METRIC APPROXIMATIONS OF UNRESTRICTED WREATH PRODUCTS WHEN THE ACTING GROUP IS AMENABLE

JAVIER BRUDE [1,2] and ROMÁN SASYK [1,3]

Abstract. We give a simple and unified proof that the unrestricted wreath product of a weak sofic, sofic, linear sofic or hyperlinear group by an amenable group is weak sofic, sofic, linear sofic or hyperlinear, respectively. By means of the Kaloujnine-Krasner theorem, this implies that group extensions with amenable quotients preserve the four aforementioned metric approximation properties.

1. INTRODUCTION

Given two groups G and H, their *unrestricted wreath product* $G \wr H$ is, by definition, the semidirect product $(\prod_H G) \rtimes_\theta H$, where H acts on the direct product $\prod_H G$ by shifting coordinates as follows: $\theta_h\big((x_{\tilde{h}})_{\tilde{h}\in H}\big) := (x_{\tilde{h}})_{h\tilde{h}\in H}$, for $(x_{\tilde{h}})_{\tilde{h}\in H} \in \prod_H G$. The purpose of this article is to provide a simple and unified proof of the following statement.

1.1. **Theorem.** *Let H be an amenable group and let G be a group.*

(1) If G is weak sofic, $G \wr H$ is weak sofic.
(2) If G is sofic, $G \wr H$ is sofic.
(3) If G is linear sofic, $G \wr H$ is linear sofic.
(4) If G is hyperlinear, $G \wr H$ is hyperlinear.

By means of the Kaloujnine-Krasner theorem, [9], and by keeping in mind that these four metric approximation properties pass to subgroups and are preserved under taking direct products, it is easy to see that Theorem 1.1 and the next result are equivalent.

1.2. **Corollary** (Extension Theorem). *Let G be a group with a normal subgroup N such that the quotient G/N is amenable. If N is weak sofic, sofic, linear sofic or hyperlinear; then G is weak sofic, sofic, linear sofic, or hyperlinear, respectively.*

The sofic and hyperlinear cases of Theorem 1.1 have been proved by Arzhantseva, Berlai, Finn-Sell and Glebsky in [2], where they also gave the application of the Kaloujnine-Krasner theorem mentioned above. Some of the extension results are older. Indeed, the sofic one is due to Elek and Szabo in [5]. More recently, in [8], Holt and Rees showed that certain metric approximations on groups, including weak soficity, are preserved under taking extensions with amenable quotients. On a slightly different direction, in [7], Hayes and Sale proved that the *restricted wreath product* of a group G having one of the metric approximation properties listed in Theorem 1.1 by an acting sofic group, preserves the approximation property of G.

In [2, §4.3] the authors explained why their techniques could not deal with the weak sofic case of Theorem 1.1. The motivation of the present article was to see if ideas we used in [4, §5], some of which can be traced back to [7,8], would serve to give a direct proof of this fact, without requiring the result on extensions of [8]. Here we achieve this in a self-contained manner that also allows us to deal with the four cases of Theorem 1.1 in a unified way. What make our proof simple are, on the one hand,

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the systematic use of the notion of abstract metric approximations, considered first by Arzhantseva in 2008 (see [1]) and on the other hand, that we deal directly with metric approximations of $\prod_H G$ rather than building them locally from metric approximations of G.

2. Metric approximation in groups

In this section we give a very brief account of some basic notions of metric approximation in groups needed in this article. Given a group G, a map $\delta : G \setminus \{1\} \to (0, \infty)$ is called a *weight function* for G.

2.1. **Definition.** *Let G be a group with a weight function δ and let K be a group with a bi-invariant metric d. Given $F \subseteq G$, $\varepsilon > 0$ and a map $\phi : G \to K$ such that $\phi(1) = 1$, we say that*

(1) ϕ is (F, ε, d)-multiplicative if $d(\phi(g)\phi(g'), \phi(gg')) < \varepsilon$ for all $g, g' \in F$;
(2) ϕ is (F, δ, d)-injective if $d(\phi(g), 1) \geq \delta(g)$ for all $g \in F \setminus \{1\}$;
(3) ϕ is (F, ε, d)-free if $d(\phi(g), 1) \geq \rho(\varepsilon)$, for all $g \in F \setminus \{1\}$, where $\rho : (0, b) \to \mathbb{R}_{>0}$ is a function.

2.2. **Definition** ([12, Definition 1.6])**.** *Let $\mathcal{C}$ be a class of groups with bi-invariant metrics. A group G is $\mathcal{C}$-approximable or has the $\mathcal{C}$-approximation property, if it has a weight function δ such that, for each finite set $F \subseteq G$ and each $\varepsilon > 0$ there exist $(K, d) \in \mathcal{C}$ and an (F, ε, d)-multiplicative function $\phi : G \to K$ which is also (F, δ, d)-injective.*

A *countable* group is $\mathcal{C}$-approximable if and only if it is embeddable in a metric ultraproduct of groups in $\mathcal{C}$. Hence, the importance of this notion comes from its relation to major open challenges in mathematics like the Connes embedding problem for group von Neumann algebras and the Gottschalk's surjunctivity conjecture, (see [12, Proposition 1.8] and references therein).

2.3. *Examples.* We list the four metric approximation properties studied in this article.

(1) A group is *weak sofic*, [6], if it is $\mathcal{C}$-approximable when $\mathcal{C}$ is the class of finite groups with bi-invariant metrics.
(2) A group is *sofic*, [5, 13], if it is $\mathcal{C}$-approximable when $\mathcal{C}$ is the class of finite symmetric groups endowed with the normalized Hamming distance
$$d_{\mathrm{Hamm}}(\sigma, \tau) := \frac{1}{|A|} |\{a \in A : \sigma(a) \neq \tau(a)\}| \text{ , for } \sigma, \tau \in Sym(A).$$
(3) A group is *linear sofic*, [3], if it is $\mathcal{C}$-approximable when $\mathcal{C}$ is the class of invertible matrices on a field K endowed with the rank distance given by
$$d_{\mathrm{rk}}(A, B) := \frac{1}{n} \mathrm{rank}(A - B) \text{ , for } A, B \in GL_n(K).$$
(4) A group is *hyperlinear*, [10, 11], if it is $\mathcal{C}$-approximable when $\mathcal{C}$ is the class of unitary matrices on a finite dimensional Hilbert space endowed with the Hilbert-Schmidt distance, d_{HS}, that is induced by the Hilbert-Schmidt norm
$$\|A\|_2 := \sqrt{\frac{1}{n} \sum_{i,j=1}^{n} |\langle Av_i, v_j \rangle|^2} \text{ , for } A \in \mathcal{B}(\mathcal{H}), \text{ where } \{v_1, \dots, v_n\} \text{ is an ONB of } \mathcal{H}.$$

In these examples, (F, δ, d)-injectivity can be replaced by a more manageable condition. Rather than requiring a weight function δ that depends on G, this gets replaced in each case as follows:

(1) *weak sofic:* ϕ is (F, ε, d)-free for a constant function $\rho(\varepsilon) := \alpha \in \mathbb{R}_{>0}$. By scaling the metric d, it can be assumed that $\alpha = 1$ and $\mathrm{diam}(K) = 1$;
(2) *sofic:* ϕ is $(F, \varepsilon, d_{\mathrm{Hamm}})$-free for the function $\rho(\varepsilon) := 1 - \varepsilon$;
(3) *linear sofic:* ϕ is $(F, \varepsilon, d_{\mathrm{rk}})$-free for the function $\rho(\varepsilon) := 1/4 - \varepsilon$;

(4) *hyperlinear:* ϕ is $(F, \varepsilon, d_{\mathrm{HS}})$-trace preserving, namely $|tr(\phi(g))| \leq \varepsilon$ for all $g \in F \setminus \{1\}$, where, for $A \in \mathcal{B}(\mathcal{H})$ and $\{v_1, \ldots, v_n\}$ an ONB of $\mathcal{H}$, $tr(A) := \frac{1}{n}\sum_{i=1}^{n}\langle Av_i, v_i\rangle$.

One of the advantages of having these alternative definitions is that each one is a uniform condition, in the sense that, unlike a weight function, it does not depend on the particular group G that is $\mathcal{C}$-approximable. For instance, it is easy to show that the four metric approximations properties given in Examples 2.3 are preserved under taking finite direct products. But then, freeness implies that they are also preserved under taking direct products.

3. The main technical result

Given a group K with a bi-invariant metric d for which $\operatorname{diam}(K) \leq 1$ and a finite set B, consider the *permutational wreath product* $K \wr_B Sym(B)$. In few occasions we will denote with a dot "$\cdot$" the permutational action of $Sym(B)$ on $\bigoplus_B K$. In [8, §5] (see also [7, Proposition 2.9]) it is shown that the following function

$$\tilde{d}(((x_b)_{b\in B}, \tau), ((y_b)_{b\in B}, \rho)) := d_{\mathrm{Hamm}}(\tau, \rho) + \frac{1}{|B|}\sum_{\substack{b\in B\\ \tau(b)=\rho(b)}} d(x_{\tau(b)}, y_{\tau(b)}) \tag{3.1}$$

is a bi-invariant metric in $K \wr_B Sym(B)$, and with this metric $\operatorname{diam}(K \wr_B Sym(B)) = 1$.

3.1. **Proposition.** *Let H be an amenable group and let G be a group with the property that for every $\varepsilon > 0$ and for every finite set $F_{\prod_H G} \subseteq \prod_H G$, there exist a group $(K, d) \in \mathcal{C}$ of diameter bounded by 1 and a function $\varphi : \prod_H G \to K$ with $\varphi(1) = 1$ that is $(F_{\prod_H G}, \varepsilon, d)$-multiplicative and $(F_{\prod_H G}, \varepsilon, d)$-free, for certain function ρ independent of K.*

Then given $\varepsilon > 0$ and a finite set $F \subseteq G \wr\wr H$, there exist a finite set $B \subseteq H$, a group $(K, d) \in \mathcal{C}$ and a function $\Phi : G \wr\wr H \to K \wr_B Sym(B)$ that is $(F, \varepsilon, \tilde{d})$-multiplicative and $(F, \varepsilon, \tilde{d})$-free, for the function $\tilde{\rho}(\varepsilon) := \min(1 - \varepsilon/3, \rho(\varepsilon/3))$.

3.2. *Remark.* The most natural hypothesis on G would have been that $\prod_H G$ is $\mathcal{C}$-approximable. However, with this condition, given the weight function for $\prod_H G$, our proof would only produce something like a weight function for $G \wr\wr H$ that also depends on ε, and this is not enough to conclude quasi-injectivity. In any case, the hypothesis on G in Proposition 3.1 is satisfied if G is $\mathcal{C}$-approximable, when $\mathcal{C}$ is one of the first three items of Examples 2.3.

Proof of Proposition 3.1. Call $\mathrm{proj}_{\prod_H G}$ and proj_H the projection maps from $G \wr\wr H$ to $\prod_H G$ and H, respectively. Let $F \subseteq G \wr\wr H$ be a finite subset and let $\varepsilon > 0$. Define $F_0 := F \cup \{1\} \cup F^{-1}$ and let $F_H := \mathrm{proj}_H(F_0)$. Since H is amenable, there exists a finite subset $B \subseteq H$ such that

$$\frac{|hB \triangle B|}{|B|} \leq \varepsilon/6\,, \quad \text{for all } h \in F_H^2 := F_H \cdot F_H \tag{3.2}$$

Let $\sigma : H \to Sym(B)$ be defined as

$$\sigma(h)b := \begin{cases} hb & \text{if } hb \in B \\ \gamma_h(hb) & \text{if not, where } \gamma_h : hB \setminus B \to B \setminus hB \text{ is a fixed bijection.} \end{cases} \tag{3.3}$$

It is easy to see that σ is $(F_H, \varepsilon/3, d_{\mathrm{Hamm}})$-multiplicative and $(F_H, \varepsilon/3, d_{\mathrm{Hamm}})$-free.

Let θ denote the shift action of H on $\prod_H G$. By the hypothesis on the group $\prod_H G$, given the finite set

$$F_{\prod_H G} := \bigcup_{\substack{x\in \mathrm{proj}_{\prod_H G}(F_0)\\ b\in B}} \theta_{b^{-1}}(x), \tag{3.4}$$

there exist a group $(K,d)\in\mathcal{C}$ of $\mathrm{diam}(K)\leq 1$ and a function $\varphi:\prod_H G\to K$ with $\varphi(1)=1$ that is $(F_{\prod_H G},\varepsilon/3,d)$-multiplicative and $(F_{\prod_H G},\varepsilon/3,d)$-free for the given function ρ, independent of K. Define

$$\begin{aligned}\Phi: G\wr H &\to K\wr_B Sym(B)\\ (x,h)&\mapsto ((\varphi\theta_{b^{-1}}(x))_{b\in B},\sigma(h))\end{aligned}$$

Claim: Φ is $(F_0,\varepsilon,\tilde{d})$-multiplicative and $(F_0,\varepsilon,\tilde{d})$-free for the function $\tilde{\rho}(\varepsilon)=\min(1-\varepsilon/3,\rho(\varepsilon/3))$.

We will first prove that Φ is $(F_0,\varepsilon,\tilde{d})$-multiplicative. To that end, take $(x,h),(x',h')\in F_0$. On the one hand

$$\Phi\left((x,h)(x',h')\right)=\Phi\left(x\theta_h(x'),hh'\right)=\Big(\left(\varphi\left(\theta_{b^{-1}}(x)\theta_{b^{-1}h}(x')\right)\right)_{b\in B},\sigma(hh')\Big)$$

and on the other hand

$$\begin{aligned}\Phi(x,h)\Phi(x',h')&=\Big(\left(\varphi\left(\theta_{b^{-1}}(x)\right)\right)_{b\in B},\sigma(h)\Big)\Big(\left(\varphi\left(\theta_{b^{-1}}(x')\right)\right)_{b\in B},\sigma(h')\Big)\\ &=\Big(\left(\varphi\left(\theta_{b^{-1}}(x)\right)\varphi\left(\theta_{(\sigma(h)^{-1}b)^{-1}}(x')\right)\right)_{b\in B},\sigma(h)\sigma(h')\Big).\end{aligned}$$

Then

(3.5)

$$\begin{aligned}\tilde{d}\big(\Phi((x,h)(x',h')),\Phi(x,h)\Phi(x',h')\big)&=d_{\mathrm{Hamm}}(\sigma(hh'),\sigma(h)\sigma(h'))\quad +\\ +\ \frac{1}{|B|}\sum_{\substack{b\in B\\ \sigma(hh')b=\sigma(h)\sigma(h')b}} d\left(\varphi(\theta_{(\sigma(hh')b)^{-1}}(x)\theta_{(\sigma(h)\sigma(h')b)^{-1}h}(x')),\varphi\left(\theta_{(\sigma(hh')b)^{-1}}(x)\right)\varphi(\theta_{(\sigma(h')b)^{-1}}(x'))\right).\end{aligned}$$

Since σ is $(F_H,\varepsilon/3,d_{\mathrm{Hamm}})$-multiplicative, it follows that $d_{\mathrm{Hamm}}(\sigma(hh'),\sigma(h)\sigma(h'))<\varepsilon/3$. It only remains to bound the second summand of (3.5). To that end, we will partition the set B in two disjoint subsets, one in which we can control de sum because all its summands are small, and another in which we can control the sum because the subset itself is small and $\mathrm{diam}(K)\leq 1$.

On the one hand, if $h\sigma(h')b\in B$ then by the definition of σ given in (3.3), we have that $\sigma(h)\sigma(h')b=h\sigma(h')b$, and so $(\sigma(h)\sigma(h')b)^{-1}h=(\sigma(h')b)^{-1}$. Since, by (3.4), $\theta_{(\sigma(hh')b)^{-1}}(x),\theta_{(\sigma(h')b)^{-1}}(x')\in F_{\prod_H G}$, we conclude that for all $b\in\sigma(h')^{-1}(B\cap h^{-1}B)$

$$d\left(\varphi(\theta_{(\sigma(hh')b)^{-1}}(x)\theta_{(\sigma(h)\sigma(h')b)^{-1}h}(x')),\varphi\left(\theta_{(\sigma(hh')b)^{-1}}(x)\right)\varphi(\theta_{(\sigma(h')b)^{-1}}(x'))\right)<\varepsilon/3. \tag{3.6}$$

On the other hand, if $b\in\sigma(h')^{-1}(B\setminus h^{-1}B)$, this subset is small because by (3.2) we have that

$$|\sigma(h')^{-1}(B\setminus h^{-1}B)|=|B\setminus h^{-1}B|<\frac{\varepsilon}{6}|B|. \tag{3.7}$$

Splitting the set B in the subsets $\sigma(h')^{-1}(B\cap h^{-1}B)$ and $\sigma(h')^{-1}(B\setminus h^{-1}B)$ and replacing (3.6) and (3.7) in (3.5) gives $\tilde{d}\big(\Phi((x,h)(x',h')),\Phi(x,h)\Phi(x',h')\big)<5/6\,\varepsilon$.

Let us now prove that Φ is $(F_0,\varepsilon,\tilde{d})$-free for the function $\tilde{\rho}$. If $h\in F_H\setminus\{1\}$ then

$$\tilde{d}(\Phi(x,h),1)\geq d_{\mathrm{Hamm}}(\sigma(h),1)\geq 1-\varepsilon/3\geq\tilde{\rho}(\varepsilon).$$

We are left to show that Φ is $(F_0,\varepsilon,\tilde{d})$-free for the function $\tilde{\rho}$ in the case when $(x,1)\in F_0\setminus\{1\}$. Since $x\neq 1$, then $\theta_{b^{-1}}(x)\in F_{\prod_H G}\setminus\{1\}$. Hence $\tilde{d}(\Phi(x,1),1)=\frac{1}{|B|}\sum_{b\in B}d(\varphi\theta_{b^{-1}}(x),1)\geq\rho(\varepsilon/3)\geq\tilde{\rho}(\varepsilon)$. □

3.3. *Remark.* In the proof we never used that θ is the shift action of H on $\prod_H G$. In fact the very same statement and proof holds true if we replace $\prod_H G$ by a group, call it E, and we replace $G\wr H$ by $E\rtimes_\theta H$. As it was discussed in the introduction, the Kaloujnine-Krasner theorem implies that both statements are equivalent. We opted to formulate Proposition 3.1 in terms of unrestricted wreath products since this is better suited for the purpose of this article.

4. Proof of Theorem 1.1

In order to prove Theorem 1.1 we apply Proposition 3.1 to the classes $\mathcal{C}$ of Examples 2.3, together with group homomorphisms built in each case to carry the metric structure from $(K \wr_B Sym(B), \tilde{d})$ to a group in the class $\mathcal{C}$ in a controlled manner.

Proof of Theorem 1.1(1). If G is weak sofic, then $\prod_H G$ is weak sofic and it verifies the hypothesis of Proposition 3.1 when ρ is equal to the constant function 1. Hence, given $\varepsilon > 0$ and a finite set $F \subseteq G \wr H$, there exist a finite set $B \subseteq H$, a finite group K with a bi-invariant metric d, $\operatorname{diam}(K) = 1$, and a function $\Phi : G \wr H \to K \wr_B Sym(B)$ that is $(F, \varepsilon, \tilde{d})$-multiplicative and $(F, \varepsilon, \tilde{d})$-free for $\tilde{\rho}(\varepsilon) = 1 - \varepsilon/3 \geq 2/3$. Since $K \wr_B Sym(B)$ is a finite group, this concludes the proof. □

4.1. The sofic case.

We will need the next lemma, a variant of which is present in [7].

4.1. Lemma. *Let A, B be finite sets. The function*

$$\psi : (Sym(A) \wr_B Sym(B), \tilde{d}) \to (Sym(A \times B), d_{\text{Hamm}})$$
$$\psi(\alpha, \beta)(a, b) := (\alpha_{\beta(b)}(a), \beta(b))$$

is an isometric group homomorphism.

Proof. For $(\alpha, \beta), (\alpha', \beta') \in Sym(A) \wr_B Sym(B)$ and $(a, b) \in A \times B$, the following identities show that ψ is a homomorphism:

$$\psi(\alpha, \beta)\psi(\alpha', \beta')(a, b) = \psi(\alpha, \beta)(\alpha'_{\beta'(b)}(a), \beta'(b)) = (\alpha_{\beta(\beta'(b))}(\alpha'_{\beta'(b)}(a)), \beta\beta'(b));$$
$$\psi(\alpha(\beta \cdot \alpha'), \beta\beta')(a, b) = ((\alpha(\beta \cdot \alpha'))_{\beta\beta'(b)}(a), \beta\beta'(b)) = (\alpha_{\beta\beta'(b)}\alpha'_{\beta'(b)}(a), \beta\beta'(b)).$$

ψ is an isometry because

$$d_{\text{Hamm}}(\psi(\alpha, \beta), \psi(\alpha', \beta')) = \frac{1}{|A||B|}|\{(a, b) \in A \times B : (\alpha_{\beta(b)}(a), \beta(b)) \neq (\alpha'_{\beta'(b)}(a), \beta'(b))\}|$$
$$= \frac{1}{|B|} \sum_{\{b \in B : \beta(b) \neq \beta'(b)\}} \frac{1}{|A|}|\{a \in A : (\alpha_{\beta(b)}(a), \beta(b)) \neq (\alpha'_{\beta'(b)}(a), \beta'(b))\}|$$
$$+ \frac{1}{|B|} \sum_{\{b \in B : \beta(b) = \beta'(b)\}} \frac{1}{|A|}|\{a \in A : \alpha_{\beta(b)}(a) \neq \alpha'_{\beta(b)}(a)\}|$$
$$= d_{\text{Hamm}}(\beta, \beta') + \frac{1}{|B|} \sum_{\{b \in B : \beta(b) = \beta'(b)\}} d_{\text{Hamm}}(\alpha_{\beta(b)}, \alpha'_{\beta(b)}) = \tilde{d}((\alpha, \beta), (\alpha', \beta')).$$ □

Proof of Theorem 1.1(2). If G is sofic, then $\prod_H G$ is sofic and it verifies the hypothesis of Proposition 3.1 when $\rho(\varepsilon) = 1 - \varepsilon$. Hence, given $\varepsilon > 0$ and a finite set $F \subseteq G \wr H$, there exist a finite set $B \subseteq H$, a finite permutational group $Sym(A)$ endowed with the normalized Hamming distance and a function $\Phi : G \wr H \to Sym(A) \wr_B Sym(B)$ that is $(F, \varepsilon, \tilde{d})$-multiplicative and $(F, \varepsilon, \tilde{d})$-free, for $\tilde{\rho}(\varepsilon) = 1 - \varepsilon/3 \geq 1 - \varepsilon$. Lemma 4.1 implies that $\psi \circ \Phi : G \wr H \to Sym(A \times B)$ is a $(F, \varepsilon, d_{\text{Hamm}})$-sofic approximation of $G \wr H$. □

4.2. The linear sofic case.

Take a finite set B. Consider $M_{|B|n}(K)$ and identify it with $M_{|B|}(M_n(K))$. Hence, for $A \in M_{|B|n}(K)$ and $b', b \in B$, $A_{(b', b)} \in M_n(K)$ denotes the block entry of A at the coordinates (b', b). Define

$$\psi : M_n(K) \wr_B Sym(B) \to M_{|B|n}(K) \tag{4.1}$$
$$\psi(U, \tau)_{(b', b)} = \begin{cases} 0 & \text{if } b' \neq \tau(b) \\ U_{\tau(b)} & \text{if } b' = \tau(b). \end{cases}$$

4.2. **Lemma.** *ψ is a group homomorphism between $GL_n(K) \wr_B Sym(B)$ and $GL_{|B|n}(K)$ and verifies that*

$$\frac{1}{2}\tilde{d}\left((U,\tau),(U',\tau')\right) \leq d_{\mathrm{rk}}\left(\psi(U,\tau),\psi(U',\tau')\right) \leq \tilde{d}\left((U,\tau),(U',\tau')\right). \tag{4.2}$$

Proof. It is obvious that ψ maps $GL_n(K) \wr_B Sym(B)$ to $GL_{|B|n}(K)$. A routine matrix computation shows that it is a group homomorphism. In order to bound $d_{\mathrm{rk}}\left(\psi(U,\tau),\psi(U',\tau')\right)$, first observe that the matrix $\psi(U,\tau)-\psi(U',\tau') \in M_{|B|}(M_n(K))$ has at most two nonzero block-entries in each column and in each row. Moreover, the only columns with one nonzero block-entry are the columns $b \in B$ for which $\tau(b) = \tau'(b)$ and the only rows with one nonzero block-entry are the rows $\tilde{b} \in B$ such that $\tau^{-1}(\tilde{b}) = \tau'^{-1}(\tilde{b})$. With this in mind, let A_1 be the submatrix of $\psi(U,\tau)-\psi(U',\tau')$ obtained after removing the rows and columns with exactly two nonzero block-entries and let A_2 be the submatrix of $\psi(U,\tau)-\psi(U',\tau')$ obtained after removing the rows and columns with exactly one nonzero block-entry. Then

$$d_{\mathrm{rk}}\left(\psi(U,\tau),\psi(U',\tau')\right) = \frac{1}{|B|n}\mathrm{rank}(\psi(U,\tau)-\psi(U',\tau')) = \frac{1}{|B|n}\mathrm{rank}(A_1) + \frac{1}{|B|n}\mathrm{rank}(A_2), \tag{4.3}$$

and

$$\frac{1}{|B|n}\mathrm{rank}(A_1) = \frac{1}{|B|}\sum_{\substack{b\in B\\ \tau(b)=\tau'(b)}} \frac{1}{n}\mathrm{rank}\left(U_{\tau(b)} - U'_{\tau(b)}\right) = \frac{1}{|B|}\sum_{\substack{b\in B\\ \tau(b)=\tau'(b)}} d_{\mathrm{rk}}\left(U_{\tau(b)}, U'_{\tau(b)}\right). \tag{4.4}$$

If we regard A_2 as a block matrix in $M_{|\{b\in B:\tau(b)\neq\tau'(b)\}|}(M_n(K))$ then all its nonzero entries are in $GL_n(K)$; each column b has exactly two nonzero entries, the ones corresponding to the rows $\tau(b)$ and $\tau'(b)$; and each row $\tilde{b}$ has exactly two nonzero entries, the ones corresponding to the columns $\tau^{-1}(\tilde{b})$ and $\tau'^{-1}(\tilde{b})$. Then

$$\frac{1}{2}n\left|\{b\in B:\tau(b)\neq\tau'(b)\}\right| \leq \mathrm{rank}(A_2) \leq n\left|\{b\in B:\tau(b)\neq\tau'(b)\}\right|, \tag{4.5}$$

where the first inequality follows from the simple fact that if a matrix $A \in M_r(K)$ has exactly two nonzero entries in each column and in each row, then $\mathrm{rank}(A) \geq r/2$. Replacing (4.4) and (4.5) in (4.3) yield the desired result. $\square$

Proof of Theorem 1.1(3). If G is linear sofic, then $\prod_H G$ is linear sofic, and it verifies the hypothesis of Proposition 3.1 when $\rho(\varepsilon) = 1/4-\varepsilon$. The proof proceeds as in the sofic case. $\square$

4.3. **The hyperlinear case.** Identify $\mathcal{B}(\mathcal{H})$ with $M_n(K)$ (with $K=\mathbb{R}$ or $K=\mathbb{C}$) via the matrix representation in an orthonormal basis $\{v_1,\dots,v_n\}$. It is clear that the function ψ defined in (4.1) can be regarded as $\psi : \mathcal{B}(\mathcal{H}) \wr_B Sym(B) \to \mathcal{B}(\bigoplus_B \mathcal{H})$ and that $||\psi(U,\tau)||_2^2 = \frac{1}{|B|}\sum_{b\in B}||U_b||_2^2$. Recall that the diameter of the unitary group in the Hilbert-Schmidt metric is 2. Then the appropriate metric in $\mathcal{U}(\mathcal{H}) \wr_B Sym(B)$ is the one obtained by scaling the second summand in (3.1) by 1/2. We still call this metric $\tilde{d}$.

4.3. **Lemma.** *ψ is a group homomorphism between $\mathcal{U}(\mathcal{H}) \wr_B Sym(B)$ and $\mathcal{U}(\bigoplus_B \mathcal{H})$ and verifies that*

$$\tilde{d}\left((U,\tau),(U',\tau')\right) \leq d_{\mathrm{HS}}\left(\psi(U,\tau),\psi(U',\tau')\right) \leq 2\sqrt{\tilde{d}\left((U,\tau),(U',\tau')\right)}. \tag{4.6}$$

Proof. It is obvious that ψ maps $\mathcal{U}(\mathcal{H}) \wr_B Sym(B)$ to $\mathcal{U}(\bigoplus_B \mathcal{H})$. Moreover, on the one hand

$$\begin{aligned}||\psi(U,\tau)-\psi(U',\tau')||_2^2 &= \frac{1}{|B|}\sum_{\substack{b\in B\\ \tau(b)\neq\tau'(b)}} \left\|U_{\tau(b)}\right\|_2^2 + \left\|U'_{\tau'(b)}\right\|_2^2 \\ &\quad + \frac{1}{|B|}\sum_{\substack{b\in B\\ \tau(b)=\tau'(b)}} \left\|U_{\tau(b)}-U'_{\tau(b)}\right\|_2^2 \\ &= 2d_{\mathrm{Hamm}}(\tau,\tau') + \frac{1}{|B|}\sum_{\substack{b\in B\\ \tau(b)=\tau'(b)}} d_{\mathrm{HS}}\left(U_{\tau(b)},U'_{\tau(b)}\right)^2 \\ &\leq 4\tilde{d}(\psi(U,\tau),\psi(U',\tau')),\end{aligned}$$

on the other hand, by the Cauchy-Schwarz inequality, we have that

$$\begin{aligned}&2d_{\mathrm{Hamm}}(\tau,\tau') + \frac{1}{|B|}\sum_{\substack{b\in B\\ \tau(b)=\tau'(b)}} d_{\mathrm{HS}}\left(U_{\tau(b)},U'_{\tau(b)}\right)^2 \geq 2d_{\mathrm{Hamm}}(\tau,\tau') + \left(\frac{1}{|B|}\sum_{\substack{b\in B\\ \tau(b)=\tau'(b)}} d_{\mathrm{HS}}\left(U_{\tau(b)},U'_{\tau(b)}\right)\right)^2 \\ &\geq 2d_{\mathrm{Hamm}}(\tau,\tau')^2 + 2\left(\frac{1}{2|B|}\sum_{\substack{b\in B\\ \tau(b)=\tau'(b)}} d_{\mathrm{HS}}\left(U_{\tau(b)},U'_{\tau(b)}\right)\right)^2 \geq \tilde{d}(\psi(U,\tau),\psi(U',\tau'))^2.\end{aligned}$$ □

For hyperlinear groups, quasi-injectivity can not be expressed in terms of $(F,\varepsilon,d_{\mathrm{HS}})$-freeness, so in principle one could not use Proposition 3.1. However, minor adjustments to its proof are enough to accommodate the notion of $(F,\varepsilon,d_{\mathrm{HS}})$-trace preserving needed in the hyperlinear case. We sketch them in the next proof.

Proof of Theorem 1.1(4). Consider F_0, F_H, B, σ and $F_{\prod_H G}$ as in the proof of Proposition 3.1, but in this case we take $\varepsilon^2/24$ in (3.2) so that σ becomes an $(F_H,\varepsilon^2/12)$-sofic approximation of H. Since $\prod_H G$ is hyperlinear, there exist a finite dimensional Hilbert space $\mathcal{H}$ and a function $\varphi : \prod_H G \to \mathcal{U}(\mathcal{H})$ with $\varphi(1) = 1$ that is $(F_{\prod_H G},\varepsilon^2/12,d_{\mathrm{HS}})$-multiplicative and $(F_{\prod_H G},\varepsilon^2/12,d_{\mathrm{HS}})$-trace preserving.

Keeping in mind that the metric in $\mathcal{U}(\mathcal{H})\wr_B Sym(B)$ is obtained by multiplying the second summand in equation (3.1) by 1/2, the same proof of Proposition 3.1 shows that the function $\Phi : G \wr H \to \mathcal{U}(\mathcal{H}) \wr_B Sym(B)$ is $(F_0,\varepsilon^2/4,\tilde{d})$-multiplicative. Then, the second inequality in (4.6) implies that $\psi\circ\Phi : G \wr H \to \mathcal{U}(\bigoplus_B \mathcal{H})$ is $(F_0,\varepsilon,d_{\mathrm{HS}})$-multiplicative. It remains to show that $\psi\circ\Phi$ is $(F_0,\varepsilon,d_{\mathrm{HS}})$-trace preserving. The basic observation that the trace of a block-matrix is equal to the sum of the traces of its block-diagonal entries, yields that $tr(\psi((U_b)_{b\in B},\tau)) = \frac{1}{|B|}\sum_{b\in B:\tau(b)=b} tr(U_b)$. Hence

$$|tr(\psi\circ\Phi(x,h))| = \frac{1}{|B|}\left|\sum_{b\in B:\sigma(h)b=b} tr(\varphi\theta_{b^{-1}}(x))\right| \leq \frac{1}{|B|}\left|\{b\in B : \sigma(h)b=b\}\right| = 1-d_{\mathrm{Hamm}}(\sigma(h),1).$$

It follows that $|tr(\psi\circ\Phi(x,h))| < \varepsilon^2/12 < \varepsilon$, whenever $h \in F_H\setminus\{1\}$. On the other hand, if $(x,1) \in F_0\setminus\{1\}$ then $\theta_b^{-1}(x) \in F_{\prod_H G}\setminus\{1\}$. Thus $|tr(\psi\circ\Phi(x,1))| = \frac{1}{|B|}\left|\sum_{b\in B} tr(\varphi\theta_{b^{-1}}(x))\right| < \varepsilon^2/12 < \varepsilon$. □

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[1]Instituto Argentino de Matemática Alberto P. Calderón-CONICET, Saavedra 15, Piso 3 (1083), Buenos Aires, Argentina.

[2]Departamento de Matemática, Facultad de Ciencias Exactas, Universidad Nacional de la Plata, Argentina.

[3]Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Argentina.

E-mail address: jbrude@mate.unlp.edu.ar
E-mail address: rsasyk@dm.uba.ar