On the Radon Transform of Stationary Random Fields

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Abstract. Let $\mathscr{X}$ be a wide sense stationary random field. It is known that $\mathscr{X}$ has no square integrable sample paths, therefore its Radon transform is not well defined as an ordinary function. However, it is possible to define its Radon transform as a generalized function a.s.. We shall see that this transform is the limit of a sequence of -ordinary- Radon transforms of truncated paths of $\mathscr{X}$. We will prove how this sequence of transforms relates to the spectral representation of $\mathscr{X}$.

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1 Introduction

The Radon transform is a very important tool in practical applications such as Computed Tomography and Aperture Synthesis [3]. They are also of interest its relations with partial differential equations [6] and other areas of analysis. It is basically developed for integrable and square integrable functions. On the other hand wide sense stationary random (w.s.s.) fields are also widely used in applications related to image and signal processing among others. However, these random fields has not a.s. square integrable sample paths so that a generalized harmonic analysis is adopted for them instead of calculating ordinary Fourier transforms over its sample paths. Similar questions arise when considering other transforms. Here we deal, partially, with the problem of adapting the Radon transform to w.s.s. fields. Some few attempts in this direction have been made with modified definitions of the Radon transform (e.g.[10, 11]). Here we shall see that, at least, one can define the Radon transform of a sample path in the sense of generalized functions a.s.. We will show that in some cases it is possible to find a closed analytical expression under this weak sense formulation. For a more practical point of view, we can prove that this transform is the sequential limit, in a distributional sense, of the ordinary Radon transform of truncated sample paths a.s.. Moreover, we prove how these sequences relates to the spectral representation of the random field in a partial analogy to the usual property relating the ordinary Fourier and Radon transforms of integrable functions (see section 3).

2 Some definitions and notation

If $A \subseteq \mathbb{R}^d$, its interior will be denoted by $\overset{\circ}{A}$, its boundary by ∂A and its Lebesgue measure by $|A|$. The indicator function of A will be $\mathbf{1}_A$. Finally if $x \in \mathbb{C}^d$ for its euclidean norm we will write $|x|$ (the difference with the Lebesgue measure of a set will be clear from the context). In addition, if $y \in \mathbb{C}^d$ the inner product with x is $x.y$. $\mathbb{S}^{d-1}$ will denote the unit sphere in $\mathbb{R}^d$ and $\mathbb{S}^{d-1} \times \mathbb{R}$ the cylider in $\mathbb{R}^{d+1}$.

2.1 Radon and Fourier transforms.

Generally, the basic formulations for the Radon transform are made over a suitable space of test functions K. The extension to the dual space K' is rather canonical, e.g. Chapter I of [4] (Also [3] or [6] for more details). In accordance with our further developments we begin defining the Radon transform for test functions in $K = \mathcal{D}(\mathbb{R}^d)$ the space of $C^\infty(\mathbb{R}^d)$ with compact support. From this, one obtains a definition of the transform over its dual $\mathcal{D}'(\mathbb{R}^d)$ the space of *distributions*. The same construction can be made over the class of Schwartz functions $K = \mathcal{S}(\mathbb{R}^d)$.

Definition 2.1. *Let $f \in \mathcal{D}(\mathbb{R}^d)$, its Radon Transform $\mathbf{R}f : \mathbb{S}^{d-1} \times \mathbb{R} \longrightarrow \mathbb{R}$ is defined by:*

$$\mathbf{R}f(\xi, p) = \int_{\{x:\xi.x=p\}} f(x)dx = \int_{\mathbb{R}^d} \delta(p - \xi.x)f(x)dx, \quad \xi \in \mathbb{S}^{d-1}, \, p \in \mathbb{R}.$$

Briefly, the Radon transform is defined by the integrals of f over the hyperplanes $\{x : \xi.x = p\}$. Obviously, $\mathbf{R}$ is a linear transformation and it maps $\mathcal{D}(\mathbb{R}^d)$ or $\mathcal{S}(\mathbb{R}^d)$ into the space of test functions on the cylinder $\mathcal{D}(\mathbb{S}^{d-1} \times \mathbb{R})$ (or $\mathcal{S}(\mathbb{S}^{d-1} \times \mathbb{R})$ respectively). A basic result about the Radon transform is the *shift property* i.e. if $a \in \mathbb{R}^d$:

$$\mathbf{R}f(\,.\, - a)(\xi, p) = \mathbf{R}f(\xi, p - \xi.a) = (\tau_a \mathbf{R}f)(\xi, p)\,.$$

For $\Phi \in \mathcal{S}(\mathbb{S}^{d-1} \times \mathbb{R})$ we define (for more details see e.g. [3])

$$\Psi\Phi(\xi, p) = \begin{cases} C_d \frac{\partial^{d-1}}{\partial p^{d-1}} \Phi(\xi, p) \;\; if \;\; d \;\; is \; odd \\ -iC_d \mathcal{H} \frac{\partial^{d-1}}{\partial p^{d-1}} \Phi(\xi, p) \;\; if \;\; d \;\; is \; even \end{cases},$$

where $\mathcal{H}$ denotes the one dimensional Hilbert transform in the p variable and $C_d = \frac{(2\pi i)^{1-d}}{2}$. If $(. , .)$ and $[. , .]$ denote the usual inner products of $L^2(\mathbb{R}^d)$ and $L^2(\mathbb{S}^{d-1} \times \mathbb{R})$ respectively, then the Radon transform verifies a Plancherel like formula [3, 4] for every $f, g \in \mathcal{D}(\mathbb{R}^d)$:

$$(f, g) = \int\limits_{\mathbb{R}^d} f(x)\overline{g(x)}dx = \int\limits_{\mathbb{S}^{d-1}} \int\limits_{\mathbb{R}} \mathbf{R}f(\xi, p)\overline{\mathbf{\Psi R}g(\xi, p)}dp \, d\Omega(\xi) \tag{1}$$

$$= [\mathbf{R}f, \Psi\mathbf{R}g].$$

This formula extends for $f, g \in L^2(\mathbb{R}^2)$. From this departs a definition of the Radon transform to elements of $\mathcal{D}'(\mathbb{R}^d)$ [3, 4]. Briefly, for fixed f note that equation 1, if $\Phi = \Psi\mathbf{R}g$, can be rewritten as

$$(f, g) = [\mathbf{R}f, \Phi], \tag{2}$$

defining the Radon transform as a functional. With some abuse in keeping the same notation, as the line integral of definition 2.1 may not be defined for $f \in \mathcal{D}'(\mathbb{R}^d)$, equation 2 extends the definition of $\mathbf{R}$ to $\mathcal{D}'(\mathbb{R}^d)$. If $f \in \mathcal{D}'(\mathbb{R}^d)$ is *fixed*, we shall call the Radon transform of f to the functional defined by equation 2 for every $\Phi \in \Psi\mathbf{R}(\mathcal{D}(\mathbb{R}^d))$ or equivalently $\Phi = \Psi\mathbf{R}g$, $g \in \mathcal{D}(\mathbb{R}^d)$). The Fourier transform of $f \in L^1(\mathbb{R}^d)$ is given by:

$$\mathcal{F}_d f(\lambda) = \widehat{f}(\lambda) = \int\limits_{\mathbb{R}^d} f(x)e^{-i\lambda . x}dx \, .$$

The suffix d will remark that the Fourier transform is taken over all the d independent variables of f. We will also consider -partial- Fourier transforms. In fact, we will consider the Fourier transform of functions $\Phi(\xi, p)$, $\xi \in \mathbb{S}^{d-1}$, $p \in \mathbb{R}$ with respect to p:

$$\mathcal{F}_1\Phi(\xi, \, . \,)(s) = \int\limits_{\mathbb{R}} \Phi(\xi, p)e^{-isp}dp \, .$$

2.2 Wide sense stationary processes

In this work we shall assume that $\mathscr{X}(x)$, $x \in \mathbb{R}^d$ is a real, wide sense stationary and mean square continuous random field (For more details on these topics see [4] or [9, 8]) over a probability space $(\Omega, \mathscr{F}, \mathbf{P})$. With no loss of generality we assume that its mean is zero. Now we describe briefly the main properties of these random fields. Firstly, on the forthcoming, from the stochastic continuity of $\mathscr{X}$ we will assume that we are working with a measurable equivalent version of it (see e.g. Theorem 1, p, 157 [5]). We shall no make more mention of this underlying device. In the stationary case, $\mathscr{X}$ admits an spectral representation in the form of a stochastic integral with respect an *orthogonal random measure* M :

$$\mathscr{X}(x) = \int_{\mathbb{R}^d} e^{i\lambda.x}\, dM(\lambda)\,. \tag{3}$$

If $\mathbf{E}(\,.\,)$ denotes the expectation of a random variable, the *spectral measure* μ of $\mathscr{X}$ is the unique non negative finite Borel measure such that $\mathbf{E}(M(A))^2 = \mu(A)$. Under the stationarity hypothesis, the covariance of $\mathscr{X}$ verifies $R_{\mathscr{X}}(x - x') = \mathbf{E}(\mathscr{X}(x)\mathscr{X}(x'))$. Moreover, the covariance is the Fourier transform of the spectral measure, i.e. $R_{\mathscr{X}} = \hat{\mu}$. A key result in the theory of w.s.s. process is that the Hilbert space $H(\mathscr{X}) = \overline{span}\{\mathscr{X}(x), x \in \mathbb{R}^d\}$ (in $L^2(\Omega, \mathscr{F}, \mathbf{P})$) is isometrically isomorphic to $L^2(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \mu)$. Moreover, if $\varphi \in L^2(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \mu)$ this isomorphism is established by the stochastic integral with respect to M:

$$\varphi \longmapsto I(\varphi) = \int_{\mathbb{R}^d} \varphi(\lambda) dM(\lambda)\ ,\ \ \mathbf{E}\,|I(\varphi)|^2 = \|\varphi\|^2_{L^2(\mu)}\ . \tag{4}$$

Conversely, if μ is a finite Borel measure, there exists a w.s.s. stationary random field with μ as its spectral measure. If μ is absolutely continuous with respect to the Lebesgue measure then its R-N derivative $S_{\mathscr{X}}$ will be called the *spectral density* of the field $\mathscr{X}$. For the range of validity of these formulations and details we refer to [5, 8]. Finally, we state a result on the interchange of an stochastic integral with an ordinary integral with respect to the Lebesgue measure. This simple lemma is an adaptation for convenience the reader of one presented in [5]. So the proof is left to the reader. The measures M and μ are as described before.

Lemma **2.2.** *Let $g(\lambda, x)$ and $h(x)$ be two Borel measurable functions such that:*

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |g(\lambda, x)|^2 d\mu(\lambda) dx < \infty \quad \textit{and} \quad h \in L^2(\mathbb{R}^d), \tag{5}$$

then:

$$\int_{\mathbb{R}^d} h(x) \left(\int_{\mathbb{R}^d} g(\lambda, x) dM(\lambda) \right) dx = \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} h(x) g(\lambda, x) dx \right) dM(\lambda)\ a.s.$$

3 Truncated Radon Transform.

For $\mathscr{X}$ a w.s.s. process and $N \in \mathbb{N}$, we define $\mathscr{X}_N(x) = \mathscr{X}(x)\mathbf{1}_{[-N,N]^d}(x)$. A direct, application of Fubini's theorem gives that $\mathscr{X}_N \in L^2(\mathbb{R}^d)$ a.s., in fact:

$$\mathbf{E}\left(\int\limits_{\mathbb{R}^d} |\mathscr{X}_N(x)|^2 dx\right) = R_{\mathscr{X}}(0)(2N)^d \,.$$

Moreover, $\mathscr{X}_N \in L^1(\mathbb{R}^d)$, thus $\mathbf{R}\mathscr{X}_N$ are well defined (in the sense of definition 2.1).

THEOREM **3.1.** *For each $N \in \mathbb{N}$, $\mathbf{R}\mathscr{X}_N$ is well defined as the transform of integrable function a.s., $\mathbf{R}\mathscr{X}$ is well defined as the transform of an element of $\mathcal{D}'(\mathbb{R}^d)$ a.s. and $\mathbf{R}\mathscr{X}_N \longrightarrow \mathbf{R}\mathscr{X}$ when $N \longrightarrow \infty$, in the sense that:*

$$\mathbf{E}|[\mathbf{R}\mathscr{X}_N, \Phi] - [\mathbf{R}\mathscr{X}, \Phi]|^2 \underset{N\longrightarrow\infty}{\longrightarrow} 0\,,$$

for every $\Phi \in \Psi\mathbf{R}(\mathcal{D}(\mathbb{R}^d))$.

Proof. Let $K \subset \mathbb{R}^d$ be a compact subset then by the Cauchy-Schwartz inequality and Fubini's theorem:

$$\mathbf{E}\left(\int\limits_{K} |\mathscr{X}(x)| dx\right)^2 \leq R_{\mathscr{X}}(0)|K| < \infty$$

and thus $\mathscr{X} \subset L^1_{loc}(\mathbb{R}^d) \subset \mathcal{D}'(\mathbb{R}^d)$ a.s. Define, the events:

$$\Omega_N = \left\{\mathscr{X}_N \in L^2(\mathbb{R}^d),\ \text{and}\ \mathscr{X} \in \mathcal{D}'(\mathbb{R}^d)\right\}\,, \Omega' = \bigcap_{N\in\mathbb{N}} \Omega_N\,.$$

Is immediate that $\mathbf{P}(\Omega') = 1$, therefore we can consider the following modifications of $\mathscr{X}$ and $\mathscr{X}_N$, say: $\mathscr{X}' = \mathscr{X}\mathbf{1}_{\Omega'}$ and $\mathscr{X}'_N = \mathscr{X}_N\mathbf{1}_{\Omega'}$. As a consequence $\mathbf{R}\mathscr{X}'$ and $\mathbf{R}\mathscr{X}'_N$ are well defined as transforms of elements of $\mathcal{D}'(\mathbb{R}^d)$ and $L^2(\mathbb{R}^d)$ respectively. Noting, that $\mathbf{R}\mathscr{X}'$ is defined by equation 2. It will be sufficient to check that

$$\mathbf{E}|[\mathbf{R}\mathscr{X}'_N, \Phi] - (\mathscr{X}', g)|^2 \underset{N\longrightarrow\infty}{\longrightarrow} 0\,,$$

for every $g \in \mathcal{D}(\mathbb{R}^d)$ and $\Phi = \Psi\mathbf{R}g$. Recall that $[\mathbf{R}\mathscr{X}'_N, \Phi] = (\mathscr{X}', g)$ and moreover as $g \in L^2(\mathbb{R}^d)$ we can take $g(\lambda, x) = e^{i\lambda.x}$ and then by lemma 2.2:

$$(\mathscr{X}', g) = \int\limits_{\mathbb{R}^d} \mathscr{X}'(x)g(x)dx = \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} e^{i\lambda.x}g(x)dM(\lambda)dx = \int\limits_{\mathbb{R}^d} \widehat{\overline{g}}(\lambda)dM(\lambda)\ \ a.s.$$

Similarly,

$$(\mathscr{X}'_N, g) = \int\limits_{\mathbb{R}^d} \overline{\mathcal{F}_d(g\mathbf{1}_{[-N,N]^d})}(\lambda) dM(\lambda) \ a.s.$$

thus

$$\mathbf{E}|[\mathbf{R}\mathscr{X}'_N, \Phi] - (\mathscr{X}', g)|^2 = \mathbf{E}|(\mathscr{X}'_N, g) - (\mathscr{X}', g)|^2$$

$$= \int\limits_{\mathbb{R}^d} |\widehat{g}(\lambda) - \mathcal{F}(g\mathbf{1}_{[-N,N]^d})(\lambda)|^2 d\mu(\lambda) = \int\limits_{\mathbb{R}^d} \left| \int\limits_{\mathbb{R}^d \setminus [-N,N]^d} g(x) e^{i\lambda . x} dx \right|^2 d\mu(\lambda)$$

$$\leq \mu(\mathbb{R}^d) \left(\int\limits_{\mathbb{R}^d \setminus [-N,N]^d} |g(x)| dx \right)^2 \underset{N \longrightarrow \infty}{\longrightarrow} 0 \ .$$

□

Note that $\mathbf{R}\mathscr{X}$ is defined as a random functional or generlaized random field [4]. We have the following interesting property.

Lemma **3.2.** *If $\mathscr{X}$ is a w.s.s. random field then the (generalized) random field $\mathbf{R}\mathscr{X}$ is stationary with respect to the action of the shifts τ_a, $a \in \mathbb{R}^d$.*

Proof. In fact, if $f, g \in \mathcal{D}(\mathbb{R}^d)$ it follows from the stationarity of $\mathscr{X}$ that [4]:

$$\mathbf{E}((\mathscr{X}, f)(\mathscr{X}, g)) = \mathbf{E}((\mathscr{X}, f(\ . + a))(\mathscr{X}, g(\ . + a)))\,, \tag{6}$$

but we have a.s. that $(\mathscr{X}, f) = [\mathbf{R}\mathscr{X}, \Psi\mathbf{R}f]$ and by the shift property of the Radon transform we also have $(\mathscr{X}, f(\ . + a)) = [\mathbf{R}\mathscr{X}, \Psi\tau_a\mathbf{R}f]$ a.s.. The same identity holds replacing g in these equations and then from equation 6:

$$\mathbf{E}([\mathbf{R}\mathscr{X}, \Psi\mathbf{R}f][\mathbf{R}\mathscr{X}, \Psi\mathbf{R}g]) = \mathbf{E}([\mathbf{R}\mathscr{X}, \Psi\tau_a\mathbf{R}f][\mathbf{R}\mathscr{X}, \Psi\tau_a\mathbf{R}g])\,.$$

□

The preceeding theorem 3.1 establishes the existence of $\mathbf{R}\mathscr{X}$ a.s., we shall see a case when we can obtain a closed analytical expression for $\mathbf{R}\mathscr{X}$ with some additional conditions.

3.0.1 Example.

Suppose that $\mathscr{X}$ admits the following convolution or moving average representation:

$$\mathscr{X}(x) = \int_{\mathbb{R}^d} f(x-y)dW(y)\,, \tag{7}$$

where $f \in \mathcal{S}(\mathbb{R}^d)$ (the Schwartz class of functions) and W is a Brownian random measure (see e.g. p. 23 of [2]). In particular $\mathscr{X}$ is a w.s.s. Gaussian random field. We shall prove that $\mathbf{R}\mathscr{X}$ is the limit of the sequence

$$\Xi_N(\xi,p) = \int_{[-N,N]^d} \mathbf{R}f(\xi, p-\xi.y)dW(y)\,.$$

In the sense that $\lim\limits_{N\longrightarrow\infty} \mathbf{E}|[\Xi_N,\Phi] - [\mathbf{R}\mathscr{X},\Phi]|^2 = 0$, for all $\Phi \in \Psi\mathbf{R}(\mathcal{D}(\mathbb{R}^d))$. Note that for each N, $\{\Xi_N(\xi,p),\ (\xi,p) \in \mathbb{S}^{d-1}\times\mathbb{R}\}$ defines an ordinary Gaussian process on the cylinder $\mathbb{S}^{d-1}\times\mathbb{R}$. For simplicity, we shall additionally assume that d is odd, and only describe the main steps of the argument. From the previous result we know that $\mathbf{R}\mathscr{X}$ must verify a.s. the following equation for every $g \in \mathcal{D}(\mathbb{R}^d)$, or equivalently $\Phi = \mathbf{R}g \in \Psi\mathbf{R}(\mathcal{D}(\mathbb{R}^d))$:

$$[\mathbf{R}\mathscr{X},\Psi\mathbf{R}g] = (\mathscr{X},g) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x-y)dW(y)g(x)dx$$

$$= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x-y)g(x)dxdW(y)\,.$$

Where the last equality holds by lemma 2.2, since $\int_{\mathbb{R}^d}\int_K |f(x-y)|^2dxdy = |K|\,\|f\|^2_{L^2(\mathbb{R}^d)} < \infty$, where K is the support of g, with $g \in L^2(\mathbb{R}^d)$. Now, we can apply equation 1 to the inner integral. In this case, recalling the shift property of the Radon transform [3, 4] we can write:

$$\int_{\mathbb{R}^d} f(x-y)g(x)dx = (f(\,.\,-y),g) = [\mathbf{R}f(\,.\,-y),\Psi\mathbf{R}g]$$

$$= \int_{\mathbb{S}^{d-1}}\int_{\mathbb{R}} \mathbf{R}f(\xi,p-\xi.y)\overline{\mathbf{\Psi}\mathbf{R}g(\xi,p)}dp\,d\Omega(\xi)\,,$$

and therefore,

$$[\mathbf{R}\mathscr{X},\Psi\mathbf{R}g] = \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}}\int_{\mathbb{R}} \mathbf{R}f(\xi,p-\xi.y)\overline{\mathbf{\Psi}\mathbf{R}g(\xi,p)}dp\,d\Omega(\xi)dW(y) \tag{8}$$

$$= \int_{\mathbb{R}^d} [\mathbf{R}f(\,.\,-y), \Psi\mathbf{R}g] dW(y)\,.$$

On the other hand,

$$[\Xi_N, \Phi] = \int_{\mathbb{S}^{d-1}} \int_{\mathbb{R}} \int_{[-N,N]^d} \mathbf{R}f(\xi, p - \xi.y) dW(y) \overline{\mathbf{\Psi}\mathbf{R}g(\xi,p)} dp\, d\Omega(\xi)$$

The result will follows if we can change the order of integration. If d is odd then $\mathbf{\Psi}\mathbf{R}g(\xi,p) = C_d \frac{\partial^{d-1}}{\partial p^{d-1}} \mathbf{R}g(\xi,p)$. Moreover, since g has compact support, there exists $R_0 > 0$ such that $\frac{\partial^{d-1}}{\partial p^{d-1}} \mathbf{R}g(\xi,p) = \mathbf{R}g(\xi,p) = 0$ for all ξ and $|p| > R_0$. We can adapt straightforwardly lemma 2.2 to this case, and then can first check that $\int_{\mathbb{S}^{d-1}} \int_{\{|p| \leq R_0\}} |\mathbf{R}g(\xi,p)|^2 dp\, d\Omega(\xi) < \infty$ and secondly:

$$\int_{\mathbb{S}^{d-1}} \int_{\{|p| \leq R_0\}} \int_{[-N,N]^d} |\mathbf{R}f(\xi, p - \xi.y)|^2 dy dp\, d\Omega(\xi) < \infty\,. \tag{9}$$

The first integral is finite since $g \in \mathcal{D}(\mathbb{R}^d)$ and then $\mathbf{R}g$ is bounded. The same happens with $\mathbf{R}f$, and then as the integral of equation 9 is also taken over a finite measure set, it is finte as well. Thus

$$[\Xi_N, \Phi] = \int_{[-N,N]^d} \int_{\mathbb{S}^{d-1}} \int_{\mathbb{R}} \mathbf{R}f(\xi, p - \xi.y) \overline{\mathbf{\Psi}\mathbf{R}g(\xi,p)} dp\, d\Omega(\xi) dW(y)$$

$$= \int_{[-N,N]^d} [\mathbf{R}f(\,.\,-y), \Psi\mathbf{R}g] dW(y)\,. \tag{10}$$

At this point we recall that $\mathbf{E}\left|\int_{\mathbb{R}^d} f(x) dW(x)\right|^2 = \|f\|^2_{L^2(\mathbb{R}^d)}$ for all $f \in L^2(\mathbb{R}^d)$ [2]. Thus, combining equations 8 and 10,

$$\mathbf{E}|[\Xi_N, \Phi] - [\mathbf{R}\mathscr{X}, \Phi]|^2 = \int_{\mathbb{R}^d \setminus [-N,N]^d} |[\mathbf{R}f(\,.\,-y), \Psi\mathbf{R}g]|^2\, dy$$

$$= \int_{\mathbb{R}^d \setminus [-N,N]^d} |(f(\,.\,-y), g)|^2\, dy = \int_{\mathbb{R}^d \setminus [-N,N]^d} \left|(\widetilde{f} * g)(y)\right|^2 dy \longrightarrow 0\,,$$

if $\widetilde{f} = f(-\,.\,)$, when $N \longrightarrow \infty$, since $\left\| \widetilde{f} * g \right\|_{L^2(\mathbb{R}^d)} \leq \|g\|_{L^1(\mathbb{R}^d)} \|f\|_{L^2(\mathbb{R}^d)} < \infty$. To illustrate, if for example in equation 7 we take $f(x) = \exp(-|x|^2)$, recalling that $\mathbf{R}f(\xi, p) = \sqrt{\pi}\exp(-p^2)$ [3, 4], then $\mathbf{R}\mathscr{X}$ is the limit of the sequence:

$$\Xi_N(\xi, p) = \int\limits_{[-N,N]^d} \sqrt{\pi}\exp(-(p - \xi.y)^2) dW(y)\,.$$

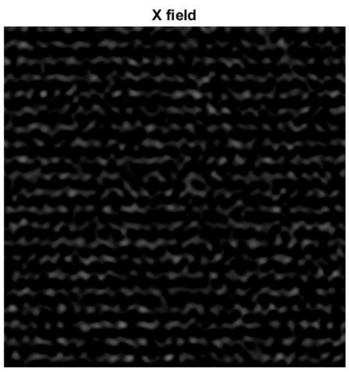
To arrive to this closed formula we made some assumptions on $\mathscr{X}$. In a general setting, it could be of more practical value the second statement of Theorem 3.1 on the existence of a sequence of truncated Radon transforms converging to $\mathbf{R}\mathscr{X}$. It justifies the use of known numerical methods for ordinary functions with compact support to approximate $\mathbf{R}\mathscr{X}$.

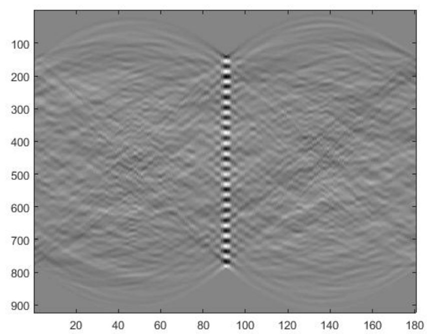
3.0.2 Example.

Let $\lambda_0 > \lambda_1$, Z_1, $Z_2 \sim \mathcal{N}(0, \sigma^2)$, $\sigma^2 \approx 0.4$ independent and also independent of W a Brownian random measure. Define $\mathscr{X}$ the random field indexed by $(x_1, x_2) \in \mathbb{R}^2$ as:

$$\mathscr{X}(x_1, x_2) = \int\limits_{[-\lambda_0,\lambda_0]^2} e^{i\,x.\lambda} dW(\lambda) + Z_1\cos(\lambda_1 x_1) + Z_2\sin(\lambda_1 x_1)\,.$$

This is a w.s.s. field with spectral measure given by $M = \mathbf{1}_{[-\lambda_0,\lambda_0]^2} W + \frac{Z_1 + iZ_2}{2}\delta_{(\lambda_1,0)} + \frac{Z_1 - iZ_2}{2}\delta_{(-\lambda_1,0)}$. In the graphics below we can see a simulation of $\mathscr{X}$ (left) and the truncated Radon transform of it $\mathbf{R}\mathscr{X}_N$ (right), for large N (in this case $N \approx 500$). The -sinogram- graphic of $\mathbf{R}\mathscr{X}_N(\xi, p)$, as usual, is made as a function of p and the angle $0 \leq \theta \leq \pi$ (in degrees), considering $\xi(\theta) = (\cos\theta, \sin\theta)$.





4 Relation of the Radon transform with the spectral representation.

We recall one of the basic relations of the Radon transform with the Fourier transform of functions [3, 4], if $f \in L^1(\mathbb{R}^d)$ and $(\xi, s) \in \mathbb{S}^{d-1} \times \mathbb{R}$:

$$\mathcal{F}_d(s\xi) = \mathcal{F}_1 \mathbf{R} f(\xi, \, . \,)(s) \, . \tag{11}$$

A function, or generalized function f, is uniquely determined by its Fourier transform $\widehat{f}$ and the previous result recovers $\widehat{f}$ from $\mathbf{R}f$. In the case of w.s.s. random fields this role is played by the random measure M. We shall prove an analogue of equation 11 for w.s.s. fields. Also recall that an appropriate Fourier inversion works for w.s.s. random fields [8, 9] similarly to Wiener's generalized harmonic analysis [1]. In this direction, we shall see that the validity of the result presented here depends on the μ measure of the boundaries of the cubes Q considered. In particular, one can recover $M(Q)$ if $\mathscr{X}$ admits an spectral density. We shall also see that the the result remains valid for the discrete part of the spectrum.

THEOREM **4.1.** *Let $\mathscr{X}$ be a w.s.s. random field.*

i) If $Q = \prod_{j=1}^d I_j$, $I_j = \{a_j, b_j\}$ (I_j with open or closed borders indistinctly) and

$$M_N(Q) = \int\limits_{Q \setminus \{0\}} (\mathcal{F}_1 \mathbf{R} \mathscr{X}_N) \left(\frac{\lambda}{|\lambda|}, \, . \right) (|\lambda|) d\lambda \, ,$$

then $\lim\limits_{N \longrightarrow \infty} \mathbf{E}|M_N(Q) - M(Q)|^2 \leq \mu(\partial Q)$.
ii) If $\lambda_0 \in \mathbb{R}^d$ and

$$M'_N(\{\lambda_0\}) = \frac{1}{(2N)^d} (\mathcal{F}_1 \mathbf{R} \mathscr{X}_N) \left(\frac{\lambda_0}{|\lambda_0|}, \, . \right) (|\lambda_0|),$$

then $\lim\limits_{N \longrightarrow \infty} \mathbf{E}|M'_N(\{\lambda_0\}) - M(\{\lambda_0\})|^2 = 0 \, .$

Remark.

We shall not repeat the details, but as in theorem 3.1 we will consider the versions of $\mathscr{X}_N$ for which the Radon transform is defined.

Proof. Note that

$$M_N(Q) = \int\limits_{Q \setminus \{0\}} (\mathcal{F}_1 \mathbf{R} \mathscr{X}_N) \left(\frac{\lambda}{|\lambda|}, \, . \right) (|\lambda|) d\lambda = \int\limits_{Q \setminus \{0\}} (\mathcal{F}_d \mathscr{X}_N)(\lambda) d\lambda$$

$$= \int_Q \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i(\omega-\lambda).x} \mathbf{1}_{[-N,N]^d}(x) dM(\omega) dx d\lambda \ a.s.$$

To prove i) we can use lemma 2.2 twice. First, we can take $g(\omega, x) = e^{i(\omega-\lambda).x}$ and $h(x) = \mathbf{1}_{[-N,N]^d}(x)$, thus

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i(\omega-\lambda).x} \mathbf{1}_{[-N,N]^d}(x) dM(\omega) dx = \int_{\mathbb{R}^d} \int_{[-N,N]^d} e^{i(\omega-\lambda).x} dx dM(\omega) \,.$$

We can apply again the lemma with $g(\omega, \lambda) = \int_{[-N,N]^d} e^{i(\omega-\lambda).x} dx$ and $h(\lambda) = \mathbf{1}_Q(\lambda)$ and then by Fubini's theorem:

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i(\omega-\lambda).x} \mathbf{1}_{[-N,N]^d}(x) dM(\omega) dx = \int_{\mathbb{R}^d} \int_{[-N,N]^d} e^{i\omega.x} \widehat{\mathbf{1}_Q}(x) dx dM(\omega) \,.$$

If we denote $S_N \mathbf{1}_Q(\omega) = \int_{[-N,N]^d} e^{i\omega.x} \widehat{\mathbf{1}_Q}(x) dx$ to the partial sum Fourier inversion operator, where $\widehat{\mathbf{1}_Q}(x) = \prod_{j=1}^{d} \widehat{\mathbf{1}_{I_j}}(x_j)$, then:

$$M_N(Q) = \int_{\mathbb{R}^d} S_N \mathbf{1}_Q(\omega) dM(\omega) \,. \tag{12}$$

Note that $S_N \mathbf{1}_Q(\omega) = \prod_{j=1}^{d} \int_{[-N,N]} \widehat{\mathbf{1}}_{I_j}(x_j) e^{ix_j\omega_j} dx_j$ and as the partial Fourier sums of indicators of intervals are bounded (page 115 [7]):

$$\left| \int_{[-N,N]} \widehat{\mathbf{1}}_{I_j}(x_j) e^{ix_j\omega_j} dx_j \right| \leq 3 \text{ for each } j \text{and for } \omega_j \in \mathbb{R} \,,$$

then $|S_N \mathbf{1}_Q(\omega)| \leq 3^d$. Moreover,

$$\lim_{N \longrightarrow \infty} \int_{[-N,N]} \widehat{\mathbf{1}}_{I_j}(x_j) e^{ix_j\omega_j} dx_j = \mathbf{1}_{(a_j,b_j)}(\omega_j) + \frac{1}{2}(\mathbf{1}_{\{a_j\}}(\omega_j) + \mathbf{1}_{\{a_j\}}(\omega_j)) \,,$$

thus by Lebesgue's theorem :

$$\lim_{N \longrightarrow \infty} \mathbf{E}|M_N(Q) - M(Q)|^2$$

$$= \int_{\mathbb{R}^d} \left| \mathbf{1}_Q(\lambda) - \left(\prod_{j=1}^{d} (\mathbf{1}_{(a_j,b_j)}(\lambda_j) + \frac{1}{2}\mathbf{1}_{\{a_j\}}(\lambda_j) + \frac{1}{2}\mathbf{1}_{\{a_j\}}(\lambda_j)) \right) \right|^2 d\mu(\lambda) \leq \mu(\partial Q)$$

since $Q = (Q \setminus \overset{\circ}{Q}) \bigcup \overset{\circ}{Q}$.
We shall only sketch the proof of (ii) since is very similar to (i). Again after an appropriate interchange of integrals one can prove that:

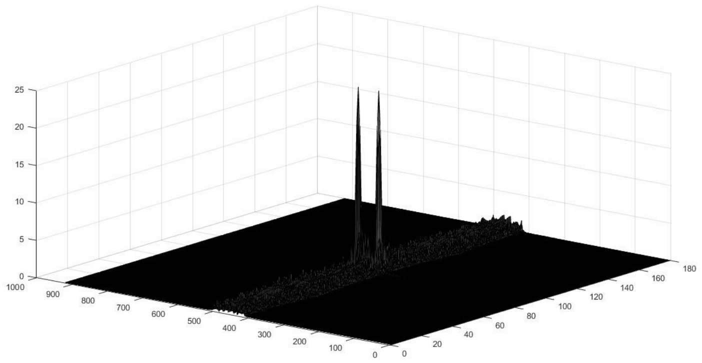
$$\frac{1}{(2N)^d}(\mathcal{F}_1 \mathbf{R} \mathscr{X}_N)\left(\frac{\lambda_0}{|\lambda_0|}, .\right)(|\lambda_0|) = \int_{\mathbb{R}^d} \prod_{j=1}^{d} \frac{\sin(N(\lambda_i - \lambda_{i\,0}))}{N(\lambda_i - \lambda_{i\,0})} dM(\lambda) \; a.s.$$

The result follows in this case noting that $\lim_{N \longrightarrow \infty} \prod_{j=1}^{d} \frac{\sin(N(\lambda_i - \lambda_{i\,0}))}{N(\lambda_i - \lambda_{i\,0})} = \mathbf{1}_{\{\lambda_0\}}(\lambda)$.

□

4.0.1 Example.

Let us consider the same random field $\mathscr{X}$ of Example 3.0.2. In the following graphic we can see the absolute value of $M'_N(\{s\xi\})$ ($N \approx 500$), $\xi(\theta) = (\cos\theta, \sin\theta)$, as a function of $s \in \mathbb{R}$ and $\theta \in [0, \pi]$ (in degrees). Note, the two peaks corresponding to the frequencies λ_1 and $-\lambda_1$.



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