcal{B}(\mathcal{H}), A^* = A, B^* = B \text{ and } iAV - iVB = \mathcal{V}\}, \quad (8)$$

where as is usual $\|(A, B)\| = \max\{\|A\|, \|B\|\}$ (i.e., the C^* -norm in $\mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H})$).

Remark 5.1. Clearly, definitions (7) and (8) make

$$\mathcal{I}(\mathcal{H}) \rightarrow \Delta \cdot \epsilon_V, \quad V \mapsto \epsilon_V$$

an isometric diffeomorphism.

Before we proceed, we state the following results, which are elementary and known, and will be used thoroughly. The first fact, is that if one deals with $n \times n$ (block) operator matrices, the diagonal map

$$E\left(\begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \dots & \dots & \dots & \dots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{pmatrix}\right) = \begin{pmatrix} A_{11} & 0 & \dots & 0 \\ 0 & A_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & A_{nn} \end{pmatrix} \quad (9)$$

is positive, linear and contractive. The second fact is the following:

Lemma 5.2. *Let A, P in $\mathcal{B}(\mathcal{H})$, P an orthogonal projection. Regard A as a 2×2 matrix in terms of P :*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{matrix} R(P) \\ N(P) \end{matrix}.$$

Then

$$\left\| \begin{pmatrix} 0 & A_{12} \\ A_{21} & 0 \end{pmatrix} \right\| \leq \left\| \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \right\|.$$

Proof. Note that

$$\|A\|^2 = \|A^*A\| = \left\| \begin{pmatrix} A_{11}^*A_{11} + A_{21}^*A_{21} & A_{11}^*A_{12} + A_{21}^*A_{22} \\ A_{12}^*A_{11} + A_{22}^*A_{21} & A_{12}^*A_{12} + A_{22}^*A_{22} \end{pmatrix} \right\| \geq \|E(A^*A)\|,$$

where E is the linear map given in (9),

$$\|E(A^*A)\| = \left\| \begin{pmatrix} A_{11}^*A_{11} + A_{21}^*A_{21} & 0 \\ 0 & A_{12}^*A_{12} + A_{22}^*A_{22} \end{pmatrix} \right\|$$

Since clearly

$$\begin{pmatrix} A_{11}^*A_{11} + A_{21}^*A_{21} & 0 \\ 0 & A_{12}^*A_{12} + A_{22}^*A_{22} \end{pmatrix} \geq \begin{pmatrix} A_{21}^*A_{21} & 0 \\ 0 & A_{12}^*A_{12} \end{pmatrix}$$

we get

$$\|A\|^2 \geq \left\| \begin{pmatrix} A_{21}^*A_{21} & 0 \\ 0 & A_{12}^*A_{12} \end{pmatrix} \right\| = \left\| \begin{pmatrix} 0 & A_{21}^* \\ A_{12}^* & 0 \end{pmatrix} \begin{pmatrix} 0 & A_{12} \\ A_{21} & 0 \end{pmatrix} \right\| = \left\| \begin{pmatrix} 0 & A_{12} \\ A_{21} & 0 \end{pmatrix} \right\|^2$$

□

Our next goal is to compare $|\mathcal{V}|_V$ with the usual operator norm $\|\mathcal{V}\|$ of $\mathcal{V} \in (\mathcal{TI}(\mathcal{H}))_V$, for $V \in \mathcal{I}(\mathcal{H})$. In fact, we shall see that for each fixed $V \in \mathcal{I}(\mathcal{H})$, $|\cdot|_V$ and $\|\cdot\|$ are equivalent in $(\mathcal{TI}(\mathcal{H}))_V$. To do this task, we shall need a classical result by M.C. Krein [17], known as the extension problem for symmetric transformations (see also the excellent text [23], Section 125, or also [12], [20] for more nuanced developements on this subject). We state this result in the following remark, adapted to our particular problem:

Remark 5.3. Let

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{12}^* & * \end{pmatrix}$$

be an incomplete operator matrix, with $A_{11}^* = A_{11}$. Then there exist (non unique) selfadjoint completions

$$\mathbf{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{pmatrix}$$

$$\text{with } \|\mathbf{A}\| = \left\| \begin{pmatrix} A_{11} \\ A_{12}^* \end{pmatrix} \right\| = \left\| \begin{pmatrix} A_{11} & A_{12} \end{pmatrix} \right\|.$$

Theorem 5.4. Let $V \in \mathcal{I}(\mathcal{H})$ and $\mathcal{V} \in (\mathcal{TI}(\mathcal{H}))_V$. Then

$$|\mathcal{V}|_V \leq \|\mathcal{V}\| \leq 2|\mathcal{V}|_V$$

Proof. Let us first consider the case of a balanced partial isometry, $n(V) = r^\perp(V)$. Let U, W be unitaries such that $UP_0W^* = V$, for an orthogonal projection P_0 . Clearly, pulling back \mathcal{V} with the left action of the pair (U, W) , it suffices to reason in the case $V = P_0$. To this effect, note that the action of (U, W) is isometric both for the Finsler norm $|\cdot|_V$ and the operator norm $\|\cdot\|$. Let (A, B) be a lifting for \mathcal{V} , i.e., $A^* = A$, $B^* = B$ and $iAP_0 - iP_0B = \mathcal{V}$. Note that, in matrix form in terms of P_0 ,

$$\mathcal{V} = i \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - i \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{12}^* & B_{22} \end{pmatrix} = i \begin{pmatrix} A_{11} - B_{11} & B_{12} \\ A_{12}^* & 0 \end{pmatrix}$$

Let us alter the lifting (A, B) . Put

$$B_0 = \begin{pmatrix} 0 & B_{12} \\ B_{12}^* & 0 \end{pmatrix}$$

and

$$A_0 = \begin{pmatrix} A_{11} - B_{11} & A_{12} \\ A_{12}^* & Z \end{pmatrix}$$

where $Z = Z^*$ is such that A_0 is a solution of Krein's extension problem for the symmetric incomplete matrix

$$\begin{pmatrix} A_{11} - B_{11} & A_{12} \\ A_{12}^* & * \end{pmatrix}$$

Straightforward computations show that (A_0, B_0) is also a lifting of \mathcal{V} : $A_0^* = A_0$, $B_0^* = B_0$ and $iA_0P_0 - iP_0B_0 = \mathcal{V}$. Then

$$\begin{aligned} \|A_0\| &= \left\| \begin{pmatrix} A_{11} - B_{11} \\ A_{12}^* \end{pmatrix} \right\| = \left\| \begin{pmatrix} A_{11} - B_{11} & 0 \\ A_{12}^* & 0 \end{pmatrix} \right\| = \left\| \begin{pmatrix} A_{11} - B_{11} & B_{12} \\ A_{12}^* & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\| \\ &= \|\mathcal{V}P_0\| \leq \|\mathcal{V}\|. \end{aligned}$$

On the other hand

$$\left\| \begin{pmatrix} 0 & B_{12} \\ B_{12}^* & 0 \end{pmatrix} \right\| = \left\| \begin{pmatrix} 0 & B_{12} \\ 0 & 0 \end{pmatrix} \right\| = \left\| \begin{pmatrix} A_{11} - B_{11} & B_{12} \\ A_{12}^* & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\| = \|\mathcal{V}P_0^\perp\| \leq \|\mathcal{V}\|.$$

Thus, we have found a lifting (A_0, B_0) of \mathcal{V} such that

$$\|(A_0, B_0)\| = \max\{\|A_0\|, \|B_0\|\} \leq \|\mathcal{V}\|.$$

It follows that $|\mathcal{V}|_{P_0} \leq \|\mathcal{V}\|$.

Let us now consider the general case. Consider the Hilbert space $\mathcal{H} \times \mathcal{H}$. Note that $V \oplus 0$ in $\mathcal{H} \times \mathcal{H}$ defined as $V(\xi, \eta) = (V\xi, 0)$ is a partial isometry with $n(V \oplus 0) = r^\perp(V \oplus 0) = +\infty$. Similarly, if $\mathcal{V} \in (\mathcal{I}(\mathcal{H}))_V$, then $\mathcal{V} \oplus 0 \in (\mathcal{I}(\mathcal{H} \times \mathcal{H}))_{V \oplus 0}$. Note that

$$|\mathcal{V} \oplus 0|_{V \oplus 0} \leq |\mathcal{V}|_V.$$

Indeed, any lifting (A, B) of \mathcal{V} , provides a lifting $(A \oplus 0, B \oplus 0)$ of $\mathcal{V} \oplus 0$. On the other hand, if

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{pmatrix}, \begin{pmatrix} B_{11} & B_{12} \\ B_{12}^* & B_{22} \end{pmatrix}$$

is a lifting of $\mathcal{V} \oplus 0$. i.e.,

$$\mathcal{V} \oplus 0 = \begin{pmatrix} \mathcal{V} & 0 \\ 0 & 0 \end{pmatrix} = i \begin{pmatrix} A_{11}V - VB_{11} & -VB_{12} \\ A_{12}^*V & 0 \end{pmatrix},$$

then, in particular (A_{11}, B_{11}) is a lifting for \mathcal{V} . Since

$$\|A_{11}\| \leq \left\| \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{pmatrix} \right\| \quad \text{and} \quad \|B_{11}\| \leq \left\| \begin{pmatrix} B_{11} & B_{12} \\ B_{12}^* & B_{22} \end{pmatrix} \right\|,$$

it follows that $|\mathcal{V}|_V \leq |\mathcal{V} \oplus 0|_{V \oplus 0}$. Then, by the first case,

$$\|\mathcal{V}\| = \|\mathcal{V} \oplus 0\| \leq |\mathcal{V} \oplus 0|_{V \oplus 0} = |\mathcal{V}|_V.$$

The other inequality is trivial:

$$\|\mathcal{V}\| = \|AV - VB\| \leq \|AV\| + \|VB\| \leq \|A\| + \|B\| \leq 2 \max\{\|A\|, \|B\|\},$$

for any lifting (A, B) of \mathcal{V} .

□

Remark 5.5. The inequality $|\mathcal{V}|_V \leq \|\mathcal{V}\|$ may be strict. The problem of finding completions of 2×2 matrices with minimal norm has been studied for non-selfadjoint matrix operators (see for instance [12]). Namely, applied in our context and following the notations of the above theorem, given the incomplete (non-selfadjoint) matrix operator

$$\begin{pmatrix} A_{11} - B_{11} & B_{12} \\ A_{12}^* & * \end{pmatrix}$$

There exists a completion $C = \begin{pmatrix} A_{11} - B_{11} & B_{12} \\ A_{12}^* & Y \end{pmatrix}$ with minimal norm, that is

$$\|C\| = \max\left\{\left\|\begin{pmatrix} A_{11} - B_{11} & B_{12} \end{pmatrix}\right\|, \left\|\begin{pmatrix} A_{11} - B_{11} \\ A_{12}^* \end{pmatrix}\right\|\right\}.$$

The row $\begin{pmatrix} A_{11} - B_{11} & B_{12} \end{pmatrix}$ is the first row of the incomplete matrix $\begin{pmatrix} A_{11} - B_{11} & B_{12} \\ B_{12}^* & * \end{pmatrix}$ which can be completed with minimal norm to the selfadjoint operator $\begin{pmatrix} A_{11} - B_{11} & B_{12} \\ B_{12}^* & Z' \end{pmatrix}$, with

$$\left\|\begin{pmatrix} A_{11} - B_{11} & B_{12} \end{pmatrix}\right\| = \left\|\begin{pmatrix} A_{11} - B_{11} & B_{12} \\ B_{12}^* & Z' \end{pmatrix}\right\|.$$

By Lemma 5.2,

$$\left\|\begin{pmatrix} A_{11} - B_{11} & B_{12} \\ B_{12}^* & Z' \end{pmatrix}\right\| \geq \left\|\begin{pmatrix} 0 & B_{12} \\ B_{12}^* & 0 \end{pmatrix}\right\|.$$

Similarly, reasoning with the first column $\begin{pmatrix} A_{11} - B_{11} \\ A_{12}^* \end{pmatrix}$, we get that there is a selfadjoint

completion $\begin{pmatrix} A_{11} - B_{11} & A_{12} \\ A_{12}^* & Z \end{pmatrix}$ such that

$$\left\|\begin{pmatrix} A_{11} - B_{11} \\ A_{12}^* \end{pmatrix}\right\| = \left\|\begin{pmatrix} A_{11} - B_{11} & A_{12} \\ A_{12}^* & Z \end{pmatrix}\right\|.$$

It follows that

$$\|C\| \geq \max\left\{\left\|\begin{pmatrix} 0 & B_{12} \\ B_{12}^* & 0 \end{pmatrix}\right\|, \left\|\begin{pmatrix} A_{11} - B_{11} & A_{12} \\ A_{12}^* & Z \end{pmatrix}\right\|\right\},$$

which is the norm of a lifting (A_0, B_0) of \mathcal{V} (as in the first part of the proof of the above Theorem). That is, $\|C\| \geq |\mathcal{V}|_V$. Now, C and $-i\mathcal{V}$ are both completions of the same incomplete (non-selfadjoint) matrix. Since C has minimal norm among these completions, one has $\|C\| \leq \|\mathcal{V}\|$. Moreover, it is known that, in general, putting 0 in the 2, 2 place is not the optimal solution (see [12], [20]): there are examples where $\|C\| < \|\mathcal{V}\|$. Then, for such \mathcal{V} , we have $|\mathcal{V}|_V < \|\mathcal{V}\|$.

In order to establish the existence of minimal liftings in $\mathcal{I}(\mathcal{H})$, we need the next lemma.

Lemma 5.6. *Let $V \in \mathcal{I}(\mathcal{H})$ and $\mathbf{A} = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ be selfadjoint in $\mathcal{H} \times \mathcal{H}$. Let*

$$\mathbf{X} = \begin{pmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{pmatrix} \in \{\epsilon_V\}', \quad \mathbf{X}^* = \mathbf{X},$$

such that

$$\|\mathbf{A} + \mathbf{X}\| \leq \|\mathbf{A} + \mathbf{Y}\| \text{ for all } \mathbf{Y}^* = \mathbf{Y} \in \{\epsilon_V\}'$$

(which exists, recall Example 4.4). Then $\mathbf{X}_0 := \begin{pmatrix} X_{11} & 0 \\ 0 & X_{22} \end{pmatrix}$ satisfies

- \mathbf{X}_0 commutes with ϵ_V , in particular, $i\mathbf{X}_0 \in \iota_V$;
- $\|\mathbf{A} + \mathbf{X}_0\| \leq \|\mathbf{A} + \mathbf{Z}\|$ for all $i\mathbf{Z} \in \iota_V$.

Proof. Clearly, $\mathbf{X}_0^* = \mathbf{X}_0$. The fact that \mathbf{X} commutes with ϵ_V , means that

$$\begin{cases} X_{12}V^* = VX_{12}^*, \\ X_{12}^*V = V^*X_{12}, \\ X_{11}V = VX_{22} \end{cases}$$

and thus also \mathbf{X}_0 commutes with ϵ_V , in particular $X_{11}V = VX_{22}$, which means that $i\mathbf{X}_0 \in \iota_V$.

By the same argument, it is also clear that if $\mathbf{Z} = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{12}^* & Z_{22} \end{pmatrix}$ commutes with ϵ_V , then also

$\mathbf{Z}_0 = \begin{pmatrix} Z_{11} & 0 \\ 0 & Z_{22} \end{pmatrix}$ also commutes with ϵ_V . Then

$$\|\mathbf{A} + \mathbf{X}\| \leq \|\mathbf{A} + \mathbf{Z}_0\|.$$

On the other hand, the linear map $E : \mathcal{B}(\mathcal{H} \times \mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H} \times \mathcal{H})$ given by

$$E\left(\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}\right) = \begin{pmatrix} T_{11} & 0 \\ 0 & T_{22} \end{pmatrix}$$

is contractive. Then

$$\|\mathbf{A} + \mathbf{X}_0\| = \|E(\mathbf{A} + \mathbf{X})\| \leq \|\mathbf{A} + \mathbf{X}\| \leq \|\mathbf{A} + \mathbf{Z}_0\|, \quad (10)$$

which completes the proof. \square

We shall call $\mathbf{A}_0 := \mathbf{A} + \mathbf{X}_0$ a *diagonal minimal lifting*. Note that the above Lemma states that any vector tangent to $\Delta \cdot \epsilon_V$ has a diagonal minimal lifting \mathbf{A}_0 . Applying Theorem 4.3 [13] we get

Theorem 5.7. *Let $V \in \mathcal{I}(\mathcal{H})$ and $\mathcal{V} = iXV - iVY \in (\mathcal{I}(\mathcal{H}))_V$. Let $\mathbf{V} = \begin{pmatrix} 0 & \mathcal{V} \\ \mathcal{V}^* & 0 \end{pmatrix} \in (T\Delta \cdot \epsilon_V)_V$, and pick \mathbf{A}_0 a diagonal minimal lifting for \mathbf{V} . Then the curve*

$$\delta(t) = e^{it\mathbf{A}_0} \epsilon_V e^{-it\mathbf{A}_0}$$

which satisfies that $\delta(0) = \epsilon_V$ and $\dot{\delta}(0) = \mathbf{V}$, has minimal length along its path in $\Delta \cdot \epsilon_V$, for $|t| \leq \frac{\pi}{2|\mathbf{V}|_V}$. Moreover, it also has minimal length among curves in bigger manifold \mathcal{O}_{ϵ_V} , in the same time interval.

Proof. The proof of the above Lemma, in fact shows that $\mathbf{A} + \mathbf{X}_0$ is a minimal lifting in the bigger quotient space. \square

Therefore, if we consider the quotient left invariant metric in $\mathcal{I}(\mathcal{H})$, we obtain:

Corollary 5.8. *Let $V \in \mathcal{I}(\mathcal{H})$ and $\mathcal{V} = iXV - iVY \in (\mathcal{TI}(\mathcal{H}))_V$. Then there exist $X_0^* = X_0$, $Y_0^* = Y_0$ with $\mathcal{V} = iX_0V - iVY_0$, such that the curve*

$$\delta(t) = e^{itX_0} V e^{-itY_0}$$

which satisfies $\delta(0) = V$ and $\dot{\delta}(0) = \mathcal{V}$, has minimal length along its path in $\mathcal{I}(\mathcal{H})$, for $|t| \leq \frac{\pi}{2|\mathcal{V}|_V}$.

6 Initial and final projections

If $V \in \mathcal{I}(\mathcal{H})$, denote by $\alpha(V) = V^*V$ and $\omega(V) = VV^*$ the initial and final projections of V . Denote by $\mathcal{P}(\mathcal{H})$ the space of (orthogonal) projections of $\mathcal{B}(\mathcal{H})$. The space of projections of a C^* -algebra has been well studied, as a complemented submanifold of the algebra, and as an homogeneous space of the inner action of the unitary group of the algebra ($u \cdot p = upu^*$, if u is unitary and p is a projection). It has also been studied as a Finsler metric space, where each tangent space is endowed with the usual norm of the algebra (see [21] and [11]). In the specific case of the algebra $\mathcal{B}(\mathcal{H})$, existence of minimal geodesics with given initial conditions or with given endpoints, have been characterized (see the references above, or [6] for the specific case of the algebra $\mathcal{B}(\mathcal{H})$).

Clearly, the maps

$$\alpha : \mathcal{I}(\mathcal{H}) \rightarrow \mathcal{P}(\mathcal{H}), \quad \alpha(V) = V^*V \quad \text{and} \quad \omega : \mathcal{I}(\mathcal{H}) \rightarrow \mathcal{P}(\mathcal{H}), \quad \omega(V) = VV^*$$

are C^∞ . Let us show that, if $\mathcal{P}(\mathcal{H})$ is given the above mentioned Finsler metric, i.e., the usual norm at every tangent space, and $\mathcal{I}(\mathcal{H})$ is considered with the quotient metric studied here, then both maps α and ω decrease distances. This fact is based in Lemma 5.2. As said above, we consider $\mathcal{I}(\mathcal{H})$ and $\mathcal{P}(\mathcal{H})$ as metric spaces, with their given Finsler metrics. Recall how a Finsler metric in the tangent spaces induces a metric in the original space: if M is a manifold with a Finsler metric $|\cdot|_m$ at $(TM)_m$ (for $m \in M$), then

$$d_M(m_1, m_2) = \inf \{ \ell(\gamma) : \gamma(t) \in M, t \in [a, b], \gamma \text{ is smooth}, \gamma(a) = m_1, \gamma(b) = m_2 \},$$

where

$$\ell(\gamma) = \int_a^b |\dot{\gamma}(t)|_{\gamma(t)} dt.$$

Proposition 6.1. *The maps*

$$\alpha : \mathcal{I}(\mathcal{H}) \rightarrow \mathcal{P}(\mathcal{H}), \quad \alpha(V) = V^*V$$

and

$$\omega : \mathcal{I}(\mathcal{H}) \rightarrow \mathcal{P}(\mathcal{H}), \quad \omega(V) = VV^*$$

are distance decreasing, i.e., if $V_1, V_2 \in \mathcal{I}(\mathcal{H})$, $E_i = \alpha(V_i)$, $F_i = \omega(V_i)$, $i = 1, 2$, then

$$d_{\mathcal{P}(\mathcal{H})}(E_1, E_2) \leq d_{\mathcal{I}(\mathcal{H})}(V_1, V_2) \quad \text{and} \quad d_{\mathcal{P}(\mathcal{H})}(F_1, F_2) \leq d_{\mathcal{I}(\mathcal{H})}(V_1, V_2).$$

Proof. We reason with the map α (the argument with ω is similar). It suffices to show that the tangent maps $(T\alpha)_V : (\mathcal{TI}(\mathcal{H}))_V \rightarrow (\mathcal{TP}(\mathcal{H}))_{\alpha(V)}$,

$$(T\alpha)_V(\mathcal{V}) = \mathcal{V}^*V + V^*\mathcal{V}$$

are contractive. Pick a pair (iX, iY) , $X^* = X, Y^* = Y$ which lifts \mathcal{V} , i.e., $\mathcal{V} = iXV - iVY$. Then

$$(T\alpha)_V(\mathcal{V}) = (-iV^*X + iYV^*)V + V^*(iXV - iVY) = iY\alpha(V) - i\alpha(V)Y = i[Y, \alpha(V)].$$

Note that the matrix of $[Y, \alpha(V)]$ in terms of the projection $\alpha(V)$ is

$$\begin{pmatrix} Y_{11} & Y_{12} \\ Y_{12}^* & Y_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{12}^* & Y_{22} \end{pmatrix} = \begin{pmatrix} 0 & -Y_{12} \\ Y_{12}^* & 0 \end{pmatrix},$$

whose norm equals the norm of

$$\begin{pmatrix} 0 & Y_{12} \\ Y_{12}^* & 0 \end{pmatrix} = \begin{pmatrix} 0 & -Y_{12} \\ Y_{12}^* & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -Y_{12} \\ Y_{12}^* & 0 \end{pmatrix} (2\alpha(V) - 1),$$

because $2\alpha(V) - 1$ is a unitary operator. By Lemma 5.2

$$\left\| \begin{pmatrix} 0 & -Y_{12} \\ Y_{12}^* & 0 \end{pmatrix} \right\| \leq \|Y\| \leq \max\{\|X\|, \|Y\|\} = \|(X, Y)\|.$$

Since this holds for any pair (X, Y) which lifts \mathcal{V} , we have that

$$|\mathcal{V}|_V = \inf\{\|(X, Y)\| : iXV - iVY = \mathcal{V}\} \geq \|(T\alpha)_V(\mathcal{V})\|,$$

as claimed. □

6.1 Balanced isometries

Recall that $V \in \mathcal{I}(\mathcal{H})$ is called *balanced* if $n(V) = r^\perp(V)$.

In some special directions, more can be said about minimal liftings and geodesics at balanced isometries.

Definition 6.2. Let $V \in \mathcal{I}(\mathcal{H})$ be balanced, with initial space $R(\alpha(V)) = \mathcal{S}_i$ and final space $R(\omega(V)) = \mathcal{S}_f$. We call a tangent vector $\mathcal{V} \in (\mathcal{TI}(\mathcal{H}))_V$ orthogonal at V if \mathcal{V} as an operator in \mathcal{H} , satisfies that $\mathcal{V}(\mathcal{S}_i) \perp \mathcal{S}_f$. Equivalently, $\omega(V)\mathcal{V}\alpha(V) = 0$

Let us construct explicit minimal liftings for tangent vectors which are othogonal to V .

1. Let $\mathcal{V} \in (\mathcal{TI}(\mathcal{H}))_V$ such that $\omega(V)V\alpha(V) = 0$. First, using the left action of $\mathcal{U}(\mathcal{H}) \times \mathcal{U}(\mathcal{H})$, we may suppose without loss of generality that $V = P_0$ is an orthogonal projection. Indeed, there exist $U, W \in \mathcal{U}(\mathcal{H})$ such that $V = UP_0W^*$. Then $\mathcal{V}_0 := U^*\mathcal{V}W \in (\mathcal{TI}(\mathcal{H}))_{P_0}$ satisfies

$$\begin{aligned} 0 = \omega(V)\mathcal{V}\alpha(V) &= VV^*\mathcal{V}V^*V = UP_0W^*(UP_0W^*)^*\mathcal{V}(UP_0W^*)^*UP_0W^* = UP_0U^*\mathcal{V}WP_0W^* \\ &= UP_0\mathcal{V}_0P_0W^*, \end{aligned}$$

and thus, $P_0\mathcal{V}_0P_0 = 0$. Suppose that we find (A, B) a minimal lifting for \mathcal{V}_0 at P_0 , i.e., $iAP_0 - iP_0B = \mathcal{V}_0$ with

$$\|(A, B)\| \leq \|(A, B) + (X, Y)\| \quad \text{for all } (iX, iY) \in \iota_{P_0}.$$

Then (UAU^*, W^*BW) is a lifting for $U\mathcal{V}_0W^* = \mathcal{V}$ at V :

$$iUAU^*V - iVW^*BW = U\{iAU^*VW^* - iU^*VWB\}W^* = U\{iAP_0 - iP_0B\}W^* = U\mathcal{V}_0W^* = \mathcal{V},$$

which is minimal

$$\|(UAU^*, W^*BW)\| = \|(A, B)\| \leq \|(A, B) + (X, Y)\| = \|(UAU^*, W^*BW) + (UXU^*, W^*YW)\|,$$

where (UXU^*, W^*YW) parametrizes all elements in $Ad(U, W)(\iota_{P_0}) = \iota_{UP_0W^*} = \iota_V$.

2. Let us construct a minimal lifting for \mathcal{V}_0 at P_0 (with $P_0\mathcal{V}_0P_0 = 0$). Let (A, B) , $A^* = A$, $B^* = B$ such that $iAP_0 - iP_0B = \mathcal{V}_0$. Then $P_0AP_0 = P_0BP_0$. Pick

$$A_0 = P_0AP_0^\perp + P_0^\perp AP_0 \quad \text{and} \quad B_0 = P_0BP_0^\perp + P_0^\perp BP_0.$$

Clearly $A_0^* = A_0$ and $B_0^* = B_0$. Also, since $P_0AP_0 = P_0BP_0$, after elementary computations,

$$\mathcal{V}_0 = iAP_0 - iP_0B = iA_0P_0 - iP_0B_0.$$

Finally, if (A', B') is another lifting of \mathcal{V}_0 , then

$$P_0\mathcal{V}_0P_0^\perp = P_0(iA'P_0 - P_0B'P_0^\perp)P_0^\perp = iP_0B'P_0^\perp,$$

i.e., $P_0B'P_0^\perp = P_0B_0P_0^\perp$, and therefore also

$$P_0^\perp B'P_0 = (P_0B'P_0^\perp)^* = (P_0B_0P_0^\perp)^* = P_0^\perp B_0P_0.$$

That is, in matrices in terms of P_0 , B' and B_0 have the same off-diagonal entries. Clearly the same happens for A' and A_0 . By Lemma 5.2, since A_0 and B_0 are codiagonal,

$$\|A_0\| \leq \|A'\| \quad \text{and} \quad \|B_0\| \leq \|B'\|,$$

i.e., (A_0, B_0) is a minimal lifting.

These special (co-diagonal, minimal) liftings just exhibited for these special velocities, have the following property:

Proposition 6.3. *Let $V \in \mathcal{I}(\mathcal{H})$ and $\mathcal{V} \in (\mathcal{TI}(\mathcal{H}))_V$ such that \mathcal{V} is orthogonal to V (i.e., $\omega(V)\mathcal{V}\alpha(V) = 0$). Pick (A_0, B_0) a codiagonal minimal lifting of \mathcal{V} as above. Then the curve $\delta(t) = e^{itA_0}V e^{-itB_0}$ (minimal along its path up to $|t| \leq \frac{\pi}{2\|\mathcal{V}\|}$), verifies that the initial and final projection curves*

$$\alpha(\delta), \omega(\delta) \in \mathcal{P}(\mathcal{H})$$

are minimal along their paths in $\mathcal{P}(\mathcal{H})$, for $|t| \leq \frac{\pi}{2\|B_0\|}$ and $|t| \leq \frac{\pi}{2\|A_0\|}$, respectively.

Proof. Note that

$$\alpha(\delta)(t) = \delta^*(t)\delta(t) = (e^{itA_0}V e^{-itB_0})^* e^{itA_0}V e^{-itB_0} = e^{-itB_0}V^*V e^{-itB_0} = e^{-itB_0}\alpha(V)e^{-itB_0},$$

with B_0 co-diagonal with respect to $\alpha(V)$. Indeed, with the same argument as above, it suffices to consider $V = P_0$, in which case it is evident. Therefore (see [21]), $\alpha(\delta)$ is minimal in $\mathcal{P}(\mathcal{H})$ for $|t| \leq \frac{\pi}{2\|B_0\|}$. The argument with $\omega(\delta)$ is analogous. \square

In other words, for balanced isometries, and velocities which are orthogonal to V , locally, moving from V_0 to V_1 optimally in $\mathcal{I}(\mathcal{H})$, involves the optimal paths for the initial and final spaces of V_0 and V_1 .

Recall from Theorem 5.4 the comparison between the Finsler norm of \mathcal{V} at V and the ambient norm of \mathcal{V} : $|\mathcal{V}|_V \leq \|\mathcal{V}\|$. Note that, for velocities which are orthogonal, at balanced partial isometries, both norms coincide:

Remark 6.4. Let $V \in \mathcal{I}(\mathcal{H})$ and $\mathcal{V} \in (\mathcal{TI}(\mathcal{H}))_V$ such that \mathcal{V} is orthogonal to V . Then

$$|\mathcal{V}|_V = \|(A_0, B_0)\| = \max\{\|A_0\|, \|B_0\|\} = \|\mathcal{V}\|,$$

its norm as an element in $\mathcal{B}(\mathcal{H})$. Indeed, again it suffices to reason in the case $V = P_0$. As seen in the discussion preceding the above proposition, \mathcal{V} has co-diagonal matrix in terms of P_0 :

$$\mathcal{V} = \begin{pmatrix} 0 & iB_0 \\ iA_0 & 0 \end{pmatrix}.$$

Then

$$\|\mathcal{V}\|^2 = \left\| \begin{pmatrix} 0 & iB_0 \\ iA_0 & 0 \end{pmatrix}^* \begin{pmatrix} 0 & iB_0 \\ iA_0 & 0 \end{pmatrix} \right\| = \left\| \begin{pmatrix} A_0^2 & 0 \\ 0 & B_0^2 \end{pmatrix} \right\| = \max\{\|A_0\|^2, \|B_0\|^2\}.$$

If $\gamma(t) \in \mathcal{I}(\mathcal{H})$, $t \in I$ is smooth, denote by $\ell_\infty(\gamma)$ the length of γ with the metric induced by the ambient norm of $\mathcal{B}(\mathcal{H})$:

$$\ell_\infty(\gamma) = \int_I \|\dot{\gamma}(t)\| dt. \quad (11)$$

Corollary 6.5. *Let $V \in \mathcal{I}(\mathcal{H})$ be balanced, and $\mathcal{V} \in (\mathcal{TI}(\mathcal{H}))_V$ orthogonal to V . Let (A, B) be a minimal lifting for \mathcal{V} . Then*

$$\delta(t) = e^{itA}V e^{-itB}$$

is minimal along its path, for $|t| \leq \frac{\pi}{2\|\mathcal{V}\|}$, when the lengths of curves are measured as in (11), with the ℓ_∞ functional.

Proof. By Corollary 5.8, we know that δ is minimal for the ℓ functional, for $|t| \leq \frac{\pi}{2\|\mathcal{V}\|_V} = \frac{\pi}{2\|\mathcal{V}\|}$, because $|\mathcal{V}|_V = \|\mathcal{V}\|$. For an arbitrary smooth curve γ in $\mathcal{I}(\mathcal{H})$, Theorem 5.4 implies that

$$\ell_\infty(\gamma) = \int_I \|\dot{\gamma}(t)\| dt \leq \int_I |\dot{\gamma}(t)|_{\gamma(t)} dt = \ell(\gamma).$$

Note that

$$\dot{\delta}(t) = e^{itA} \{iAV - iVB\} e^{-itB} = e^{itA} \mathcal{V} e^{-itB}.$$

Then, the facts that the metric $|\cdot|_V$ is invariant under the action of $\mathcal{U}(\mathcal{H}) \times \mathcal{U}(\mathcal{H})$, and that \mathcal{V} is orthogonal, imply that

$$|\dot{\delta}(t)|_{\delta(t)} = |\mathcal{V}|_V = \|\mathcal{V}\| = \|e^{itA} \mathcal{V} e^{-itB}\| = \|\dot{\delta}(t)\|,$$

and therefore $\ell(\delta) = \ell_\infty(\delta)$. □

Clearly $\mathcal{P}(\mathcal{H})$ is a complemented submanifold of $\mathcal{I}(\mathcal{H})$. We shall prove another consequence of Remark 6.4: if $P_0, P_1 \in \mathcal{P}(\mathcal{H})$ are regarded as points in $\mathcal{I}(\mathcal{H})$, and they can be joined by a minimal geodesic in $\mathcal{P}(\mathcal{H})$, then this path is minimal between P_0 and P_1 in $\mathcal{I}(\mathcal{H})$.

Before, let us recall the necessary and sufficient condition that P_0, P_1 must satisfy in order that they can be joined by a minimal geodesic of $\mathcal{P}(\mathcal{H})$ (see [6]):

Remark 6.6. Let $P_0, P_1 \in \mathcal{P}(\mathcal{H})$, then there exists a minimal geodesic of $\mathcal{P}(\mathcal{H})$ (or in fact, any geodesic) joining P_0 and P_1 if and only if

$$\dim(R(P_0) \cap N(P_1)) = \dim(R(P_1) \cap N(P_0)). \quad (12)$$

Corollary 6.7. Let $P_0, P_1 \in \mathcal{P}(\mathcal{H})$ satisfy condition (12). Let $\delta(t) \in \mathcal{P}(\mathcal{H})$ be a geodesic joining $\delta(0) = P_0$ and $\delta(1) = P_1$, and $\gamma(t) \in \mathcal{I}(\mathcal{H})$ be any other smooth curve joining P_0 and P_1 . Then

$$\ell(\delta) \leq \ell(\gamma).$$

Proof. First, note that if P is a projection and $\mathcal{V} \in (\mathcal{TP}(\mathcal{H}))_P$, then \mathcal{V} is P -co-diagonal: $P\mathcal{V}P = 0$ (or, in the notation employed here, \mathcal{V} is *orthogonal at P* , regarded as an element in $\mathcal{I}(\mathcal{H})$). This basic fact is well known in the geometry of $\mathcal{P}(\mathcal{H})$ (see [11]): if $P(t)$ is a smooth curve in $\mathcal{P}(\mathcal{H})$ with $P(0) = P$ and $\dot{P}(0) = \mathcal{V}$, then differentiating $P^2(t) = P(t)$ yields (at $t = 0$)

$$\mathcal{V}P + P\mathcal{V} = \mathcal{V},$$

which implies $P\mathcal{V}P = P^\perp \mathcal{V} P^\perp = 0$. Therefore, by Remark 6.4, $|\mathcal{V}|_P = \|\mathcal{V}\|$. It follows that $\ell_\infty(\delta) = \ell(\delta)$. On the other hand, by Proposition 6.1,

$$\ell_\infty(\alpha(\gamma)) \leq \ell(\gamma);$$

since δ is minimal in $\mathcal{P}(\mathcal{H})$,

$$\ell(\delta) = \ell_\infty(\delta) \leq \ell_\infty(\alpha(\gamma)),$$

and the proof follows. □

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