A note on geodesics of projections in the Calkin algebra

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Abstract

Let $\mathcal{C}(\mathcal{H})=\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ be the Calkin algebra ($\mathcal{B}(\mathcal{H})$ the algebra of bounded operators on the Hilbert space $\mathcal{H}$, $\mathcal{K}(\mathcal{H})$ the ideal of compact operators, and $\pi:\mathcal{B}(\mathcal{H})\to\mathcal{C}(\mathcal{H})$ the quotient map), and $\mathcal{P}_{\mathcal{C}(\mathcal{H})}$ the differentiable manifold of selfadjoint projections in $\mathcal{C}(\mathcal{H})$. A projection p in $\mathcal{C}(\mathcal{H})$ can be lifted to a projection $P\in\mathcal{B}(\mathcal{H})$: $\pi(P)=p$. We show that given $p,q\in\mathcal{P}_{\mathcal{C}(\mathcal{H})}$, there exists a *minimal geodesic* of $\mathcal{P}_{\mathcal{C}(\mathcal{H})}$ which joins p and q if and only if there exist lifting projections P and Q such that either both $N(P-Q\pm 1)$ are finite dimensional, or both infinite dimensional. The minimal geodesic is *unique* if $p+q-1$ has trivial anhihilator. Here the assertion that a geodesic is *minimal* means that it is shorter than any other piecewise smooth curve $\gamma(t)\in\mathcal{P}_{\mathcal{C}(\mathcal{H})}$, $t\in I$, joining the same endpoints, where the length of γ is measured by $\displaystyle\int_I\|\dot\gamma(t)\|dt$.

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1 Introduction

If $\mathcal{A}$ is a C*-algebra. let $\mathcal{P}_{\mathcal{A}}$ denote the set of (selfadjoint) projections in $\mathcal{A}$. $\mathcal{P}_{\mathcal{A}}$ has a rich geometric structure, see for instante the papers [6] by H. Porta and L.Recht and [2] by G. Corach, H. Porta and L. Recht. In these works, it was shown that $\mathcal{P}_{\mathcal{A}}$ is a differentiable (C^∞) complemented submanifold of $\mathcal{A}_s$, the set of selfadjoint elements of $\mathcal{A}$, and has a natural linear connection, whose geodesics can be explicitly computed. A metric is introduced, called in this context a Finsler metric: since the tangent spaces of $\mathcal{P}_{\mathcal{A}}$ are closed (complemented) linear subspaces of $\mathcal{A}_s$, they can be endowed with the norm metric. With this Finsler metric, Porta and Recht [6] showed that two projections $p,q\in\mathcal{P}_{\mathcal{A}}$ which satisfy that $\|p-q\|<1$ can be joined by a unique geodesic, which is minimal for the metric (i.e., it is shorter than any other smooth curve in $\mathcal{P}_{\mathcal{A}}$ joining the same endpoints).

In general, two projections p,q in $\mathcal{A}$ satisfy that $\|p-q\|\le 1$, so that what remains to consider is what happens in the extremal case $\|p-q\|=1$: under what conditions does there exist a geodesic, or a minimal geodesic, joining them. For general C*-algebras, this is too vast a question. In this note we shall consider it for the case of the Calkin algebra $\mathcal{A}=\mathcal{C}(\mathcal{H})=\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$,

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where $\mathcal{B}(\mathcal{H})$ is the algebra of bounded linear operators in a Hilbert space $\mathcal{H}$ and $\mathcal{K}(\mathcal{H})$ is the ideal of compact operators. Denote by $\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{C}(\mathcal{H})$ the quotient $*$-homomorphism.

Let $c \in \mathcal{C}(\mathcal{H})$ be a selfadjoint element. We say that c has *trivial anhihilator* if $cx = 0$ for $x \in \mathcal{C}(\mathcal{H})$ implies $x = 0$. Note that since $c^* = c$, this is equivalent to $xc = 0$ implies $x = 0$.

Clearly, if $p-q\pm 1$ have trivial anhilators, then for any lifting projecions P and Q (of p and q, respectively), $\dim N(P-Q\pm 1) < \infty$: if $\dim N(P-Q\pm 1) = +\infty$, then $(P-Q-1)P_{N(P-Q-1)} = 0$ so that $(p-q-1)\pi(P_{N(P-Q-1)}) = 0$ with $\pi(P_{N(P-Q-1)}) \neq 0$ (and similarly for $P_Q + 1)$. Also, these conditions are weaker than $p - q \pm 1$ being invertible.

Given $p, q \in \mathcal{P}_{\mathcal{C}(\mathcal{H})}$, throughout this note we denote $b = p + q$. We find a necessary and sufficient condition for the existence of a geodesic joining p and q in $\mathcal{P}_{\mathcal{C}(\mathcal{H})}$ (**Theorem 3.6**): there exists a geodesic joining p and q if and only if there exist lifting projections P and Q of p and q, such that $N(P - Q - 1)$ and $N(P - Q + 1$ are both finite dimensonal or both infinite dimensional.

Moreover, in the case of existence, also *minimal* a minimal geodesic exists.

With respect to *uniqueness* of minimal geodesics, we find a sufficient condition described in terms of b: there exists a unique minimal geodesics if $b - 1$ has trivial anhihilator (**Theorem 3.7**).

Note the following elementary relation, put $a = p - q$: $a^2 + b^2 = 2b$, so that $(b - 1)^2 = (1 - a)(1 + a)$. Thus, if $b - 1$ has trivial anhihilator, then $(b - 1)^2$ also has this property, and therefore both $a \pm 1$ have this property, which shows that the uniqueness sufficient condition is stronger than the existence condition. Also note that if $\|p - q\| < 1$, then $a \pm 1$ and thus also $b - 1$ are invertible, so that the conditions of the above theorems are weaker than the condition $\|p - q\| < 1$.

Section 2 contains preliminary facts. In Section 3 the main results are stated.

2 Preliminaries

The space $\mathcal{P}_{\mathcal{A}}$ is sometimes called the Grassmann manifold of $\mathcal{A}$. In the case when $\mathcal{A} = \mathcal{B}(\mathcal{H})$, $\mathcal{P}_{\mathcal{B}(\mathcal{H})}$ parametrizes the set of closed subspaces of $\mathcal{H}$: to each closed subspace $\mathcal{S} \subset \mathcal{H}$ corresponds the orthogonal projection $P_{\mathcal{S}}$ onto $\mathcal{S}$. Let us describe below the main features of the geometry of $\mathcal{P}_{\mathcal{A}}$ in the general case.

2.1 Homogeneous structure

Denote by $\mathcal{U}_{\mathcal{A}} = \{u \in \mathcal{A} : u^*u = uu^* = 1\}$ the unitary group of $\mathcal{A}$. It is a Banach-Lie group, whose Banach-Lie algebra is $\mathcal{A}_{as} = \{x \in \mathcal{A} : x^* = -x\}$. This group acts on $\mathcal{P}_{\mathcal{A}}$ by means of $u \cdot p = upu^*$, $u \in \mathcal{U}_{\mathcal{A}}$, $p \in \mathcal{P}_{\mathcal{A}}$. The action is smooth and locally transitive. It is known (see [6], [2]) that $\mathcal{P}_{\mathcal{A}}$ is what in differential geometry is called a homogeneous space of the group $\mathcal{U}_{\mathcal{A}}$. The local structure of $\mathcal{P}_{\mathcal{A}}$ is described using this action. For instance, the tangent space $(T\mathcal{P}_{\mathcal{A}})_p$ of $\mathcal{P}_{\mathcal{A}}$ at p is given by $(T\mathcal{P}_{\mathcal{A}})_p = \{x \in \mathcal{A}_s : x = px + xp\}$.

The isotropy subgroup of the action at p, i.e., the elements of $\mathcal{U}_{\mathcal{A}}$ which fix a given p, is $\mathcal{I}_p = \{v \in \mathcal{U}_{\mathcal{A}} : vp = pv\}$. The isotropy algebra $\mathfrak{I}_p$ at p is its Banach-Lie algebra $\mathfrak{I}_p = \{y \in \mathcal{A}_{as} : yp = py\}$.

2.2 Reductive structure

Given an homogeneous space, a *reductive* structure is a smooth distribution $p \mapsto \mathbf{H}_p \subset \mathcal{A}_{as}$, $p \in \mathcal{P}_\mathcal{A}$, of supplements of $\mathfrak{I}_p$ in $\mathcal{A}_{as}$, which is invariant under the action of $\mathcal{I}_p$. That is, a distribution $\mathbf{H}_p$ of closed linear subspaces of $\mathcal{A}_{as}$ verifying that $\mathbf{H}_p \oplus \mathfrak{I}_p = \mathcal{A}_{as}$; $v\mathbf{H}_p v^* = \mathbf{H}_p$ for all $v \in \mathcal{I}_p$; and the map $p \mapsto \mathbf{H}_p$ is smooth.

In the case of $\mathcal{P}_\mathcal{A}$, the choice of the (so called) *horizontal* subspaces $\mathbf{H}_p$ is fairly natural, if one represents elements of $\mathcal{A}$ as matrices. Having fixed a base point $p \in \mathcal{P}_\mathcal{A}$, any element $a \in \mathcal{A}$ can be represented as a 2×2 matrix. Then, it is clear that

$$\mathcal{I}_p = \{\begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix} : v_1^* v_1 = v_1 v_1^* = p, v_2^* v_2 = v_2 v_2^* = p^\perp\} \ , \quad \mathfrak{I}_p = \{\begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix} : y_i^* = -y_i\}.$$

Thus, the natural choice for $\mathbf{H}_p$ defined in [2] is $\mathbf{H}_p = \{\begin{pmatrix} 0 & z \\ -z^* & 0 \end{pmatrix} : z \in p\mathcal{A}p^\perp\}$.

It is worth noting that tangent vectors at p have *selfadjoint* codiagonal matrices.

As in classical differential geometry, a reductive structure on a homogeneous space defines a linear connection, and all the invariants of the linear connection (covariant derivative, torsion and curvature tensors, etc) can be computed in terms of horizontal elements [2], [6]. We shall focus here on *geodesics*. Given a base point $p \in \mathcal{P}_\mathcal{A}$, and a tangent vector $\mathbf{x} = \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} \in (T\mathcal{P}_\mathcal{A})_p$, the unique geodesic δ of $\mathcal{P}_\mathcal{A}$ with $\delta(0) = p$ and $\dot\delta(0) = \mathbf{x}$ is given by

$$\delta(t) = e^{tz_\mathbf{x}} p e^{-tz_\mathbf{x}},$$

where $z_\mathbf{x} := \begin{pmatrix} 0 & -x \\ x^* & 0 \end{pmatrix}$.

2.3 Finsler metric

As we mentioned above, one endows each tangent space $(T\mathcal{P}_\mathcal{A})_p$ with the usual norm of $\mathcal{A}$. We emphasize that this (constant) distribution of norms is not a Riemannian metric (the C*-norm is not given by an inner product), neither is it a Finsler metric in the classical sense (the map $a \mapsto \|a\|$ is non differentiable). Therefore the minimality result which we describe below does not follow from general considerations. It was proved in [6] using ad-hoc techniques.

1. Given $p \in \mathcal{P}_\mathcal{A}$ and $\mathbf{x} \in (T\mathcal{P}_\mathcal{A})_p$, normalized so that $\|\mathbf{x}\| \leq \pi/2$, then the geodesic δ remains minimal for all t such that $|t| \leq 1$.

2. Given $p, q \in \mathcal{P}_\mathcal{A}$ such that $\|p - q\| < 1$, there exists a unique minimal geodesic δ such that $\delta(0) = p$ and $\delta(1) = q$.

We shall call these geodesics (with initial speed $\|\mathbf{x}\| \leq \pi/2$) *normalized* geodesics.

Remark 2.1. One does not have to deal with homogeneous reductive spaces in order to understand (at least partially) these results. Using the C*-norm at every tangent space, means that a curve δ as above (never mind calling it a geodesic) has the following property: for every piecewise differentiable curve $\gamma(t) \in \mathcal{P}_\mathcal{A}$ whose endpoints are $\gamma(t_0) = p$ and $\gamma(t_1) = q$, it holds that

$$\int_{t_0}^{t_1} \|\dot\gamma(t)\| dt \geq \int_0^1 \|\dot\delta(t)\| dt = \|\mathbf{x}\|.$$

2.4 Projections in $\mathcal{B}(\mathcal{H})$

Let us finish this preliminary section by recalling the case $\mathcal{A} = \mathcal{B}(\mathcal{H})$. Given $P, Q \in \mathcal{P}_{\mathcal{B}(\mathcal{H})}$ (operators in Hilbert space will be denoted with upper case letters, this inconsistency will prove useful later, when we deal with the Calkin algebra), in [1] it was proved that there exists a geodesic of the linear connection just described, which joins P and Q, if and only if $\dim R(P) \cap N(Q) = \dim N(P) \cap R(Q)$.

The geodesic can be chosen minimal. There is a *unique* minimal geodesic joining these points if and only if these dimensions equal zero.

Note the fact that the condition $\|P - Q\| < 1$ implies that these dimensions are zero, but the converse is not true.

3 Projections in the Calkin algebra

From now on, we consider the case of $\mathcal{A} = \mathcal{C}(\mathcal{H})$. Tipically, upper case letters P, A, X will denote elements of $\mathcal{B}(\mathcal{H})$ and their lower case counterparts $p = \pi(P), a = \pi(A), x = \pi(X)$ the corresponding elements of $\mathcal{C}(\mathcal{H})$.

Let us point out the following elementary facts:

Lemma 3.1. *Let $p \in \mathcal{P}_{\mathcal{C}(\mathcal{H})}$, $p \neq 0, 1$. Then there exists $P \in \mathcal{P}_{\mathcal{B}(\mathcal{H})}$ such that $\pi(P) = p$*

Proof. Let $T \in \mathcal{B}(\mathcal{H})$ such that $\pi(T) = p$. Since $p^* = p$, $\pi(\frac{1}{2}(T + T^*)) = p$, i.e., we can suppose that $T^* = T$. Clearly $T^2 - T \in \mathcal{K}(\mathcal{H})$, and thus the spectrum of T accumulates only (eventualy) at 0 and 1. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a continuous function which is equal to 1 on an interval I containing 1 in its interior, $0 \notin I$, such that φ is zero in $\sigma(T) \cap (\mathbb{R} \setminus I)$ (note that the spectrum $\sigma(T)$ is countable). Then $P = \varphi(T)$ is a selfadjoint projection, and

$$\pi(P) = \varphi(\pi(T)) = \varphi(p) = p,$$

$\square$

Remark 3.2. Note that any pair of proper projections p, q ($\neq 0, 1$) in $\mathcal{C}(\mathcal{H})$ are unitarilly equivalent. Indeed, let $P, Q \in \mathcal{P}_{\mathcal{B}(\mathcal{H})}$ such that $\pi(P) = p$ and $\pi(Q) = q$. Thus, $P, P^\perp, Q$ and $Q^\perp$ have infinite rank ($p, q \neq 0, 1$). Then P and Q are unitarilly equivalent: there exists $U \in \mathcal{U}_{\mathcal{B}(\mathcal{H})}$ such that $UPU^* = Q$. Then $u = \pi(U)$ is a unitary element in $\mathcal{C}(\mathcal{H})$ such that $upu^* = q$.

If $\Delta(t) = e^{tZ} P e^{-tZ}$ is a geodesic in $\mathcal{P}_{\mathcal{B}(\mathcal{H})}$ with $\|Z\| \leq \pi/2$, then it is clear that $\delta(t) = \pi(\Delta(t))$ is a geodesic in $\mathcal{P}_{\mathcal{C}(\mathcal{H})}$ with $\|z\| \leq \pi/2$. Indeed,

$$\delta(t) = e^{tz} p e^{-tz},$$

where $\pi(P) = p$, and $z = \pi(Z)$ satifies $z^* = -z$ and

$$pzp = p^\perp z p^\perp = \pi(PZP) = \pi(P^\perp Z P^\perp) = 0.$$

Moreover, $\|z\| \leq \|Z\| \leq \pi/2$. The next result shows that there is a converse for this statement: any geodesic δ of $\mathcal{P}_{\mathcal{C}(\mathcal{H})}$ with initial speed $\|z\| \leq \pi/2$ lifts to a geodesic Δ with initial speed $\|Z\| \leq \pi/2$. It is based on the following elementary observation, which is an excercise and certainly well known. We include a proof.

Lemma 3.3. *Let $x = x^* \in \mathcal{C}(\mathcal{H})$. Then there exists $X = X^* \in \mathcal{B}(\mathcal{H})$ such that $\pi(X) = x$ and $\|X\| = \|x\|$.*

Proof. Clearly, there exists $X_0 = X_0^* \in \mathcal{B}(\mathcal{H})$ such that $\pi(X_0) = x$. Recall the Weyl-von Neumann theorem (see for instance [3]), which states that there exists a diagonalizable selfadjoint operator X_d and a compact operator K such that $X_0 = X_d + K$. Thus, we may suppose that X_0 is diagonalizable, and let us fix the orthonormal basis of $\mathcal{H}$ which diagonalizes X_0. Denote by $\mathbf{d} = \{d_n\}$ the sequence of entries of X_0. It suffices to find a sequence $\mathbf{k} \in c_0$ (the space of sequences which converge to zero) such that

$$\|\mathbf{d} + \mathbf{k}\|_\infty \leq \ \|\mathbf{d} + \mathbf{k}'\|_\infty$$

for any other sequence $\mathbf{k}' \in c_0$. Indeed, denote by $\mathcal{D} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ the linear positive contraction which assigns to T the diagonal operator $\mathcal{D}(T)$ with the same diagonal entries as T. Then, if K_0 denotes the diagonal compact operator with $\mathbf{k}$ in its diagonal, and K' is any other compact operator (with diagonal $\mathbf{k}'$)

$$\|X_0 + K_0\| = \|\mathbf{d} + \mathbf{k}\|_\infty \leq \|\mathbf{d} + \mathbf{k}'\|_\infty = \|\mathcal{D}(X_0 + K')\| \leq \|X_0 + K'\|.$$

In order to find an optimal sequence $\mathbf{k} \in c_0$ such that $\|\mathbf{d}+\mathbf{k}\|_\infty$ is as small as possible, note that

$$\inf\{\|\mathbf{d} + \mathbf{k}'\|_\infty : \mathbf{k}' \in c_0\} = \limsup |\mathbf{d}|.$$

Indeed, on one hand, there can only be finitely many d_n such that $d_n < \limsup \mathbf{d}$, so that $\inf\{\|\mathbf{d} + \mathbf{k}'\|_\infty : \mathbf{k}' \in c_0\} \leq \limsup |\mathbf{d}|$. On the other hand, for any $\mathbf{k}' = \{k'_n\} \in c_0$,

$$\limsup |\mathbf{d}| = \limsup |\mathbf{d} + \mathbf{k}'| \leq \sup_n |d_n + k'_n| = \|\mathbf{d} + \mathbf{k}'\|_\infty.$$

Next, note that there exists an optimal $\mathbf{k} \in c_0$ such that $\|\mathbf{d} + \mathbf{k}\|_\infty = \limsup |\mathbf{d}|$. Clearly it suffices to reason with sequences of non negative numbers. Define

$$\mathbf{d}_0 = \begin{cases} \limsup \mathbf{d} & \text{if} \quad d_n > \limsup \mathbf{d} \\ d_n & \text{if} \quad d_n \leq \limsup \mathbf{d} \end{cases}$$

and

$$\mathbf{k}_0 = \begin{cases} d_n - \limsup \mathbf{d} & \text{if} \quad d_n > \limsup \mathbf{d} \\ 0 & \text{if} \quad d_n \leq \limsup \mathbf{d} \end{cases}.$$

Clearly $\mathbf{d} + \mathbf{k} = \mathbf{d}_0 + \mathbf{k}_0$ and $\|\mathbf{d}_0\|_\infty = \limsup \mathbf{d}$. Finally, $\mathbf{k}_0 \in c_0$: any subsequence of of $\mathbf{k}_0$ has a converging sebsequence, which can only converge to zero. $\square$

Proposition 3.4. *Let $\delta(t) = e^{tz} p e^{-tz}$ be a (normalized) geodesic of $\mathcal{P}_{\mathcal{C}(\mathcal{H})}$: $z^* = -z$, $pzp = p^\perp z p^\perp = 0$ and $\|z\| \leq \pi/2$. Then there exists a normalized geodesic $\Delta(t) = e^{tZ} P e^{-tZ}$ which lifts δ: $Z^* = -Z$, $PZP = P^\perp Z P^\perp = 0$, $\|Z\| \leq \pi/2$, $\pi(P) = p$ and $\pi(Z) = z$*

Proof. Given $z^* = -z$, by the above lemma (iz is selfadjoint), there exists $Z_0 \in \mathcal{B}(\mathcal{H})$ such that $Z_0^* = -Z_0$, $\pi(Z_0) = z$ and $\|Z_0\| = \|z\|$. Pick $P \in \mathcal{P}_{\mathcal{B}(\mathcal{H})}$ such that $\pi(P) = p$. Consider

$$Z = P Z_0 P^\perp + P^\perp Z_0 P.$$

Clearly $\pi(Z) = pzp^\perp + p^\perp zp = z$, because z is p-codiagonal. Also, Z is P-codiagonal, and

$$Z^* = (PZ_0P^\perp)^* + (P^\perp Z_0 P)^* = -P^\perp Z_0 P - PZ_0P^\perp = -Z.$$

In matrix form, $Z = \begin{pmatrix} 0 & PZ_0P^\perp \\ P^\perp Z_0 P & 0 \end{pmatrix}$, so that

$$\|Z\| = \|PZ_0P^\perp\| = \|P^\perp Z_0 P\| \leq \|Z_0\|.$$

Since $\|Z_0\| = \|z\| \geq \|Z\|$, equality holds. $\square$

Remark 3.5. Note that when lifting a geodesic δ of $\mathcal{P}_{\mathcal{C}(\mathcal{H})}$ to a geodesic Δ of $\mathcal{P}_{\mathcal{B}(\mathcal{H})}$, the initial point $\Delta(0) = P$ can be chosen to be any projection in the fiber of $\pi^{-1}(\delta(0))$ of $\delta(0)$.

Our main result follows.

Theorem 3.6. *Let p, q be proper projections in $\mathcal{C}(\mathcal{H})$ (i.e., $p, q \neq 0, 1$). Then there exists a geodesic of $\mathcal{P}_{\mathcal{C}(\mathcal{H})}$ joining p and q if and only there exist liftings projections P and Q of p and q ($\pi(P) = p$, $\pi(Q) = q$) such that $N(P - Q - 1)$ and $N(P - Q + 1)$ are both finite dimensional, or else, are both infinite dimensional.*

In either case, the geodesic can be chosen minimal.

Proof. Suppose first that there exists a geodesic $\delta(t) = e^{tz}pe^{-tz}$ such that $\delta(1) = q$. Then δ lifts to $\Delta(t) = e^{tZ}Pe^{-tZ}$. Denote $Q = \Delta(1)$. The existence of Δ implies that (see[1]) $\dim R(P) \cap N(Q) = \dim N(P) \cap R(Q)$. It is easy to see that

$$R(P) \cap N(Q) = \{\xi \in \mathcal{H} : P\xi = \xi \text{ and } Q\xi = 0\} = \{\xi \in \mathcal{H} : (P - Q)\xi = \xi\} = N(P - Q - 1).$$

Similarly, $N(P) \cap R(Q) = N(P - Q + 1)$. Thus the necessary part is clear.

Conversely, let P, Q be projections in $\mathcal{B}(\mathcal{H})$ such that $\pi(P) = p$ and $\pi(Q) = q$. If $P - Q + 1$ and $P - Q - 1$ have finite dimensional nullspaces. Let us consider the following 5-space decomposition of $\mathcal{H}$ which is customary when dealing with two subspaces ([4], [5]):

$$\mathcal{H} = R(P) \cap R(Q) \;\oplus\; N(P) \cap N(Q) \;\oplus\; R(P) \cap N(Q) \;\oplus\; N(P) \cap R(Q) \;\oplus\; \mathcal{H}_0,$$

where $\mathcal{H}_0$, the orthogonal complement of the sum of the first four, is usually called the *generic part* of P and Q. It is known, that these subspaces reduce both P and Q, and that the reduction of our problem to the generic part has a positive solution: on the generic part, there exists a unique (minimal) geodesic joining these projections (see [1]). Since we are currently supposing that $R(P) \cap N(Q)$ and $N(P) \cap R(Q)$ are finite dimensional, we can replace P and Q, which in this decomposition are

$$P = 1 \oplus 0 \oplus 1 \oplus 0 \oplus P_0 \quad \text{and} \quad Q = 1 \oplus 0 \oplus 0 \oplus 1 \oplus Q_0$$

with P', Q' given by

$$P' = 1 \oplus 0 \oplus 0 \oplus 0 \oplus P_0 \quad \text{and} \quad Q' = 1 \oplus 0 \oplus 0 \oplus 0 \oplus Q_0,$$

which are indeed projections, and differ from P, Q on a finite dimensional subspace. Thus $\pi(P') = p$ and $\pi(Q') = q$. Note that

$$R(P') \cap N(Q') = N(P') \cap R(Q') = \{0\}.$$

Therefore there exists a geodesic $\Delta(t) = e^{tZ}P'e^{-tZ}$ joining P' and Q' in $\mathcal{P}_{\mathcal{B}(\mathcal{H})}$. Thus $\delta = \pi(\Delta)$ is a geodesic in $\mathcal{P}_{\mathcal{C}(\mathcal{H})}$ joining p and q.

If $R(P) \cap N(Q)$ and $N(P) \cap R(Q)$ are infinite dimensional, there exists a normalized geodesic of $\mathcal{P}(\mathcal{H})$ joining P and Q, and thus the claim follows similarly in this case.

In either case, the geodesics are given by exponents z or Z with norm less or equal than $\pi/2$, so that they are minimal. $\square$

For the uniqueness of normalized geodesics, we have the following sufficient condition in terms of $b = p + q$:

Theorem 3.7. *If $b-1$ has trivial anhililator, then there exists a unique minimal geodesic between $p, q \in \mathcal{P}_{\mathcal{C}(\mathcal{H})}$.*

Proof. Suppose that $b - 1$ has trivial anhihilator. Then, as remarked in the Introduction, $\dim N(P - Q \pm 1) < \infty$. Let δ and δ' be two normalized minimal geodesics joining p and q: $\delta(t) = e^{tz}pe^{-tz}$, $\delta'(t) = e^{tz'}pe^{-tz'}$, $\|z\| = \|z'\| \leq \pi/2$, with z, z' p-codiagonal, and $\delta(1) = \delta'(1) = q$. Then, as in the proof of Theorem 3.6, there exist projections P and Q, relative to the geodesic δ, such that $R(P) \cap N(Q) = N(P) \cap R(Q) = \{0\}$, and $Z^* = -Z$ P-codiagonal with $\|Z\| = \|z\|$ such that $\Delta(t) = e^{tZ}Pe^{-tZ}$ lifts δ. Denote $Q = \Delta(1)$. Similarly, let P', Q', Z' the data corresponding to δ', with $R(P') \cap N(Q') = N(P') \cap R(Q') = \{0\}$. Note that $P - P'$ and $Q - Q'$ are compact. The exponents Z and Z' are uniquely determined by P, Q and P', Q' respectively. Let us recall from [1] how they are constructed. The operator $B - 1 = P + Q - 1$ is selfadjoint and has trivial nullspace. Indeed, since $B - 1 = P - Q^\perp$ is a difference of projections, its nullspace is trivial

$$N(B - 1) = R(P) \cap R(Q^\perp) \oplus N(P) \cap N(Q^\perp) = R(P) \cap N(Q) \oplus N(Q) \cap R(P) = \{0\}.$$

Then, if $B - 1 = V|B - 1|$ is the polar decomposition, $|B - 1|$ has trivial nullspace and V is a symmetry ($V^* = V^{-1} = V$). Then, it was shown in [1] that

$$e^Z = V(2P - 1). \tag{1}$$

In fact, this formula was shown for projections in generic position. In the case at hand, we have also to take the subspace $R(P) \cap R(Q) \oplus N(P) \cap N(Q)$ into account. On this subspace, P and Q coincide, so that $P + Q - 1$ equals $2P - 1$, thus $|P + Q - 1|$ is the identity and therefore V coincides with $2P - 1$. On the other hand, since P and Q coincide here, the exponent Z of the minimal geodesic is 0. Thus, on $R(P) \cap R(Q) \oplus N(P) \cap N(Q)$, equation (1) is $e^0 = (2P - 1)(2P - 1) = 1$, which is trivial.

Equation (1) determines Z, as the unique anti-Hermitian logarithm of $V(2P - 1)$ with spectrum in the interval $[-\pi/2, \pi/2]$. Analogously, we have that $e^{iZ'} = V'(2P' - 1)$, where V' is the unitary part in the polar decompostion of $B' - 1 = P' + Q' - 1$. Thus, in order to show that $z = z'$, it suffices to show that $e^z = e^{z'}$, i.e., that $\pi(V')(2p - 1) = \pi(V')(2p - 1)$. Since $2p - 1$ is invertible, we have to show that $\pi(V) = \pi(V')$. Note that $b = \pi(B) = \pi(B')$, and since π is a $*$-homomorphism, $\pi(|B - 1|) = |\pi(B - 1)| = |b - 1|$. Thus $b - 1 = \pi(V)|b - 1| = \pi(V')|b - 1|$, or equivalently, $\pi(V)(b - 1) = \pi(V')(b - 1)$. Then $(\pi(V) - \pi(V'))(b - 1) = 0$, and since $b - 1$ has trivial anhihilator, $\pi(V) = \pi(V')$. $\square$

Proposition 3.8. *Let $p, q \in \mathcal{P}_{\mathcal{C}(\mathcal{H})}$. Suppose that there exist lifting projections P and Q such that* $\dim N(P - Q \pm 1) = +\infty$. *Then there exist infinitely many minimal geodesics joining p and q in $\mathcal{P}_{\mathcal{C}(\mathcal{H})}$.*

Proof. Let $P, Q \in \mathcal{P}_{\mathcal{B}(\mathcal{H})}$ such that $\pi(P) = p$ and $\pi(Q) = q$. Then $R(P) \cap N(Q)$ and $N(P) \cap R(Q)$ are infinite dimensional. Then there exist isometries

$$V, V' : N(P) \cap R(Q) \to R(P) \cap N(Q)$$

such that $V - V'$ is not compact. For instance, let V be such an isometry, and pick a unitary U acting in $R(P) \cap N(Q)$ such that $U - 1$ is not compact, and put $V' = UV$. There are infinitely many unitaries U with this property, such that $\pi(U)$ are different: the unitary group of $\mathcal{C}(R(P) \cap N(Q))$ is infinite. Then (see for instance [1]), one constructs geodesics between P and Q determining its velocity vectors as follows:

$$Z = 0 \oplus 0 \oplus \{i\pi/2(V + V^*)\} \oplus Z_0,$$

in the (four space) decomposition

$$\mathcal{H} = R(P) \cap R(Q) \ \oplus N(P) \cap N(Q) \ \oplus \{R(P) \cap N(Q) \ \oplus \ N(P) \cap R(Q)\} \ \oplus \mathcal{H}_0,$$

and Z' defined analogously, with V'. The part Z_0 acting in $\mathcal{H}_0$ ($\|Z_0\| \leq \pi/2$) is uniquely determined. Then clearly $Z - Z'$ is non compact. It follows that if $z = \pi(Z)$ and $z' = \pi(Z')$, then $\delta(t) = e^{tz} p e^{-tz}$ and $\delta'(t) = e^{tz'} p e^{-tz'}$ are two geodesics joining p and q, with

$$\dot{\delta}(0) - \dot{\delta}'(0) = z - z' \neq 0,$$

i.e., $\delta \neq \delta'$. □

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