Semiclosed projections and applications

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Abstract

We characterize the semiclosed projections and apply them to compute the Schur complement of a selfadjoint operator with respect to a closed subspace. These projections occur naturally when dealing with weak complementability.

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1. Introduction

A linear subspace $\mathcal{E}$ of a Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ is *semiclosed* if there exists an inner product $\langle\cdot,\cdot\rangle'$ such that $(\mathcal{E}, \langle\cdot,\cdot\rangle')$ is complete and continuously included in $\mathcal{H}$. The notion of semiclosed subspace was introduced by Kaufman but subspaces of this like appear in the literature before his seminal paper [28]. For instance, in the form of *contractively included subspaces* in the theory of the de Branges-Rovnyak spaces [18, 19], as the *operator ranges* of Fillmore and Williams [23] and, under the name of *para-closed* subspaces, in the work of Foiaș on the lattice of invariant subspaces [24]. Amongst the semiclosed subspaces of $\mathcal{H} \times \mathcal{H}$, Kaufman pays much attention to those that are graphs of linear operators in $\mathcal{H}$, the so-called *semiclosed* operators. In fact, the family of all such operators is the main object of analysis in his aforementioned account of semiclosed subspaces and operators. In the light of Kaufman's study on semiclosed operators we recognize a semiclosed operator V in the factorization $A = BV$ given by the earlier Douglas' Lemma for closed densely defined operators A, B satisfying the operator range inclusion $\mathcal{R}(A) \subseteq \mathcal{R}(B)$, as well as the Banach space version of the concept of semiclosed operator in a previous work by Caradus [12].

An operator E with domain $\mathcal{D}(E)$ and range $\mathcal{R}(E)$ in $\mathcal{H}$ is a *projection* provided $\mathcal{R}(E) \subseteq \mathcal{D}(E)$ and $E^2x = Ex$ for all $x \in \mathcal{D}(E)$. A semiclosed projection E densely defined in a Hilbert space $\mathcal{H}$ occurs when we are given a closed subspace $\mathcal{S}$ of $\mathcal{H}$ and a selfadjoint operator B everywhere defined on $\mathcal{H}$ such that B is $\mathcal{S}$-weakly complementable. Indeed, in this case the Schur complement $B_{/\mathcal{S}}$ of B to $\mathcal{S}$ exists and $B_{/\mathcal{S}} = (I - E)B$ for some (as a matter of fact, any) semiclosed densely defined projection E in an appropriate class. This fact (cf. [14]) drew our attention to study Hilbert space projections that are densely defined and semiclosed.

Closed projections were studied by Ôta in [33], where he showed that a projection is closed if and only if its nullspace and range are both closed subspaces. We generalize the result to semiclosed projections obtaining the corresponding assertion. Further in-depth investigations on closed projections were carried

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on by Ando [4]. He proved that a closed densely defined projection E with nullspace $\mathcal{N}$ and range $\mathcal{M}$ is distinctively represented as $E = (\Gamma^{-1}P_{\mathcal{M}})^*\Gamma^{-1}$ where $\Gamma := (P_{\mathcal{M}} + P_{\mathcal{N}})^{1/2}$ with $P_{\mathcal{M}}$ and $P_{\mathcal{N}}$ the orthogonal projectors onto $\mathcal{M}$ and $\mathcal{N}$, respectively. Moreover, he established that the well defined operator $\Gamma^{-1}E\Gamma$ is a bounded orthogonal projection. We obtain the analogous result for semiclosed projections with $P_{\mathcal{M}}$ and $P_{\mathcal{N}}$ replaced by A_1 and A_2, respectively, where (A_1, A_2) is any pair of positive semidefinite operators such that $\mathcal{M} = \mathcal{R}(A_1)$ and $\mathcal{N} = \mathcal{R}(A_2)$. However, in this case, it is not possible to obtain a "distinguished" one.

Ando also gave a 2×2 block matrix representation of a closed densely defined projection. The analog for a semiclosed densely defined projection E can be obtained under some extra condition on $\mathcal{D}(E)$. The very same condition allows us to define the Moore-Penrose pseudoinverse $E^\dagger$. The case when E is closed was considered in [16]. Therein it was shown that the inverse gives a bijective correspondence between the products of pairs of orthogonal projections and the set of closed densely defined projections. More generally, we prove that the Moore-Penrose of a semiclosed projection can be related to an operator which is a product of an orthogonal projection times a positive operator. The set of products PA, with P an orthogonal projection and A a positive semidefinite operator was studied in [7].

The semiclosed projections are fundamental when studying weak complementability, a concept introduced in [6] for operators in Hilbert spaces, that is a generalization of the concept of complementability, introduced by Ando for matrices, [2]. The semiclosed projections arisen in this context, when given a selfadjoint operator B on $\mathcal{H}$ and a closed subspace $\mathcal{S}$ of $\mathcal{H}$ such that B is $\mathcal{S}$-weakly complementable, are studied and fully characterized. On the other hand, we study the set of quasi-complementable pairs $(B, \mathcal{S})$, i.e., the set of B-symmetric closed projections onto a prescribed subspace $\mathcal{S}$. The relation between the notions of weak complementability and quasi-complementability is analyzed to establish whether they are comparable and to what extent.

Finally, we give a formula of the Schur complement $B_{/\mathcal{S}}$ of a selfadjoint operator B to $\mathcal{S}$ in terms of semiclosed projections. Also, we characterize $B_{/\mathcal{S}}$ as the maximum of a set, when a generalization of the minus order is considered, using again semiclosed projections, see [6] and [8].

The paper has five sections including this one. Section 2 is a brief expository introduction to semiclosed subspaces and operators, and serves to set the notation and give some other results that are needed in the following sections. Section 3 is entirely devoted to the study of the class of semiclosed densely defined projections. We also deal with the semiclosed densely defined projections having Moore-Penrose inverses, in particular, those with closed nullspaces. In Section 4 we are concerned with B-symmetric projections while in Section 5 we study the notions of weak and quasi complementability and give some applications.

2. Preliminaries

We assume that all Hilbert spaces are complex and separable. If $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces, by an *operator* from $\mathcal{H}$ to $\mathcal{K}$ we mean a linear function from a subspace of $\mathcal{H}$ to $\mathcal{K}$. The domain, range, nullspace and graph of any given operator A are denoted by $\mathcal{D}(A)$, $\mathcal{R}(A)$, $\mathcal{N}(A)$ and $gr(A)$, respectively. Given a subset $\mathcal{T} \subseteq \mathcal{K}$, the preimage of $\mathcal{T}$ under A is $A^{-1}(\mathcal{T}) := \{x \in \mathcal{H} : \ Ax \in \mathcal{T}\}$. $L(\mathcal{H}, \mathcal{K})$ stands for the space of the bounded linear operators everywhere defined on $\mathcal{H}$ to $\mathcal{K}$. When $\mathcal{H} = \mathcal{K}$ we write, for short, $L(\mathcal{H})$; $CR(\mathcal{H})$ denotes the subset of $L(\mathcal{H})$ of closed range operators.

The direct sum of two subspaces $\mathcal{M}$ and $\mathcal{N}$ of $\mathcal{H}$ is represented by $\mathcal{M} \dotplus \mathcal{N}$. If, moreover $\mathcal{M} \perp \mathcal{N}$, their orthogonal sum is denoted by $\mathcal{M} \oplus \mathcal{N}$. The symbol $\mathcal{Q}$ indicates the subset of the oblique projections in $L(\mathcal{H})$, namely, $\mathcal{Q} := \{Q \in L(\mathcal{H}) : Q^2 = Q\}$ and $\mathcal{P}$ the subset of all the orthogonal projections in $L(\mathcal{H})$, $\mathcal{P} := \{P \in L(\mathcal{H}) : P^2 = P = P^*\}$; for a closed subspace $\mathcal{M}$, $P_{\mathcal{M}}$ denotes the element in $\mathcal{P}$ with range $\mathcal{M}$.

Denote by $L(\mathcal{H})^s$ the set of selfadjoint operators in $L(\mathcal{H})$, $GL(\mathcal{H})$ the group of invertible operators in $L(\mathcal{H})$, $L(\mathcal{H})^+$ the cone of positive semidefinite operators in $L(\mathcal{H})$ and set $GL(\mathcal{H})^+ := GL(\mathcal{H}) \cap L(\mathcal{H})^+$. Given two operators $S, T \in L(\mathcal{H})$, the notation $T \leq S$ signifies that $S - T \in L(\mathcal{H})^+$. Given any $T \in L(\mathcal{H})$, $|T| := (T^*T)^{1/2}$ is the modulus of T and $T = U|T|$ is the polar decomposition of T, with U the partial isometry such that $\mathcal{N}(U) = \mathcal{N}(T)$.

Given $B \in L(\mathcal{H})^s$ and a (non necessarily closed) subspace $\mathcal{S}$ of $\mathcal{H}$, the B-orthogonal complement of $\mathcal{S}$ is $\mathcal{S}^{\perp_B} := \{x \in \mathcal{H} : \langle\, Bx, y \,\rangle = 0, \text{ for every } y \in \mathcal{S}\} = B^{-1}(\mathcal{S}^\perp) = B(\mathcal{S})^\perp$.

The next result, due to Fillmore and Williams, characterizes the sum and the intersection of operator ranges as operator ranges.

Theorem 2.1 ([23, Theorem 2.2, Corollary 2])**.** *Let $A, B \in L(\mathcal{H})$. Then*

1. $\mathcal{R}(A) + \mathcal{R}(B) = \mathcal{R}((AA^* + BB^*)^{1/2})$.
2. *There exist $X, Y \in L(\mathcal{H})$ such that $\mathcal{R}(A) \cap \mathcal{R}(B) = \mathcal{R}((AXA^*)^{1/2}) = \mathcal{R}((BYB^*)^{1/2})$.*

Given $B \in L(\mathcal{H})^s$ and $\mathcal{S}$ a closed subspace of $\mathcal{H}$, we say that $\mathcal{S}$ is B-*positive* if $\langle Bs, s \rangle > 0$ for every $s \in \mathcal{S},\ s \neq 0$. B-*nonnegative*, B-*neutral*, B-*negative* and B-*nonpositive* subspaces are defined analogously. If $\mathcal{S}$ and $\mathcal{T}$ are two closed subspaces of $\mathcal{H}$, the notation $\mathcal{S} \ \oplus_B \ \mathcal{T}$ is used to indicate the orthogonal direct sum of $\mathcal{S}$ and $\mathcal{T}$ when, in addition, $\langle Bs, t \rangle = 0$ for every $s \in \mathcal{S}$ and $t \in \mathcal{T}$.

The following is a consequence of the spectral theorem for Hilbert space selfadjoint operators.

Lemma 2.2. *Let $B \in L(\mathcal{H})^s$ and $\mathcal{S}$ be a closed subspace of $\mathcal{H}$. Then the Grammian of B $G_B := P_{\mathcal{S}} B P_{\mathcal{S}}$ can be represented as*

$$G_B = B_1 - B_2, \tag{2.1}$$

where $B_1, B_2 \in L(\mathcal{H})^+$ and $\mathcal{R}(B_1) \perp \mathcal{R}(B_2)$. Also, if $\mathcal{S}_+ := \overline{\mathcal{R}(B_1)}$ and $\mathcal{S}_- := \overline{\mathcal{R}(B_2)} \oplus \mathcal{S} \cap \mathcal{N}(G_B)$, then $\mathcal{S}$ can be represented as

$$\mathcal{S} = \mathcal{S}_+ \ \oplus_B \ \mathcal{S}_-, \tag{2.2}$$

where $\mathcal{S}_+$ is B-positive and $\mathcal{S}_-$ is B-nonpositive.

The next lemma characterizes the positive operators in terms of its matrix decomposition, see [1].

Lemma 2.3. *Let $\mathcal{S} \subseteq \mathcal{H}$ be a closed subspace and $B \in L(\mathcal{H})^s$ with matrix decomposition*

$$B = \begin{array}{c} \begin{bmatrix} a & b \\ b^* & c \end{bmatrix} \end{array} \begin{array}{c} \mathcal{S} \\ \mathcal{S}^\perp \end{array}.$$

Then $B \in L(\mathcal{H})^+$ if and only if

$$a \geq 0,\ b = b^*,\ \mathcal{R}(b) \subseteq \mathcal{R}(a^{1/2}),\ \text{and } c = f^*f + t,$$

with f the reduced solution of the equation $b = a^{1/2}x$ and $t \geq 0$.

Semiclosed subspaces and operators

The notions of *semiclosed subspace* and *semiclosed operator* were formally introduced by Kaufman [28], though these notions were considered by other authors before, as we pointed out in the Introduction.

Definition. A subspace $\mathcal{S}$ of $\mathcal{H}$ is *semiclosed* if $\mathcal{S}$ is a (not necessarily closed) subspace for which there exists an inner product $\langle \cdot, \cdot \rangle'$ such that $(\mathcal{S}, \langle \cdot, \cdot \rangle')$ is a Hilbert space which is *continuosly included* in $\mathcal{H}$, i.e., there exists $b > 0$ such that $\langle x, x \rangle \leq b \langle x, x \rangle'$ for every $x \in \mathcal{S}$.

As only an infinite dimensional subspace can be semiclosed, but not closed, only infinite dimensional complex Hilbert spaces are considered.

Operator ranges are semiclosed subspaces: in fact, if $T \in L(\mathcal{H})$ define

$$\|u\|_T := \|T^\dagger u\| \text{ for } u \in \mathcal{R}(T),$$

where $T^\dagger$ denotes the (possibly unbounded) Moore-Penrose inverse of T, see [32]. Then,

$$\|u\| = \|TT^\dagger u\| \leq \|T\| \|T^\dagger u\| = \|T\| \|u\|_T \text{ for } u \in \mathcal{R}(T). \tag{2.3}$$

See [3, 11].

The space $\mathcal{R}(T)$ equipped with the Hilbert space structure $\|\cdot\|_T$ is denoted by

$$\mathcal{M}(T) := (\mathcal{R}(T), \|\cdot\|_T).$$

The Hilbert spaces $\mathcal{M}(T)$ play a significant role in many areas, in particular in the de Branges complementation theory [3].

In fact, these are all the semiclosed subspaces: Fillmore and Williams proved that $\mathcal{S}$ is a semiclosed subspace of $\mathcal{H}$ if and only if $\mathcal{S}$ is the range of a closed operator T on $\mathcal{H}$. Moreover, the operator T can be chosen to be bounded and positive (semidefinite), see [23, Theorem 1.1]. Furthermore, if T is a contraction, i.e. $\|T\| \leq 1$, then $\mathcal{S}' := \mathcal{M}((I - TT^*)^{1/2})$ is its *de Branges complement* and $\mathcal{S} + \mathcal{S}' = \mathcal{H}$ [3, Corollary 3.8], where the last sum need not be direct [11, Proposition 3.4].

Given two operators $T_1, T_2 \in L(\mathcal{H})$, by Theorem 2.1, the subspace $R(T_1) + R(T_2)$ is the range of $T := (T_1T_1^* + T_2T_2^*)^{1/2}$. This shows that the sum of semiclosed subspaces is again semiclosed. The following interesting result by Ando compares the norm $\|\cdot\|_T$ with the norms $\|\cdot\|_{T_1}$ and $\|\cdot\|_{T_2}$.

Theorem 2.4 ([3, Corollary 3.8]). *For $T_1, T_2 \in L(\mathcal{H})$, let $T := (T_1T_1^* + T_2T_2^*)^{1/2}$. Then $\|u_1 + u_2\|_T^2 \leq \|u_1\|_{T_1}^2 + \|u_2\|_{T_2}^2$, for $u_1 \in \mathcal{R}(T_1)$ and $u_2 \in \mathcal{R}(T_2)$, and for any $u \in \mathcal{R}(T)$, there are unique $u_1 \in \mathcal{R}(T_1)$ and $u_2 \in \mathcal{R}(T_2)$ such that $u = u_1 + u_2$ and*

$$\|u_1 + u_2\|_T^2 = \|u_1\|_{T_1}^2 + \|u_2\|_{T_2}^2.$$

Applying again Theorem 2.1 it follows that the family of semiclosed subspaces is closed under intersection, see also [10, Proposition 4, Proposition 6]. The set of semiclosed subspaces is the lattice of domains of closed operators in $\mathcal{H}$ [23]. Also, if $\mathcal{S}$ is a semiclosed subspace of $\mathcal{H}$ then all inner products $\langle \cdot, \cdot \rangle'$ such that $(\mathcal{S}, \langle \cdot, \cdot \rangle')$ is a Hilbert space which is continuosly included in $\mathcal{H}$, generate the same topology on $\mathcal{S}$. See [29, 30] and [10, Theorem 11].

Definition ([28]). An operator $C : \mathcal{D}(C) \subseteq \mathcal{H} \to \mathcal{K}$ is a *semiclosed operator* if its graph $gr(C)$ is a semiclosed subspace of $\mathcal{H} \times \mathcal{K}$.

Denote by $SC(\mathcal{H}, \mathcal{K})$ the set of all semiclosed operators with domain in $\mathcal{H}$ to $\mathcal{K}$ and set $SC(\mathcal{H}) := SC(\mathcal{H}, \mathcal{H})$. The following is a characterization of $SC(\mathcal{H})$, see [28, Theorem 1].

Theorem 2.5. *Given an operator $C : \mathcal{D}(C) \subseteq \mathcal{H} \to \mathcal{H}$, the following are equivalent:*

i) $C \in SC(\mathcal{H})$,

ii) $\mathcal{D}(C)$ *is a semiclosed subspace of* $\mathcal{H}$ *and* $C \in L(\mathcal{D}(C), \mathcal{H})$,

ii) there exist $A, D \in L(\mathcal{H})$ *such that* $C = AD^\dagger|_{\mathcal{R}(D)}$.

iv) there exist $A \in L(\mathcal{H})$ *and* $D \in L(\mathcal{H})^+$ *such that* $C = AD^\dagger|_{\mathcal{R}(D)}$ *and* $\mathcal{N}(D) \subseteq \mathcal{N}(A)$.

Corollary 2.6. *Let $C : \mathcal{D}(C) \subseteq \mathcal{H} \to \mathcal{H}$ be a given operator. Then $C \in SC(\mathcal{H})$ if and only if there exists $D \in L(\mathcal{H})^+$ such that $\mathcal{D}(C) = \mathcal{R}(D)$ and $CD \in L(\mathcal{H})$.*

The set $SC(\mathcal{H})$ is closed under addition, multiplication, inversion and restriction to semiclosed subspaces of $\mathcal{H}$, [28]. Also, if $T_1, T_2 \in SC(\mathcal{H}, \mathcal{K})$ are such that T_1 and T_2 coincide on $\mathcal{D}(T_1) \cap \mathcal{D}(T_2)$, then the operator $T : \mathcal{D}(T_1) + \mathcal{D}(T_2) \to \mathcal{K}$ coinciding with T_1 on $\mathcal{D}(T_1)$ and with T_2 on $\mathcal{D}(T2)$ is a semiclosed operator, [21].

Remark. In [22, Theorem 2], Douglas proved that given A, B densely defined closed operators on $\mathcal{H}$ such that $\mathcal{R}(A) \subseteq \mathcal{R}(B)$, there exist an operator V on $\mathcal{H}$ with $\mathcal{D}(V) = \mathcal{D}(A)$ and a number $M \geq 0$ such that

$$A = BV \text{ and } \|Vx\|^2 \leq M(\|x\|^2 + \|Ax\|^2), \quad \text{for every } x \in \mathcal{D}(V). \tag{2.4}$$

The operator V is semiclosed: in fact, define

$$\langle (x, Vx), (y, Vy) \rangle' := \langle x, y \rangle + \langle Ax, Ay \rangle \text{ for } x, y \in \mathcal{D}(V).$$

Then $(gr(V), \langle \cdot, \cdot \rangle')$ is a Hilbert space because A is closed. On the other hand, since (2.4) holds,

$$\langle (x, Vx), (x, Vx) \rangle \leq (M + 1) \langle (x, Vx), (x, Vx) \rangle' \text{ for every } x \in \mathcal{D}(V).$$

Hence $(gr(V), \langle \cdot, \cdot \rangle')$ is continuously included in $\mathcal{H} \times \mathcal{H}$.

3. Semiclosed projections

A linear operator E acting in $\mathcal{H}$ is a *projection* if

$$\mathcal{R}(E) \subseteq \mathcal{D}(E) \text{ and } E^2x = Ex \text{ for every } x \in \mathcal{D}(E).$$

Theorem 3.1 ([33, Lemma 3.5])**.** *Let E be a projection in $\mathcal{H}$ then*

$$\mathcal{R}(E) \dotplus \mathcal{N}(E) = \mathcal{D}(E).$$

Conversely, given two subspaces $\mathcal{N}, \mathcal{M}$ of $\mathcal{H}$ such that $\mathcal{N} \cap \mathcal{M} = \{0\}$, there exists a projection E with $\mathcal{R}(E) = \mathcal{M}$ and $\mathcal{N}(E) = \mathcal{N}$.

Write $E = P_{\mathcal{M}//\mathcal{N}}$ to denote the projection with $\mathcal{R}(E) = \mathcal{M}$ and $\mathcal{N}(E) = \mathcal{N}$. If E is a densely defined projection in $\mathcal{H}$, then E^* is a (non necessarily densely defined) closed projection (see [33, Proposition 3.4]) with $\mathcal{N}(E^*) = \mathcal{R}(E)^\perp$ and $\mathcal{R}(E^*) = \mathcal{N}(E)^\perp$. The last equality follows from the former and the fact that $I - E$ is a projection with domain $\mathcal{D}(E)$ so that $\mathcal{R}(E^*) = \mathcal{N}(I - E^*) = \mathcal{R}(I - E)^\perp = \mathcal{N}(E)^\perp$. Then, by Theorem 3.1,

$$\mathcal{D}(E^*) = \mathcal{N}(E)^\perp \dotplus \mathcal{R}(E)^\perp.$$

Also, E is a closed projection if and only if $\mathcal{R}(E)$ and $\mathcal{N}(E)$ are closed subspaces of $\mathcal{H}$, see [33, Lemma 3.5]. More generally,

Proposition 3.2. *Let $E : \mathcal{D}(E) \subseteq \mathcal{H} \to \mathcal{H}$ be a projection. Then E is semiclosed if and only if $\mathcal{R}(E)$ and $\mathcal{N}(E)$ are semiclosed subspaces.*

Proof. If $\mathcal{R}(E)$ and $\mathcal{N}(E)$ are semiclosed subspaces of $\mathcal{H}$, there exist $A_1, A_2 \in L(\mathcal{H})^+$ such that $\mathcal{R}(E) = \mathcal{R}(A_1)$ and $\mathcal{N}(E) = \mathcal{R}(A_2)$, then, by Theorem 2.1, $\mathcal{D}(E) = \mathcal{R}(E) \dotplus \mathcal{N}(E)$ is semiclosed. Let us see that $E \in L(\mathcal{D}(E), \mathcal{H})$. Consider $\Gamma = (A_1^2 + A_2^2)^{1/2}$. Then, by Theorem 2.1, $\mathcal{R}(\Gamma) = \mathcal{R}(A_1) + \mathcal{R}(A_2) = \mathcal{D}(E)$. Let $u \in \mathcal{D}(E)$. Then, by Theorem 2.4, there exist uniquely $m \in \mathcal{R}(E)$ and $n \in \mathcal{N}(E)$ such that $u = m+n$ and $\|u\|_\Gamma^2 = \|m\|_{A_1}^2 + \|n\|_{A_2}^2$. Then, using (2.3),

$$\|Eu\|^2 = \|m\|^2 \leq \|A_1\|^2 \|m\|_{A_1}^2 \leq \|A_1\|^2 (\|m\|_{A_1}^2 + \|n\|_{A_2}^2) = \|A_1\|^2 \|u\|_\Gamma^2.$$

Then $E \in L(\mathcal{D}(E), \mathcal{H})$ and, by Theorem 2.5, $E \in SC(\mathcal{H})$.

Conversely, suppose that E is a semiclosed projection. Then, by Theorem 2.5, $\mathcal{D}(E)$ is a semiclosed subspace of $\mathcal{H}$ and, by [28, Theorem 2], $\mathcal{R}(E) = E(\mathcal{D}(E))$ is also a semiclosed subspace of $\mathcal{H}$. Since the set $SC(\mathcal{H})$ is closed under addition, $E \in SC(\mathcal{H})$ if and only if $I - E \in SC(\mathcal{H})$. Hence, $\mathcal{N}(E) = \mathcal{R}(I - E)$ is a semiclosed subspace of $\mathcal{H}$, where we used again [28, Theorem 2]. □

In [4, Theorem 2.2], Ando proved that if $E = P_{\mathcal{M}//\mathcal{N}}$ is a closed projection and $\Gamma := (P_\mathcal{M} + P_\mathcal{N})^{1/2}$ then $\mathcal{D}(E) = \mathcal{R}(\Gamma)$ and E admits the following representation:

$$E = (\Gamma^{-1} P_\mathcal{M})^* \Gamma^{-1}.$$

Moreover, the well defined operator $P := \Gamma^{-1} E \Gamma$ is an orthogonal projection, see [4, Theorem 2.3]. Analogous results can be obtained for densely defined semiclosed projections.

Let $E = P_{\mathcal{M}//\mathcal{N}}$ be a densely defined semiclosed projection. Since, by Proposition 3.2, $\mathcal{M}$ and $\mathcal{N}$ are semiclosed subspaces, then there exist $A_1, A_2 \in L(\mathcal{H})^+$ such that $\mathcal{M} = \mathcal{R}(A_1)$ and $\mathcal{N} = \mathcal{R}(A_2)$. Define the operator $\Gamma = \Gamma(A_1, A_2)$ as

$$\Gamma := (A_1^2 + A_2^2)^{1/2}. \tag{3.1}$$

The operator Γ is positive. By Theorem 2.1, $\mathcal{R}(\Gamma) = \mathcal{R}(A_1) + \mathcal{R}(A_2) = \mathcal{D}(E)$. Then $\mathcal{R}(\Gamma)$ is dense, or, equivalently, Γ is injective.

Proposition 3.3 (c.f. Theorem 2.2 [4])**.** *Let E be a densely defined semiclosed projection with $\mathcal{R}(E) = \mathcal{R}(A_1)$, $\mathcal{N}(E) = \mathcal{R}(A_2)$, $A_1, A_2 \in L(\mathcal{H})^+$ and Γ as in (3.1). Then E admits the representation*

$$E = (\Gamma^{-1} A_1^2)^* \Gamma^{-1}.$$

Proof. Since $\mathcal{R}(A_1^2) \subseteq \mathcal{R}(A_1) \subseteq \mathcal{R}(\Gamma)$, by Douglas's Lemma [22], there exists a unique $D \in L(\mathcal{H})$ such that $A_1^2 = \Gamma D^* = D\Gamma$, then $D^* = \Gamma^{-1}A_1^2$ and $D = (\Gamma^{-1}A_1^2)^*$. Write $\tilde{E} = D\Gamma^{-1}$. Since $\mathcal{D}(E) = \mathcal{R}(\Gamma) = \mathcal{D}(\tilde{E})$, for the proof of the assertion it suffices to show that $E\Gamma x = \tilde{E}\Gamma x = Dx$ for every $x \in \mathcal{H}$. Since both $E\Gamma$ and D are bounded (for the boundedness of $E\Gamma$ see Theorem 2.5) and $\mathcal{R}(\Gamma)$ is dense in $\mathcal{H}$, the equality is guaranteed if the operators $E\Gamma$ and D coincide on this dense subspace. It is $E\Gamma^2 x = E(A_1^2 + A_2^2)x = A_1^2 x = D\Gamma x$. Therefore $E\Gamma = D$ or $E = \tilde{E} = D\Gamma^{-1}$ in $\mathcal{R}(\Gamma)$. □

Corollary 3.4 (c.f. Theorem 2.3 [4])**.** *Let E be a densely defined operator in $\mathcal{H}$. Then E is a (densely defined) semiclosed projection if and only if there exists $\Gamma \in L(\mathcal{H})^+$ with $\mathcal{R}(\Gamma) = \mathcal{D}(E)$ such that*

$$\Gamma^{-1}E\Gamma \in \mathcal{P}.$$

Proof. Suppose that E is a densely defined semiclosed projection with $\mathcal{R}(E) = \mathcal{R}(A_1)$, $A_1 \in L(\mathcal{H})^+$, $\mathcal{N}(E) = \mathcal{R}(A_2)$, $A_2 \in L(\mathcal{H})^+$. Let $D := (\Gamma^{-1}A_1^2)^*$, with Γ as in (3.1). Then, by Proposition 3.3, $E\Gamma = D$. Let $P_\Gamma := \Gamma^{-1}E\Gamma = \Gamma^{-1}D$, then P_Γ is a bounded projection. In fact, since $\mathcal{R}(D) \subseteq \mathcal{R}(E) \subseteq \mathcal{D}(E) = \mathcal{R}(\Gamma)$, by Douglas' Lemma, the only solution X_0 of the equation $D = \Gamma X$ is given by $X_0 = \Gamma^{-1}D \in L(\mathcal{H})$. Also, since $\mathcal{R}(A_1) \subseteq \mathcal{R}(\Gamma)$, by Douglas's Lemma again, there exists a unique $D' \in L(\mathcal{H})$ such that $A_1 = \Gamma D'^* = D'\Gamma$, then $D'^* = \Gamma^{-1}A_1$ and $D' = (\Gamma^{-1}A_1)^*$. From $A_1^2 = \Gamma(D'^*\Gamma D'^*) = \Gamma D^*$ and the fact that Γ is inyective, it follows that $D = D'\Gamma D'$. Then $P_\Gamma = \Gamma^{-1}D = \Gamma^{-1}D'\Gamma D' = \Gamma^{-1}A_1 D' = (D')^* D' \in L(\mathcal{H})$ and P_Γ is selfadjoint.

Conversely, suppose that E is a densely defined operator in $\mathcal{H}$ such that there exists $\Gamma \in L(\mathcal{H})^+$ with $\mathcal{R}(\Gamma) = \mathcal{D}(E)$ and $\Gamma^{-1}E\Gamma := P_\Gamma \in \mathcal{P}$. Then $E\Gamma = \Gamma P_\Gamma$. Since Γ is injective, $\mathcal{R}(E) = E(\mathcal{D}(E)) = E(\mathcal{R}(\Gamma)) = \mathcal{R}(E\Gamma) = \mathcal{R}(\Gamma P_\Gamma) \subseteq \mathcal{R}(\Gamma) = \mathcal{D}(E)$ and $E = \Gamma P_\Gamma \Gamma^{-1}$. Then E is a densely defined projection and, by Theorem 2.5, $E \in SC(\mathcal{H})$. □

Since $E\Gamma = \Gamma P_\Gamma$, it follows that $P_\Gamma = P_{\Gamma^{-1}(\mathcal{R}(E))}$. Since $\mathcal{R}(E) = \Gamma(\mathcal{R}(P_\Gamma))$, $\mathcal{N}(E) = \Gamma(\mathcal{N}(P_\Gamma))$ and P_Γ is orthogonal, it holds that $\Gamma^{-1}(\mathcal{R}(E)) \perp \Gamma^{-1}(\mathcal{N}(E))$.

If $\Gamma, \Gamma' \in L(\mathcal{H})^+$ are as in Corollary 3.4. Then $\mathcal{R}(\Gamma) = \mathcal{D}(E) = \mathcal{R}(\Gamma')$ and, by Douglas' Lemma, there exists $G \in GL(\mathcal{H})$ such that $\Gamma' = \Gamma G = G^*\Gamma$. Moreover, if $P_\Gamma = P_{\Gamma^{-1}(\mathcal{R}(E))}$ and $P_{\Gamma'} = P_{\Gamma'^{-1}(\mathcal{R}(E))}$ then

$$P_{\Gamma'} = G^{-1}P_\Gamma G.$$

In fact, the projection $G^{-1}P_\Gamma G$ is bounded, $\mathcal{R}(G^{-1}P_\Gamma G) = G^{-1}(\mathcal{R}(P_\Gamma)) = (\Gamma G)^{-1}(\mathcal{R}(E)) = \Gamma'^{-1}(\mathcal{R}(E))$ $= \mathcal{R}(P_{\Gamma'})$ and $\mathcal{N}(G^{-1}P_\Gamma G) = G^{-1}(\mathcal{N}(P_\Gamma)) = (\Gamma G)^{-1}(\mathcal{N}(E)) = \Gamma'^{-1}(\mathcal{N}(E)) = \mathcal{N}(P_{\Gamma'})$.

3.1. On the Moore-Penrose inverse of semiclosed projections with closed nullspace

In order to define the Moore-Penrose inverse of a densely defined projection E in a satisfactory fashion we need an extra condition on its domain. This condition guarantees the existence of an orthogonal complement of $\mathcal{N}(E)$ relative to $\mathcal{D}(E)$.

Lemma 3.5. *Let $E = P_{\mathcal{M}//\mathcal{N}}$ be a densely defined projection. Then the following statements are equivalent:*

i) $\mathcal{M} \subseteq \mathcal{N}^\perp \oplus \mathcal{N}$,

ii) $\mathcal{D}(E) = P_{\mathcal{N}^\perp}(\mathcal{M}) \oplus \mathcal{N}$,

iii) $\mathcal{D}(E) = \mathcal{D}(E) \cap \mathcal{N}^\perp \oplus \mathcal{N}$.

In this case, $\mathcal{M} \cap \overline{\mathcal{N}} = \{0\}$ and therefore E admits an extension to $P_{\mathcal{M}//\overline{\mathcal{N}}}$.

Proof. *i)* $\Rightarrow$ *ii)* : Let $x \in \mathcal{M}$. Then $x = P_{\mathcal{N}^\perp}x + n$, for some $n \in \mathcal{N}$. Therefore $x \in P_{\mathcal{N}^\perp}(\mathcal{M}) \oplus \mathcal{N}$, $\mathcal{M} \subseteq P_{\mathcal{N}^\perp}(\mathcal{M}) \oplus \mathcal{N}$ and $\mathcal{D}(E) = \mathcal{M} \dotplus \mathcal{N} \subseteq P_{\mathcal{N}^\perp}(\mathcal{M}) \oplus \mathcal{N}$. To see the other inclusion, if $y \in P_{\mathcal{N}^\perp}(\mathcal{M})$ then there exists $m \in \mathcal{M}$ such that $y = P_{\mathcal{N}^\perp}m$. Since $m \in \mathcal{M}$, there exists $t \in \mathcal{N}^\perp$ and $n \in \mathcal{N}$ such that $m = t + n$. Then $y = m - (I - P_{\mathcal{N}^\perp})m = m - n \in \mathcal{M} \dotplus \mathcal{N}$. Then $P_{\mathcal{N}^\perp}(\mathcal{M}) \subseteq \mathcal{D}(E)$ and $P_{\mathcal{N}^\perp}(\mathcal{M}) \oplus \mathcal{N} \subseteq \mathcal{D}(E)$.

$ii) \Rightarrow iii)$: Clearly, $P_{\mathcal{N}^\perp}(\mathcal{M}) \subseteq \mathcal{D}(E) \cap \mathcal{N}^\perp$. On the other hand, let $x \in \mathcal{D}(E) \cap \mathcal{N}^\perp$. Then $x = m + n$, for some $m \in \mathcal{M}$ and $n \in \mathcal{N}$ and $x = P_{\mathcal{N}^\perp} x = P_{\mathcal{N}^\perp} m \in P_{\mathcal{N}^\perp}(\mathcal{M})$. Therefore $P_{\mathcal{N}^\perp}(\mathcal{M}) = \mathcal{D}(E) \cap \mathcal{N}^\perp$ and $\mathcal{D}(E) = \mathcal{D}(E) \cap \mathcal{N}^\perp \oplus \mathcal{N}$.

$iii) \Rightarrow i)$: It follows from the fact that $\mathcal{M} \subseteq \mathcal{D}(E) = \mathcal{D}(E) \cap \mathcal{N}^\perp \oplus \mathcal{N} \subseteq \mathcal{N}^\perp \oplus \mathcal{N}$.

In this case, from $\mathcal{M} \subseteq \mathcal{N} \oplus \mathcal{N}^\perp$, we have that $P_{\overline{\mathcal{N}}}(\mathcal{M}) \subseteq \mathcal{N}$. Therefore $\mathcal{M} \cap \overline{\mathcal{N}} = \{0\}$. In fact, if $x \in \mathcal{M} \cap \overline{\mathcal{N}}$. Then $x = P_{\overline{\mathcal{N}}} x \in \mathcal{N} \cap \mathcal{M} = \{0\}$. □

Given E a densely defined projection, condition $i)$ on $\mathcal{R}(E)$ and $\mathcal{N}(E)$ need not hold: the example on page 278 of [23] shows that there exist subspaces $\mathcal{M}, \mathcal{N}$ of $\mathcal{H}$ such that $\overline{\mathcal{M} + \mathcal{N}} = \mathcal{H}$, but $\mathcal{M} \cap \overline{\mathcal{N}} \neq \{0\}$.

We begin by applying Lemma 3.5 to obtain a matrix decomposition of a given projection. Every closed projection $P_{\mathcal{M}//\mathcal{N}}$ admits a matrix representation according to the orthogonal decomposition $\mathcal{H} = \mathcal{N}^\perp \oplus \mathcal{N}$ on $\mathcal{D}(E) = P_{\mathcal{N}^\perp}(\mathcal{M}) \oplus \mathcal{N}$, see [4, Theorem 2.6]. We generalize this result for a densely defined projection $E = P_{\mathcal{M}//\mathcal{N}}$ satisfying the conditions of Lemma 3.5.

Proposition 3.6 (c.f. [4, Theorem 2.6]). *Let $E = P_{\mathcal{M}//\mathcal{N}}$ be a densely defined projection such that $\mathcal{M} \subseteq \mathcal{N} \oplus \mathcal{N}^\perp$. According to the orthogonal decomposition $\mathcal{H} = \mathcal{N}^\perp \oplus \overline{\mathcal{N}}$, E admits the matrix representation*

$$E = \begin{bmatrix} I & 0 \\ P_{\overline{\mathcal{N}}}(P_{\mathcal{N}^\perp} P_{\overline{\mathcal{M}}}|_{\mathcal{M}})^{-1} & 0 \end{bmatrix} \text{ on } P_{\mathcal{N}^\perp}(\mathcal{M}) \oplus \mathcal{N}.$$

Proof. By Lemma 3.5, the operator $P_{\mathcal{N}^\perp} P_{\overline{\mathcal{M}}}|_{\mathcal{M}}$ is injective and dense in $\mathcal{N}^\perp$. In fact, if $P_{\mathcal{N}^\perp} P_{\overline{\mathcal{M}}} x = 0$ for $x \in \mathcal{M}$, then $x \in \mathcal{M} \cap \overline{\mathcal{N}} = \{0\}$. Also, $\mathcal{N}^\perp = P_{\mathcal{N}^\perp}(\overline{\mathcal{M} + \mathcal{N}}) \subseteq \overline{P_{\mathcal{N}^\perp}(\mathcal{M})} \subseteq \mathcal{N}^\perp$. So that $\overline{\mathcal{R}(P_{\mathcal{N}^\perp} P_{\overline{\mathcal{M}}}|_{\mathcal{M}})} = \overline{P_{\mathcal{N}^\perp}(\mathcal{M})} = \mathcal{N}^\perp$ and, the operator $P_{\overline{\mathcal{N}}}(P_{\mathcal{N}^\perp} P_{\overline{\mathcal{M}}}|_{\mathcal{M}})^{-1}$ is a linear operator from $P_{\mathcal{N}^\perp}(\mathcal{M}) \subseteq \mathcal{N}^\perp$ to $\mathcal{N}$.

By Lemma 3.5 again, $\mathcal{D}(E) = P_{\mathcal{N}^\perp}(\mathcal{M}) \oplus \mathcal{N}$ and $P_{\mathcal{N}^\perp}(\mathcal{M}) = \mathcal{D}(E) \cap \mathcal{N}^\perp$. Clearly, $EP_{\overline{\mathcal{N}}}|_{\mathcal{N}} = 0$ and $P_{\mathcal{N}^\perp} E P_{\mathcal{N}^\perp}|_{\mathcal{D}(E) \cap \mathcal{N}^\perp} = I_{\mathcal{N}^\perp}|_{\mathcal{D}(E) \cap \mathcal{N}^\perp}$. Finally, $P_{\overline{\mathcal{N}}} E P_{\mathcal{N}^\perp}|_{\mathcal{D}(E) \cap \mathcal{N}^\perp} = P_{\overline{\mathcal{N}}}(P_{\mathcal{N}^\perp} P_{\overline{\mathcal{M}}}|_{\mathcal{M}})^{-1}$ on $P_{\mathcal{N}^\perp}(\mathcal{M})$. □

Let $E = P_{\mathcal{M}//\mathcal{N}}$ be a densely defined projection such that $\mathcal{M} \subseteq \mathcal{N}^\perp \oplus \mathcal{N}$. By Lemma 3.5, $\mathcal{D}(E) \cap \mathcal{N}^\perp = P_{\mathcal{N}^\perp}(\mathcal{M})$. In this case, the Moore-Penrose inverse $E^\dagger$ of E is well defined (see [32]): $E^\dagger : \mathcal{M} \oplus \mathcal{M}^\perp \to P_{\mathcal{N}^\perp}(\mathcal{M}) \subseteq P_{\mathcal{N}^\perp}(\mathcal{M}) \oplus \overline{\mathcal{N}}$,

$$E^\dagger = \begin{cases} 0 & \text{if } x \in \mathcal{M}^\perp \\ (E|_{P_{\mathcal{N}^\perp}(\mathcal{M})})^{-1} x & \text{if } x \in \mathcal{M}. \end{cases}$$

The operators $EE^\dagger$ and $E^\dagger E$ are well defined and they are densely defined projections. In fact, $EE^\dagger = P_{\overline{\mathcal{M}}}$ on $\mathcal{D}(E^\dagger) = \mathcal{M} \oplus \mathcal{M}^\perp$ and $E^\dagger E = P_{\mathcal{N}^\perp}$ on $\mathcal{D}(E) = P_{\mathcal{N}^\perp}(\mathcal{M}) \oplus \mathcal{N}$.

Proposition 3.7. *Let $E = P_{\mathcal{M}//\mathcal{N}}$ be a densely defined projection such that $\mathcal{M} \subseteq \mathcal{N}^\perp \oplus \mathcal{N}$. Then*

$$E^\dagger = (P_{\mathcal{M}//\overline{\mathcal{N}}})^\dagger.$$

Proof. By Lemma 3.5, $\tilde{E} := P_{\mathcal{M}//\overline{\mathcal{N}}}$ is a densely defined projection such that $E \subseteq \tilde{E}$. Since $E|_{P_{\mathcal{N}^\perp}(\mathcal{M})} = \tilde{E}|_{P_{\mathcal{N}^\perp}(\mathcal{M})}$, it follows that $E^\dagger = \tilde{E}^\dagger$. □

In view of Proposition 3.7, we focus our attention on the Moore-Penrose inverse of densely defined projections with closed nullspace.

Proposition 3.8. *Let $E = P_{\mathcal{M}//\mathcal{N}}$ be a densely defined projection with closed nullspace. Then*

$$E^\dagger = P_{\mathcal{N}^\perp} P_{\overline{\mathcal{M}}} \text{ on } \mathcal{D}(E^\dagger) = \mathcal{M} \oplus \mathcal{M}^\perp.$$

Proof. If $x \in \mathcal{M}^\perp$ then $E^\dagger x = 0$. On the other hand, if $m \in \mathcal{M}$, let $y := (E|_{P_{\mathcal{N}^\perp}(\mathcal{M})})^{-1} m$, then $y \in P_{\mathcal{N}^\perp}(\mathcal{M})$ and $Ey = m$; but $EP_{\mathcal{N}^\perp} m = Em = m$. Therefore

$$E^\dagger m = (E|_{P_{\mathcal{N}^\perp}(\mathcal{M})})^{-1} m = P_{\mathcal{N}^\perp} m = P_{\mathcal{N}^\perp} P_{\overline{\mathcal{M}}} m.$$

□

If $E = P_{\mathcal{M}//\mathcal{N}}$ is a densely defined closed projection, then $E^\dagger = P_{\mathcal{N}^\perp} P_{\mathcal{M}} \in \mathcal{P} \cdot \mathcal{P}$. Moreover, the map $E \mapsto E^\dagger$, from the set of densely defined closed projections onto $\mathcal{P} \cdot \mathcal{P}$ is a bijection, see [16].

To study the semiclosed case, consider the set

$$\mathcal{P} \cdot L(\mathcal{H})^+ := \{T \in L(\mathcal{H}) : T = PA \text{ with } P \in \mathcal{P} \text{ and } A \in L(\mathcal{H})^+\}.$$

This set was studied in [7], where it was showed that any $T \in \mathcal{P} \cdot L(\mathcal{H})^+$ can be factored as $T = P_T A$, where $P_T := P_{\overline{\mathcal{R}(T)}}$ and $A \in L(\mathcal{H})^+$ is such that $\mathcal{N}(T) = \mathcal{N}(A)$, though this factorization may not be unique. We say that $A \in L(\mathcal{H})^+$ is *optimal* for T if $T = P_T A$ and $\mathcal{N}(T) = \mathcal{N}(A)$. A description of the set of optimal operators for T can be found in [7, Remark 4.2 and Proposition 4.4].

Proposition 3.9. *If $E = P_{\mathcal{M}//\mathcal{N}}$ is a densely defined semiclosed projection with closed nullspace, then there exists $\Gamma \in L(\mathcal{H})^+$ such that $\mathcal{R}(\Gamma) = \mathcal{D}(E^\dagger)$ and $E^\dagger \Gamma \in \mathcal{P} \cdot L(\mathcal{H})^+$.*

Proof. If $A \in L(\mathcal{H})^+$ is such that $\mathcal{R}(A) = \mathcal{M}$, by Proposition 3.8, $E^\dagger = P_{\mathcal{N}^\perp} P_{\overline{\mathcal{M}}}$ on $\mathcal{D}(E^\dagger)$. Take $\Gamma := \begin{array}{c} \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix} \end{array} \begin{array}{c} \overline{\mathcal{M}} \\ \mathcal{M}^\perp \end{array}$. Then $\Gamma \in L(\mathcal{H})^+$ and $\mathcal{R}(\Gamma) = \mathcal{R}(A) \oplus \mathcal{M}^\perp = \mathcal{D}(E^\dagger)$. Hence $E^\dagger \Gamma = P_{\mathcal{N}^\perp} P_{\overline{\mathcal{M}}} \Gamma = P_{\mathcal{N}^\perp} A \in \mathcal{P} \cdot L(\mathcal{H})^+$. $\square$

This generalizes the fact that if E is a densely defined closed projection then $E^\dagger \in \mathcal{P} \cdot \mathcal{P}$, since in this case, the operator Γ can be chosen to be the identity. From Proposition 3.9 it follows that every densely defined semiclosed projection E with closed nullspace has an associated set in $\mathcal{P} \cdot L(\mathcal{H})^+$, namely, $\{T = P_{\mathcal{N}(E)^\perp} A : \ A \in L(\mathcal{H})^+, \ \mathcal{R}(A) = \mathcal{R}(E)\}$.

On the other hand, every $T \in \mathcal{P} \cdot L(\mathcal{H})^+$ has an associated set of semiclosed projections.

Proposition 3.10. *Let $T \in \mathcal{P} \cdot L(\mathcal{H})^+$. If $A \in L(\mathcal{H})^+$ is optimal for T then*

$$\overline{\mathcal{R}(A) \dotplus \mathcal{R}(T)^\perp} = \mathcal{H}.$$

Proof. Consider $T \in \mathcal{P} \cdot L(\mathcal{H})^+$ and write $T = P_T A$ with $A \in L(\mathcal{H})^+$ optimal. Observe that, $\mathcal{R}(A) \cap \mathcal{R}(T)^\perp = \{0\}$. In fact, if $x \in \mathcal{R}(A) \cap \mathcal{R}(T)^\perp$ then $x = Ay$ for some $y \in \mathcal{H}$ and $0 = P_T x = P_T A y = Ty$. Then $y \in \mathcal{N}(T) = \mathcal{N}(A)$ and $x = Ay = 0$. Also, $\mathcal{R}(A) \dotplus \mathcal{R}(T)^\perp = P_T(\mathcal{R}(A)) \oplus \mathcal{R}(T)^\perp = \mathcal{R}(T) \oplus \mathcal{R}(T)^\perp$ is dense in $\mathcal{H}$. $\square$

Given $T \in \mathcal{P} \cdot L(\mathcal{H})^+$, define the set

$$\Phi(T) := \{E = P_{\mathcal{R}(A)//\mathcal{R}(T)^\perp} \text{ such that } A \text{ is optimal for } T\}.$$

By Proposition 3.10, every $E \in \Phi(T)$ is a densely defined semiclosed projection with closed nullspace. Moreover, $\mathcal{D}(E) = \mathcal{R}(T) \oplus \mathcal{R}(T)^\perp$ and $\overline{\mathcal{R}(E)} = \overline{\mathcal{R}(T^*)}$. Also, there exists a unique $E \in \Phi(T)$ if and only if $\overline{\mathcal{R}(T^*)} \cap \mathcal{N}(T^*) = \{0\}$, see [7, Proposition 4.1].

4. *B*-symmetric projections

A densely defined operator T is *symmetric* if $T \subset T^*$ and it is *selfadjoint* if $T = T^*$, i.e., T is symmetric and $\mathcal{D}(T) = \mathcal{D}(T^*)$.

Definition. Let $B \in L(\mathcal{H})^s$ and E be a densely defined projection. We say that E is *B-symmetric* if BE is symmetric and it is *B-selfadjoint* if BE is selfadjoint.

Since B is bounded, $\mathcal{D}(BE) = \mathcal{D}(E)$ and $(BE)^* = E^* B$. Therefore E is B-symmetric if and only if $BEx = E^* Bx$ for every $x \in \mathcal{D}(E)$ and E is B-selfadjoint if and only if $BE = E^* B$.

Proposition 4.1. *Let $B \in L(\mathcal{H})^s$ and E be a densely defined projection in $\mathcal{H}$. Then E is B-symmetric if and only if $\mathcal{N}(E) \subseteq B(\mathcal{R}(E))^\perp$.*

Proof. Suppose that E is B-symmetric. Let $x \in \mathcal{N}(E)$ and $y \in \mathcal{D}(E)$. Then

$$\langle\, x, BEy \,\rangle = \langle\, x, E^*By \,\rangle = \langle\, Ex, By \,\rangle = 0$$

so, $\mathcal{N}(E) \subseteq B(\mathcal{R}(E))^\perp$. Conversely, suppose that $\mathcal{N}(E) \subseteq B(\mathcal{R}(E))^\perp$. Then $B(\mathcal{N}(E)) \subseteq \mathcal{R}(E)^\perp$ and $\overline{B(\mathcal{R}(E))} \subseteq \mathcal{N}(E)^\perp$. Then $B(\mathcal{D}(E)) = B(\mathcal{R}(E) \dotplus \mathcal{N}(E)) = B(\mathcal{R}(E)) + B(\mathcal{N}(E)) \subseteq \overline{B(\mathcal{R}(E))} + \mathcal{R}(E)^\perp \subseteq \mathcal{N}(E)^\perp + \mathcal{R}(E)^\perp = \mathcal{D}(E^*)$. Also, $\mathcal{R}(I-E) = \mathcal{N}(E) \subseteq B^{-1}(\mathcal{R}(E)^\perp) = B^{-1}(\mathcal{N}(E^*)) = \mathcal{N}(E^*B)$. Then $E^*B(I-E)x = E^*Bx - E^*BEx = 0$ for every $x \in \mathcal{D}(E)$. Therefore, $E^*Bx = E^*BEx = BEx$, for every $x \in \mathcal{D}(E)$, where we used $B(\mathcal{R}(E)) \subseteq \mathcal{R}(E^*)$. Then E is B-symmetric. $\square$

Proposition 4.2. *Let $B \in L(\mathcal{H})^s$ and $\mathcal{S}$ a (not necessarily closed) subspace. There exists a B-symmetric projection onto $\mathcal{S}$ if and only if*

$$\mathcal{H} = \overline{\mathcal{S} + B(\mathcal{S})^\perp}.$$

In this case, $\mathcal{S} \cap B(\mathcal{S})^\perp = \mathcal{S} \cap N(B)$.

Proof. Suppose that E is a B-symmetric projection onto $\mathcal{S}$. Then, by Proposition 4.1, $\mathcal{N}(E) \subseteq B(\mathcal{S})^\perp$ and $\mathcal{D}(E) = \mathcal{R}(E) \dotplus \mathcal{N}(E) \subseteq \mathcal{S} + B(\mathcal{S})^\perp$. Therefore $\mathcal{H} = \overline{\mathcal{D}(E)} \subseteq \overline{\mathcal{S} + B(\mathcal{S})^\perp}$.

Conversely, suppose that $\mathcal{H} = \overline{\mathcal{S} + B(\mathcal{S})^\perp}$. Then $\overline{\overline{\mathcal{S}} + B(\mathcal{S})^\perp} = \mathcal{H}$. Let $\mathcal{L}' = \overline{\mathcal{S}} \cap B(\mathcal{S})^\perp$, then $\overline{\mathcal{S}} + B(\mathcal{S})^\perp = \overline{\mathcal{S}} \dotplus B(\mathcal{S})^\perp \cap \mathcal{L}'^\perp$, hence $\mathcal{S} \cap (B(\mathcal{S})^\perp \cap \mathcal{L}'^\perp) = \{0\}$. Define $E := P_{\mathcal{S}//B(\mathcal{S})^\perp \cap \mathcal{L}'^\perp}$. Then $\overline{\mathcal{D}(E)} = \overline{\mathcal{S} \dotplus (B(\mathcal{S})^\perp \cap \mathcal{L}'^\perp)} = \overline{\overline{\mathcal{S}} \dotplus B(\mathcal{S})^\perp \cap \mathcal{L}'^\perp} = \overline{\overline{\mathcal{S}} + B(\mathcal{S})^\perp} = \mathcal{H}$. Therefore E is a densely defined projection with (closed) nullspace contained in $B(\mathcal{S})^\perp$ and, by Proposition 4.1, E is B-symmetric.

Finally, the inclusion $\mathcal{S} \cap \mathcal{N}(B) \subseteq \mathcal{S} \cap B(\mathcal{S})^\perp$ always holds. On the other hand, let $x \in \mathcal{S} \cap B(\mathcal{S})^\perp$ and $y \in \mathcal{D}(E)$ then

$$\langle\, Bx, y \,\rangle = \langle\, BEx, y \,\rangle = \langle\, x, BEy \,\rangle = \langle\, Bx, Ey \,\rangle = 0.$$

Then $Bx \in \mathcal{D}(E)^\perp = \{0\}$ and then $x \in \mathcal{S} \cap \mathcal{N}(B)$. $\square$

When the projection E is semiclosed, the B-symmetry can be given in terms of bounded operators.

Proposition 4.3. *Let $B \in L(\mathcal{H})^s$ and E be a densely defined semiclosed projection of $\mathcal{H}$. Then E is B-symmetric if and only if P_Γ commutes with $\Gamma B \Gamma$, where $\Gamma \in L(\mathcal{H})^+$ and $P_\Gamma \in \mathcal{P}$ are as in Corollary 3.4.*

Proof. Suppose that E is B-symmetric, then $B\Gamma P_\Gamma x = BE\Gamma x = E^*B\Gamma x$ for every $x \in \mathcal{H}$. Then $\Gamma B \Gamma P_\Gamma x = \Gamma E^* B \Gamma x$ for every $x \in \mathcal{H}$. Now, since $E\Gamma = \Gamma P_\Gamma$, it follows that $\Gamma E^* \subset (E\Gamma)^* = P_\Gamma^* \Gamma$. Therefore $\Gamma B \Gamma P_\Gamma x = P_\Gamma^* \Gamma B \Gamma x$ for every $x \in \mathcal{H}$, i.e., P_Γ is $\Gamma B \Gamma$-selfadjoint and since P_Γ is selfadjoint, then P_Γ commutes with $\Gamma B \Gamma$.

Conversely, if P_Γ commutes with $\Gamma B \Gamma$, by [17, Lemma 3.2], $\mathcal{N}(P_\Gamma) \subseteq (\Gamma B \Gamma)^{-1}(\mathcal{R}(P_\Gamma)^\perp)$. Therefore

$$\mathcal{N}(E) = \Gamma(\mathcal{N}(P_\Gamma)) \subseteq \mathcal{R}(\Gamma) \cap B^{-1}(\mathcal{R}(\Gamma P_\Gamma)^\perp) = \mathcal{R}(\Gamma) \cap B^{-1}(\mathcal{R}(E)^\perp) \subseteq B^{-1}(\mathcal{R}(E)^\perp).$$

Then, by Proposition 4.1, E is B-symmetric. $\square$

We devote the last part of this section to characterize the B-symmetric closed projections. Some of these results where stated in [15], for a positive weight B. To extend these results to the selfadjoint case the notion of semiclosed projections turns out to be useful.

Proposition 4.4. *Let $B \in L(\mathcal{H})^s$ and E be a densely defined closed projection of $\mathcal{H}$ onto $\mathcal{S}$. If E is B-symmetric, then BE admits a bounded selfadjoint extension to $\mathcal{H}$. Moreover, if $B = B_1 - B_2$ and $\mathcal{S} = \mathcal{S}_1 \oplus_B \mathcal{S}_2$ are any decompositions as in* (2.1) *and* (2.2), *respectively, then*

$$\overline{BE} = (BE)^* = B_1^{1/2} P_{\mathcal{M}_1} B_1^{1/2} - B_2^{1/2} P_{\mathcal{M}_2} B_2^{1/2},$$

where $\mathcal{M}_i = \overline{B_i^{1/2}(\mathcal{S}_i)}$, for $i = 1, 2$.

To prove this proposition we need the following lemma.

Lemma 4.5. *Let $B \in L(\mathcal{H})^s$ and E be a densely defined closed projection of $\mathcal{H}$ onto $\mathcal{S}$. Suppose that $B = B_1 - B_2$ and $\mathcal{S} = \mathcal{S}_1 \oplus_B \mathcal{S}_2$ are any decompositions as in* (2.1) *and* (2.2) *respectively. If E is B-symmetric, then E admits a factorization $E = E_1 + E_2$, where E_1 and E_2 are semiclosed projections, with $\mathcal{D}(E) = \mathcal{D}(E_1) = \mathcal{D}(E_2)$, $\mathcal{R}(E_1) = \mathcal{S}_1$, $\mathcal{R}(E_2) = \mathcal{S}_2$, E_1 is B_1-symmetric and E_2 is B_2-symmetric.*

Proof. Since E is B-symmetric, $E = P_{\mathcal{S}//\mathcal{T}}$, with $\mathcal{T}$ a closed subspace such that $\mathcal{T} \subseteq B^{-1}(\mathcal{S}^\perp)$ and $\mathcal{D}(E) = \mathcal{S} \dotplus \mathcal{T} = \mathcal{S}_1 \dotplus \mathcal{S}_2 \dotplus \mathcal{T}$. Let $E_1 := P_{\mathcal{S}_1//\mathcal{T}+S_2}$ and $E_2 := P_{\mathcal{S}_2//\mathcal{T}+S_1}$. Then $E_i = P_{\mathcal{S}_i}E$ in $\mathcal{D}(E)$ and E_i is B_i-symmetric for $i = 1, 2$. Let us prove that for the case $i = 1$; the other case is similar. First observe that $\mathcal{D}(P_{\mathcal{S}_1}E) = \mathcal{D}(E)$ and $\mathcal{R}(P_{\mathcal{S}_1}E) \subseteq \mathcal{S}_1 \subseteq \mathcal{D}(E)$. Then, $(P_{\mathcal{S}_1}E)^2 = P_{\mathcal{S}_1}EP_{\mathcal{S}_1}E = P_{\mathcal{S}_1}E$ so that $P_{\mathcal{S}_1}E$ is a densely defined projection. On the other hand $\mathcal{R}(P_{\mathcal{S}_1}E) = P_{\mathcal{S}_1}\mathcal{R}(E) = \mathcal{S}_1$ and $\mathcal{N}(P_{\mathcal{S}_1}E) = \mathcal{N}(E) + \mathcal{S} \cap \mathcal{S}_1^\perp = \mathcal{T} + \mathcal{S}_2$. Therefore $E_1 = P_{\mathcal{S}_1}E$ in $\mathcal{D}(E)$ and, since $\mathcal{N}(E_1)$ and $\mathcal{R}(E_1)$ are semiclosed subspaces, E_1 is a semiclosed projection. Also, $\mathcal{D}(E) = \mathcal{D}(E_1) = \mathcal{D}(E_2)$ and $E = P_{\mathcal{S}}E = P_{\mathcal{S}_1}E + P_{\mathcal{S}_2}E = E_1 + E_2$ with $E_1E_2 = E_2E_1 = 0$. Finally, let us see that E_1 is B_1-symmetric. If $x \in \mathcal{D}(E_1) = \mathcal{D}(E)$ then

$$BE_1x = BEE_1x = E^*BE_1x = E^*BP_{\mathcal{S}_1}Ex = E^*P_{\mathcal{S}_1}BEx = (P_{\mathcal{S}_1}E)^*E^*Bx =$$

$$= E_1^*E^*Bx = (EE_1)^*Bx = E_1^*Bx.$$

Then $BE_1 \subset E_1^*B$. Now, since $BE_1 = B_1E_1$, it follows that

$$B_1E_1 \subset E_1^*B \subset (BE_1)^* = (B_1E_1)^* = E_1^*B_1.$$

Hence E_1 is B_1-symmetric. □

Proof of Proposition 4.4. By Lemma 4.5, E admits a factorization in the form $E = E_1 + E_2$, where E_1 is a B_1-symmetric densely defined semiclosed projection with range $\mathcal{S}_1$ and E_2 is a B_2-symmetric densely defined semiclosed projection with range $\mathcal{S}_2$. Let us show that $P_{\mathcal{M}_1}B_1^{1/2} = B_1^{1/2}E_1$ in $\mathcal{D}(E_1)$. In fact, if $x \in \mathcal{D}(E_1)$ then $P_{\mathcal{M}_1}B_1^{1/2}x = P_{\mathcal{M}_1}B_1^{1/2}E_1x + P_{\mathcal{M}_1}B_1^{1/2}(I - E_1)x$. Since $E_1x \in \mathcal{S}_1$, $P_{\mathcal{M}_1}B_1^{1/2}E_1x = B_1^{1/2}E_1x$. Also, $P_{\mathcal{M}_1}B_1^{1/2}(I - E_1)x = 0$ because $B_1^{1/2}(I - E_1)x \in \mathcal{R}(B_1^{1/2}(I - E_1)) = B^{1/2}\mathcal{N}(E_1) \subseteq B_1^{1/2}(B_1^{-1/2}(\mathcal{S}_1^\perp)) = \mathcal{R}(B_1^{1/2}) \cap B^{-1/2}(\mathcal{S}^\perp) \subseteq \mathcal{M}_1^\perp$, where we used Proposition 4.1. Therefore, $P_{\mathcal{M}_1}B_1^{1/2} = B_1^{1/2}E_1$ in $\mathcal{D}(E_1)$. In a similar way, it can be proved that $P_{\mathcal{M}_2}B_2^{1/2} = B_2^{1/2}E_2$ in $\mathcal{D}(E_2)$. Therefore, if $x \in \mathcal{D}(E) = \mathcal{D}(E_1) = \mathcal{D}(E_2)$, then

$$BEx = BE_1x - BE_2x = B_1E_1x - B_2E_2x = (B_1^{1/2}P_{\mathcal{M}_1}B_1^{1/2} - B_2^{1/2}P_{\mathcal{M}_2}B_2^{1/2})x.$$

Define $S := B_1^{1/2}P_{\mathcal{M}_1}B_1^{1/2} - B_2^{1/2}P_{\mathcal{M}_2}B_2^{1/2}$, then $S \in L(\mathcal{H})^s$ and, $BE \subseteq S = S^* \subseteq (BE)^*$. Therefore $(BE)^* = S$. □

5. Quasi and weak complementability

The complementability of an operator $B \in L(\mathcal{H})$ with respect to two given closed subspaces $\mathcal{S}$ and $\mathcal{T}$ of $\mathcal{H}$ was studied for matrices by Ando [2] and extended to operators in Hilbert spaces by Carlson and Haynsworth [13].

Definition. Let $B \in L(\mathcal{H})^s$ and $\mathcal{S} \subseteq \mathcal{H}$ be a closed subspace. Then B is $\mathcal{S}$*-complementable* if

$$\mathcal{H} = \mathcal{S} + B(\mathcal{S})^\perp.$$

In [17] it was shown that B is $\mathcal{S}$-complementable if and only if there exists a B-selfadjoint projection onto $\mathcal{S}$; i.e., the set

$$\mathcal{P}(B, \mathcal{S}) := \{Q \in \mathcal{Q} : \mathcal{R}(Q) = \mathcal{S},\ BQ \in L(\mathcal{H})^s\}$$

is not empty.

Proposition 5.1 ([17])**.** *Let $B \in L(\mathcal{H})^s$ and $\mathcal{S} \subseteq \mathcal{H}$ be a closed subspace. If the matrix decomposition of B is given by*

$$B = \begin{bmatrix} a & b \\ b^* & c \end{bmatrix} \begin{array}{c} \mathcal{S} \\ \mathcal{S}^\perp \end{array}, \tag{5.1}$$

then B is $\mathcal{S}$-complementable if and only if $\mathcal{R}(b) \subseteq \mathcal{R}(a)$.

One way of generalizing the concept of complementability is to consider B-symmetric densely defined closed projections onto $\mathcal{S}$.

Definition. Let $B \in L(\mathcal{H})^s$ and $\mathcal{S} \subseteq \mathcal{H}$ be a closed subspace. We say that the pair $(B, \mathcal{S})$ is *quasi-complementable* if there exists a B-symmetric densely defined closed projection onto $\mathcal{S}$.

The set of quasi-complementable pairs was studied in [15] for a positive weight B.

Proposition 5.2. *Let $B \in L(\mathcal{H})^s$ and $\mathcal{S} \subseteq \mathcal{H}$ be a closed subspace. The pair $(B, \mathcal{S})$ is quasi-complementable if and only if $\overline{B\mathcal{S}} \cap \mathcal{S}^\perp = \{0\}$.*

Proof. It follows from the definition of quasi-complementability and Proposition 4.2. □

Let E be a densely defined closed projection with range $\mathcal{S}$. Consider the matrix decomposition of E with respect to $\mathcal{H} = \mathcal{S} \oplus \mathcal{S}^\perp$

$$E = \begin{bmatrix} I & x \\ 0 & 0 \end{bmatrix} \begin{array}{c} \mathcal{S} \\ \mathcal{S}^\perp \end{array}, \tag{5.2}$$

where $x : \mathcal{D}(x) \subseteq \mathcal{S}^\perp \to \mathcal{S}$ is a densely defined linear operator.

Proposition 5.3. *Let $B \in L(\mathcal{H})^s$, $\mathcal{S} \subseteq \mathcal{H}$ be a closed subspace and E be a densely defined closed projection onto $\mathcal{S}$. Then E is B-symmetric if and only if*

$$ax \subset b,$$

where a, b are as in (5.1) *and x is as in* (5.2)*.*

Proof. This result follows by similar arguments as those found in [15, Proposition 2.2]. □

Corollary 5.4. *Let $B \in L(\mathcal{H})^s$, $\mathcal{S} \subseteq \mathcal{H}$ be a closed subspace and E be a densely defined closed projection onto $\mathcal{S}$. Then $(B, \mathcal{S})$ is quasi-complementable if and only if $b^* = x^* a$, where a, b are as in* (5.1) *and x is as in* (5.2)*.*

A different way of extending the concept of complementability was given in [6], where Antezana et al. defined the notion of *weak complementability* to study the Schur complement in this context. We use these ideas when $\mathcal{S} = \mathcal{T}$ and $B \in L(\mathcal{H})^s$.

Definition. [6] Let $\mathcal{S} \subseteq \mathcal{H}$ be a closed subspace, and $B \in L(\mathcal{H})^s$ with representation as in (5.1). Then B is *$\mathcal{S}$-weakly complementable* if

$$\mathcal{R}(b) \subseteq \mathcal{R}(|a|^{1/2}).$$

The notion of weak complementability is distinct to the notion of complementability only in the infinite dimensional setting. Every positive operator B is $\mathcal{S}$-weakly complementable, and if B is $\mathcal{S}$-weakly complementable for every closed subspace $\mathcal{S} \subseteq \mathcal{H}$ then B is semidefinite, see [14, Proposition 3.1].

The next proposition gives an operator characterization of the $\mathcal{S}$-weak complementability as the solution of a Riccati type equation [5].

Proposition 5.5. *Let $B \in L(\mathcal{H})^s$ and $\mathcal{S}$ be a closed subspace of $\mathcal{H}$. Then B is $\mathcal{S}$-weakly complementable if and only if there exists a positive solution of the equation*

$$BP_{\mathcal{S}}B = XP_{\mathcal{S}}X^*.$$

Proof. Suppose that the matrix decomposition of B induced by $\mathcal{S}$ is as in (5.1) and $a = u|a|$ is the polar decomposition of a. Let $f \in L(\mathcal{S}^\perp, \mathcal{S})$ be the reduced solution of $b = |a|^{1/2}x$.

Let $A = \begin{array}{c c} \begin{bmatrix} |a| & u|a|^{1/2}f \\ f^*|a|^{1/2}u & f^*f \end{bmatrix} & \begin{array}{c} \mathcal{S} \\ \mathcal{S}^\perp \end{array} \end{array}$ then, by Lemma 2.3, $A \geq 0$, because uf is the reduced solution of $|a|^{1/2}x = u|a|^{1/2}f$ and $f^*f = f^*uuf$. It easily follows that $BP_{\mathcal{S}}B = AP_{\mathcal{S}}A$.

Conversely, suppose that $BP_{\mathcal{S}}B = AP_{\mathcal{S}}A$ with $A \geq 0$ and $A = \begin{array}{c c} \begin{bmatrix} a_{11} & a_{12} \\ a_{12}^* & a_{22} \end{bmatrix} & \begin{array}{c} \mathcal{S} \\ \mathcal{S}^\perp \end{array} \end{array}$. Then, we have the following equations

$$a^2 = a_{11}^2 \tag{5.3}$$
$$ab = a_{11}a_{12} \tag{5.4}$$
$$b^*b = a_{12}^*a_{12}. \tag{5.5}$$

From (5.3), $a_{11} = |a|$. From (5.4), $ab = ua_{11}b = a_{11}a_{12}$, then $a_{11}b = ua_{11}a_{12} = a_{11}ua_{12}$ and $\mathcal{R}(b - ua_{12}) \subseteq \mathcal{N}(a_{11}^{1/2}) \subseteq \mathcal{N}(a_{12}^*) = \mathcal{N}(ua_{12}^*)$, where we used $A \geq 0$ and Lemma 2.3. Then, from (5.5), $a_{12}^*a_{12} = bb^* = (b - ua_{12} + ua_{12})^*(b - ua_{12} + ua_{12}) = (b - ua_{12})^*(b - ua_{12}) + a_{12}^*a_{12}$. Therefore, $|b - ua_{12}| = 0$, or $b = ua_{12}$. Since $A \geq 0$, again by Lemma 2.3, there exists $x_0 \in L(\mathcal{H})$ such that $a_{12} = a_{11}^{1/2}x_0$, thus, $b = ua_{12} = ua_{11}^{1/2}x_0 = a_{11}^{1/2}ux_0$ and $\mathcal{R}(b) \subseteq \mathcal{R}(a_{11}^{1/2}) = \mathcal{R}(|a|^{1/2})$. □

Let $\mathcal{S}$ be a closed subspace of $\mathcal{H}$ and let E be a densely defined projection with $\mathcal{N}(E) = \mathcal{S}^\perp$. Then, by Proposition 3.6, the matrix representation of E according to the orthogonal decomposition $\mathcal{H} = \mathcal{S} \oplus \mathcal{S}^\perp$ is

$$E = \begin{bmatrix} I & 0 \\ y & 0 \end{bmatrix} \text{ on } \mathcal{D}(E) = \mathcal{D}(y) \oplus \mathcal{S}^\perp, \tag{5.6}$$

with $\mathcal{D}(y) = P_{\mathcal{S}}(\mathcal{R}(E))$ dense in $\mathcal{S}$ and $y = P_{\mathcal{S}^\perp}(P_{\mathcal{S}}P_{\overline{\mathcal{R}(E)}}|_{\mathcal{R}(E)})^{-1}$.

To characterize the $\mathcal{S}$-weak complementability of B in terms of projections we need the following lemma.

Lemma 5.6. *Let $B \in L(\mathcal{H})^s$, $\mathcal{S}$ be a closed subspace of $\mathcal{H}$ and E be a densely defined projection with $\mathcal{N}(E) = \mathcal{S}^\perp$. Suppose that the matrix decomposition of B and E are as in* (5.1) *and* (5.6)*, respectively. Then the following are equivalent:*

i) $EB \in L(\mathcal{H})^s$,

ii) $ya = b^*$ *and* $yb \in L(\mathcal{S}^\perp)^s$,

iii) $\mathcal{R}(B) \subseteq \mathcal{D}(E)$ *and* $B\mathcal{S} \subseteq \mathcal{R}(E)$.

Proof. *i)* $\Rightarrow$ *ii)* : Suppose that $EB \in L(\mathcal{H})^s$. Then, $EB = \begin{bmatrix} a & b \\ ya & yb \end{bmatrix} = (EB)^* = \begin{bmatrix} a & (ya)^* \\ b^* & (yb)^* \end{bmatrix}$. Therefore $ya = b^*$ and $yb \in L(\mathcal{S}^\perp)^s$.

ii) $\Rightarrow$ *iii)* : Since $\mathcal{R}(a) \subseteq \mathcal{D}(y)$, $\mathcal{R}(b) \subseteq \mathcal{D}(y)$ and $\mathcal{S}^\perp \subseteq \mathcal{D}(E)$, it follows that $\mathcal{R}(B) \subseteq \mathcal{R}(a) + \mathcal{R}(b) \oplus \mathcal{S}^\perp \subseteq \mathcal{D}(y) \oplus \mathcal{S}^\perp = \mathcal{D}(E)$. On the other hand, let $z \in B\mathcal{S}$. Then $z = as + b^*s$ for some $s \in \mathcal{S}$. Then, $(I - E)z = -yas + b^*s = 0$. Hence $z \in \mathcal{N}(I - E) = \mathcal{R}(E)$.

iii) $\Rightarrow$ *i)* : Let $x \in \mathcal{H}$. Then $x = s + t$ for $s \in \mathcal{S}$ and $t \in \mathcal{S}^\perp$. Then

$$EBx = E(Bs + Bt) = Bs + E(bt + ct) = Bs + bt = \begin{bmatrix} a & b \\ b^* & 0 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} a & b \\ b^* & 0 \end{bmatrix} x.$$

Therefore, $EB = \begin{bmatrix} a & b \\ b^* & 0 \end{bmatrix} \in L(\mathcal{H})^s$. □

Corollary 5.7. *Let $B \in L(\mathcal{H})^s$, $\mathcal{S}$ be a closed subspace of $\mathcal{H}$ and E be a densely defined projection with $\mathcal{N}(E) = \mathcal{S}^\perp$. If $EB \in L(\mathcal{H})^s$ then $B\mathcal{S} \cap \mathcal{S}^\perp = \{0\}$.*

Corollary 5.8. *Let $B \in L(\mathcal{H})^s$, $\mathcal{S}$ be a closed subspace of $\mathcal{H}$ and E be a densely defined semiclosed projection with $\mathcal{N}(E) = \mathcal{S}^{\perp}$. Suppose that the matrix decomposition of B and E are as in* (5.1) *and* (5.6)*, respectively. Then, $EB \in L(\mathcal{H})^s$ and $\mathcal{R}(|a|^{1/2}) \subseteq \mathcal{D}(E)$ if and only if $ya = b^*$ and $\mathcal{R}(|a|^{1/2}) \subseteq \mathcal{D}(y)$.*

Proof. Suppose that $EB \in L(\mathcal{H})^s$ and $\mathcal{R}(|a|^{1/2}) \subseteq \mathcal{D}(E)$. Then, by Lemma 5.6, $ya = b^*$. Also, since $\mathcal{D}(E) = \mathcal{D}(y) \oplus \mathcal{S}^{\perp}$, it follows that $\mathcal{R}(|a|^{1/2}) \subseteq \mathcal{D}(y)$.

Conversely, suppose that $ya = b^*$ and $\mathcal{R}(|a|^{1/2}) \subseteq \mathcal{D}(y)$. Then, $\mathcal{R}(|a|^{1/2}) \subseteq \mathcal{D}(y) \oplus \mathcal{S}^{\perp} = \mathcal{D}(E)$. On the other hand, since $E \in SC(\mathcal{H})$ we get that $y \in SC(\mathcal{S}, \mathcal{S}^{\perp})$ and, by similar arguments as those found in the proof of Theorem 5.9, we have that $y|a|^{1/2} \in L(\mathcal{S}, \mathcal{S}^{\perp})$. Then, if $a = u|a|$ is the polar decomposition of a,

$$yb = y(ya)^* = y(y|a|^{1/2}u|a|^{1/2})^* = y|a|^{1/2}u(y|a|^{1/2})^*.$$

Hence $yb \in L(\mathcal{S}^{\perp})^s$ and, by Lemma 5.6, $EB \in L(\mathcal{H})^s$. □

The next theorem characterizes the $\mathcal{S}$-weak complementability of a selfadjoint operator in terms of semiclosed projections. See also [14, Theorem 3.14].

Theorem 5.9. *Let $B \in L(\mathcal{H})^s$ and $\mathcal{S}$ be a closed subspace of $\mathcal{H}$. Suppose that the matrix decomposition of B is as in* (5.1)*. Then B is $\mathcal{S}$-weakly complementable if and only if there exists a densely defined semiclosed projection E with $\mathcal{N}(E) = \mathcal{S}^{\perp}$ such that $EB \in L(\mathcal{H})^s$ and $\mathcal{R}(|a|^{1/2}) \subseteq \mathcal{D}(E)$.*

Proof. Suppose that B is $\mathcal{S}$-weakly complementable. If the matrix decomposition of B induced by $\mathcal{S}$ is as in (5.1), let f be the reduced solution of $b = |a|^{1/2}x$ and $a = u|a|$ be the polar decomposition of a. Write $(|a|^{1/2})^{\dagger}$ for the Moore-Penrose inverse of $|a|^{1/2}$ and set

$$E := \begin{bmatrix} I & 0 \\ f^*u(|a|^{1/2})^{\dagger} & 0 \end{bmatrix}.$$

Then $\mathcal{D}(E) = \mathcal{D}((|a|^{1/2})^{\dagger}) \oplus \mathcal{S}^{\perp}$ is a semiclosed subspace of $\mathcal{H}$ (because is the sum of two semiclosed subspaces), E is a densely defined projection with $\mathcal{N}(E) = \mathcal{S}^{\perp}$ and $\mathcal{R}(|a|^{1/2}) \subseteq \mathcal{D}(E)$. On the other hand, since $\mathcal{R}(B) \subseteq \mathcal{R}(|a|^{1/2}) \oplus \mathcal{S}^{\perp}$, the product $(I-E)B$ is well defined. Moreover

$$\begin{aligned}(I-E)B &= \begin{bmatrix} 0 & 0 \\ -f^*u(|a|^{1/2})^{\dagger} & I \end{bmatrix} \begin{bmatrix} |a|^{1/2}u|a|^{1/2} & |a|^{1/2}f \\ f^*|a|^{1/2} & c \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & c - f^*uf \end{bmatrix} \in L(\mathcal{H})^s.\end{aligned}$$

So that $EB \in L(\mathcal{H})^s$.

Finally,

$$E \begin{bmatrix} |a|^{1/2} & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} |a|^{1/2} & 0 \\ f^*u & 0 \end{bmatrix} \in L(\mathcal{H}).$$

Then $E \in L(\mathcal{D}(E), \mathcal{H})$ and, by Theorem 2.5, E is semiclosed.

Conversely, suppose that there exists a densely defined semiclosed projection E with $\mathcal{N}(E) = \mathcal{S}^{\perp}$ such that $EB \in L(\mathcal{H})^s$ and $\mathcal{R}(|a|^{1/2}) \subseteq \mathcal{D}(E)$. By Corollary 2.6, there exists an operator $D \in L(\mathcal{H})^+$ such that $\mathcal{D}(E) = \mathcal{R}(D)$ and $ED \in L(\mathcal{H})$. Then, if $\mathcal{R}(|a|^{1/2}) \subseteq \mathcal{D}(E) = \mathcal{R}(D)$, by Douglas's Lemma, $|a|^{1/2}P_{\mathcal{S}} = DX_0$ for some $X_0 \in L(\mathcal{H})$. Therefore, $E|a|^{1/2}P_{\mathcal{S}} = EDX_0 \in L(\mathcal{H})$.

Suppose that the matrix decomposition of E is in (5.6). Then, by Lemma 5.6, $ya = b^*$ and $yb \in L(\mathcal{S}^{\perp})^s$. From the fact that $E|a|^{1/2}P_{\mathcal{S}} \in L(\mathcal{H})$, we have that $y|a|^{1/2} \in L(\mathcal{S}, \mathcal{S}^{\perp})$ and, since $ya = b^*$, we also have that $y|a|^{1/2}u|a|^{1/2} = b^*$. Then $b = |a|^{1/2}(y|a|^{1/2}u)^*$, $\mathcal{R}(b) \subseteq \mathcal{R}(|a|^{1/2})$ and B is $\mathcal{S}$-weakly complementable. □

Suppose that B is $\mathcal{S}$-weakly complementable. If the matrix decomposition of B induced by $\mathcal{S}$ is as in (5.1), let f be the reduced solution of $b = |a|^{1/2}x$ and $a = u|a|$ be the polar decomposition of a. Set

$$E_0 := \begin{bmatrix} I & 0 \\ y_0 & 0 \end{bmatrix}, \tag{5.7}$$

with $y_0 := f^*u(|a|^{1/2})^\dagger$. Then, by the proof of Theorem 5.9, E_0 is a densely defined semiclosed projection such that $\mathcal{N}(E_0) = \mathcal{S}^\perp$, $E_0B \in L(\mathcal{H})^s$ and $\mathcal{D}(E_0) = \mathcal{D}((|a|^{1/2})^\dagger) \oplus \mathcal{S}^\perp$. Define

$$\mathcal{P}^*(B,\mathcal{S}) := \{E \text{ semiclosed projection} : \overline{\mathcal{D}(E)} = \mathcal{H},\ \mathcal{N}(E) = \mathcal{S}^\perp,\ EB \in L(\mathcal{H})^s \text{ and } \mathcal{R}(|a|^{1/2}) \subseteq \mathcal{D}(E)\}.$$

Let $E \in \mathcal{P}^*(B,\mathcal{S})$ with matrix decomposition as in (5.6). Then, it can be proved that

$$\mathcal{R}(y_0) \subseteq \mathcal{R}(y).$$

Furthermore, $y|a|^{1/2} = y_0|a|^{1/2}$.

In what follows, we characterize the subset of projections with fixed domain $\mathcal{D}(E_0)$. More precisely, consider

$$\mathcal{P}_0^*(B,\mathcal{S}) := \{E \text{ semiclosed projection} : \mathcal{D}(E) = \mathcal{D}(E_0),\ \mathcal{N}(E) = \mathcal{S}^\perp \text{ and } B(\mathcal{S}) \subseteq R(E)\}.$$

Clearly, by Lemma 5.6, $\mathcal{P}_0^*(B,\mathcal{S}) \subseteq \mathcal{P}^*(B,\mathcal{S})$. Also, $\mathcal{P}_0^*(B,\mathcal{S})$ is not empty because $E_0 \in \mathcal{P}_0^*(B,\mathcal{S})$.

Theorem 5.10. *Let $B \in L(\mathcal{H})^s$ and $\mathcal{S}$ be a closed subspace of $\mathcal{H}$ such that B is $\mathcal{S}$-weakly complementable. Then*

$$\mathcal{P}_0^*(B,\mathcal{S}) = E_0 + \{W \in SC(\mathcal{H}) : \mathcal{D}(E_0) \subseteq \mathcal{D}(W),\ \mathcal{R}(W) \subseteq \mathcal{S}^\perp \text{ and } B\mathcal{S} \dotplus \mathcal{S}^\perp \subseteq \mathcal{N}(W)\}.$$

Proof. Let $E = E_0 + W$, where $W \in SC(\mathcal{H})$ such that $\mathcal{D}(E_0) \subseteq \mathcal{D}(W)$, $\mathcal{R}(W) \subseteq \mathcal{S}^\perp$ and $B\mathcal{S} \dotplus \mathcal{S}^\perp \subseteq \mathcal{N}(W)$. Then $E_0 + W \in SC(\mathcal{H})$ and $\mathcal{D}(E_0+W) = \mathcal{D}(E_0) \cap \mathcal{D}(W) = \mathcal{D}(E_0)$. Observe that, $\mathcal{R}(E_0+W) \subseteq \mathcal{R}(E_0) \dotplus \mathcal{S}^\perp = \mathcal{D}(E_0) = \mathcal{D}(E_0+W)$. Also, since $\mathcal{R}(W) \subseteq \mathcal{N}(W) \cap \mathcal{N}(E_0)$, $W^2 = 0$, $E_0W = 0$ and, from $\mathcal{R}(I-E_0) \subseteq \mathcal{N}(W)$, $WE_0 = W$. Hence $W + E_0$ is a semiclosed densely defined projection. Furthermore, $\mathcal{N}(E_0+W) = \mathcal{S}^\perp$. In fact, it is clear that $\mathcal{S}^\perp \subseteq \mathcal{N}(E_0+W)$ and if $h \in \mathcal{N}(E_0+W) \subseteq \mathcal{D}(E_0+W) = \mathcal{D}(E_0)$ then $E_0h = -Wh$, so that $E_0h \in \mathcal{R}(E_0) \cap \mathcal{R}(W) \subseteq \mathcal{R}(E_0) \cap \mathcal{S}^\perp = \{0\}$. Therefore $h \in N(E_0) = \mathcal{S}^\perp$ and $\mathcal{N}(E_0+W) \subseteq \mathcal{S}^\perp$. Finally, $\mathcal{R}(B) \subseteq \mathcal{D}(E_0) = \mathcal{D}(E_0+W)$ and $B\mathcal{S} \subseteq \mathcal{R}(E_0+W)$. In fact, if $h \in B\mathcal{S}$, by Lemma 5.6, $h \in \mathcal{R}(E_0) \subseteq \mathcal{D}(E_0) = \mathcal{D}(W+E_0)$. Hence, $(W+E_0)h = Wh + E_0h = h$, because $h \in \mathcal{N}(W)$. Then, again by Lemma 5.6, $(E_0+W)B \in L(\mathcal{H})^s$ and $E_0 + W \in \mathcal{P}_0^*(B,\mathcal{S})$.

Conversely, let $E \in \mathcal{P}_0^*(B,\mathcal{S})$ and define $W := E - E_0$. Then $W \in SC(\mathcal{H})$ and $\mathcal{D}(W) = \mathcal{D}(E) \cap \mathcal{D}(E_0) = \mathcal{D}(E_0)$. Also, $\mathcal{R}(W) = \mathcal{R}((I-E_0)+(E-I)) \subseteq \mathcal{S}^\perp$. It is clear that $\mathcal{S}^\perp \subseteq \mathcal{N}(W)$ and, if $h \in B\mathcal{S}$, by Lemma 5.6, $h \in \mathcal{R}(E) \cap \mathcal{R}(E_0)$. Then $Wh = Eh - E_0h = h - h = 0$. Therefore $B\mathcal{S} \dotplus \mathcal{S}^\perp \subseteq \mathcal{N}(W)$ and $E = E_0 + W$. □

5.1. Comparison between the notions of quasi and weak complementability

The next examples show that quasi-complementability does not imply weak complementability and viceversa.

Example 1. Let $\mathcal{S} \subseteq \mathcal{H}$ be a closed subspace such that $dim(\mathcal{S}) = dim(\mathcal{S}^\perp) = \infty$. Take $a : \mathcal{S} \to \mathcal{S}$ such that $a \in L(\mathcal{S})^+$, $\mathcal{R}(a)$ is not closed and $\overline{\mathcal{R}(a)} = \mathcal{S}$ and, take $b : \mathcal{S}^\perp \to \mathcal{S}$ such that $b \in GL(\mathcal{S}^\perp, \mathcal{S})$. Let $y : \mathcal{D}(y) \subseteq \mathcal{S} \to \mathcal{S}^\perp$ be defined by $y := b^*a^\dagger$ in $\mathcal{D}(y) = \mathcal{R}(a)$. Then y is a closed operator. In fact, let $\{x_n\}_{n\geq 1} \subseteq \mathcal{R}(a)$ be such that $x_n \to x_0 \in \mathcal{S}$ and $b^*a^\dagger x_n \to y_0 \in \mathcal{S}$. Then, $x_n = ab^{*-1}(b^*a^\dagger x_n) \to ab^{*-1}y_0$, because $ab^{*-1} \in L(\mathcal{S})$. Hence, $x_0 = ab^{*-1}y_0$. So that $x_0 \in \mathcal{R}(a) = \mathcal{D}(y)$ and $y(x_0) = b^*a^\dagger ab^{*-1}y_0 = y_0$. Let $x := y^* = (b^*a^\dagger)^* = a^\dagger b$, $E = \begin{array}{c c} \begin{bmatrix} I & x \\ 0 & 0 \end{bmatrix} & \begin{array}{c} \mathcal{S} \\ \mathcal{S}^\perp \end{array} \end{array}$ in $\mathcal{D}(E) = \mathcal{S} \oplus b^{-1}(\mathcal{R}(a))$ and $B = \begin{array}{c c} \begin{bmatrix} a & b \\ b^* & c \end{bmatrix} & \begin{array}{c} \mathcal{S} \\ \mathcal{S}^\perp \end{array} \end{array}$ for some $c \in L(\mathcal{S}^\perp)^s$. Then E is B-symmetric, because

$$ax = aa^\dagger b = b \text{ in } \mathcal{D}(x) = b^{-1}(\mathcal{R}(a)).$$

Observe that $x^* = (y^*)^* = y$, $\mathcal{D}(x^*) = \mathcal{D}(y) = \mathcal{R}(a) \subsetneqq \mathcal{R}(a^{1/2})$ and then $\mathcal{R}(a^{1/2}) \not\subseteq \mathcal{D}(x^*) \subseteq \mathcal{D}(E^*)$. Then, by Proposition 5.11, B is not $\mathcal{S}$-weakly complementable.

Example 2. See [15, Example 2.14]. Let $B \in L(\mathcal{H})^+$ be such that $\mathcal{R}(B)$ is not closed. Let $h \in \overline{\mathcal{R}(B)} \setminus \mathcal{R}(B)$ and define the closed subspace $\mathcal{S} = span\ \{x\}$. Clearly, B is $\mathcal{S}$-weakly complementable. On the other hand, $(B\mathcal{S})^\perp = B^{-1}(\mathcal{S}^\perp) = B^{-1}(span\ \{x\}) = B^{-1}(span\ \{x\} \cap \mathcal{R}(B)) = B^{-1}(\{0\}) = \mathcal{N}(B)$. Then $\overline{B\mathcal{S}} = \overline{\mathcal{R}(B)}$ and $\mathcal{S}^\perp \cap \overline{B\mathcal{S}} \neq \{0\}$. Therefore, by Proposition 5.2, the pair $(B,\mathcal{S})$ is not quasi complementable.

Proposition 5.11. *Let $B \in L(\mathcal{H})^s$ and $\mathcal{S} \subseteq \mathcal{H}$ be a closed subspace such that B is $\mathcal{S}$-weakly complementable and the matrix decomposition of B is as in* (5.1). *If E is a densely defined closed B-symmetric projection on $\mathcal{S}$ then $E^* \in \mathcal{P}^*(B, \mathcal{S})$.*

Proof. If E is a densely defined closed B-symmetric projection on $\mathcal{S}$ then, clearly E^* is a densely defined closed projection with nullspace $\mathcal{S}^\perp$ and, by Proposition 4.4, $E^*B \in L(\mathcal{H})^s$.

On the other hand, by Proposition 4.1, if the matrix decomposition of E is as in (5.2). Then $ax \subset b$. Since B is $\mathcal{S}$-weakly complementable, $b = |a|^{1/2}f$, with f the reduced solution of the equation $b = |a|^{1/2}h$. Then, if $a = u|a|$ is the polar decomposition of a, $(|a|^{1/2}f)z = (|a|^{1/2}|a|^{1/2}ux)z$ for every $z \in \mathcal{D}(x)$. Then $(|a|^{1/2}ux - f) \in \overline{\mathcal{R}(|a|^{1/2})} \cap \mathcal{N}(|a|^{1/2}) = \{0\}$, so that, $(|a|^{1/2}x)z = (uf)z$ for every $z \in \mathcal{D}(x)$. Therefore $|a|^{1/2}x \subset uf$. Then $(uf)^* \subset (|a|^{1/2}x)^* = x^*|a|^{1/2}$ and $x^*|a|^{1/2} \in L(\mathcal{S}, \mathcal{S}^\perp)$. Hence $\mathcal{R}(|a|^{1/2}) \subseteq \mathcal{D}(E^*)$. □

Proposition 5.12. *Let $B \in L(\mathcal{H})^s$ and $\mathcal{S} \subseteq \mathcal{H}$ be a closed subspace such that $B\mathcal{S} \subseteq \mathcal{S} + (B\mathcal{S})^\perp$. Then B is $\mathcal{S}$-weakly complementable and the pair $(B, \mathcal{S})$ is quasi-complementable.*

Proof. Observe that $B\mathcal{S} \oplus (B\mathcal{S})^\perp \subseteq \mathcal{S} + (B\mathcal{S})^\perp$. Therefore the pair $(B, \mathcal{S})$ is quasi-complementable. Now we are going to show that B is $\mathcal{S}$-weakly complementable.

Consider $P_\mathcal{S}$ and $P_{\overline{B(\mathcal{S})}}$. Then

$$(P_\mathcal{S} - P_{B(\mathcal{S})^\perp})^2 = P_\mathcal{S} P_{\overline{B\mathcal{S}}} P_\mathcal{S} + P_{\mathcal{S}^\perp} P_{(B\mathcal{S})^\perp} P_{\mathcal{S}^\perp}.$$

Then

$$(P_\mathcal{S} - P_{B(\mathcal{S})^\perp})^2 = \begin{array}{cc} \begin{bmatrix} P_\mathcal{S} P_{\overline{B\mathcal{S}}} P_\mathcal{S} & 0 \\ 0 & P_{\mathcal{S}^\perp} P_{(B\mathcal{S})^\perp} P_{\mathcal{S}^\perp} \end{bmatrix} & \begin{matrix} \mathcal{S} \\ \mathcal{S}^\perp \end{matrix} \end{array}$$

and

$$|P_\mathcal{S} - P_{B(\mathcal{S})^\perp}| = |P_{\overline{B(\mathcal{S})}} - P_{\mathcal{S}^\perp}| = \begin{array}{cc} \begin{bmatrix} (P_\mathcal{S} P_{\overline{B\mathcal{S}}} P_\mathcal{S})^{1/2} & 0 \\ 0 & (P_{\mathcal{S}^\perp} P_{(B\mathcal{S})^\perp} P_{\mathcal{S}^\perp})^{1/2} \end{bmatrix} & \begin{matrix} \mathcal{S} \\ \mathcal{S}^\perp \end{matrix} \end{array}.$$

Let $E = P_{\mathcal{S}//\mathcal{N}}$, with $\mathcal{N} \subseteq (B\mathcal{S})^\perp$, see Proposition 4.1. Then $E^* = P_{\mathcal{N}^\perp//\mathcal{S}^\perp}$ and, by Proposition 4.4, $E^*B \in L(\mathcal{H})$. Let $\Gamma_{E^*} = (P_{\mathcal{S}^\perp} + P_{\mathcal{N}^\perp})^{1/2}$. Then, by Proposition 3.4, $E^*\Gamma_{E^*} \in L(\mathcal{H})$ and,

$$\mathcal{R}((P_\mathcal{S} P_{\overline{B\mathcal{S}}} P_\mathcal{S})^{1/2}) \subseteq \mathcal{R}(|P_{\overline{B(\mathcal{S})}} - P_{\mathcal{S}^\perp}|) = \mathcal{R}(P_{\overline{B(\mathcal{S})}} - P_{\mathcal{S}^\perp}) \subseteq \overline{B\mathcal{S}} + \mathcal{S}^\perp \subseteq \mathcal{N}^\perp + \mathcal{S}^\perp = \mathcal{R}(\Gamma_{E^*}).$$

But $\mathcal{R}(|P_\mathcal{S} B P_\mathcal{S}|) = \mathcal{R}(P_\mathcal{S} B P_\mathcal{S}) \subseteq \mathcal{R}(P_\mathcal{S} P_{\overline{B\mathcal{S}}} P_\mathcal{S})$. In fact, if $y \in \mathcal{R}(P_\mathcal{S} B P_\mathcal{S})$ then, there exists $x \in \mathcal{H}$ such that $y = P_\mathcal{S} B P_\mathcal{S} x = P_\mathcal{S} P_{\overline{B\mathcal{S}}}(B P_\mathcal{S} x)$. Since $B\mathcal{S} \subseteq \mathcal{S} + (B\mathcal{S})^\perp$, there exists $s \in \mathcal{S}$ and $t \in (B\mathcal{S})^\perp$ such that $BP_\mathcal{S} x = s + t$. Then $y = P_\mathcal{S} P_{\overline{B\mathcal{S}}}(s + t) = P_\mathcal{S} P_{\overline{B\mathcal{S}}} P_\mathcal{S} s$. Then, by Douglas's Lemma,

$$\mathcal{R}(|P_\mathcal{S} B P_\mathcal{S}|^{1/2}) \subseteq \mathcal{R}((P_\mathcal{S} P_{\overline{B\mathcal{S}}} P_\mathcal{S})^{1/2}) \subseteq \mathcal{R}(\Gamma_{E^*}).$$

This gives, $E^*|P_\mathcal{S} B P_\mathcal{S}|^{1/2} \in L(\mathcal{H})$. Then, B is $\mathcal{S}$-weakly complementable. □

Corollary 5.13. *Let $B \in L(\mathcal{H})^s$ and $\mathcal{S} \subseteq \mathcal{H}$ be a closed subspace. Suppose that the matrix decomposition of B is as in* (5.1). *If the pair $(B, \mathcal{S})$ is quasi-complementable and $\mathcal{R}(|a|^{1/2}) \subseteq \mathcal{R}(P_\mathcal{S} P_{\overline{B\mathcal{S}}})$ then B is $\mathcal{S}$-weakly complementable.*

Proof. Observe that $\mathcal{R}(|a|^{1/2}) \subseteq \mathcal{R}(P_\mathcal{S} P_{\overline{B\mathcal{S}}})$ if and only if $\mathcal{R}(|a|^{1/2}) \subseteq \mathcal{R}((P_\mathcal{S} P_{\overline{B\mathcal{S}}} P_\mathcal{S})^{1/2})$. Then the result follows from Proposition 5.12. □

5.2. Applications: Schur complements of selfadjoint operators

We recall the definition of Schur complement for an $\mathcal{S}$-weakly complementable selfadjoint operator.

Definition. Let $B \in L(\mathcal{H})^s$ and $\mathcal{S} \subseteq \mathcal{H}$ be a closed subspace such that B is $\mathcal{S}$-weakly complementable. When B is as in (5.1), let f be the reduced solution of $b = |a|^{1/2}x$ and $a = u|a|$ the polar decomposition of a. The *Schur complement* of B to $\mathcal{S}$ is defined as

$$B_{/\mathcal{S}} := \begin{bmatrix} 0 & 0 \\ 0 & c - f^*uf \end{bmatrix}.$$

$B_{\mathcal{S}} := B - B_{/\mathcal{S}}$ is the $\mathcal{S}$-*compression* of B.

When $B \in L(\mathcal{H})^+$ this formula gives the usual Schur complement, see [1, Theorem 3].

When the operator B is $\mathcal{S}$-complementable, the Schur complement can be written as $B_{/\mathcal{S}} = (I-F)B$, for any bounded projection with $\mathcal{N}(F) = \mathcal{S}^\perp$ such that $(FB)^* = FB$. In fact, from [14, Corollary 3.12] it suffices to take $F = Q^*$, for any $Q \in \mathcal{P}(B, \mathcal{S})$.

A similar formula for $B_{/\mathcal{S}}$ can be given when B is $\mathcal{S}$-weakly complementable. In this case the projection need not be bounded, but it is a semiclosed densely defined projection with closed nullspace.

Theorem 5.14 (c.f. [14, Theorem 3.14])**.** *Let $B \in L(\mathcal{H})^s$ and $\mathcal{S}$ be a closed subspace of $\mathcal{H}$. Suppose that B is $\mathcal{S}$-weakly complementable then*

$$B_{/\mathcal{S}} = (I - E)B,$$

for every $E \in \mathcal{P}^*(B, \mathcal{S})$.

Proof. Suppose that the matrix decomposition of E is as in (5.2) and that of B is as in (5.1). Then, by Lemma 5.6, $ya = b^*$ and since $\mathcal{R}(|a|^{1/2}) \subseteq \mathcal{D}(E)$, by Theorem 5.9, $y|a|^{1/2} \in L(\mathcal{S}, \mathcal{S}^\perp)$. Hence

$$(I-E)B = \begin{bmatrix} 0 & 0 \\ -y & I \end{bmatrix} \begin{bmatrix} a & b \\ b^* & c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -ya + b^* & c - yb \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & c - yb \end{bmatrix}.$$

Let f be the reduced solution of $b = |a|^{1/2}x$ and $a = u|a|$ the polar decomposition of a. Then $yb = f^*uf$. In fact, since $ya = b^*$ we have that $y|a| = f^*|a|^{1/2}u = f^*u|a|^{1/2}$. Then $y|a|^{1/2} = f^*u$ on $\mathcal{R}(|a|^{1/2})$, and since $y|a|^{1/2}$ is bounded, $y|a|^{1/2} = f^*u$ on $\overline{\mathcal{R}(|a|^{1/2})}$. Then $yb = y|a|^{1/2}f = f^*uf$ because $\mathcal{R}(f) \subseteq \overline{\mathcal{R}(|a|^{1/2})}$. Hence $(I-E)B = B_{/\mathcal{S}}$.

□

In particular, Theorem 5.14 gives a formula for the Schur complement of any positive operator B to $\mathcal{S}$ in terms of semiclosed projections.

Different (but equivalent) definitions where given for minus order, for example, using generalized inverses in the matrix case, see [31]. We give the following definition, equivalent to those appearing in [36] and [20].

Definition. Let $A, B \in L(\mathcal{H})$, we write $A \overset{-}{\leq} B$ if there exist projections $P, Q \in \mathcal{Q}$ such that $A = PB$ and $A^* = QB^*$.

It was proved in [36] and [20] that $\overset{-}{\leq}$ is a partial order, known as the minus order for operators. In [6], it was shown that $A \overset{-}{\leq} B$ if and only if the sets $\overline{R(A)} \dot{+} \overline{R(B-A)}$ and $\overline{R(A^*)} \dot{+} \overline{R(B^*-A^*)}$ are closed. In [20, Theorem 3.3], another characterization of the minus order in terms of the range additivity property was given:

$$A \overset{-}{\leq} B \text{ if and only if } R(B) = R(A) \dot{+} R(B-A) \text{ and } R(B^*) = R(A^*) \dot{+} R(B^* - A^*).$$

In view of this equivalence, the left minus order was defined in [20] for operators in $L(\mathcal{H})$. This notion is weaker than the minus order, in the infinite dimensional setting.

Definition. Let $A, B \in L(\mathcal{H})$, we write $A \ {}_{-}\!\leq B$ if $R(B) = R(A) \dot{+} R(B-A)$.

The relation ${}_{-}\!\leq$ is a partial order, see [20]. The following is a characterization of the left minus order in terms of a semiclosed projection.

Proposition 5.15 (c.f. [20, Proposition 3.13])**.** *Let $A, B \in L(\mathcal{H})$. Then $A \,{}_{-}\!\leq B$ if and only if there exists a densely defined semiclosed projection P such that $A = PB$ and $R(A) \subseteq R(B)$.*

Proof. Suppose that $A \,{}_{-}\!\leq B$ then $R(B) = R(A) \dot{+} R(B-A)$ so that $R(A) \subseteq R(B)$. Define $P = P_{R(A)//R(B-A) \oplus N(B^*)}$. Then P is a densely defined semiclosed projection and it is easy to check that $A = PB$.

Conversely, if $A = PB$, for P a densely defined (semiclosed) projection and $R(A) \subseteq R(B)$ then, by [9, Proposition 2.4], $R(B) = R(A) + R(B-A)$. The sum is direct because $R(A) \subseteq R(P)$ and $R(B-A) \subseteq N(P)$. $\square$

A different generalization of the minus order was introduced by Arias et al. in [8]:

Definition. Given $A, B \in L(\mathcal{H})$, we write $A \prec B$ if there exists two densely defined projections Q, P with closed ranges such that $A = QB$ and $A^* = PB^*$.

The relation $\prec$ is a partial order in $L(\mathcal{H})$ (see [8, Lemma 4.5]). In [8, Lemma 4.4], it was proved that $A \prec B$ if and only if $\overline{\mathcal{R}(A)} \cap \mathcal{R}(B-A) = \{0\}$ and $\overline{\mathcal{R}(A^*)} \cap \mathcal{R}(B^* - A^*) = \{0\}$. Hence, $A \prec B$ if and only if there exists Q, P two densely defined semiclosed projections with closed ranges such that $A = QB$ and $A^* = PB^*$.

Let $B \in L(\mathcal{H})^s$ and $\mathcal{S}$ be a closed subspace of $\mathcal{H}$. Suppose that the matrix decomposition of B is as in (5.1). Denote by $\tilde{\mathcal{Q}}$ the set of densely defined closed range projections and define

$$\mathcal{M}^{\prec}(B, \mathcal{S}) := \{X \in L(\mathcal{H})^s : R(X) \subseteq \mathcal{S}^{\perp} \text{ and, for some } Q \in \tilde{\mathcal{Q}},\ X = QB \text{ and } R(|a|^{1/2}) \subseteq \mathcal{D}(Q)\}.$$

Theorem 5.16. *Let $B \in L(\mathcal{H})^s$ and $\mathcal{S}$ be a closed subspace of $\mathcal{H}$ such that B is $\mathcal{S}$-weakly complementable. Then*

$$B_{/\mathcal{S}} = \max_{\prec}\ \mathcal{M}^{\prec}(B, \mathcal{S}).$$

Proof. Let $E \in \mathcal{P}^*(B, \mathcal{S})$. Then, by Theorem 5.14, $B_{/\mathcal{S}} = (I-E)B \in L(\mathcal{H})^s$ and $\mathcal{R}(B_{/\mathcal{S}}) \subseteq \mathcal{S}^{\perp}$. Set $Q := (I-E)$, then $Q \in \tilde{\mathcal{Q}}$, $R(|a|^{1/2}) \subseteq \mathcal{D}(Q)$ and $B_{/\mathcal{S}} = QB$. So that $B_{/\mathcal{S}} \in \mathcal{M}^{\prec}(B, \mathcal{S})$.

On the other hand, let $X \in \mathcal{M}^{\prec}(B, \mathcal{S})$. Then $X \in L(\mathcal{H})^s$, $R(X) \subseteq \mathcal{S}^{\perp}$ and and there exists $Q \in \tilde{\mathcal{Q}}$ such that $X = QB$ and $R(|a|^{1/2}) \subseteq \mathcal{D}(Q)$. Suppose that the matrix decomposition of B is as in (5.1) and, let f be the reduced solution of $b = |a|^{1/2}x$ and $a = u|a|$ the polar decomposition of a. Hence,

$$B = \begin{bmatrix} a & |a|^{1/2}f \\ f^*|a|^{1/2} & c \end{bmatrix} = \begin{bmatrix} |a|^{1/2} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} u & f \\ f^* & c \end{bmatrix} \begin{bmatrix} |a|^{1/2} & 0 \\ 0 & I \end{bmatrix}.$$

Set $\Gamma := \begin{bmatrix} |a|^{1/2} & 0 \\ 0 & I \end{bmatrix}$ and $U := \begin{bmatrix} u & f \\ f^* & c \end{bmatrix}$. Since Q is semiclosed, by similar arguments as those found in the proof of Theorem 5.9, $Q\Gamma \in L(\mathcal{H})$. Therefore $X = QB = Q\Gamma U \Gamma = \Gamma U (Q\Gamma)^*$. Let $E \in \mathcal{P}^*(B, \mathcal{S})$ then, again by the proof of Theorem 5.9, $(I-E)\Gamma \in L(\mathcal{H})$ and

$$X = (I-E)X = (I-E)\Gamma U (Q\Gamma)^* = Q\Gamma U((I-E)\Gamma)^* = Q((I-E)\Gamma U \Gamma)^* = Q((I-E)B)^* = QB_{/\mathcal{S}},$$

where we used Theorem 5.14. Then $X \prec B_{/\mathcal{S}}$. $\square$

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