Supports for Minimal Hermitian Matrices

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Abstract

We study certain pairs of subspaces V and W of $\mathbb{C}^n$ we call supports that consist of eigenspaces of the eigenvalues $\pm\|M\|$ of a minimal hermitian matrix M ($\|M\| \leq \|M + D\|$ for all real diagonals D).

For any pair of orthogonal subspaces we define a non negative invariant δ called the adequacy to measure how close they are to form a support and to detect one. This function δ is the minimum of another map F defined in a product of spheres of hermitian matrices. We study the gradient, Hessian and critical points of F in order to approximate δ. These results allow us to prove that the set of supports has interior points in the space of flag manifolds.

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1. Introduction

Let $\mathbb{C}^n$ denote as usual the vector space of n-tuples of complex numbers and $M_n(\mathbb{C})$ the $n \times n$ complex matrices. Let $M_n^h(\mathbb{C})$ (respectively $M_n^{ah}(\mathbb{C})$) be the set of hermitian or self-adjoint (respectively anti-hermitian) matrices and $\| \; \|$ the spectral norm in $M_n(\mathbb{C})$.

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We call $Z \in M_n^h(\mathbb{C})$ a minimal matrix if

$$\|Z\| \leq \|Z + D\| \ , \text{ for every real diagonal } \ D, \tag{1.1}$$

(a similar definition can be given for antihermitian matrices and pure imaginary diagonals). Minimal matrices allow the description of short length curves in the homogeneous space $\mathcal{P} = U(n)/U(\mathcal{D}_n)$, where $\mathcal{D}_n$ denotes the diagonal $n \times n$ complex matrices, $U(\mathcal{D}_n)$ the unitary diagonal matrices and $U(n)$ the unitary group of $M_n(\mathbb{C})$. More precisely, we consider the homogeneous space $\mathcal{P}$, with the left action $L_U(\rho) = U\rho$, for $U \in U(n)$, $\rho \in \mathcal{P}$ (where the action is performed on any element of the class ρ). Then the space $\mathcal{P}$ is provided with the invariant Finsler metric defined by the quotient norm in $M_n(\mathbb{C})^{ah}/\mathcal{D}_n^{ah}$, the tangent space of $\mathcal{P}$ at ρ. This structure allows the definition of a natural distance $\mathrm{d}(\rho_1, \rho_2)$ in $\mathcal{P}$ as the infimum of the length of curves in $\mathcal{P}$ joining ρ_1 and ρ_2 (see [2, 3] for details).

The following result is a restatement of Theorem I of [3] in the present context.

Theorem 1. *Let $\rho \in \mathcal{P} = U(n)/U(\mathcal{D}_n)$, $X \in (T\mathcal{P})_\rho \simeq M_n(\mathbb{C})^{ah}/\mathcal{D}_n^{ah}$ and $Z \in M_n^{ah}(\mathbb{C})$ a minimal matrix which projects to X ($X = Z\rho - \rho Z$). Then the curve given by $\gamma(t) = L_{e^{tZ}}\rho = e^{tZ}\rho e^{-tZ}$ satisfies $\gamma(0) = \rho$, $\dot{\gamma}(0) = X$ and has minimal length among all curves in $\mathcal{P}$ joining $\gamma(0)$ to $\gamma(t)$ for each t with $|t| \leq \frac{\pi}{2\|Z\|}$.*

Observe that from all the cases covered by Theorem I of [3], the homogeneous space $\mathcal{P} = U(n)/U(\mathcal{D}_n)$ we are considering here is probably the simplest non commutative non trivial case.

This result motivates the study of minimal matrices in the spectral norm. Some particular properties have been studied already but in the present work we focus on the particular and rich structure of a spectral pair of eigenspaces related to a minimal matrix.

If $Z \in M_n^h(\mathbb{C})$ is a minimal matrix, then $\pm\|Z\|$ must be eigenvalues of Z. Nevertheless, this condition is not enough. If $\pm\|Z\| \in \sigma(Z)$, then Z is minimal if and only if there exist orthogonal corresponding eigenspaces V_+ and V_- (ranges of the spectral projections P_{V_+} and P_{V_-} corresponding to the eigenvalues $\pm\|Z\|$) that satisfy

$$Z = \|Z\|P_{V_+} - \|Z\|P_{V_-} + R \tag{1.2}$$

(where $R \in M_n^h(\mathbb{C})$, its range is orthogonal to $V_+ \oplus V_-$, and $\|R\| \leq \|Z\|$) and such that V_+ and V_- satisfy the following property:

Condition 1. There exist orthonormal sets $\{v_i\}_{i=1}^p \subset V_+$ and $\{w_j\}_{j=q+1}^{p+q} \subset V_-$ such that

$$\operatorname{co}\big(\{v_i \circ \overline{v_i}\}_{i=1}^r\big) \cap \operatorname{co}\left(\{w_j \circ \overline{w_j}\}_{j=r+1}^{r+s}\right) \neq \emptyset \tag{1.3}$$

where $\circ$ denotes the Hadamard or entrywise product and $\operatorname{co}(A)$ the convex hull of A (see Corollary 3 in [1]).

In Theorem 3 it is proved that Condition 1 is equivalent to the following property held by two orthogonal subspaces V and W.

Definition 1. *Given two orthogonal subspaces V and $W \subset \mathbb{C}^n$ we call the pair (V, W) a* **support** *if there exist non trivial subsets $\{v^1, v^2, \dots, v^p\}$ of V and $\{w^1, w^2, \dots, w^q\}$ of W with coordinates in the canonical basis given by $v^i = (v_1^i, v_2^i, \dots, v_n^i)$, for $i = 1, \dots, p$ and $w^j = (w_1^j, w_2^j, \dots, w_n^j)$, for $j = 1, \dots, q$ such that*

$$\left\{\begin{array}{ccc} |v_1^1|^2 + |v_1^2|^2 + \cdots + |v_1^p|^2 & = & |w_1^1|^2 + |w_1^2|^2 + \cdots + |w_1^q|^2 \\ |v_2^1|^2 + |v_2^2|^2 + \cdots + |v_2^p|^2 & = & |w_2^1|^2 + |w_2^2|^2 + \cdots + |w_2^q|^2 \\ \vdots & \vdots & \vdots \\ |v_n^1|^2 + |v_n^2|^2 + \cdots + |v_n^p|^2 & = & |w_n^1|^2 + |w_n^2|^2 + \cdots + |w_n^q|^2 \end{array}\right. \tag{1.4}$$

or equivalently $\sum_{i=1}^p v^i \circ \overline{v^i} = \sum_{j=1}^q w^j \circ \overline{w^j}$, where $\circ$ denotes the Hadamard product and $\overline{x}$ the vector formed by the conjugated coordinates of x.

This definition can also be stated choosing orthogonal vectors $\{v^i\}_{i=1}^t$ and $\{w^j\}_{j=1}^h$ (see Theorem 2).

Remark 1. *The previous discussion implies that Z is a minimal matrix with a decomposition as in* (1.2) *if and only if the pair of subspaces (V_+, V_-) is a support (see also Theorem 3).*

We will denote with $\mathcal{S}_{(r,s)}$ the set of supports of $\mathbb{C}^n$ with corresponding dimensions r and s:

$$\begin{aligned} \mathcal{S}_{(r,s)} = \{&(V, W) \in \mathbb{C}^n \times \mathbb{C}^n : (V, W) \text{ is a support} \\ &\text{with } \dim(V) = r \text{ and } \dim(W) = s\} \end{aligned} \tag{1.5}$$

Remark 2. *The definition of the set of supports suggests that it might have the structure of a real algebraic set. As expected, $\mathcal{S}_{(r,s)}$ turns out to be closed (see Proposition 2). Nevertheless, the fact that for every $n \in \mathbb{N}_{n\geq 3}$ there exist interior points in $\mathcal{S}_{(r,s)}$ in the ambient of a flag manifold of $\mathbb{C}^n$ is a surprising result (see Section 9). It would be interesting to find out if $\mathcal{S}_{(r,s)}$ is a semi-algebraic set.*

The previous comments allow us to state the following result.

Remark 3. *There exists a function between the set of minimal matrices Z with eigenspaces V_+ and V_- corresponding to the eigenvalues $\pm\|Z\|$ with $\dim(V_+) = r$, $\dim(V_-) = s$ onto $\mathcal{S}_{(r,s)}$ that maps Z to the support (V_+, V_-).*

Consider the equivalence class of a matrix $M \in M_n^h(\mathbb{C})$ defined by $[M] = \{N \in M_n^h(\mathbb{C}) : (M - N) \in \mathcal{D}_n\}$. The relation between $[M]$ and the support determined by its corresponding minimal matrix (or matrices) is a work in progress that will studied elsewhere.

Supports are a fundamental aspect of the description of minimal matrices. In this work we are going to analyze the structure of the set of supports $\mathcal{S}_{(r,s)}$ as a subset of the flag manifold $\mathcal{F}_{(r,s)}$ (see (2.1)) under the identification of $(V, W) \in \mathcal{S}_{(r,s)}$ with $V \oplus W \oplus (\mathbb{C}^n \ominus V \ominus W)$. The authors consider that the study of $\mathcal{S}_{(r,s)} \subset \mathcal{F}_{(r,s)}$ is interesting by itself.

In order to measure how far are two subspaces V and W to become a support we define in (5.3) a number $\delta(V, W) \geq 0$ we call the adequacy of V and W that satisfies $\delta(V, W) = 0$ if and only if (V, W) is a support. The adequacy is a natural tool to achieve this and can be computed as the minimum of a function F defined on the product of certain spheres $S_V \times S_W$ of linear maps (see (5.2) and (5.5)). We study the gradient, Hessian and critical points of this F (see Section 5) to allow the approximation of the adequacy. Some of the formulas obtained are used in the appendices to obtain numerical examples of particular supports that are interior points of flag manifolds in low dimensions. These results are used to prove in Theorem 10 that there exist open neighborhoods of supports (formed by supports in $\mathbb{C}^n$ for every $n \in \mathbb{N}_{n\geq 3}$) in the flag manifold $\mathcal{F}_{(\dim(V),\dim(W))}$.

We also consider a geometric interpretation of the adequacy in Section 6 describing a new space of parameters to calculate it. This perspective allows the characterization of some critical points of the map F whose global minimum is the adequacy in sections 7 and 8.

2. Preliminaries and notation

Here we introduce some notation used throughout the article. $M_n(\mathbb{C})$ will denote the $n \times n$ matrices with coefficients in $\mathbb{C}$, $M_n^h(\mathbb{C})$ the hermitian matrices and $M_n^{ah}(\mathbb{C})$ the anti-hermitian matrices. The expression $\operatorname{diag}(a_1, a_2, \ldots, a_n)$ denotes the diagonal matrix in $M_n(\mathbb{C})$ with the elements $a_1, a_2, \ldots, a_n \in \mathbb{C}$ in its principal diagonal, and $\Phi : M_n(\mathbb{C}) \to \mathcal{D}_n \subset M_n(\mathbb{C})$ the conditional expectation such that $\Phi(x)$ is the diagonal matrix formed with the diagonal entries of x.

As usual $GL(n,\mathbb{C})$ denotes the general group of invertible matrices in $C^{n\times n}$. And $\mathrm{Gr}(k,\mathbb{C}^n)$ will denote the Grassmannian manifold of all k-dimensional subspaces of $\mathbb{C}^n$.

We denote with $v\circ w$ the Hadamard (or Schur) entrywise product of two vectors $v,\ w\in\mathbb{C}^n$, where $v\circ w\in\mathbb{C}^n$, and $(v\circ w)_i=v_iw_i$, for $i=1,\dots n$. Similarly $A\circ B$ is the Hadamard (or Schur) product of two matrices $A,B\in M_n(\mathbb{C})$.

We use $\mathcal{F}_{(r,s)}$ to represent the set

$$\begin{aligned}\mathcal{F}_{(r,s)}=\{(V,W): V\perp W \text{ are subspaces of } \mathbb{C}^n,\\ \dim(V)=r,\dim(W)=s\}.\end{aligned}\tag{2.1}$$

Observe that the pair $(V,W)\in\mathcal{F}_{(r,s)}$ can be identified with the element $\{0\}\subset V\subset V\oplus W\subset\mathbb{C}^n$ in a classic flag manifold $F(r,r+s,n)$ which is isomorphic to the homogeneous space $U(n)/\big(U(r)\times U(s)\times U(n-r-s)\big)$. Therefore, $\mathcal{F}_{(r,s)}$ can be identified with the flag manifold $F(r,r+s,n)$.

3. Properties of a support (V,W)

Given $v^1,v^2,\dots,v^p\in V$, for V a subspace of $\mathbb{C}^n$, we denote with

$$\underline{v}=(v^1,v^2,\dots,v^p)=\begin{pmatrix} v_1^1 & v_1^2 & \dots & v_1^p\\ \vdots & \vdots & \dots & \\ v_n^1 & v_n^2 & \dots & v_n^p\end{pmatrix}\tag{3.1}$$

either the $n\times p$ matrix or the $\mathbb{C}^p\to\mathbb{C}^n$ linear map. Let $\Phi: M_n(\mathbb{C})\to\mathcal{D}_n$ be the map such that $\Phi(x)$ is the diagonal of x. Then the n-tuples that appear in (1.4) can be written

$$\sum_{i=1}^p v^i\circ\overline{v^i}=\big(|v_1^1|^2+\dots+|v_1^p|^2,\ \dots\ ,|v_n^1|^2+\dots+|v_n^p|^2\big)\simeq\Phi(\underline{v}\ \underline{v}^*)\tag{3.2}$$

where we identified the vector with the diagonal matrix $\Phi(\underline{v}\ \underline{v}^*)$ of $\underline{v}\ \underline{v}^*\in M_n^h(\mathbb{C})$.

Then using the singular value decomposition of $\underline{v}=usx^*$, with $u\in U(n)$, $x\in U(p)$ and s the $n\times p$ diagonal matrix of the singular values of $\underline{v}$ in the $s_{j,j}$

entries. Let us denote them with s_j, $j = 1, \dots, p$. Now consider the column vectors $u^i \in \mathbb{C}^n$, $i = 1, \dots, n$, from the unitary matrix u. Note that these u^i are eigenvectors of $\underline{v}\ \underline{v}^*$.

Let e_i, for $i = 1, \dots, n$, be the i^{th} element of the canonical basis of $\mathbb{C}^n$. Then the i^{th} diagonal elements of $\Phi(\underline{v}\ \underline{v}^*)$ are

$$\langle \underline{v}\ \underline{v}^* e_i, e_i\rangle = \langle x s^t u^* e_i, x s^t u^* e_i\rangle = \sum_{j=1}^{p} s_j^2 |u_i^j|^2 = \sum_{j=1}^{p} \langle s_j u_i^j, s_j u_i^j\rangle. \tag{3.3}$$

Therefore if we consider the $n \times p$ matrix given by its columns

$$\widetilde{\underline{v}} = (s_1 u^1, s_2 u^2, \dots, s_p u^p) \tag{3.4}$$

then the computation made in (3.3) proves that

$$\Phi(\underline{v}\ \underline{v}^*) = \Phi(\widetilde{\underline{v}}\ \widetilde{\underline{v}}^*). \tag{3.5}$$

Moreover, these columns generate the same subspace as the original $\{v^j\}_{j=1}^p$.

Let $K \subset \{1, \dots, p\}$ be the subset of indexes such that $s_k \neq 0$ if and only if $k \in K$ and let $t = \#(K)$. Then the vectors $\{s_k u^k\}_{k \in K} \subset \mathbb{C}^n$ are orthogonal to each other and generate the same subspace than the original columns v^j of $\underline{v}$ for $j = 1, \dots, p$.

Therefore, if we consider the $n \times t$ matrix with columns $s_k u^k$, $k \in K$

$$\widetilde{\underline{v'}} = (s_{k_1} u^{k_1}, s_{k_2} u^{k_2}, \dots, s_{k_t} u^{k_t}) \tag{3.6}$$

then its columns form an orthogonal basis of the subspace generated by $\{v^j\}_{j=1}^r$ and it is apparent that also $\Phi\left(\widetilde{\underline{v'}}\ \widetilde{\underline{v'}}^*\right) = \Phi(\widetilde{\underline{v}}\ \widetilde{\underline{v}}^*) = \Phi(\underline{v}\ \underline{v}^*)$.

Therefore, we have proved the following result.

Theorem 2. *If (V, W) is a support in $\mathbb{C}^n$ as in Definition 1 then there exists not null orthogonal vectors $\{v^i\}_{i=1}^t \subset V$ and $\{w^j\}_{j=1}^h \subset W$ that satisfy equation (1.4), or equivalently $\sum_{i=1}^t v^i \circ \overline{v^i} = \sum_{j=1}^h w^j \circ \overline{w^j}$.*

Remark 4. *Observe that in Definition 1 the vectors $\{v^i\}_{i=1,\dots,r}$ of the subspace V are not required to be linearly independent nor generators of V, and similarly for $\{w^j\}_{j=1,\dots,s}$ in W, but the previous theorem states that orthonormal vectors can be chosen. Moreover, these vectors can be taken bounded in norm with a fixed constant C after multiplying all of them by $\frac{C}{\|x\|}$ where $\|x\|$ is the greatest norm of all the vectors considered.*

Definition 2. *If $\{v^1, \dots, v^p\}$ is a system of p vectors in $\mathbb{C}^n$, then the diagonal matrix (or corresponding vector) $\Phi(\underline{v}\ \underline{v}^*)$ will be called the **moment** of the system $\{v^1, \dots, v^p\}$ (with the notation of $\underline{v} \in \mathbb{C}^{n\times p}$ detailed in (3.1)).*

Therefore the previous discussion also proved the following result.

Proposition 1. *If $\{v^1, \dots, v^p\}$ is a system of p linearly independent vectors in V, then there is an orthogonal basis $\{c^1, \dots, c^p\}$ of V with the same moment as that of $\{v^1, \dots, v^p\}$.*

Remark 5. *Observe that if for $V \perp W$ subspaces of $\mathbb{C}^n$, there exist $\{v^1, \dots, v^p\} \subset V$, $\{w^1, \dots, w^q\} \subset W$, and we define $\underline{v} \in \mathbb{C}^{n\times p}$ and $\underline{w} \in \mathbb{C}^{n\times q}$ as in (3.1), then the equality*

$$\Phi(\underline{v}\ \underline{v}^*) = \Phi(\underline{w}\ \underline{w}^*) \tag{3.7}$$

is equivalent to the fact that (V, W) is a support in $\mathbb{C}^n$ (see (3.2)).

Given a support (V, W) of $\mathbb{C}^n$ Proposition 1 implies that there exists an orthogonal set $\{v^i\}_{i=1}^p$ for V and $\{w^j\}_{j=1}^q$ for W that satisfy (1.4). Now consider the orthonormal corresponding set after normalizing each vector. Now adding all the equations in (1.4) we obtain that $\sum_{i=1}^p \|v^i\|^2 = \sum_{j=1}^q \|w^j\|^2$, and then

$$\sum_{i=1}^{p} \frac{\|v^i\|^2}{\sum_{k=1}^p \|v^k\|^2} \left(\frac{v^i}{\|v^i\|} \circ \frac{\overline{v^i}}{\|v^i\|} \right) = \sum_{j=1}^{q} \frac{\|w^j\|^2}{\sum_{k=1}^q \|w^k\|^2} \left(\frac{w^j}{\|w^j\|} \circ \frac{\overline{w^j}}{\|w^j\|} \right)$$

which in turns implies that Condition 1 stated in (1.3) holds. Then statement (b) of Corollary 3 in [1] is fulfilled and $M = \lambda\ P_V - \lambda\ P_W + R \in M_n^h(\mathbb{C})$ (with $P_V R = P_W R = 0$, $R \in M_n^h(\mathbb{C})$, $\|R\| \le \lambda > 0$) is a minimal matrix in the sense that $\|M\| \le \|M + D\|$ for all real diagonal matrices $D \in M_n^h(\mathbb{C})$ and $\|\ \|$ the spectral norm (see [1]). Then a support allows the construction of a minimal matrix, and vice versa. In the following theorem we collect some statements that are equivalent to the definition of a support.

Theorem 3. *Let V, W be two non trivial orthogonal subspaces of $\mathbb{C}^n$, then the following statements are equivalent.*

1. *(V, W) is a support, that is, there exist non trivial subsets $\{v^1, v^2, \dots, v^p\}$ of V and $\{w^1, w^2, \dots, w^q\}$ of W such that (1.4) holds.*
2. *The hermitian matrix $M = \lambda(P_V - P_W) + R$ is minimal (see (1.1)) for every $\lambda \in \mathbb{R}_{>0}$, $R \in M_n^h(\mathbb{C})$, $\|R\| \le \lambda$, $R(P_V + P_W) = 0$.*

3. *There exist non trivial subsets* $\{v^1, v^2, \dots, v^p\}$ *of* V *and* $\{w^1, w^2, \dots, w^q\}$ *of* W *such that*
$$\Phi(\underline{v}\ \underline{v}^*) = \Phi(\underline{w}\ \underline{w}^*)$$
with $\underline{v}$ *and* $\underline{w}$ *defined as in* (3.1) *and* $\Phi(m)$ *the diagonal of* m.
4. *The sets* $\sigma_V = \{c \in M_n^h(\mathbb{C}) : P_V c = c \geq 0,\ \operatorname{Tr}(c) = 1\}$ *and* $\sigma_W = \{d \in M_n^h(\mathbb{C}) : P_W d = d \geq 0,\ \operatorname{Tr}(d) = 1\}$*, satisfy*
$$\Phi(\sigma_V) \cap \Phi(\sigma_W) \neq \emptyset.$$

Proof. 1. $\Leftrightarrow$ 2. follows after the previous discussion.
2. $\Leftrightarrow$ 4. is proved using the comments following the proof of Corollary 3 in [1] or the property mentioned in (6.3).
1. $\Leftrightarrow$ 3. is Remark 5. □

Proposition 2. *The set of supports* $\mathcal{S}_{(r,s)}$ *is closed in the flag manifold* $\mathcal{F}_{(r,s)}$.

Proof. Consider a sequence of supports given by $\{(V_k, W_k)\}_{k\in\mathbb{N}} \subset \mathcal{S}_{(r,s)}$ and such that its corresponding orthogonal projections converge. It is apparent that there exist V and W subspaces of $\mathbb{C}^n$ such that $\dim(V) = r$, $\dim(W) = s$, $V \perp W$ and satisfy $\lim_{k\to\infty} P_{V_k} = P_V$ and $\lim_{k\to\infty} P_{W_k} = P_W$, that is, $(V, W) \in \mathcal{F}_{(r,s)}$. We only need to prove that the condition (1.4) holds.
Consider for each pair (V_k, W_k) a pair of matrices $(\underline{v}_k, \underline{w}_k)$ that satisfy
$$\Phi(\underline{v}_k\ \underline{v}_k^*) = \Phi(\underline{w}_k\ \underline{w}_k^*) \tag{3.8}$$
as in Remark 5.
Note that as mentioned in Remark 4 we can choose the column vectors of the matrices $\underline{v}_k$ and $\underline{w}_k$ with norm less or equal than one. Then using compacity arguments and after taking subsequences we can suppose that the matrices $\underline{v}_k$ are of the same size, and their columns converge to vectors in V that form a matrix $\underline{v}$. Similar arguments can be used for $\underline{w}_k$ to obtain a matrix $\underline{w}$. Since for all k equality (3.8) holds, then $\lim_{k\to\infty} \Phi(\underline{v}_k\ \underline{v}_k^*) = \lim_{k\to\infty} \Phi(\underline{w}_k\ \underline{w}_k^*)$ which is $\Phi(\underline{v}\ \underline{v}^*) = \Phi(\underline{w}\ \underline{w}^*)$. Since this is equivalent to the equalities (1.4) (see Remark 5) then (V, W) is a support. □

4. Symplectic interpretation of the map Φ

Consider the manifold $M = (\mathbb{C}^n)^r$ composed of matrices $\underline{v}$ defined in (3.1). We denote by $\underline{v}_k$, $k = 1, \dots, n$ the rows of $\underline{v}$ considered as vectors in $\mathbb{C}^r$.

Since $\mathbb{C}^n$ carries a natural symplectic form, so does M (the product form). In this way, M becomes a symplectic manifold. We consider next the left operation action of the unitary group $U(n)$ on M. This operation is symplectic. Now we identify the Lie algebra $u(n)$ of $U(n)$ with its dual $u^*(n)$ using the inner product $\langle A, B\rangle = tr(AB^*)$.

In this context the moment map $\mu : M \to u^*(n)$ can be computed explicitly:

$$\mu(\underline{v}) = \frac{i}{2}\underline{v}\ \underline{v}^* \quad , \text{for} \quad \underline{v} \in M. \tag{4.1}$$

Observe that the entries of the matrix $\mu(\underline{v})$ are

$$(\mu(\underline{v}))_{k,l} = \frac{i}{2}\langle \underline{v}_k,\ \underline{v}_l\rangle \quad , \text{for } k = 1\ldots, n \text{ and } l = 1, \ldots, n. \tag{4.2}$$

and the entries of the diagonal are

$$(\mu(\underline{v}))_{k,k} = \frac{i}{2}\|\underline{v}_k\|^2 \quad , \text{for } k = 1, \ldots, n. \tag{4.3}$$

Finally, observe that for the induced left action of the diagonal unitary matrices $\mathbb{T}^n \subset U(n)$ on M, the corresponding moment map μ_d is obtained as follows $\mu_d : M \to (t^n)^*$

$$\mu_d(\underline{v}) = \frac{i}{2}\begin{pmatrix} \|\underline{v}_1\|^2 \\ \vdots \\ \|\underline{v}_n\|^2 \end{pmatrix},$$

which is exactly $i/2$ times what was called the *moment of the system* $\{v^1, \ldots, v^r\}$ in Definition 2.

5. Adequacy of a pair of orthogonal subspaces

Recall that with $\mathcal{F}_{(r,s)}$ we denote the space pairs of orthogonal subspaces (V, W) of $\mathbb{C}^n$ with $r = \dim V$ and $s = \dim W$. Also consider systems $\underline{v} = (v^1, \ldots, v^r) : \mathbb{C}^r \to C^n$ as in (3.1) and similarly $\underline{w} = (w^1, \ldots, w^s) : \mathbb{C}^s \to C^n$ of vectors in $\mathbb{C}^n$ such that $\mathrm{Im}(\underline{v}) \subset V$ and $\mathrm{Im}(\underline{w}) \subset W$.

Recall that in Definition 2 we called $\Phi(\underline{v}\ \underline{v}^*)$ the moment of the system $\underline{v}$ where Φ is the conditional expectation that associates to any $n \times n$ matrix its diagonal part. This map takes its values in the subalgebra $\mathcal{D}_n$ of diagonal

$n \times n$ matrices which will sometimes be identified with $\mathbb{C}^n$. Observe that the map $\underline{v} \mapsto \Phi(\underline{v}\ \underline{v}^*)$ is homogeneous in the following sense:

$$\Phi\left((\alpha\underline{v})(\alpha\underline{v})^*\right) = |\alpha|^2\Phi(\underline{v}\ \underline{v}^*), \text{ for } \alpha \in \mathbb{C}. \tag{5.1}$$

Recall that with this notation (V, W) is called a support (see 1.4 and Theorem 2 and Proposition 1) if there is a non trivial pair $(\underline{v}, \underline{w})$ with $\mathrm{Im}(\underline{v}) \subset V$, $\mathrm{Im}(\underline{w}) \subset W$ such that

$$\Phi(\underline{v}\ \underline{v}^*) = \Phi(\underline{w}\ \underline{w}^*)$$

(here non trivial refers to $\underline{v} \neq (0, \dots 0)$ and $\underline{w} \neq (0, \dots 0)$).

Observe that if there is a non trivial pair $(\underline{v}, \underline{w})$ as before such that $\Phi(\underline{v}\ \underline{v}^*)$ and $\Phi(\underline{w}\ \underline{w}^*)$ are only linearly dependent then choosing $\alpha \in \mathbb{R}$ appropriately we can get $\Phi\left((\alpha\underline{v})(\alpha\underline{v})^*\right) = \Phi(\underline{w}\ \underline{w}^*)$, with $\mathrm{Im}(\alpha\underline{v}) \subset V$ so that (V, W) is a support.

The objective of this section is to define and compute a "numerical obstruction" for the pair (V, W) to be a support, i.e. a non negative invariant of (V, W) which vanishes if and only if the pair (V, W) is a support. We will call this obstruction the *adequacy* of (V, W).

Note that if (1.4) holds for the vector columns of $\underline{v}$ and $\underline{w}$ then $\mathrm{Tr}(\underline{v}\ \underline{v}^*) = \mathrm{Tr}(\underline{w}\ \underline{w}^*)$ follows. Then the remark made in (5.1) about the homogeneous nature of ω allow us to restrict to the space of pairs $(\underline{v}, \underline{w})$ that are "normalized" in the sense that

$$\mathrm{Tr}(\underline{v}\ \underline{v}^*) = 1 \quad \text{and} \quad \mathrm{Tr}(\underline{w}\ \underline{w}^*) = 1.$$

Observe that in the space $\hom(\mathbb{C}^r, V)$ we have a natural norm given by $\mathrm{Tr}(\underline{v}\ \underline{v}^*)^{1/2}$ and the same holds for $\hom(\mathbb{C}^s, W)$. Therefore if we denote with

$$S_V \text{ and } S_W \text{ the unit spheres of } \hom(\mathbb{C}^r, V) \text{ and } \hom(\mathbb{C}^s, W) \tag{5.2}$$

respectively, then the selected pairs $(\underline{v}, \underline{w})$ belong to $S_V \times S_W$.

Finally we define the adequacy of the pair (V, W).

Definition 3. *Given a pair of non trivial orthogonal subspaces $V, W \subset \mathbb{C}^n$, its* ***adequacy*** *is defined as the number*

$$\delta(V, W) = \inf\left\{||\Phi(\underline{v}\ \underline{v}^*) - \Phi(\underline{w}\ \underline{w}^*)||^2 : (\underline{v}, \underline{w}) \in S_V \times S_W\right\} \tag{5.3}$$

with S_V and S_W defined in (5.2).

Since $S_V \times S_W$ is compact there always exist $(\underline{v}, \underline{w})$ in $S_V \times S_W$ such that $\delta(V, W)$ is attained. Note that $\delta(V, W) = 0$ implies that the subspaces V and W form a support (see Definition 1).

Next, in order to compute $\delta(V, W)$ we introduce convenient parameters.

- First we fix two isometries

$$\mathcal{V} : \mathbb{C}^r \to V, \qquad \mathcal{W} : \mathbb{C}^s \to W.$$

  Observe that in particular, $P_V = \mathcal{V}\mathcal{V}^*$ and $P_W = \mathcal{W}\mathcal{W}^*$ are the orthogonal projections in $\mathbb{C}^n$ onto V and W respectively.

- Then any morphism $f : \mathbb{C}^r \to V$ is of the form $f = \mathcal{V}g$ for $g : \mathbb{C}^r \to \mathbb{C}^r$ a linear map. If we write the polar form $g = au$ where $a \geq 0$ and u is unitary, we have $f = \mathcal{V}au$. Therefore we observe, in relation to the problem of parametrization:

  1. $\mathrm{Tr}(ff^*) = \mathrm{Tr}(\mathcal{V}a^2\mathcal{V}^*) = \mathrm{Tr}(\mathcal{V}^*\mathcal{V}a^2) = \mathrm{Tr}(a^2)$ so that $f \in S_V$ if and only if $\mathrm{Tr}(a^2) = 1$.
  2. And we have $\Phi(ff^*) = \Phi(\mathcal{V}a^2\mathcal{V}^*)$.

Similar considerations can be done for $\mathcal{W}$ and S_W.

In view of these remarks we parametrize the problem of finding the minimum of $\delta(V, W)$ as follows.

The parameter space will be $\Sigma = \Sigma_r \times \Sigma_s$, where

$$\Sigma_r = \{a \in M_r^h(\mathbb{C}) : \mathrm{Tr}(a^2) = 1\} \;\text{ and }\; \Sigma_s = \{b \in M_s^h(\mathbb{C}) : \mathrm{Tr}(b^2) = 1\} \quad (5.4)$$

are the unit spheres of the self-adjoint matrices (positive or not) of sizes $r \times r$ and $s \times s$.

The function we have to minimize is $F : \Sigma \to [0, +\infty)$, defined by

$$F(a, b) = ||\Phi(\mathcal{V}a^2\mathcal{V}^*) - \Phi(\mathcal{W}b^2\mathcal{W}^*)||^2 \quad (5.5)$$

where the norm is given by $\|x\| = \sqrt{Tr(x^*x)}$. Its minimum value is the adequacy

$$\delta(V, W) = \min_{(a,b)\in\Sigma} F(a, b). \quad (5.6)$$

In the next computations, in order to alleviate the notation, we will write

$$\Delta = \Delta(a, b) = \Phi(\mathcal{V}a^2\mathcal{V}^*) - \Phi(\mathcal{W}b^2\mathcal{W}^*) \quad (5.7)$$

5.1. The gradient of F

Now we let a vary as a function of a real parameter t and independently b vary as a function of u. Then

$$\frac{\partial F}{\partial t} = 2\langle \frac{\partial \Phi(\mathcal{V}a^2\mathcal{V}^*)}{\partial t}, \Delta\rangle \qquad \text{and} \qquad \frac{\partial F}{\partial u} = -2\langle \frac{\partial \Phi(\mathcal{W}b^2\mathcal{W}^*)}{\partial u}, \Delta\rangle.$$

If we denote with $\frac{da}{dt} = X$, $\frac{db}{du} = Y$, then

$$\frac{\partial F}{\partial t} = 2\langle \Phi(\mathcal{V}(aX + Xa)\mathcal{V}^*), \Delta\rangle \quad \text{and} \quad \frac{\partial F}{\partial u} = -2\langle \Phi(\mathcal{W}(bY + Yb)\mathcal{W}^*), \Delta\rangle.$$

Here the inner products are traces of products, so using that Δ is diagonal we can write

$$\frac{\partial F}{\partial t} = 2\,\mathrm{Tr}(\mathcal{V}(aX + Xa)\mathcal{V}^*\Delta) \quad \text{and} \quad \frac{\partial F}{\partial u} = -2\,\mathrm{Tr}(\mathcal{W}(bY + Yb)\mathcal{W}^*\Delta).$$

Therefore

$$\frac{\partial F}{\partial t} = 2\langle aX + Xa, \mathcal{V}^*\Delta\mathcal{V}\rangle_{M_r(\mathbb{C})} \quad \text{and} \quad \frac{\partial F}{\partial u} = -2\langle bY + Yb, \mathcal{W}^*\Delta\mathcal{W}\rangle_{M_s(\mathbb{C})}.$$

where the inner products now involved are the natural ones in $r \times r$ and $s \times s$ matrices using the corresponding traces.

If in these algebras $M_r(\mathbb{C})$ and $M_s(\mathbb{C})$ we consider the operators

$$S_a(X) = aX + Xa, \quad \text{and} \quad S_b(Y) = bY + Yb, \tag{5.8}$$

then its adjoints (for the natural inner products) are precisely $S_{a^*} = S_a$ and $S_{b^*} = S_b$ since a, b are self-adjoint. So we can write

$$\frac{\partial F}{\partial t} = 2\langle X, S_a(\mathcal{V}^*\Delta\mathcal{V})\rangle \quad \text{and} \quad \frac{\partial F}{\partial u} = -2\langle Y, S_b(\mathcal{W}^*\Delta\mathcal{W})\rangle.$$

Therefore we obtained the following result.

Theorem 4. *The gradient of the function $F : \Sigma \to [0, +\infty)$, $F(a, b) = ||\Phi(\mathcal{V}a^2\mathcal{V}^*) - \Phi(\mathcal{W}b^2\mathcal{W}^*)||^2$ on the Riemannian manifold $\Sigma = \Sigma_r \times \Sigma_s$ at (a, b) (with Σ_r and Σ_s as in* (5.4)*) is*

$$\mathrm{grad}_{(a,b)} F = 2\left(S_a(\mathcal{V}^*\Delta\mathcal{V})_{tan}, -S_b(\mathcal{W}^*\Delta\mathcal{W})_{tan}\right) \tag{5.9}$$

where the subscript "tan" refers to the tangential component (to the sphere $\Sigma_r \times \Sigma_s$) of the corresponding vector: $X_{tan} = X - \langle X, a\rangle a$, for $X \in M_r(\mathbb{C})$ and $Y_{tan} = Y - \langle Y, b\rangle b$, for $Y \in M_s(\mathbb{C})$, Δ is defined in (5.7)*, S_a, S_b in* (5.8) *and $\mathcal{V}$, $\mathcal{W}$ are fixed isometries as in* (5)*.*

5.2. Approximation of the adequacy $\delta(V,W)$.

The previous theorem allow us to construct a gradient descent type algorithm to approximate the adequacy of a pair of orthogonal subspaces (V,W) in $\mathbb{C}^n$.

1. Starting with V and W, construct the corresponding isometries $\mathcal{V} \in \mathbb{C}^{n\times r}$ and $\mathcal{W} \in \mathbb{C}^{n\times s}$ defined in (5) (take for example an orthonormal basis of V and build the matrix $\mathcal{V}$ whose columns are the vectors of that basis, similarly for $\mathcal{W}$).
2. Choose randomly two positive definite trace one matrices $a_1 \in \mathbb{C}^{r\times r}$ and $b_1 \in \mathbb{C}^{s\times s}$.
3. Then for $i = 1, \dots, k$ calculate recursively:
   (a) $(a_i, b_i) - \operatorname{grad}_{(a_i,b_i)} F$ using the identity 5.9:

$$\begin{aligned}(a_i, b_i) - \operatorname{grad}_{(a_i,b_i)} F \;\; = \;\; &\Big(a_i - 2\Big(S_{a_i}(\mathcal{V}^*\Delta_i\mathcal{V}) - \operatorname{Tr}\left(S_{a_i}(\mathcal{V}^*\Delta_i\mathcal{V})a_i\right) a_i\Big)\;,\\ &b_i + 2\Big(S_{b_i}(\mathcal{W}^*\Delta_i\mathcal{W}) - \operatorname{Tr}\left(S_{b_i}(\mathcal{W}^*\Delta_i\mathcal{W})b_i\right) b_i\Big)\Big)\end{aligned}$$

   where $S_c(X) = cX + Xc$, and $\Delta_i = \Phi(\mathcal{V}a_i^2\mathcal{V}^*) - \Phi(\mathcal{W}b_i^2\mathcal{W}^*)$.

   (b) Then consider $(\alpha_{i+1}, \beta_{i+1}) = (a_i, b_i) - \operatorname{grad}_{(a_i,b_i)} F$, and define a_{i+1} and b_{i+1} as its modules with unit norm:

$$a_{i+1} = \frac{1}{\operatorname{Tr}(|\alpha_{i+1}|^2)^{1/2}}|\alpha_{i+1}| \quad \text{and} \quad b_{i+1} = \frac{1}{\operatorname{Tr}(|\beta_{i+1}|^2)^{1/2}}|\beta_{i+1}|$$

   (c) If $i+1 < k$ go back to step a) and continue the iteration with a_{i+1} and b_{i+1}.
4. After finishing the k iterations compute $\operatorname{Tr}(\Delta_{k+1}\Delta_{k+1})$ to approximate the adequacy $\delta(V,W)$ (see 5.5).

In Figure 1 it is shown the output of several evaluations of the adequacy using the previous procedure on a pair of orthogonal subspaces moved with the multiplication of a curve of unitary matrices.

Remark 6. *Some of the examples presented in Appendix A, Appendix B and Appendix C were obtained using the previous algorithm to approximate the adequacy.*

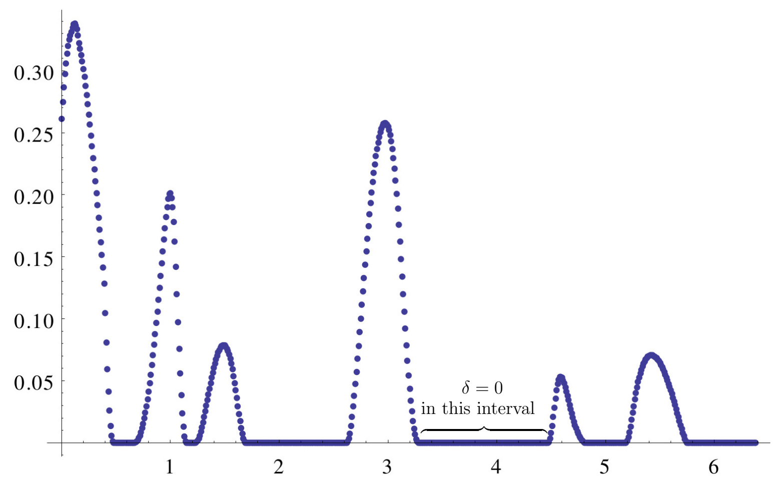

Figure 1: Plot of the points $(x_j, \delta(V_j, W_j))$, for $x_j = (j/100)$, the subspaces $V_j = e^{ix_jA}(V)$ and $W_j = e^{ix_jA}(W)$, with $j = 1, \ldots, 650$, starting with $V \perp W$, and A a self-adjoint matrix, using the algorithm mentioned in 5.2 to calculate the adequacy δ. Observe the intervals where the approximation of the adequacy is null that suggest that for those values of x_j the pairs (V_j, W_j) form a support.

5.3. The critical points of F

The point $(a, b) \in \Sigma$ is critical for F if and only if $S_a(\mathcal{V}^*\Delta\mathcal{V})$ is normal to Σ_r and $S_b(\mathcal{W}^*\Delta\mathcal{W})$ is normal to Σ_s. Then we can state the following result.

Theorem 5. *The point $(a, b) \in \Sigma = \Sigma_r \times \Sigma_s$ is critical for F if and only if*

$$\begin{cases} S_a(\mathcal{V}^*\Delta\mathcal{V}) = \lambda a \\ S_b(\mathcal{W}^*\Delta\mathcal{W}) = \mu b \end{cases}, \textit{ for } \lambda, \mu \in \mathbb{R}. \tag{5.10}$$

5.4. Analysis of the conditions (5.10)

Suppose that we have operators $c \geq 0$, u self-adjoint and $cu + uc = \eta c$ where $\eta \in \mathbb{R}$. Then the following commutation rule holds

$$\begin{cases} cu &= (\eta - u)c \\ uc &= c(\eta - u) \end{cases}.$$

Then, u commutes with c and we have $uc = cu = \frac{\eta}{2}c$. The previous comments allow us to state the next result.

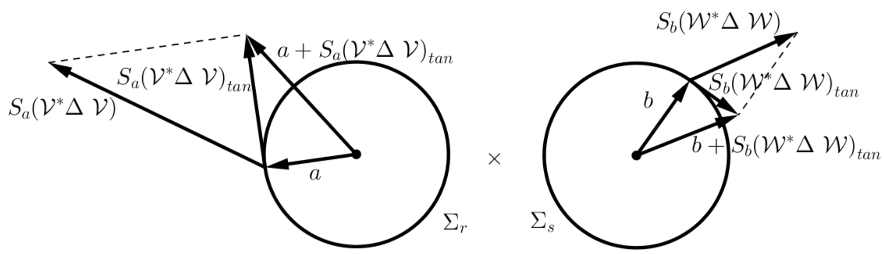

Figure 2: Increments on the tangents of both spheres Σ_r and Σ_s in the direction of the gradient used in the approximation algorithm for the adequacy $\delta(V, W)$.

Theorem 6. *In a critical point (a, b) of F as in (5.10) where $a \geq 0$ and $b \geq 0$ then $\mathcal{V}^*\Delta\mathcal{V}$ commutes with a and $(\mathcal{V}^*\Delta\mathcal{V})a = \frac{\lambda}{2}a$ and also $\mathcal{W}^*\Delta\mathcal{W}$ commutes with b and $(\mathcal{W}^*\Delta\mathcal{W})b = \frac{\mu}{2}b$.*

Remark: In these notes we are interested in the minimum value of F on Σ. Since $(a, b) \in \Sigma$ implies $(|a|, |b|) \in \Sigma$, because $a^2 = |a|^2$, $b^2 = |b|^2$ if a, b are hermitian, and $F(a, b) = F(|a|, |b|)$ it is clear that the minimum of F is attained on some (a, b) with $a \geq 0$ and $b \geq 0$.

5.5. The Hessian of the map F

Recall the expression of $\operatorname{grad}_{(a,b)} F$ obtained in (5.9) and the definition of S_a and S_b in (5.8) for $a \in \Sigma_r$ and $b \in \Sigma_s$ (see (5.4)).

We write V_{tan} to denote the tangential part of $V \in M_r^h(\mathbb{C})$ of the sphere Σ_r when V is considered as a tangent vector at a point of Σ_r (correspondingly for $W \in M_s^h(\mathbb{C})$ and Σ_s). Let us denote with $\pi_r : M_r^h(\mathbb{C}) \to T(\Sigma_r)_a$ and $\pi_s : M_s^h(\mathbb{C}) \to T(\Sigma_s)_b$

$$\pi_r(V) = V_{tan} = V - \langle V, a\rangle a, \text{ for } a \in \Sigma_r, V \in M_r^h(\mathbb{C}) \tag{5.11}$$

$$\pi_s(W) = W_{tan} = W - \langle W, b\rangle b, \text{ for } b \in \Sigma_s, W \in M_r^h(\mathbb{C}). \tag{5.12}$$

Recall also that in a Riemannian manifold, the Hessian of a function U at a critical point is given by

$$H(U)(Z, W) = \langle D_Z \operatorname{grad} U, W\rangle \tag{5.13}$$

where D denotes the covariant derivative of the Levi-Civita connection of the metric. Finally recall that the covariant derivative in our case is the tangent projection of the "ambient" derivative.

In the computations below we will need expressions for the derivatives ∂_X and ∂_Y in the directions $X \in T(\Sigma_r)_a$ and $Y \in T(\Sigma_s)_b$ respectively of the projections π_r and π_s.

Recall that in (5.9) we calculated

$$\operatorname{grad}_{(a,b)} F = 2\left(\pi_r\left(S_a(\mathcal{V}^*\Delta\mathcal{V})\right), -\pi_s\left(S_b(\mathcal{W}^*\Delta\mathcal{W})\right)\right)$$

where $\Delta = \Phi(\mathcal{V}a^2\mathcal{V}^*) - \Phi(\mathcal{W}b^2\mathcal{W}^*)$. In order to calculate (5.13) we can use that $D_{(X,Y)} = D_{(X,0)+(0,Y)} = D_{(X,0)} + D_{(0,Y)}$.

Then the covariant derivative of $\pi_r(S_a(\mathcal{V}^*\Delta\mathcal{V}))$ is given by

$$\begin{aligned} D_{(X,Y)}&(S_a(\mathcal{V}^*\Delta\mathcal{V}) - \langle S_a(\mathcal{V}^*\Delta\mathcal{V}), a\rangle a) = \\ &= S_X(\mathcal{V}^*\Delta\mathcal{V}) + S_a(\mathcal{V}^*(\partial_X\Delta + \partial_Y\Delta)\mathcal{V}) \\ &\quad - \langle S_a(\mathcal{V}^*(\partial_X\Delta + \partial_Y\Delta)\mathcal{V}), a\rangle a - 2\langle S_X(\mathcal{V}^*\Delta\mathcal{V}), a\rangle a \\ &\quad - \langle S_a\left(\mathcal{V}^*\Delta\mathcal{V}\right), a\rangle X \end{aligned} \tag{5.14}$$

where we have used that S_a and S_X are self-adjoint and $S_a(X) = S_X(a)$.

The covariant derivative of $\pi_s(S_b(\mathcal{W}^*\Delta\mathcal{W}))$ can be calculated similarly.

Observe that

$$\begin{aligned} \partial_X(\Delta) &= \partial_X\left(\Phi\left(\mathcal{V}a^2\mathcal{V}^* - \mathcal{W}b^2\mathcal{W}^*\right)\right) \\ &= \Phi(\mathcal{V}(aX + Xa)\mathcal{V}^*) = \Phi(\mathcal{V}S_a(X)\mathcal{V}^*) \end{aligned} \tag{5.15}$$

and $\partial_Y(\Delta) = -\Phi(\mathcal{W}S_b(Y)\mathcal{W}^*)$. Then using that $\partial_X\Delta + \partial_Y\Delta$ is diagonal

$$\begin{aligned} \langle S_a(\mathcal{V}^*(\partial_X\Delta + \partial_Y\Delta)\mathcal{V}), X\rangle &= \langle(\partial_X\Delta + \partial_Y\Delta), \mathcal{V}S_a(X)\mathcal{V}^*\rangle \\ &= \langle(\partial_X\Delta + \partial_Y\Delta), \Phi\left(\mathcal{V}S_a(X)\mathcal{V}^*\right)\rangle \\ &= \langle(\partial_X\Delta + \partial_Y\Delta), \partial_X\Delta\rangle \end{aligned}$$

where we have used in the last equality the formula obtained in (5.15) for $\partial_X\Delta$. Similarly we can prove that $\langle S_b(\mathcal{W}^*(\partial_X\Delta + \partial_Y\Delta)\mathcal{W}), Y\rangle = -\langle(\partial_X\Delta + \partial_Y\Delta), \partial_Y\Delta\rangle$. Then

$$\begin{aligned} \langle S_a(\mathcal{V}^*(\partial_X\Delta + \partial_Y\Delta)\mathcal{V}), X\rangle &- \langle S_b(\mathcal{W}^*(\partial_X\Delta + \partial_Y\Delta)\mathcal{W}), Y\rangle = \\ &= \langle(\partial_X\Delta + \partial_Y\Delta), \partial_X\Delta\rangle + \langle(\partial_X\Delta + \partial_Y\Delta), \partial_Y\Delta\rangle \\ &= \langle(\partial_X\Delta + \partial_Y\Delta), (\partial_X\Delta + \partial_Y\Delta)\rangle \\ &= \|\partial_X\Delta + \partial_Y\Delta\|^2 \end{aligned} \tag{5.16}$$

Finally, using the expression (5.13) for the quadratic form $H(F)\left((X,Y),(X,Y)\right)$ and (5.14) we obtain that

$$\begin{aligned}H(F)\left((X,Y),(X,Y)\right) &= \langle D_{(X,Y)}\left(\pi_r(S_a(\mathcal{V}\Delta\mathcal{V}^*))\right), X\rangle - \langle D_{(X,Y)}\left(\pi_s(S_b(\mathcal{W}^*\Delta\mathcal{W}))\right), Y\rangle \\ &= ||\partial_X\Delta + \partial_Y\Delta||^2 + 2\left(\langle \mathcal{V}^*\Delta\mathcal{V}X, X\rangle - \langle \mathcal{W}^*\Delta\mathcal{W}Y, Y\rangle\right) \\ &\quad - \langle S_a(\mathcal{V}^*\Delta\mathcal{V}), a\rangle ||X||^2 + \langle S_b(\mathcal{W}^*\Delta\mathcal{W}), b\rangle ||Y||^2\end{aligned}$$

where we have used that $a \perp X$, $b \perp Y$, $\langle S_X(\mathcal{V}^*\Delta\mathcal{V}), X\rangle = 2\langle \mathcal{V}^*\Delta\mathcal{V}X, X\rangle$ and $\langle S_Y(\mathcal{W}^*\Delta\mathcal{W}), Y\rangle = 2\langle \mathcal{W}^*\Delta\mathcal{W}Y, Y\rangle$. We could simplify the expression of the Hessian even more using that $\langle S_a(\mathcal{V}^*\Delta\mathcal{V}), a\rangle = 2\langle \mathcal{V}^*\Delta\mathcal{V}, a^2\rangle$ and $\langle S_b(\mathcal{W}^*\Delta\mathcal{W}), b\rangle = 2\langle \mathcal{W}^*\Delta\mathcal{W}, b^2\rangle$ to obtain the following result

Theorem 7. *The Hessian of the map $F : \Sigma_r \times \Sigma_s \to \mathbb{R}_{\geq 0}$ (see* (5.5) *and* (5.4)*) for $X \in T(\Sigma_r)_a$ and $Y \in T(\Sigma_s)_b$ at a critical point (a,b) can be calculated as*

$$\begin{aligned}H(F)\left((X,Y),(X,Y)\right) = ||\partial_X\Delta + \partial_Y\Delta||^2 + 2\big(&\langle \mathcal{V}^*\Delta\mathcal{V}, X^2 - a^2\|X\|^2\rangle \\ &- \langle \mathcal{W}^*\Delta\mathcal{W}, Y^2 - b^2\|Y\|^2\rangle\big).\end{aligned}$$

6. A geometric interpretation of the adequacy

Let V and W be two orthogonal subspaces of $\mathbb{C}^n$ with $\dim(V) = r$ and $\dim(W) = s$ as before.

In this section we distinguish three subalgebras of $M_n(\mathbb{C})$.

1. $\mathcal{D}_n \subset M_n(\mathbb{C})$ the subalgebra of diagonal matrices and $\Phi : M_n(\mathbb{C}) \to \mathcal{D}_n$ the conditional expectation that associates to the matrix m its diagonal part $\Phi(m)$ as before. Observe that Φ is an orthogonal projection for the natural Hilbert structure of $M_n(\mathbb{C})$. We have for $m \in M_n(\mathbb{C})$

$$\Phi(m) = \sum_{k=1}^{n} p_k m p_k \tag{6.1}$$

   where p_k is the orthogonal projection onto the k-axis of $\mathbb{C}^n$.
2. We denote with $M_n(V) \subset M_n(\mathbb{C})$ the subalgebra of the endomorphisms x of $\mathbb{C}^n$ which commute with $P_V = \mathcal{V}\mathcal{V}^*$ (for $\mathcal{V} : \mathbb{C}^r \to V \subset \mathbb{C}^n$ an isometry with range V) and verify $P_V x = x$. Observe that $M_n(V)$ is a C*-subalgebra of $M_n(\mathbb{C})$ with identity P_V.

Also the map $\mathcal{P}_V : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ defined by

$$\mathcal{P}_V(m) = P_V m P_V$$

satisfies the requirements of a conditional expectation in $M_n(\mathbb{C})$ with image $M_n(V)$, except for the fact that $\mathcal{P}_V(I) = P_V \neq I$. Finally

$$\mathcal{I}_V : M_r(\mathbb{C}) \to M_n(V) \subset M_n(\mathbb{C}) \quad , \quad \mathcal{I}_V(a) = \mathcal{V} a \mathcal{V}^*$$

defines an isomorphism of C*-algebras between $M_r(\mathbb{C})$ and $M_n(V)$.

3. Similarly we denote $M_n(W)$, P_W, $\mathcal{P}_W$ and $\mathcal{I}_W$ related to the subspace W. Notice that $M_n(V)$ and $M_n(W)$ are orthogonal in $M_n(\mathbb{C})$ for the Hilbert space structure and also in the sense that

$$ab = ba = 0 \text{ for } a \in M_n(V) \text{ and } b \in M_n(W).$$

Now we analyze the optimization problem of computing the adequacy of (V, W) in this context.

We denote with $M_n^h(V)$ the self-adjoint part of $M_n(V)$. The function $a \mapsto \mathcal{V} a^2 \mathcal{V}^*$ maps bijectively the positive part Σ_r^+ of the unit sphere $\Sigma_r = \{a \in M_r^h(\mathbb{C}) : \mathrm{Tr}(a^2) = 1\}$ (see (5.4)) onto the set

$$\sigma_V = \{c \in M_n^h(V) : c \geq 0 \text{ and } \mathrm{Tr}\, c = 1\}. \tag{6.2}$$

Note that if $c \in \sigma_V$, then $\mathcal{V}^* c^{1/2} \mathcal{V}$ lies in Σ_r^+, where $c^{1/2}$ is the positive square root of the operator c. Similar considerations apply to W and we can define the corresponding σ_W.

Recall that the minimum of the function F (the adequacy of the pair (V, W), see (5.6)) is attained, among other points, at some $(a, b) \in \Sigma_r \times \Sigma_s$ where $a \geq 0$ and $b \geq 0$. Therefore the adequacy can be obtained as the square of the distance of the set $\Phi(\sigma_V)$ to the set $\Phi(\sigma_W)$

$$\delta(V, W) = \left(\mathrm{dist}\,(\Phi(\sigma_V), \Phi(\sigma_W))\right)^2 \tag{6.3}$$

Now we describe the set $\Phi(\sigma_V)$ (and similarly $\Phi(\sigma_W)$). Clearly σ_V is a convex compact set in $M_n^h(V)$ and therefore σ_V is the convex hull of the set σ_V^e of its extremal points. Since $\mathcal{I}_V : M_r(\mathbb{C}) \to M_n(V) \subset M_n(V)$ is an isomorphism of C*-algebras, the set σ_V^e consists of the projections p of rank one in $M_n(V)$. Now these projections p are obtained as follows

$$p = uu^*, \quad \text{with } u \text{ a unit vector in } V.$$

In this case the diagonal of p coincides with $\Phi(p) = \operatorname{diag}(|u_1|^2, \dots, |u_n|^2)$.

Let us denote by $\Sigma_V \subset \mathbb{C}^n$ the unit sphere of V and correspondingly by $\Sigma_W \subset \mathbb{C}^n$ the unit sphere of W.

Also define $m : \mathbb{C}^n \to \mathbb{R}^n$ by

$$m(v) \simeq \Phi(vv^*) \tag{6.4}$$

where we identified the diagonal $\Phi(vv^*)$ with the vector $((vv^*)_{1,1}, \dots, (vv^*)_{n,n}) \in \mathbb{R}^n_{\geq 0}$. Then we can state the next result.

Theorem 8. *If m is as in (6.4), Σ_V is the unit sphere of the subspace V and $co(m(\Sigma_V))$ is the convex hull of the set $m(\Sigma_V)$, then*

$$\Phi(\sigma_V) = co(m(\Sigma_V))$$

for Φ defined in (6.1) and σ_V in (6.2).

Proof. Since Φ is linear, $\Phi(\sigma_V)$ is a convex compact set in $\mathbb{R}^n$. Therefore, $\Phi(\sigma_V)$ is the convex hull of its extremal set. But it is well known that the extremal set of $\Phi(\sigma_V)$ is contained in the image $\Phi(\sigma_V^e)$ which is $m(\Sigma_V)$. Therefore $\Phi(\sigma_V)$ is included in the convex hull of $m(\Sigma_V)$.

The inclusion $m(\Sigma_V) \subset \Phi(\sigma_V)$ implies that $co(m(\Sigma_V)) \subset \Phi(\sigma_V)$ which proves the equality. □

Remark 7. *Note that in general the set of extremal points of $\Phi(\sigma_V)$ is strictly included in $\Phi(\sigma_V^e) = m(\Sigma_V)$.*

Remark 8. *If S^{2r-1} denotes the unit sphere in $\mathbb{C}^r$ then, since $\Sigma_V = \mathcal{V}S^{2r-1}\mathcal{V}^*$, we can replace $m(\Sigma_V)$ with $m(\mathcal{V}S^{2r-1}\mathcal{V}^*)$ in the previous theorem.*

7. On the critical points of the function F

The results of the previous section motivates the study of minimum values of $F : \Sigma = \Sigma_r \times \Sigma_s \to \mathbb{R}_{\geq 0}$ (see (5.5)) attained at extremal points of the sets $\Phi(\sigma_V)$ and $\Phi(\sigma_W)$. In this section we describe critical points of F under the assumption that they are attained on pairs of one dimensional projections. This would always be the case if the sets $\Phi(\sigma_V)$ and $\Phi(\sigma_W)$ were strictly convex as seen in all the examples we examined where none of the vectors of the standard basis belong to either subspace.

We assume the following:

1. $(a, b) \in \Sigma$ is a critical point for F
2. a and b are one dimensional projections in $\mathbb{C}^r$ and $\mathbb{C}^s$ respectively.
3. We choose $\tilde{a} \in \mathbb{C}^r$ and $\tilde{b} \in \mathbb{C}^s$ unit vectors such that

$$a(x) = \langle x, \tilde{a} \rangle \tilde{a} \ , \quad x \in \mathbb{C}^r \ , \text{ and } b(y) = \langle y, \tilde{b} \rangle \tilde{b} \ , \quad y \in \mathbb{C}^s.$$

4. We denote with

$$x_k = \mathcal{V}^* e_k \ , \text{ and } y_k = \mathcal{W}^* e_k, \ \text{ for } k = 1, \ldots, n,$$

where e_k are the standard base vectors and $\mathcal{V}$, $\mathcal{W}$ some fixed isometries as in (5).

5. We denote $\alpha_k = \langle \tilde{a}, x_k \rangle$ and $\beta_k = \langle \tilde{b}, y_k \rangle$. Then, using that $\mathcal{V}\tilde{a} = \sum \langle \mathcal{V}\tilde{a}, e_k \rangle e_k$, and that therefore $\tilde{a} = \mathcal{V}^*\mathcal{V}\tilde{a} = \sum \langle \mathcal{V}\tilde{a}, e_k \rangle \mathcal{V}^* e_k$ we can conclude that $\tilde{a} = \sum \alpha_k x_k$. Similarly $\tilde{b} = \sum \beta_k y_k$ can be obtained.
6. Since $a, b \geq 0$, after some computations follows that the pair (a, b) is a critical point for the function F (see Theorem 6) if and only if

$$\sum_{k=1}^{n} (|\alpha_k|^2 - |\beta_k|^2 - \lambda/2)\alpha_k x_k = 0 = \sum_{k=1}^{n} (|\alpha_k|^2 - |\beta_k|^2 - \mu/2)\beta_k y_k. \quad (7.1)$$

Observe that $\sum |\alpha_k|^2 = 1$. In fact $\alpha_k = \langle \mathcal{V}\tilde{a}, e_k \rangle$ and since $\mathcal{V}$ is an isometry, $||\mathcal{V}\tilde{a}|| = 1$. Similarly $\sum |\beta_k|^2 = 1$.

Now we turn the analysis of equations (7.1). First notice that there exist non trivial complex combinations of the form

$$\sum_{k=1}^{n} \xi_k x_k = 0 \ , \text{ and } \sum_{k=1}^{n} \eta_k y_k = 0$$

because $r = \dim V < n$ and $s = \dim W < n$.

For each of such pairs, $\xi_1, \eta_1; \ldots; \xi_n, \eta_n$ consider the system

$$\alpha_k|\alpha_k|^2 - (\lambda/2 + |\beta_k|^2)\alpha_k - \xi_k = 0, \ \text{ and } \beta_k|\beta_k|^2 + (\mu/2 - |\alpha_k|^2)\beta_k + \eta_k = 0 \quad (7.2)$$

obtained from (7.1) identifying each coefficient with the corresponding ξ_k and η_k.

Next we multiply the first equation of (7.2) by φ_k and the second by ψ_k (with $|\varphi_k| = 1$ and $|\psi_k| = 1$) so that each $\sigma_k = \varphi_k \xi_k$ is real and $\tau_k = \psi_k \eta_k$ is real. Defining $s_k = \varphi_k \alpha_k$ and $t_k = \psi_k \beta_k$ we get from equations (7.2)

$$s_k^3 - (\lambda/2 + t_k^2)s_k - \sigma_k = 0, \ \text{ and } t_k^3 + (\mu/2 - s_k^2)t_k + \tau_k = 0 \quad (7.3)$$

and all the coefficients of these equations are real numbers.

In fact multiplying the first equation in (7.1) by $\tilde{a}$ and the second by $\tilde{b}$ we get

$$\sum_{k=1}^{n}(|\alpha_k|^2-|\beta_k|^2)|\alpha_k|^2=\lambda/2\ ,\ \text{and}\ \sum_{k=1}^{n}(|\alpha_k|^2-|\beta_k|^2)|\beta_k|^2=\mu/2 \qquad (7.4)$$

which shows that λ and μ are real and moreover $\lambda \geq \mu$ because

$$\frac{\lambda-\mu}{2}=\sum(|\alpha_k|^2-|\beta_k|^2)^2.$$

In terms of s_k and t_k equations (7.4) can be rewritten in the form

$$\sum_{k=1}^{n}(s_k^2-t_k^2)s_k^2=\lambda/2\ ,\ \text{and}\ \sum_{k=1}^{n}(s_k^2-t_k^2)t_k^2=\mu/2. \qquad (7.5)$$

The set of equations (7.3) and (7.5) form a complete system of 2n+2 equations with 2n+2 unknowns.

8. Characterization of critical points of F

Based on the discussion of the preceding paragraphs we state the following theorem.

Theorem 9. *Let $\mathcal{V}:\mathbb{C}^r\to\mathbb{C}^n$ and $\mathcal{W}:\mathbb{C}^s\to\mathbb{C}^n$ be fixed isometries such that $R(\mathcal{V})=V\perp R(\mathcal{W})=W$, $\{e_k\}_{k=1}^n$ be the standard basis of $\mathbb{C}^n$ and a,b be unidimensional projections in $\mathbb{C}^r$ and $\mathbb{C}^s$ respectively. Then the following statements are equivalent,*

i) *the pair (a,b) is a critical point of the map F (defined in (5.5)),*

ii) *there exists a pair of unitary vectors $(\tilde{a},\tilde{b})\in\mathbb{C}^r\times\mathbb{C}^s$ such that $a=\langle\cdot,\tilde{a}\rangle\tilde{a}$ and $b=\langle\cdot,\tilde{b}\rangle\tilde{b}$, and $(\tilde{a},\tilde{b})$ satisfy equations (7.1) for $\alpha_k=\langle\tilde{a},\mathcal{V}^*e_k\rangle$ and $\beta_k=\langle\tilde{b},\mathcal{W}^*e_k\rangle$ for $k=1,\dots,n$, and*

iii) *there exists a pair of unitary vectors $(\tilde{a},\tilde{b})\in\mathbb{C}^r\times\mathbb{C}^s$ such that $\tilde{a}=\sum_{k=1}^n\overline{\varphi}_k s_k\mathcal{V}^*e_k$ and $\tilde{b}=\sum_{k=1}^n\overline{\psi}_k t_k\mathcal{W}^*e_k$, where*

   (a) *$s_k,t_k\in\mathbb{R}$, for $k=1,\dots,n$ and $\sum_{k=1}^n s_k^2=1=\sum_{k=1}^n t_k^2$,*

   (b) *$\varphi_k,\psi_k\in\mathbb{C}$, $|\varphi_k|=|\psi_k|=1$, for $k=1,\dots,n$*

   (c) *$\|\sum_{k=1}^n\overline{\varphi}_k s_k\mathcal{V}^*e_k\|=1=\|\sum_{k=1}^n\overline{\psi}_k\ t_k\mathcal{W}^*e_k\|$*

(d) *there exists* $\sigma_k, \tau_k \in \mathbb{R}_{\geq 0}$ *such that* $\begin{cases} \sum_{k=1}^n \overline{\varphi}_k \sigma_k \mathcal{V}^* e_k = 0 \\ \sum_{k=1}^n \overline{\psi}_k \tau_k \mathcal{W}^* e_k = 0, \end{cases}$

(e) *and* $s_k, t_k \in \mathbb{R}$, *for* $k = 1, \dots, n$ *are solutions of the systems*

$$\begin{cases} s_k^3 - \Big(\big(\sum_{j=1}^n (s_j^2 - t_j^2) s_j^2\big) + t_k^2\Big) s_k + \sigma_k &= 0 \\ t_k^3 + \Big(\big(\sum_{j=1}^n (t_j^2 - s_j^2) t_j^2\big) - s_k^2\Big) t_k + \tau_k &= 0 \end{cases}$$

Proof. The equivalence i) $\Leftrightarrow$ ii) has been discussed in the previous section.

ii) $\Rightarrow$ iii) has also been proved at the end of the previous section.

Let us consider the implication iii) $\Rightarrow$ ii).

If we define λ and μ with $\begin{cases} \lambda/2 &= \sum_{k=1}^n (s_k^2 - t_k^2) s_k^2 \\ \mu/2 &= \sum_{k=1}^n (t_k^2 - s_k^2) t_k^2 \end{cases}$ then λ, μ, s_k, t_k (for $k = 1, \dots, n$) satisfy (7.5). Moreover, iii) (e) implies that they also satisfy (7.3).

Let us now define $\alpha_k = \overline{\varphi}_k s_k$, $\beta_k = \overline{\psi}_k t_k$, for $k = 1, \dots, n$, and observe that the equations iii) (c) $\| \sum_{k=1}^n \overline{\varphi}_k s_k \mathcal{V}^* e_k \| = 1 = \| \sum_{k=1}^n \overline{\psi}_k \, t_k \mathcal{W}^* e_k \|$ are equivalent to $\sum_{k=1}^n \overline{\varphi}_k s_k e_k = \sum_{k=1}^n \alpha_k e_k \in V$ and $\sum_{k=1}^n \overline{\psi}_k t_k e_k = \sum_{k=1}^n \beta_k e_k \in W$. Then if we define

$$\begin{aligned} \tilde{a} &= \sum_{k=1}^n \overline{\varphi}_k s_k \mathcal{V}^* e_k = \sum_{k=1}^n \alpha_k \mathcal{V}^* e_k \\ \tilde{b} &= \sum_{k=1}^n \overline{\psi}_k t_k \mathcal{W}^* e_k = \sum_{k=1}^n \beta_k \mathcal{W}^* e_k \end{aligned}$$

follows that

$$\begin{aligned} \mathcal{V}\tilde{a} &= \sum_{k=1}^n \alpha_k e_k \quad \Rightarrow \quad \alpha_k = \langle \tilde{a}, \mathcal{V}^* e_k \rangle \\ \mathcal{W}\tilde{b} &= \sum_{k=1}^n \beta_k e_k \quad \Rightarrow \quad \beta_k = \langle \tilde{b}, \mathcal{W}^* e_k \rangle \end{aligned}$$

(since $\sum_{k=1}^n \alpha_k e_k \in V$ and $\sum_{k=1}^n \beta_k e_k \in W$).

Now if we define $\xi_k = \overline{\varphi}_k \sigma_k$ and $\eta_k = \overline{\psi}_k \tau_k$, for $k = 1, \dots, n$, then iii)(e) implies that equations (7.2) are satisfied and therefore equations (7.1) are also satisfied with $\alpha_k = \langle \tilde{a}, \mathcal{V}^* e_k \rangle$ and $\beta_k = \langle \tilde{b}, \mathcal{W}^* e_k \rangle$. Then statement ii) holds. □

9. Supports that have neighborhoods of $\mathcal{S}_{(r,s)}$ in $\mathcal{F}_{(r,s)}$

Recall that with $\mathcal{F}_{(r,s)}$ we denote the set of pairs (V, W) of orthogonal subspaces V and W of $\mathbb{C}^n$ such that $\dim(V) = r$ and $\dim(W) = s$. See Section 2 for its relation with flag manifolds.

In this section we study the existence of supports $(V, W) \in \mathcal{S}_{(r,s)}$ that belong to an open neighborhood of $\mathcal{F}_{(r,s)}$ formed entirely of supports in $\mathcal{S}_{(r,s)}$.

Remark 9. *Note that in general, a support* (V, W) *of* $\mathbb{C}^n$ *in the flag* $\mathcal{F}_{(r,s)}$*, is not necessarily an interior point of* $\mathcal{F}_{(r,s)}$*. Consider for example two orthogonal one dimensional subspaces* $V = gen\{v\}$ *and* $W = gen\{w\}$ *that form a support in* $\mathbb{C}^n$ *(*$n \geq 3$*). Then their generators must satisfy* $|v_i| = |w_i|$ *for* $i = 1, \ldots, n$*. Suppose that* $v_1 \neq 0$*, and* $v_1 = \rho e^{i\theta}$*,* $w_1 = \rho e^{i\beta}$ *with* $\rho = |v_1| = |w_1|$ *and* $\theta, \beta \in [0, 2\pi)$*, Then for* $\varepsilon > 0$ *consider small perturbations* v_ε *and* w_ε *with their first coordinates* $(v_\varepsilon)_1 = \rho/(1+\varepsilon)e^{i\theta}$ *and* $(w_\varepsilon)_1 = \rho(1+\varepsilon)e^{i\beta}$ *and the rest equal to those of* v *and* w*. Then* $\langle v_\varepsilon, w_\varepsilon \rangle = \langle v, w \rangle = 0$ *but* $|(v_\varepsilon)_1| = \rho/(1+\varepsilon) \neq \rho(1+\varepsilon) = |(w_\varepsilon)_1|$ *for* $\varepsilon > 0$*. If we denote with* V_ε *and* W_ε *the subspaces generated by* v_ε *and* w_ε *respectively, the previous calculations prove that there exist pair of subspaces* $(V_\varepsilon, W_\varepsilon)$ *in the flag* $\mathcal{F}_{(1,1)}$ *that do not form a support and that they can be chosen as close to* (V, W) *as desired (taking* $\varepsilon \to 0$*). Therefore* (V, W) *is not an interior point of* $\mathcal{F}_{(1,1)}$*.*

Theorem 10. *Let* $n \in \mathbb{N}$*,* $n \geq 3$*. Then, there exists a support* (V_n, W_n) *in* $\mathbb{C}^n$ *that is an interior point of the flag* $\mathcal{F}_{(r,s)}$ *for certain* r*,* $s < n$*.*

Proof. We will use the examples described in the appendices in the cases $n = 3$, $n = 4$ and $n = 5$ where some cases of supports that are interior points of the flags $\mathcal{F}_{(2,1)}$, $\mathcal{F}_{(2,1)}$ and $\mathcal{F}_{(3,1)}$ are shown.

Consider now the supports (V, W) of $\mathbb{C}^3$, $\mathbb{C}^4$ and $\mathbb{C}^5$ described in appendices Appendix A, Appendix B and Appendix C respectively. We will also denote with V_3, V_4 and V_5 the matrices whose columns are defined with the generators of the corresponding subspaces described in each case in the mentioned appendices. $M_3 = V_3 \circ \overline{V_3}$, $M_4 = V_4 \circ \overline{V_4}$ and $M_5 = V_5 \circ \overline{V_5}$ are also the matrices defined there. Similarly W_3, W_4 and W_5 will denote the matrices whose unique column is the generator of the corresponding subspace W. In each case these supports are interior points of the corresponding flag manifolds.

Observe that for any $n \in \mathbb{N}$, $n \geq 3$, there exist $h, k, l \in \mathbb{N}$ such that $n = 3h + 4k + 5l$. Let us now fix a triple of those h, k and l and consider the subspaces V and W defined as follows. V is generated by the columns of the following $n \times n$ block matrix V_n formed with h copies of V_3, k of V_4 and l of

V_5 in the diagonal

$$V_n = \begin{pmatrix} \overbrace{\oplus_{i=1}^{h} V_3}^{3h} & \overbrace{0}^{4k} & \overbrace{0}^{5l} \\ 0 & \oplus_{i=1}^{k} V_4 & 0 \\ 0 & 0 & \oplus_{i=1}^{l} V_5 \end{pmatrix} \quad \text{where } \oplus_{i=1}^{m} V_j = \begin{pmatrix} \overbrace{V_j \quad 0 \quad \dots \quad 0}^{j\times m} \\ 0 \quad \ddots \quad \ddots \quad \vdots \\ \vdots \quad \ddots \quad \ddots \quad 0 \\ 0 \quad \dots \quad 0 \quad V_j \end{pmatrix}$$

and W_n is the $n \times 1$ matrix formed with h copies of W_3, k of W_4 and l of W_5 concatenated (where W_i, $i = 1, 2, 3$ are the subspaces used in the appendices Appendix A, Appendix B and Appendix C). The transpose of W_n is:

$$(W_n)^t = \big(\overbrace{W_3 \quad \dots \quad W_3}^{3h} \; \overbrace{W_4 \quad \dots \quad W_4}^{4k} \; \overbrace{W_5 \quad \dots \quad W_5}^{5l}\big) \in \mathbb{C}^{1\times n}$$

Now consider the subspace $V \subset \mathbb{C}^n$, generated by the columns of V_n and $W \in \mathbb{C}^n$ generated by W_n. Then it can be verified that V is orthogonal to W and (since $\dim(V_3) = \dim(V_4) = 2$ and $\dim(V_5) = 3$) that $\dim(V) = 2h + 2k + 3l$ and $\dim(W) = 1$. Moreover, considering $M_n = V_n \circ \overline{V_n} \in M_n(\mathbb{R}_+)$ it is easy to see that $\det(M_n) = \det(M_3)^h \det(M_4)^k \det(M_5)^l \neq 0$ because every factor is non-zero (see Appendix A, Appendix B and Appendix C). Then, the linear system

$$M_n X = W_n \circ \overline{W_n}$$

has a unique solution $X \in \mathbb{R}^{n\times 1}$. The concrete solution X can be found considering the examples of the Appendix, and satisfies $X_{i,1} > 0$ for all $i = 1, \dots, n$. Thus the pair $(V, W) \in \mathcal{F}_{(2h+2k+3l,1)} \subset \mathbb{C}^n = \mathbb{C}^{3h+4k+5l}$ and is a support as in Definition 1 (consider the vectors $v^i = \sqrt{X_{i,1}} v_i$ in the conditions (1.4), for $i = 1, \dots, n$, and v_i the i^{th} column of V_n).

Now consider small perturbations V' and W' of the subspaces V and W such that the dimensions of the perturbed subspaces are conserved and $V' \perp W'$ holds. That is, the pair $(V', W') \in \mathcal{F}(2h + 2k + 3l, 1)$ and is near (V, W). Then, we can choose n vectors of V' close to the ones in the columns of V_n such that they generate V'. Similarly for W'. Let us denote with V_n' and W_n' the matrices such that its columns are the respective generators mentioned. Moreover, V' and W' can be chosen in a neighborhood of V and W in such a way that the pair of matrices V_n' and W_n' satisfy that

1. $\det(V'_n \circ \overline{V'_n}) \neq 0$,
2. the unique solution $X \in \mathbb{C}^{n\times 1}$ of $(V'_n \circ \overline{V'_n})X = W'_n \circ \overline{W'_n}$ satisfies $X_{i,1} > 0$.

This implies that the pair (V', W') is a support according to Definition 1. Then (V, W) is an interior point of $\mathcal{F}_{(2h+2k+3l,1)}$. □

Remark 10. *Observe that in the decomposition used in the previous proof given by $n = 3h + 4k + 5l$, with $h, k, l \in \mathbb{N}$, the term $5l$ is only needed for $n = 5$. Every $n \in \mathbb{N} \setminus \{1, 2, 5\}$ can be written as $n = 3h + 4k$.*

Remark 11. *Note that at the end of the proof of the previous theorem if the subspaces V' and W' are not required to be orthogonal they still satisfy the conditions (1) and (2) if they are close enough to V and W.*

Here we present examples of supports in low dimensions that are interior points of flag manifolds.

Appendix A. Example of a support in $\mathbb{C}^3$ that is an interior point of $\mathcal{F}_{(2,1)}$

Let us consider the dimension 2 subspace $V \subset \mathbb{C}^3$ generated by the following norm one vectors:

$$\begin{cases} v^1 &= \left(\frac{1886514-7511450i}{\sqrt{157449642458577}}, 0, -\frac{4236005-8917684i}{\sqrt{157449642458577}}\right), \\ v^2 &= \left(-\frac{6034458+5957865i}{\sqrt{175782050184862}}, \frac{10006368+1934893i}{\sqrt{175782050184862}}, 0\right), \\ v^3 &= \left(-\frac{30683+33081i}{28\sqrt{4664715}}, \left(\frac{1537}{4} + \frac{479i}{4}\right)\sqrt{\frac{3}{1554905}}, \left(\frac{61}{7} + \frac{157i}{14}\right)\sqrt{\frac{55}{84813}}\right), \end{cases}$$

and the subspace W generated by $w = \left(\frac{5-i}{2\sqrt{15}}, \left(\frac{1}{2} - \frac{i}{2}\right)\sqrt{\frac{3}{5}}, \frac{2}{\sqrt{15}}\right)$, that is orthogonal to V.

Then, if $a_1 = \frac{115667}{303810}$, $a_2 = \frac{85199}{222794}$ and $a_3 = \frac{395794}{1670955}$ direct computations show that $\sum_{i=1}^{3} a_i = 1$ and that if $V_3 = (v^1, v^2, v^3)$ and $M_3 = V_3 \circ \overline{V_3} = (v^1 \circ \overline{v^1}, v^2 \circ \overline{v^2}, v^3 \circ \overline{v^3})$ are the 3×3 matrices with columns v^i and respectively $v^i \circ \overline{v^i}$, for $i = 1, 2, 3$, then

$$M_3 \cdot a^t = \sum_{i=1}^{3} a_i \left(v^i \circ \overline{v^i}\right) = w \circ \overline{w}. \tag{A.1}$$

where $a = (a_1, a_2, a_3)$, and a^t is its transpose (a column matrix). This proves that the pair (V, W) is a support (consider the vectors $v^i = \sqrt{a_i}\, v^i$, for

$i = 1, 2, 3$ and the definition (1.4)). Moreover, it can be checked that the matrix M_3 (the one involved in the equation (A.1)) has non-zero determinant (which proves that the numbers a_i, $i = 1, 2, 3$ are unique).

Then, it can be proved that the support (V, W) is interior in the set of $(2, 1)$ flags $\mathcal{F}_{(2,1)}$ of $\mathbb{C}^3$. This follows because continuous and small perturbations w' and $(v')^i$ of the vectors w and v^i (with the condition $w' \perp (v')^i$, for $i = 1, 2, 3$), produce a non-zero determinant of the perturbed corresponding matrix M_3' with columns $(v')^i \circ \overline{(v')^i}$ for $i = 1, 2, 3$. Then there are unique solutions $a_i' > 0$ of the corresponding equation (A.1) for the new vectors $(v')^i$ and w'. This proves that there exists a neighborhood of (V, W) in $\mathcal{F}_{(2,1)}$ such that every pair (V', W') belonging to it is a support according to Definition 1.

Appendix B. Example of a support in $\mathbb{C}^4$ that is an interior point of $\mathcal{F}_{(2,1)}$

Let $V \subset \mathbb{C}^4$ be the subspace of dimension 2 generated by the following norm one vectors:

$$\begin{cases} v^1 &= \left(-\frac{\frac{698}{3}+75i}{\sqrt{212114}}, \frac{\frac{1036}{3}+51i}{\sqrt{212114}}, \left(\frac{77}{3} - \frac{218i}{3}\right)\sqrt{\frac{2}{106057}}, \left(-\frac{113}{3} + \frac{49i}{3}\right)\sqrt{\frac{2}{106057}}\right), \\ v^2 &= \left(-\frac{530-\frac{655i}{2}}{\sqrt{1918749}}, \frac{760-\frac{173i}{2}}{\sqrt{1918749}}, \frac{\frac{219}{2}-782i}{\sqrt{1918749}}, \frac{\frac{263}{2}+552i}{\sqrt{1918749}}\right), \\ v^3 &= \left(-\frac{75+\frac{45i}{4}}{\sqrt{29729}}, \frac{54-\frac{365i}{4}}{\sqrt{29729}}, -\frac{18-\frac{243i}{4}}{\sqrt{29729}}, -\frac{\frac{169}{2}+\frac{159i}{4}}{\sqrt{29729}}\right) \\ v^4 &= \left(-\frac{\frac{1345}{2}+283i}{\sqrt{1909509}}, \frac{\frac{563}{2}+239i}{\sqrt{1909509}}, \frac{738+\frac{263i}{2}}{\sqrt{1909509}}, -\frac{782-\frac{519i}{2}}{\sqrt{1909509}}\right) \end{cases}$$

and W the subspace generated by $w = \left(\frac{\frac{1}{2}-\frac{i}{2}}{\sqrt{2}}, \frac{\frac{1}{2}-\frac{i}{2}}{\sqrt{2}}, \frac{\frac{1}{2}+\frac{i}{2}}{\sqrt{2}}, \frac{\frac{1}{2}+\frac{i}{2}}{\sqrt{2}}\right)$.

If $a_1 = \frac{20559837596768881}{124590980225106843}$, $a_2 = \frac{96813856451303497}{415303267417022810}$, $a_3 = \frac{1154873210442508}{8612279739062685}$ and $a_4 = \frac{49954131355895969}{106792268764377294}$, it can be verified that if $V_4 = (v^1, v^2, v^3, v^4)$, $M_4 = V_4 \circ \overline{V_4} = (v^1 \circ v^1, v^2 \circ v^2, v^3 \circ v^3, v^4 \circ v^4)$ and $a = (a_1, a_2, a_3, a_4)$, then

$$M_4 \cdot a^t = \sum_{i=1}^{4} a_i \left(v^i \circ \overline{v^i}\right) = w \circ \overline{w}. \tag{B.1}$$

The determinant of the matrix M_4 is non-zero and therefore similar considerations as those made in the previous example in Appendix A can be used in order to prove that (V, W) is a support of $\mathbb{C}^4$ that is included in an open subset of the flags $\mathcal{F}_{(2,1)}$.

Appendix C. Example of a support in $\mathbb{C}^5$ that is an interior point of $\mathcal{F}_{(3,1)}$

Let $V \subset \mathbb{C}^5$ be the subspace of dimension 3 generated by the rows of the matrix $M_V = \begin{pmatrix} -\frac{19}{50} - \frac{i}{50} & -\frac{2}{25} + \frac{19i}{50} & -\frac{7}{25} + \frac{3i}{25} & \frac{8}{25} + \frac{3i}{25} & \frac{8}{25} + \frac{11i}{50} \\ -\frac{1}{5} + \frac{11i}{50} & \frac{1}{10} + \frac{3i}{25} & \frac{19}{50} - \frac{i}{5} & \frac{19}{50} + \frac{3i}{10} & -\frac{21}{50} \\ \frac{29}{50} & -\frac{1}{50} + \frac{3i}{10} & -\frac{1}{10} - \frac{8i}{25} & \frac{1}{5} + \frac{9i}{50} & \frac{1}{5} - \frac{21i}{50} \end{pmatrix}$ and W the subspace generated by $w = \left(\frac{1-i}{\sqrt{10}}, \frac{1-i}{\sqrt{10}}, \frac{1+i}{\sqrt{10}}, \frac{1+i}{\sqrt{10}}, \frac{1+i}{\sqrt{10}}\right)$.

Now define the coefficient matrix $C = \begin{pmatrix} -\frac{16}{5} + \frac{53i}{10} & -\frac{4}{5} - \frac{5i}{2} & \frac{8}{5} + \frac{29i}{10} \\ 1 + \frac{i}{2} & -\frac{3}{5} & \frac{2}{5} + \frac{13i}{10} \\ -\frac{13i}{5} & -\frac{11}{5} + \frac{29i}{10} & \frac{4}{5} + i \\ 1 + \frac{3i}{5} & \frac{3}{5} + \frac{7i}{5} & \frac{1}{2} + \frac{7i}{10} \\ \frac{13}{10} + \frac{3i}{2} & -\frac{2}{5} + \frac{9i}{5} & -\frac{14}{5} - \frac{3i}{10} \end{pmatrix}$,
and consider the 5 vectors belonging to V obtained from the rows of the product $C \cdot M_V \in \mathbb{C}^{5\times 5}$. Let us denote those vectors (rows) with v^1, v^2, v^3, v^4 and v^5. Then it can be checked that for $a_1 = \frac{38245034600180718292066117}{93505493283505350729949090}$, $a_2 = \frac{8768938084324043506208 02}{9350549328350535072994909}$, $a_3 = \frac{118407893248532986294897 61}{93505493283505350729949090}$, $a_4 = \frac{11483749488079211997737796}{46752746641752675364974545}$ and $a_5 = \frac{11683232297988866306709 60}{9350549328350535072994909}$ the equality $\sum_{i=1}^{5} a_i = 1$ holds and if $V_5 = (v^1, v^2, v^3, v^4, v^5)$, $M_5 = V_5 \circ \overline{V_5} = (v^1 \circ v^1, v^2 \circ v^2, v^3 \circ v^3, v^4 \circ v^4, v^5 \circ v^5)$ and $a = (a_1, a_2, a_3, a_4, a_5)$, then

$$M_5 \cdot a^t = \sum_{i=1}^{5} a_i \; (v_i \circ \overline{v_i}) = w \circ \overline{w}. \tag{C.1}$$

The determinant of the matrix M_5 involved in equation (C.1) is non-zero and therefore similar considerations as those made in the previous examples of the Appendix can be made in order to prove that (V, W) is a support that is included in an open subset of the flags $\mathcal{F}_{(3,1)}$ of $\mathbb{C}^5$.

Remark 12. *Note that the steps used to prove that the previous example* (V_5, W_5) *is an interior point of* $\mathcal{F}_{(3,1)}$ *in* $\mathbb{C}^5$ *cannot be followed if the dimensions of the subspaces were* $\dim(V) = 2$ *and* $\dim(W) = 1$ *as in Appendix A and Appendix B. This is because if* $\dim(V) = 2$ *then* $\operatorname{rank} M_5 = \operatorname{rank}(V_5 \circ \overline{V_5}) \leq \operatorname{rank}(V_5)\operatorname{rank}(\overline{V_5}) = 4$*, and therefore* $\det(M_5) = 0$ *in this case (for any choice of* V_5*). This is not enough to asseverate that there is not a support in* $\mathbb{C}^5$ *that is an interior point of* $\mathcal{F}_{(2,1)}$*, but we have not found an example with these dimensions.*

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