MULTI-ORBITAL FRAMES THROUGH MODEL SPACES

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Abstract. We characterize the normal operators A on ℓ^2 and the elements $a^i \in \ell^2$, with $1 \leq i \leq m$, so that the sequence $\{A^n a^1, \ldots, A^n a^m\}_{n \geq 0}$ is Bessel or a frame. The characterization is given in terms of the backward shift invariant subspaces in $H^2(\mathbb{D})$.

1. Introduction

The study of the dynamical behaviour of a bounded operator A on a Hilbert space $\mathcal{H}$ consists of studying the orbits $\{A^n f : n \in \mathbb{N}_0\}$ for $f \in \mathcal{H}$. The literature is full of examples with characterizations of the operators A such that there exists an orbit satisfying a particular property, and sometimes also of the initial vector for such orbits. For instance, if $\mathcal{H}$ is separable and of infinite dimension, the orbit $\{A^n f\}$ is an orthonormal basis if and only if A is the forward shift with respect to the basis $e_n := A^n f$, for $n \geq 0$. Actually, this can be taken as the definition of the forward shift with respect to a given ordered basis $\{e_n\}_{n \geq 0}$. Moreover, it is not hard to see that the only vectors with this property are λe_0, where $\lambda \in \mathbb{C}$, with $|\lambda| = 1$.

Less restrictive requirements for an orbit is that of being a frame or even a Bessel sequence. Motivated by a time-space sampling problem, in [2] the authors characterize the diagonalizable operators A and the vectors $f \in \mathcal{H}$ such that the orbit $\{A^n f\}_{n \geq 0}$ is a frame for a Hilbert space of numerable dimension. The problem is modeled with the space $\mathcal{H} = \ell^2(\mathbb{N})$, but with the right definitions the results is valid for finite dimension.

A fortiori, in [1] it is shown that any normal operator A that admits an orbit as a frame must be diagonalizable, so the above result applies for normal operators as well. The normality of A allows the use of the spectral theorem, which in conjunction with the fact that some orbit is a frame forces the operator to be diagonalizable. Then, $\{A^n f\}$ is a frame if and only if the sequence of eigenvalues $\{\lambda_j\}$ is an interpolating sequence for the Hardy space of the disk $H^2(\mathbb{D})$, and

$$f = \{d_j (1 - |\lambda_j|^2)^{1/2} : j \in \mathbb{N}\} \in \ell^2(\mathbb{N}),$$

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where $C^{-1} \leq d_j \leq C$ for all $j \in \mathbb{N}$ and some $C > 0$ (see also (2.5) below and the subsequent comment). The result holds for finite dimension, taking $\ell^2(J)$, where $J \subset \mathbb{N}$ is finite, and accepting interpolating sequences also as those that perform finite interpolation in $H^2(\mathbb{D})$.

In [4] the authors consider the problem of characterizing the normal operators A and vectors $f_1, \ldots, f_m \in \ell^2(\mathbb{N})$, where $m \in \mathbb{N}$, such that the union of orbits $\{A^n f_j,\, n \in \mathbb{N},\, 1 \leq j \leq m\}$ is a frame. They obtained a characterization where, as before, A has to be diagonalizable, the eigenvalues form a union of at most m interpolating sequences for $H^2(\mathbb{D})$, and there are two more conditions, the last of which is not well understood and difficult to handle.

In the present paper we give a different, more intuitive and geometric characterization, which shows to what extent the pseudo-hypebolic metric of $\mathbb{D}$ plays a role in the structure of the eigenvalues and their interaction with the vectors f_j. To do so, we need some tools from the theory of $H^2(\mathbb{D})$, such as interpolating sequences, reproducing kernels and model spaces, which we establish in the next section.

Finally, in proving the above characterization we found a result of independent interest (Thm. 2.7), which gives an upper bound for the Bessel constant of the difference of normalized reproducing kernels in $H^2(\mathbb{D})$ under some geometric conditions of their base points.

1.1. **The Hardy space H^2 and the model subspaces.** Write $\varphi_0(z) = \phi_0(z) = z$ and for $\lambda_j \neq 0$,

$$\varphi_{\lambda_i}(z) = \frac{\lambda_i - z}{1 - \overline{\lambda}_i z} \quad \text{and} \quad \phi_{\lambda_i}(z) = \frac{\overline{\lambda}_i}{|\lambda_i|}\, \varphi_{\lambda_i}(z).$$

If $\{\lambda_i\}$ is a sequence in $\mathbb{D}$, the Blaschke product

$$B(z) = \prod_i \phi_{\lambda_i}(z) \text{ converges } \Leftrightarrow \sum_i (1 - |\lambda_i|^2) < \infty,$$

where the convergence is uniform on compact sets and $\{\lambda_i\}$ is called a Blaschke sequence. Every function $f \in H^2$ factorizes as $f = gB$, where $g \in H^2$ has no zeros on $\mathbb{D}$ and B is the Blasche product of the zeros of f. If the zeros $\{\lambda_j\}$ have single multiplicies, the orthogonal complement $K_B := (BH^2)^{\perp}$ of the closed subspace BH^2 of H^2 is generated by

$$k_{\lambda_j}(z) = \frac{(1 - |\lambda_j|^2)^{1/2}}{(1 - \overline{\lambda}_j z)},$$

the normalizations of the reproducing kernels $K_{\lambda_j}(z) = (1 - \overline{\lambda}_j z)^{-1}$. The name means that $\langle f, K_\lambda \rangle = f(\lambda)$ for every $f \in H^2$. Blaschke products are special cases of inner functions, which are functions $u \in H^\infty$ whose radial limit at the boundary satisfies $|u(e^{i\theta})| = 1$ for almost every $e^{i\theta} \in \partial\mathbb{D}$. The model spaces are $K_u := (uH^2)^{\perp}$, which by Beurling's theorem [3] are the closed backward shift invariant subspaces of H^2. They are called model spaces because the compression of the forward shift to K_u is a model for a broad class of contractions (see [12]).

A sequence $\{\lambda_j\}$ in $\mathbb{D}$ is called interpolating (for H^2) if

$$Ef := \{\langle f, k_{\lambda_j}\rangle\} \in \ell^2, \ \forall f \in H^2 \ \text{ and every } \ w \in \ell^2 \ \text{ is of this form.}$$

That is, $E : H^2 \to \ell^2$ is onto. Here we allow the set of indexes of ℓ^2 to be the set of natural numbers or a finite section, so finite sequences of different points will also be called interpolating.

When the above holds, $\{\lambda_i\}$ is the zero set of the Blaschke product u, and the restriction of E to the model space $E_u : K_u \to \ell^2$ is invertible. Therefore

$$\|E_u^{-1}\|^{-2}\,\|f\|^2 \le \sum |\langle f, k_{\lambda_i}\rangle|^2 \le \|E_u\|^2\,\|f\|^2, \qquad \forall f \in K_u.$$

This is equivalent to say that $\{k_{\lambda_i}\}$ is a Riesz base for K_u, or without specifying u, that it is a Riesz sequence (see [10, Lect. 6, 1]). This means that there are C_0, $C_1 > 0$ such that

$$C_0 \sum_j |c_j|^2 \le \|\sum_j c_j k_{\lambda_j}\|^2 \le C_1 \sum_j |c_j|^2, \qquad \forall \{c_j\} \in \ell^2. \tag{1.1}$$

On the other hand, $\{\lambda_j\}$ in $\mathbb{D}$ is called interpolating (for H^∞) if

$$\forall w \in \ell^\infty(J) \ \text{ there is } \ f \in H^\infty \ \text{ such that } \ f(\lambda_j) = w_j, \ \forall j \in J$$

(again $J = \mathbb{N}$ or it is finite). The problem of characterizing interpolating sequences for H^∞ was considered by several authors until Carleson obtained the definitive version in [5]. In [11] Shapiro and Shield provided a different proof and showed that interpolating sequences are the same for all H^p, where $1 \le p \le \infty$. For a Blaschke sequence $\{\lambda_i\}$ write B for its Blaschke product and $B_j = \prod_{i:\, i\neq j} \phi_{\lambda_i}$. The sequence is interpolating if and only if

$$\delta(B) := \inf_j |B_j(\lambda_j)| > 0.$$

A sequence satisfying this condition is usually called uniformly separated. When this happens, $B_j(\lambda_j)^{-1}B_j \in H^2$ has norm $\le \delta^{-1}$, here $\delta := \delta(B)$, and

$$f_i = k_{\lambda_i} \ \text{ and } \ g_i = \frac{B_i}{B_i(\lambda_i)}\, k_{\lambda_i} \ \text{ are biorthogonal sequences in } K_B.$$

Therefore, when $\{c_i\} \in \ell^2$, the (unique) function $g \in K_B$ that interpolates $\langle g, k_{\lambda_j}\rangle = c_j$ for all j is $g(z) = \sum_i c_i g_i(z)$. Any function $F \in H^2$ satisfying $\langle F, k_{\lambda_j}\rangle = c_j$ for all j has the form $F = g + Bh$, where g is as above, $h \in H^2$, and $\|F\|^2 = \|g\|^2 + \|h\|^2$ (since $K_B = (BH^2)^\perp$ and multiplication by B is an isometry). In particular, $g \in K_B$ is the function of minimum norm that satisfies $\langle g, k_{\lambda_j}\rangle = c_j$ for all j. Consequently, [11, Lemma 3] gives us

$$\|g\|^2 \le (2/\delta^4)(1 + \log \delta^{-2}) \sum |c_i|^2, \tag{1.2}$$

where $\delta = \delta(B)$. Also, the constants C_0 and C_1 of (1.1) depend only on δ. Indeed, a more general statement will be given for C_1 in Theorem 2.3. For

C_0 notice that (1.2) together with Lemma 2.2 imply that

$$\sum_i |\langle f, g_i\rangle|^2 \le C_\delta \|f\|^2, \qquad \forall f \in K_B,$$

where C_δ is the constant of (1.2). In particular, when $f = \sum_j c_j k_{\lambda_j}$, for $\{c_j\} \in \ell^2$, we obtain the first inequality in (1.1) with $C_0 = C_\delta^{-1}$.

The pseudo-hyperbolic metric in $\mathbb{D}$ is given by $\rho(z,w) = |\varphi_z(w)|$, and we denote $\Delta(z,r) = \{w \in \mathbb{D} : \ \rho(z,w) < r\}$ for $0 < r < 1$, with similar convention for the closed ball $\Delta[z,r]$. Also, we will use that Blaschke products satisfy the Lipschitz condition $\rho(B(z), B(w)) \le \rho(z,w)$ for z, $w \in \mathbb{D}$, and the elementary equality

$$1 - |\varphi_v(z)|^2 = \frac{(1-|v|^2)(1-|z|^2)}{|1-\overline{v}z|^2}. \tag{1.3}$$

2. Basic necessary conditions

Let e_{j_0} be the j_0 element of the standard base and $a^i \in \ell^2$ for $i = 1, \dots, m$. Then

$$\begin{aligned}\sum_n \sum_{i=1}^m |\langle A^n a^i, e_{j_0}\rangle|^2 &= \sum_n |\lambda_{j_0}^2|^n \Big[|a_{j_0}^1|^2 + \cdots + |a_{j_0}^m|^2\Big] \\ &= \frac{|a_{j_0}^1|^2 + \cdots + |a_{j_0}^m|^2}{1-|\lambda_{j_0}|^2}.\end{aligned}$$

So, the lower bound for a frame implies that this is bounded below away from zero, implying that $\sum_j (1-|\lambda_j|^2) \lesssim \sum_j \sum_{i=1}^m |a_j^i|^2 < \infty$, hence λ_j is a Blaschke sequence. Additionally, the Bessel constant (the upper frame constant) on the standard base gives

$$C_0(1-|\lambda_j|^2) \le \sum_{i=1}^m |a_j^i|^2 \le C_1(1-|\lambda_j|^2).$$

Let $\tilde{a}_j^i = \{\overline{\alpha}_j^i (1-|\lambda_j|^2)^{\frac{1}{2}}\} \in \ell^2(J)$ for $i = 1, \dots, m$, where $\sum_{i=1}^m |\alpha_j^i|^2 = 1$, with J is finite or not. Then $a^i = d \cdot \tilde{a}^i$, a coordinate to coordinate product for $1 \le i \le m$, where $d \in \ell^\infty(J)$ is given by

$$\sqrt{C}_0 \le d_j = \left[\frac{|a_j^1|^2 + \cdots + |a_j^m|^2}{1-|\lambda_j|^2}\right]^{\frac{1}{2}} \le \sqrt{C}_1.$$

That is, any m vectors $a^1, \dots, a^m \in \ell^2(J)$ such that the union of the respective A-orbits satisfies the lower and upper frame bounds when tested against the standard base, write as

$$a_j^i = d_j\, \tilde{a}_j^i = d_j\, \overline{\alpha}_j^i (1-|\lambda_j|^2)^{\frac{1}{2}}, \quad \text{for } j \in J \ \text{ and } \ 1 \le i \le m, \tag{2.4}$$

where $C^{-1} \le d_j \le C$ for some $C \ge 1$ and $\sum_{i=1}^m |\alpha_j^i|^2 = 1$ for all $j \in J$. Moreover, it is clear that $\{A^n a^i : \ n \ge 0,\ 1 \le i \le m\}$ is a frame (a Bessel sequence) if and only if a^i are given by (2.4) and $\{A^n\tilde{a}^i\colon n\geq 0,\ 1\leq i\leq m\}$ is a frame (respectively, a Bessel sequence). So, from now on we work with $\tilde{a}^i$ for $1\leq i\leq m$.

Write $\tilde{b}=\{(1-|\lambda_j|^2)^{1/2}\}\in\ell^2(J)$, and let $c\in\ell^2(J)$, where J could be finite. Then

$$\begin{aligned}\sum_n|\langle A^n\tilde{b},c\rangle|^2 &= \sum_n\sum_{i,j}\lambda_i^n\overline{\lambda}_j^n(1-|\lambda_i|^2)^{1/2}(1-|\lambda_j|^2)^{1/2}\,\overline{c}_ic_j\\ &= \sum_{i,j}\sum_n\lambda_i^n\overline{\lambda}_j^n(1-|\lambda_i|^2)^{1/2}(1-|\lambda_j|^2)^{1/2}\,\overline{c}_ic_j\\ &= \sum_{i,j}\frac{(1-|\lambda_i|^2)^{1/2}(1-|\lambda_j|^2)^{1/2}}{1-\lambda_i\overline{\lambda}_j}\,\overline{c}_ic_j\\ &= \sum_i\sum_j\langle c_jk_{\lambda_j},c_ik_{\lambda_i}\rangle \;=\; \|\sum_j c_jk_{\lambda_j}\|^2.\end{aligned}$$

It follows that

$$\sum_n|\langle A^n\tilde{a}^i,c\rangle|^2=\|\sum_j\alpha_j^ic_jk_{\lambda_j}\|^2\quad\text{ for }1\leq i\leq m. \tag{2.5}$$

In particular, when $m=1$, the orbit $\{A^n\tilde{a}^1,n\geq 0\}$ is a frame if and only if $\{k_{\lambda_j}\}$ is a Riesz sequence (i.e.: a Riesz base for the subspace $K_u:=(uH^2)^\perp$, where u is the Blaschke product with zeros λ_j). By the previous section this happens if and only if $\{\lambda_j\}$ is an interpolating sequence (see [2, Thm. 3.14]).

2.1. **Carleson measures and Bessel sequences.** A positive measure μ on $\mathbb{D}$ is called a Carleson measure if

$$\int|f|^2d\mu\leq C_2^2\|f\|^2,\quad\ \forall\ f\in H^2$$

It is well known that μ is Carleson if and only if

$$\mu(Q)\leq C\ell(Q)$$

for every angular square $Q=\{re^{i\theta}:1-\ell\leq r<1,\ |\theta-\theta_0|\leq\ell\}$, where $\ell=\ell(Q)$. The smallest constant C is called the Carleson norm of μ and it is noted $\|\mu\|_*$. Also, the optimal C_2 and $\|\mu\|_*$ are equivalent quantities (see [7, I, Thm.5.6]).

Carleson measures and interpolating sequences are closely related, as we shall see in the next lemma. For a sequence $\{\lambda_j\}$ in $\mathbb{D}$ consider the purely atomic measure $\mu=\sum_j(1-|\lambda_j|^2)\delta_{\lambda_j}$, where δ_λ is the probability measure with mass concentrated at $\lambda\in\mathbb{D}$. Observe that $\mu(\mathbb{D})=\sum_j(1-|\lambda_j|^2)<\infty$ if and only if $\{\lambda_j\}$ is a Blaschke sequence. Furthermore, it is well known that

Lemma 2.1. *If $S=\{\lambda_j\}$ is a sequence in $\mathbb{D}$ and $\mu=\sum_j(1-|\lambda_j|^2)\delta_{\lambda_j}$, then*

(1) *S is a finite union of interpolating sequences if and only if μ is Carleson.*

(2) *S is interpolating if and only if μ is Carleson and $\rho(\lambda_j, \lambda_k) \geq \beta > 0$ when $j \neq k$ (i.e.: S is separated).*

When (2) holds and B is the respective Blaschke product, $\delta(B)$ can be estimated from μ and β, and viceversa.

Proof. Assertion (1) is proved in [9, Lemma 21]. Assertion (2) can be found in [7, VII, Thm.1.1], where it is also proved the equivalence with S being uniformly separated and the relations between the various parameters are established. □

The following basic and well-known result on Bessel sequences can be found, for instance, in [6, pp. 51-53].

Lemma 2.2. *For a sequence $\{f_j\}$ in $\mathcal{H}$, the next assertions are equivalent:*

(1) *$T(\{c_j\}) = \sum c_j f_j$ is a bounded operator from ℓ^2 to $\mathcal{H}$:*

$$\| \sum c_j f_j \|^2 \leq \|T\|^2 \sum |c_j|^2.$$

(2) *$T^* f = (\{\langle f, f_j \rangle\})$ is a bounded operator from $\mathcal{H}$ to ℓ^2:*

$$\sum |\langle f, f_j \rangle|^2 \leq \|T^*\|^2 \, \|f\|^2.$$

By (2.5), $\{A^n \tilde{a}^1, \dots, A^n \tilde{a}^m\}$ is a Bessel sequence with constant $\leq B^2$ if and only if $\{\alpha_j^1 k_{\lambda_j}, \dots, \alpha_j^m k_{\lambda_j}\}$ is Bessel with constant $\leq B^2$.

Theorem 2.3. *The sequence $\{A^n \tilde{a}^1, \dots, A^n \tilde{a}^m\}$ is Bessel with constant $\leq B^2$ if and only if the measure $\mu := \sum_i (1 - |\lambda_i|^2) \delta_{\lambda_i}$ is Carleson, where if $C \geq 0$ is such that*

$$\int |f|^2 d\mu \leq C^2 \|f\|^2, \qquad \text{for } \; f \in H^2,$$

then we can take $B^2 \leq C^2 \leq mB^2$.

Proof. If

$$\| \sum_j \alpha_j^1 c_j k_{\lambda_j} \|^2 + \dots + \| \sum_j \alpha_j^m c_j k_{\lambda_j} \|^2 \leq B^2 \sum_j |c_j|^2, \tag{2.6}$$

the inequality holds for each member of the left sum. Hence, Lemma 2.2 says that for all $f \in H^2$,

$$\sum_j \sum_{i=1}^m |\alpha_j^i|^2 \, |\langle f, k_{\lambda_j} \rangle|^2 = \sum_{i=1}^m \sum_j |\langle f, \alpha_j^i k_{\lambda_j} \rangle|^2 \;\leq\; B^2 \sum_{i=1}^m \|f\|^2.$$

Since $|\alpha_j^1|^2 + \dots + |\alpha_j^m|^2 = 1$, then $\mu := \sum_j (1 - |\lambda_j|^2) \delta_{\lambda_j}$ is Carleson with

$$\int |f|^2 d\mu \leq mB^2 \|f\|^2, \qquad \text{for } f \in H^2.$$

Reciprocally, if μ is Carleson with $\sum_j |\langle f, k_{\lambda_j}\rangle|^2 \leq C^2\|f\|^2$, for $f \in H^2$, by Lemma 2.2,

$$\|\sum_j \alpha_j^i c_j k_{\lambda_j}\|^2 \leq C^2 \sum_j |\alpha_j^i|^2 |c_j|^2 \quad \text{for } 1 \leq i \leq m$$

and $\{c_j\} \in \ell^2$. Adding for $1 \leq i \leq m$ we get (2.6) with $B^2 = C^2$. □

The next estimate for the distance between two normalized reproducing kernels in H^2 is sharp and can be found in [4, Lemma B7]:

$$\|k_v - k_w\|^2 \leq 2\rho(v,w)^2 \qquad \forall v, w \in \mathbb{D}. \tag{2.7}$$

Theorem 2.4. *Let $S = \{\lambda_j\}$ be a sequence such that $\sum_i (1-|\lambda_i|^2)\delta_{\lambda_i}$ is Carleson and let $m \in \mathbb{N}$. Then there is a constant $D > 0$ satisfying*

$$D^2 \sum_j |c_j|^2 \leq \sum_{i=1}^m \Big\| \sum_j \alpha_j^i c_j k_{\lambda_j} \Big\|^2 \qquad \forall c \in \ell^2. \tag{2.8}$$

if and only if

1. *there is $\beta > 0$ such that $\Delta(\lambda_j, \beta)$ contains no more than m points of S counting repetitions for all j.*
2. *there is $0 < \eta < \beta$ such that if $\lambda_{j_1}, \ldots, \lambda_{j_p}$ $(p \leq m)$ are the points of S in $\Delta(\lambda_{j_1}, \eta)$ counting repetitions, the related matrix satisfies*

$$D_0^2 \left\| \begin{bmatrix} c_{j_1} \\ \vdots \\ c_{j_p} \end{bmatrix} \right\|_{\mathbb{C}^p}^2 \leq \left\| \begin{bmatrix} \alpha_{j_1}^1 & \cdots & \alpha_{j_p}^1 \\ \vdots & & \vdots \\ \alpha_{j_1}^m & \cdots & \alpha_{j_p}^m \end{bmatrix} \begin{bmatrix} c_{j_1} \\ \vdots \\ c_{j_p} \end{bmatrix} \right\|_{C^{m\times 1}}^2 \qquad \forall (c_{j_1}, \ldots, c_{j_p}) \in \mathbb{C}^p,$$

where $D_0 > 0$ does not depend on p or the α's .

Proof of necessity for Theorem 2.4. First we show that (1) holds. Suppose otherwise that for any $\beta > 0$ there are at least m+1 points $\lambda_{j_0}, \lambda_{j_1}, \ldots, \lambda_{j_m}$ of S counting repetitions, such that

$$\lambda_{j_0}, \lambda_{j_1}, \ldots, \lambda_{j_m} \in \Delta(\lambda_{j_0}, \beta). \tag{2.9}$$

To simplify notation we assume that $j_s = s$ for $s = 0, \ldots, m$. Taking $c = (c_0, \ldots, c_m, 0, \ldots) \in \ell^2$, each summand of the right side of (2.8) is

$$\begin{aligned} &\|\sum_{j=0}^m \alpha_j^i c_j k_{\lambda_j}\|^2 = \\ &= \quad \|\Big[\sum_{j=0}^m c_j \alpha_j^i\Big] k_{\lambda_0} + c_1 \alpha_1^i [k_{\lambda_1} - k_{\lambda_0}] + \cdots + c_m \alpha_m^i [k_{\lambda_m} - k_{\lambda_0}]\|^2, \end{aligned}$$

where $1 \leq i \leq m$. If we take a normal vector $c \in \mathbb{C}^{m+1}$ such that

$$\begin{bmatrix} \alpha_0^1 & \ldots & \alpha_m^1 \\ \vdots & & \vdots \\ \alpha_0^m & \ldots & \alpha_m^m \end{bmatrix} \begin{bmatrix} c_0 \\ \vdots \\ c_m \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix},$$

by the Cauchy-Schwarz inequality and (2.7), the right side of (2.8) becomes

$$\begin{aligned}\sum_{i=1}^{m} \left\|c_1\alpha_1^i[k_{\lambda_1}-k_{\lambda_0}]+\cdots+c_m\alpha_m^i[k_{\lambda_m}-k_{\lambda_0}]\right\|^2 \leq \\ \leq \;& m\sum_{i=1}^{m}\left(\left\|c_1\alpha_1^i[k_{\lambda_1}-k_{\lambda_0}]\right\|^2+\cdots+\left\|c_m\alpha_m^i[k_{\lambda_m}-k_{\lambda_0}]\right\|^2\right) \\ \leq \;& m2\beta^2\sum_{i=1}^{m}\left(|c_1\alpha_1^i|^2+\cdots+|c_m\alpha_m^i|^2\right)\leq 2m\beta^2.\end{aligned}$$

Therefore, (2.8) applied to this particular case says that $D^2 \leq 2m\beta^2$. This means that (2.9) can't happen for $\beta < D/\sqrt{2m}$.

Proof of (2). Now let $0<\eta<\beta$, where β is given by item (1) and suppose that for $1 \leq p \leq m$, we have

$$\lambda_{j_1},\ldots,\lambda_{j_p}\in\Delta(\lambda_{j_1},\eta).$$

As before, we write $j=1,\ldots,p$ instead of $j_1,\ldots,j_p$. Then,

$$\begin{aligned}\|\sum_{j=1}^{p}\alpha_j^i c_j k_{\lambda_j}\|^2 \;&=\; \left\|\Big[\sum_{j=1}^{p}c_j\alpha_j^i\Big]k_{\lambda_1}+c_1\alpha_1^i[k_{\lambda_2}-k_{\lambda_1}]+\cdots+c_p\alpha_p^i[k_{\lambda_p}-k_{\lambda_1}]\right\|^2 \\ \leq\; m&\left[\left\|\Big[\sum_{j=1}^{p}c_j\alpha_j^i\Big]k_{\lambda_1}\right\|^2+\|c_1\alpha_1^i[k_{\lambda_2}-k_{\lambda_1}]\|^2+\cdots+\|c_p\alpha_p^i[k_{\lambda_p}-k_{\lambda_1}]\|^2\right].\end{aligned}$$

So, if $c=(c_1,\ldots,c_p,0,\ldots)\in\ell^2$, by (2.8) applied to this case and (2.7),

$$\begin{aligned}D^2\sum_{j=1}^{p}|c_j|^2 \;&\leq\; \sum_{i=0}^{m}\|\sum_{j=1}^{p}\alpha_j^i c_j k_{\lambda_j}\|^2 \\ &\leq\; \sum_{i=0}^{m} m\left(\Big|\sum_{j=1}^{p}c_j\alpha_j^i\Big|^2+2\eta^2|c_1\alpha_1^i|^2+\cdots+2\eta^2|c_p\alpha_p^i|^2\right),\end{aligned}$$

which clearly implies that

$$D^2\left\|\begin{bmatrix}c_1\\ \vdots\\ c_p\end{bmatrix}\right\|_{\mathbb{C}^p}^2 \leq m\left\|\begin{bmatrix}\alpha_1^1 & \ldots & \alpha_p^1\\ \vdots & & \vdots\\ \alpha_1^m & \ldots & \alpha_p^m\end{bmatrix}\begin{bmatrix}c_1\\ \vdots\\ c_p\end{bmatrix}\right\|_{C^{m\times 1}}^2+2m\eta^2\left\|\begin{bmatrix}c_1\\ \vdots\\ c_p\end{bmatrix}\right\|_{\mathbb{C}^p}^2$$

for every $(c_1,\ldots,c_p)\in\mathbb{C}^p$. Thus, if Υ denotes the above matrix and $c\in\mathbb{C}^p$ is normalized, when $2m\eta^2<D^2$,

$$0<D_0^2=\frac{D^2-2m\eta^2}{m}\leq\|\Upsilon c\|^2.$$

□

2.2. **Differences of normalized reproducing kernels.** For $E \subset \mathbb{D}$ and $0 < r < 1$ we write $\Omega_r(E) := \{z \in \mathbb{D} : \rho(z, E) \leq r\}$.

Lemma 2.5. *Let $\mu_0 = \sum(1 - |\lambda_j|^2)\delta_{\lambda_j}$ be a Carleson measure, $0 < r < 1$ and $\lambda'_j \in \Delta(\lambda_j, r)$. Then $\mu_r = \sum(1 - |\lambda'_j|^2)\delta_{\lambda'_j}$ is a Carleson measure such that for some constant $C(r) \geq 1$,*

$$C(r)^{-1} \|\mu_r\|_* \leq \|\mu_0\|_* \leq C(r) \|\mu_r\|_*. \tag{2.10}$$

Proof. By [7, p. 3] any $z \in \Delta[\lambda_j, r]$ satisfies $|z| \leq \varphi_{|\lambda_j|}(-r)$. Then (1.3) implies that for any angular square $Q \subset \mathbb{D}$,

$$\mu_r(\Omega_r(Q)) \geq \sum_{\lambda_j \in Q} (1-|\varphi_{|\lambda_j|}(-r)|^2) = \sum_{\lambda_j \in Q} \frac{(1 - |\lambda_j|^2)(1 - r^2)}{(1 + |\lambda_j| r)^2} \geq \frac{1 - r}{1 + r} \mu_0(Q).$$

If Q_r is the smallest angular square containing $\Omega_r(Q)$, there is $c(r) \geq 1$ such that $\ell(Q_r) \leq c(r)\ell(Q)$. Hence,

$$\left[\frac{1 - r}{1 + r}\right] \frac{\mu_0(Q)}{c(r)\ell(Q)} \leq \frac{\mu_r(Q_r)}{\ell(Q_r)},$$

implying that

$$\|\mu_0\|_* \leq \frac{1 + r}{1 - r} c(r) \|\mu_r\|_*.$$

If $C(r)$ is the constant in the right side, the lemma follows by symmetry. □

In what follows if $\{f_j\}$ is a Bessel sequence for H^2, that is,

$$\sum_j |\langle f, f_j \rangle|^2 \leq B^2 \|f\|^2 \quad \forall f \in H^2,$$

we write $\mathcal{B}^2(f_j)$ for the smallest constant B^2.

Corollary 2.6. *Let $\mu_0 = \sum(1 - |\lambda_j|^2)\delta_{\lambda_j}$ be a Carleson measure, $0 < r < 1$ and $\lambda'_j \in \Delta[\lambda_j, r]$. Then $\{k_{\lambda'_j}\}$ is Bessel with $\mathcal{B}^2(k_{\lambda'_j}) \leq C(r, \|\mu_0\|_*)$.*

Proof. By (2.10) $\mu_r = \sum(1 - |\lambda'_j|^2)\delta_{\lambda'_j}$ is a Carleson measure with $\|\mu_r\|_*$ depending on r and $\|\mu_0\|_*$. Therefore, the comments preceding Lemma 2.1 say that there is $C_r \geq 0$ depending on $\|\mu_r\|_*$ such that

$$\int |f|^2 d\mu_r \leq C_r^2 \|f\|^2, \quad \text{for} \;\; f \in H^2.$$

Thus, $\{k_{\lambda'_j}\}$ is Bessel with $\mathcal{B}(k_{\lambda'_j}) \leq C_r^2$ by Theorem 2.3 with $m = 1$. □

If $\lambda_1, \ldots, \lambda_N \in \mathbb{D}$ and $\lambda'_j \in \Delta(\lambda_j, \eta)$ for some $0 < \eta < 1$, then (2.7) implies that the Bessel constant

$$\mathcal{B}^2(k_{\lambda_j} - k_{\lambda'_j}) \leq 2N\eta^2.$$

Next we see how to control this constant for infinitely many values of λ_j. Together with Theorem 2.4, this is the main result of the paper.

Theorem 2.7. *Let $\mu_0 = \sum(1-|\lambda_j|^2)\delta_{\lambda_j}$ be a Carleson measure, and $\lambda'_j \in \Delta(\lambda_j, \eta)$ for $0<\eta<1$. Then there is a constant $C>0$ depending only on $\{\lambda_j\}$ such that*

$$\mathcal{B}(k_{\lambda_i} - k_{\lambda'_i}) \leq C\eta.$$

Proof. Since by Lemma 2.1, $\{\lambda_i\}$ is a finite union of interpolating sequences, we can assume that it is interpolating. Let B be the Blaschke product with zeros λ_i, write $B_i = B/\phi_{\lambda_i}$ (i.e.: B with the factor ϕ_{λ_i} removed) and recall that $\delta(B) = \inf_i |B_i(\lambda_i)| > 0$. We prove first the result for $f \in K_B$, which by (1.2) can be written as

$$f = \sum_i c_i \frac{B_i}{B_i(\lambda_i)} k_{\lambda_i}, \quad \text{with } c \in \ell^2 \quad \text{and} \quad \|f\| \leq C_\delta \|c\|_{\ell^2}, \tag{2.11}$$

where $C_\delta > 0$ is a constant depending only on $\delta(B)$. So,

$$\begin{aligned}\langle f, k_{\lambda_j} - k_{\lambda'_j}\rangle &= \Big\langle c_j \frac{B_j}{B_j(\lambda_j)} k_{\lambda_j}, k_{\lambda_j} - k_{\lambda'_j}\Big\rangle + \Big\langle \sum_{i:i\neq j} c_i \frac{B_i}{B_i(\lambda_i)} k_{\lambda_i}, k_{\lambda_j} - k_{\lambda'_j}\Big\rangle \\ &= c_j\left(1 - \frac{B_j(\lambda'_j)}{B_j(\lambda_j)}\langle k_{\lambda_j}, k_{\lambda'_j}\rangle\right) - \Big\langle \sum_{i:i\neq j} c_i \frac{B_i}{B_i(\lambda_i)} k_{\lambda_i}, k_{\lambda'_j}\Big\rangle \\ &= D_j - R_j. \end{aligned} \tag{2.12}$$

To estimate $D_j = \dfrac{c_j}{B_j(\lambda_j)}\left[B_j(\lambda_j) - B_j(\lambda'_j) + B_j(\lambda'_j)(1 - \langle k_{\lambda_j}, k_{\lambda'_j}\rangle)\right]$ we notice that

$$|D_j| \leq \left|\frac{c_j}{B_j(\lambda_j)}\right| \left[2\rho(\lambda_j, \lambda'_j) + \sqrt{2}\rho(\lambda_j, \lambda'_j)\right] \leq |c_j|\, 4\frac{\rho(\lambda_j, \lambda'_j)}{\delta(B)}, \tag{2.13}$$

where the first inequality comes from $\rho(B_j(\lambda_j), B_j(\lambda'_j)) \leq \rho(\lambda_j, \lambda'_j)$ and (2.7).

To estimate R_j write $B_{i,j} = B/(\phi_{\lambda_i}\phi_{\lambda_j})$. For $0<r<1$ consider the analytic function $F_j : \Delta[\lambda_j, r] \to \mathbb{C}$ given by

$$F_j(\lambda') = \sum_{i:i\neq j} \frac{c_i}{B_i(\lambda_i)} \langle B_{i,j} k_{\lambda_i}, K_{\lambda'}\rangle,$$

where $K_\lambda(z) = (1-\overline{\lambda}z)^{-1}$ is the reproducing kernel for H^2. By the maximum modulus principle F_j attains its maximum on the boundary of $\Delta[\lambda_j, r]$. That is, there is $\lambda''_j \in \partial\Delta[\lambda_j, r]$ such that $\rho(\lambda_j, \lambda''_j) = r$ and $|F_j(\lambda')| \leq |F_j(\lambda''_j)|$ for every $\lambda' \in \Delta[\lambda_j, r]$. Since $\lambda' = \varphi_{\lambda_j}(w)$ with $0 \leq |w| \leq r$, a straightforward estimate from formula (1.3) gives

$$\left[\frac{1-r}{1+r}\right](1-|\lambda_j|^2) \leq (1-|\lambda'|^2) \leq \left[\frac{1+r}{1-r}\right](1-|\lambda_j|^2),$$

implying that

$$(1-|\lambda'|^2)|F_j(\lambda')|^2 \leq C_1(r)\,(1-|\lambda''_j|^2)|F_j(\lambda''_j)|^2 \tag{2.14}$$

for some constant $C_1(r) > 0$. Since $|\phi_{\lambda_j}(\lambda''_j)| = \rho(\lambda_j, \lambda''_j) = r$, then

$$\begin{aligned} r(1-|\lambda''_j|^2)^{\frac{1}{2}}|F_j(\lambda''_j)| &= \Big|\phi_{\lambda_j}(\lambda''_j)\sum_{i:i\neq j}\frac{c_i}{B_i(\lambda_i)}\,\langle B_{i,j}k_{\lambda_i}, k_{\lambda''_j}\rangle\Big| \\ &= \Big|\sum_{i:i\neq j}\frac{c_i}{B_i(\lambda_i)}\,\langle B_i k_{\lambda_i}, k_{\lambda''_j}\rangle\Big| \\ &\leq \Big|\sum_{i}\frac{c_i}{B_i(\lambda_i)}\,\langle B_i k_{\lambda_i}, k_{\lambda''_j}\rangle\Big| + \Big|\frac{c_j}{B_j(\lambda_j)}\,\langle B_j k_{\lambda_j}, k_{\lambda''_j}\rangle\Big|. \end{aligned}$$

Consequently, the Cauchy-Schwarz inequality gives

$$\begin{aligned} r^2\sum_j(1-|\lambda''_j|^2)|F_j(\lambda''_j)|^2 &\leq 2\sum_j|\langle f, k_{\lambda''_j}\rangle|^2 + 2\sum_j\Big|\frac{c_j}{B_j(\lambda_j)}\Big|^2 \\ &\leq 2\mathcal{B}(k_{\lambda''_j})^2\|f\|^2 + 2\delta(B)^{-2}\sum_j|c_j|^2 \\ &\leq C_2(r,\|\mu_0\|_*,\delta(B))\sum_j|c_j|^2, \end{aligned} \tag{2.15}$$

where the last inequality holds by (2.11) and because by Corollary 2.6 the Bessel constant $\mathcal{B}(k_{\lambda''_j})^2$ has a bound that depends only on r and $\|\mu_0\|_*$. So, if $\rho(\lambda_j, \lambda'_j) \leq \eta \leq r$, (2.14) and (2.15) give

$$\begin{aligned} \sum_j\Big|\sum_{i:i\neq j}\frac{c_i}{B_i(\lambda_i)}\,\langle B_i k_{\lambda_i}, k_{\lambda'_j}\rangle\Big|^2 &= \sum_j|\phi_{\lambda_j}(\lambda'_j)|^2\Big|\sum_{i:i\neq j}\frac{c_i}{B_i(\lambda_i)}\,\langle B_{i,j}k_{\lambda_i}, k_{\lambda'_j}\rangle\Big|^2 \\ &= \sum_j|\phi_{\lambda_j}(\lambda'_j)|^2(1-|\lambda'_j|^2)|F_j(\lambda'_j)|^2 \\ &\leq \eta^2\,C_3(r,\|\mu_0\|_*,\delta(B))\sum_j|c_j|^2. \end{aligned} \tag{2.16}$$

Inserting inequalities (2.13) and (2.16) in (2.12), and using Cauchy-Schwarz again, we obtain

$$\sum_j|\langle f, k_{\lambda_j} - k_{\lambda'_j}\rangle|^2 \leq \eta^2\,2\Big[\frac{4^2}{\delta(B)^2} + C_3(r,\|\mu_0\|_*,\delta(B))\Big]\sum_j|c_j|^2.$$

This proves the theorem for $f \in K_B$. A general $h \in H^2$ decomposes as $h = f + Bg$, where $f \in K_B$, $g \in H^2$ and $\|h\|^2 = \|f\|^2 + \|g\|^2$. Thus,

$$\langle f + Bg, k_{\lambda_j} - k_{\lambda'_j}\rangle = \langle f, k_{\lambda_j} - k_{\lambda'_j}\rangle - B(\lambda'_j)\langle g, k_{\lambda'_j}\rangle,$$

and since $|B(\lambda'_j)| = \rho(B(\lambda_j), B(\lambda'_j)) \leq \rho(\lambda_j, \lambda'_j) \leq \eta$,

$$\begin{aligned} \sum_j|\langle f + Bg, k_{\lambda_j} - k_{\lambda'_j}\rangle|^2 &\leq 2\sum_j|\langle f, k_{\lambda_j} - k_{\lambda'_j}\rangle|^2 + 2\eta^2\sum_j|\langle g, k_{\lambda'_j}\rangle|^2 \\ &\leq \eta^2 C(r,\|\mu_0\|_*,\delta(B))\,(\|f\|^2 + \|g\|^2), \end{aligned}$$

where the last inequality uses the result for $f \in K_B$ and, as before, Corollary 2.6. Since $0 < r < 1$ is arbitrary, we can fix it (for instance $r = 1/2$) and the constant in the last inequality depends only on $\|\mu_0\|_*$ and $\delta(B)$, as claimed. □

The proof of the last theorem reduces to the case of an interpolating Blaschke product B, and then it is proved that the constant C only depends on $\|\mu_0\|_*$ and $\delta(B)$. Since in this case $\|\mu_0\|_*$ is bounded depending on $\delta(B)$ (see [7, VII, Sec.1]), the constant C for this case depends only on $\delta(B)$.

3. Separation conditions

Definition. We say that a sequence (finite or not) S in $\mathbb{D}$ is m-separated (with radius $\geq \beta$) if every pseudo-hyperbolic ball $\Delta(z, \beta)$, with $z \in S$, has no more than m points of S including repetitions.

It is clear that if we take $0 < \beta_1 < \beta$ in the above definition then S is also m-separated with radius $\geq \beta_1$. Also, $(m-1)$-separated implies m-separated, and 1-separated simply means separated, as in (2) of Lemma 2.1.

Since the order of a sequence will not be relevant in what follows, we operate with them as if they were sets with pointwise multiplicities. So, for instance, the union of two sequences has the points of both with the sum of multiplicities and some order.

Lemma 3.1. *Let S be a m-separated sequence in $\mathbb{D}$ with radius $\geq \beta$. Then S splits into at most m separated sequences (finite or not) with radius $\geq \beta/4m$.*

Proof. Consider the balls $\Delta(z_n, \beta/4m)$ including repetitions of the $z_n \in S$. Let $U \subset \bigcup_{z_n \in S} \Delta(z_n, \beta/4m)$ be a connected component. Hence, U is a union of these balls and we show that there cannot be more than m of them (including repetitions). Otherwise,

$$U \supset \Delta(z_{n(1)}, \beta/4m) \cup \ldots \cup \Delta(z_{n(m+1)}, \beta/4m),$$

where the union of balls is connected and has pseudo-hyperbolic diameter $\leq 2(m+1)\beta/4m \leq \beta$. Consequently, $\Delta(z_{n(1)}, \beta)$ contains the points $z_{n(j)}$ for $j = 1, \ldots, m+1$, contradicting the hypothesis. Therefore

$$U = \Delta(z_{n(1)}, \beta/4m) \cup \ldots \cup \Delta(z_{n(k)}, \beta/4m), \quad \text{where } k \leq m.$$

If we accept the empty set and finite sequences then $S = \bigcup_{j=1}^{m} S_j$, where $S_j = \{z_{n(j)} \in U : U \text{ componente}\}$ for $1 \leq j \leq m$. □

Lemma 3.2. *Let S be a m-separated sequence in $\mathbb{D}$. Then there is a set $J \subset \{1, \ldots, m\}$ and parameters $0 < \eta_p < \gamma_p < 1$, for $p \in J$, such that S splits into subsequences $S_p\,(p \in J)$ (finite or infinite), so that when $p \in J$:*

(i) *There is a single multiplicity sequence $S'_p \subset S_p$ such that*

$$S_p = \bigcup \{\Delta(z'_n(p), \eta_p) \cap S : \; z'_n(p) \in S'_p\},$$

where each $\Delta(z'_n(p), \eta_p)$ has p points of S_p counting multiplicities.

(ii) *$\rho(z'_n(p), z'_k(p)) > \gamma_p$ if $n \neq k$.*
(iii) *$\rho(S_p, S_k) > (4/5)\gamma_p$ if $p > k$, and $p, k \in J$.*
(iv) *Once γ_p is obtained by reverse induction we can choose $0 < \eta_p < \gamma_p$ arbitrarily small, eventually lowering the index p until $S_p \neq \emptyset$.*

Proof. By hypothesis there is a radius $\beta_m > 0$ such that S is m-separated of radius $\geq \beta_m$. Hence, by the previous lemma, S splits into at most m separated sequences $T_1, \dots, T_m$ of separation $\geq \beta_m/4m := \gamma_m$ (some could be empty or finite). Chose any $0 < \eta_m < \gamma_m/10$, set

$$S'_m = \{z'_n \in T_m : \Delta(z'_n, \eta_m) \text{ has just } m \text{ points of } S \text{ counting repetitions}\},$$

and let S_m be the sequence of all the points of S in those balls. If $S_m = \emptyset$, we reindex the parameters γ_m and η_m as γ_{m-1} and η_{m-1}, respectively, declare that $m \notin J$ and keep the process with $m-1$ instead of m. Notice that by definition, $S_m = \emptyset$ iff $S'_m = \emptyset$, which holds if $T_m = \emptyset$.

If $S_m \neq \emptyset$, we keep $m \in J$ and notice that each of the balls has one and only one point of each T_q $(1 \leq q \leq m)$, because the distance between two different points in T_q is $\geq \gamma_m$. Thus,

$$\rho(z'_n, z'_k) > \gamma_m \ \text{ if } n \neq k, \ \text{ and } \ \rho(S_m, S \setminus S_m) > \gamma_m - 2\eta_m \geq \frac{4}{5}\gamma_m. \tag{3.17}$$

If the remaining $S \setminus S_m$ is empty we are done. Otherwise it is a $(m-1)$-separated sequence (with radius $\geq \eta_m/2$). Indeed, if there is $z \in S \setminus S_m$ such that $\Delta(z, \eta_m/2)$ has at least m points of $S \setminus S_m$ counting multiplicities, then this ball has at least one point z' of T_m, and consequently $\Delta(z', \eta_m)$ has at least m points of S. In addition, since $\eta_m < \gamma_m/10 < \beta_m$, it cannot have more than m points of S. Thus, $\Delta(z', \eta_m)$ has exactly m points of S implying that all the points of S in $\Delta(z', \eta_m)$ are in S_m, a contradiction.

Therefore, we can repeat the process above with $S \setminus S_m$ instead of S, $m-1$ instead of m and $\beta_{m-1} := \eta_m/2$. So, again there is $\gamma_{m-1} > 0$ analogously defined and we can choose $0 < \eta_{m-1} < \gamma_{m-1}/10$, otherwise arbitrary, to define analogous S'_{m-1} and S_{m-1} as before, just observing that they could be empty, in which case $m-1 \notin J$ and we lower the index from $m-1$ to $m-2$.

We keep this process going until we exhausts all the points of S. Since the construction repeats condition (3.17) for each $p \in J$, we get that if $p \in J$,

$$\rho(z'_n, z'_k) > \gamma_p \ \text{ for } z'_n, z'_k \in S'_p \ \text{ with } n \neq k$$

and

$$\rho(S_p, S \setminus \bigcup\{S_q : q \in J,\, q \geq p\}) > \gamma_p - 2\eta_p \geq \frac{4}{5}\gamma_p.$$

Thus, if $p, k \in J$, with $p > k$, then $S_k \subset S \setminus \bigcup\{S_q : q \in J,\, q \geq p\}$, and consequently

$$\rho(S_p, S_k) \geq \frac{4}{5}\gamma_p.$$

Therefore (ii) and (iii) hold. Also, (i) and (iv) hold by construction. □

Suppose that $S = \{\lambda_j\}$ is m-separated and that $\sum(1 - |\lambda_j|^2)\delta_{\lambda_j}$ is a Carleson measure. By Lemma 3.1 and Lemma 2.1, S is the union of at most m interpolating sequences. More importantly for our purpose, each $S_p = \{\lambda_j(p) : j \geq 1\}$ of the decomposition given by Lemma 3.2 is a finite union of interpolating sequences. Let B_p be the Blaschke product whose zeros are S_p counting multiplicities, or $B_p \equiv 1$ if $S_p = \emptyset$. It is then known (see [8]) that

$$\inf\{|B(z)| : \rho(z, S_p) \geq \beta\} > 0 \quad \text{for any } \beta > 0. \tag{3.18}$$

That is, B_p is bounded below away from zero at any fixed positive distance of its zeros, which fails for arbitrary Blaschke products. Now define

$$A_q := \prod_{p=1,\, p\neq q}^{m} B_p \quad \text{and} \quad f_q^i = \sum_j c_j(q)\alpha_j^i(q) k_{\lambda_j(q)} \in K_{B_q},$$

for $1 \leq i, q \leq m$ and $\{c_j(q)\}_j \in \ell^2$, where $|\alpha_j^1(q)|^2 + \cdots + |\alpha_j^m(q)|^2 = 1$ for all $1 \leq q \leq m$ and $j \geq 1$. Also, notice that $f_q^i = 0 = c_j(q)$ if $B_q \equiv 1$.

Since the zeros of A_q are $\bigcup_{p=1,\, p\neq q}^{m} S_p$ and by (iii) of Lemma 3.2,

$$\rho\Big(\bigcup_{p=1,\, p\neq q}^{m} S_p, S_q\Big) > 0,$$

it follows from (3.18) that

$$0 < \varepsilon_q = \inf\{|A_q(\lambda_j(q))| : j \geq 1\}.$$

We also need an elementary fact about Toeplitz operators. If $g \in H^\infty(\mathbb{D})$, it is easy to prove that the normalized reproducing kernels for H^2 are eigenvalues of the Toeplitz operator $T_{\overline{g}}$ such that $T_{\overline{g}} k_\lambda = \overline{g(\lambda)} k_\lambda$.

The next proposition will allow us to reduce the proof of sufficiency of Theorem 2.4 to a particular case. We keep the above notations.

Proposition 3.3. *If*

$$\sum_{i=1}^{m} \|f_q^i\|^2 \geq D_q^2 \sum_j |c_j(q)|^2 \tag{3.19}$$

for all $1 \leq q \leq m$ *and* $\{c_j(q)\}_j \in \ell^2$, *then*

$$m \sum_{i=1}^{m} \|f_1^i + \cdots + f_m^i\|^2 \geq \min_{1\leq q\leq m}\{D_q^2 \varepsilon_q^2\} \sum_{q=1}^{m} \sum_j |c_j(q)|^2.$$

Proof. Since $T_{\overline{A_q}}$ is a contraction on H^2, for each $1 \leq i \leq m$,

$$\begin{aligned} \|f_1^i + \cdots + f_m^i\|^2 &\geq \|T_{\overline{A_q}}(f_1^i + \cdots + f_m^i)\|^2 = \|T_{\overline{A_q}} f_q^i\|^2 \\ &= \|\sum_j c_j(q)\alpha_j^i(q)\overline{A_q(\lambda_j(q))} k_{\lambda_j(q)}\|^2 \end{aligned}$$

for $1 \le q \le m$. So, by (3.19),

$$\sum_{i=1}^{m} \|f_1^i + \cdots + f_m^i\|^2 \ge D_q^2 \sum_j |c_j(q)|^2 \, |A_q(\lambda_j(q))|^2 \ge D_q^2 \sum_j |c_j(q)|^2 \varepsilon_q^2.$$

Adding these inequalities for $1 \le q \le m$, we obtain

$$m \sum_{i=1}^{m} \|f_1^i + \cdots + f_m^i\|^2 \ge \sum_{q=1}^{m} D_q^2 \varepsilon_q^2 \sum_j |c_j(q)|^2 \ge \min_{1 \le q \le m} \{D_q^2 \varepsilon_q^2\} \sum_{q=1}^{m} \sum_j |c_j(q)|^2.$$

□

3.1. **The sufficiency of Theorem 2.4.** Finally, the proof of Theorem 2.4 needs the following elementary inequality.

Lemma 3.4. *If $x, y_1, \ldots, y_p \in H^2$, with $p \le m$, then*

$$\Big\| x + \sum_{k=1}^{p} y_k \Big\|^2 \ge \frac{\|x\|^2}{2} - m \sum_{k=1}^{p} \|y_k\|^2. \tag{3.20}$$

Proof. This follows using Cauchy-Schwarz's inequality twice. For $y \in H^2$, $\|x\|^2 \le 2\|x+y\|^2 + 2\|y\|^2$, so $\|x+y\|^2 \ge \|x\|^2/2 - \|y\|^2$. Then,

$$\Big\| x + \sum_{k=1}^{p} y_k \Big\|^2 \ge \frac{\|x\|^2}{2} - \Big\| \sum_{k=1}^{p} y_k \Big\|^2 \ge \frac{\|x\|^2}{2} - p \sum_{k=1}^{p} \|y_k\|^2.$$

□

Proof of sufficiency for Theorem 2.4. Since by (1) of the theorem, $S = \{\lambda_j\}$ is m-separated and $\mu = \sum_i (1 - |\lambda_i|^2)\delta_{\lambda_i}$ is a Carleson measure, Lemma 3.2 and the comments that follow the lemma apply to S. Therefore, we have the decomposition of the lemma

$$S = \bigcup_{p \in J} S_p, \quad \text{with } J \subset \{1, \ldots, m\},$$

where by (iv) we can choose $\eta_p < \eta$ for all $p \in J$ (here η is the parameter that appears in (2) of Theorem 2.4). This guarantees that the sequences S_p satisfy (2) of the theorem for each of the balls appearing in (i) of Lemma 3.2.

Furthermore, Proposition 3.3 reduces the problem of proving (2.8) for the sequence S to prove it for each sequence S_p from the above decomposition. Therefore we fix an arbitrary p between 1 and m. The subsequence S'_p in (i) of Lemma 3.2 is separated with radius of separation $\ge \gamma_p$, so Lemma 2.1 says that it is itself interpolating.

We re-index S_p as follows: write $S'_p = \{\lambda_j(1) : j \ge 1\}$ and $\{\lambda_j(\nu) : 1 \le \nu \le p\}$ for the p elements of S_p that are in $\Delta(\lambda_j(1), \eta_p)$. Therefore

$$S_p = \{\lambda_j(\nu) : 1 \le \nu \le p, \ 1 \le j\},$$

and the respective parameters of Theorem 2.4 write as $c_j(\nu)$ and $\alpha_j^i(\nu)$, for $1 \leq \nu \leq p$, $1 \leq j$ and $1 \leq i \leq m$. We see that for each $1 \leq i \leq m$,

$$\begin{aligned}
&\Big\| \sum_{j\geq 1}\sum_{\nu=1}^{p} c_j(\nu)\alpha_j^i(\nu) k_{\lambda_j(\nu)}\Big\|^2 = \\
&\qquad = \Big\| \sum_{j\geq 1}\Big[\Big(\sum_{\nu=1}^{p} c_j(\nu)\alpha_j^i(\nu)\Big) k_{\lambda_j(1)} + \sum_{\nu=2}^{p} c_j(\nu)\alpha_j^i(\nu)\left(k_{\lambda_j(\nu)} - k_{\lambda_j(1)}\right)\Big]\Big\|^2 \\
&\text{(by (3.20))} \qquad \geq \frac{1}{2}\Big\| \sum_{j\geq 1}\Big(\sum_{\nu=1}^{p} c_j(\nu)\alpha_j^i(\nu)\Big) k_{\lambda_j(1)}\Big\|^2 - \\
&\qquad - m\sum_{\nu=2}^{p}\Big\| \sum_{j\geq 1} c_j(\nu)\alpha_j^i(\nu)\left(k_{\lambda_j(\nu)} - k_{\lambda_j(1)}\right)\Big\|^2. \qquad (3.21)
\end{aligned}$$

If B is the Blaschke product with zeros $\{\lambda_j(1)\}_j$, Lemma 2.1 says that $\delta(B)$ is estimated depending on γ_p and the measure μ associated to S. Thus, the comments that follow (1.2) imply that $\{k_{\lambda_j(1)} : j \geq 1\}$ is a Riesz sequence with lower constant D_1^2 independent of η_p (only depends on γ_p and μ). Hence,

$$\Big\| \sum_{j\geq 1}\big(\sum_{\nu=1}^{p} c_j(\nu)\alpha_j^i(\nu)\big) k_{\lambda_j(1)}\Big\|^2 \geq D_1^2 \sum_{j\geq 1}\Big|\sum_{\nu=1}^{p} c_j(\nu)\alpha_j^i(\nu)\Big|^2,$$

and having into account that with the new indexation, the last condition of the theorem rewrites as

$$D_0^2 \left\| \begin{bmatrix} c_j(1) \\ \vdots \\ c_j(p) \end{bmatrix} \right\|_{\mathbb{C}^p}^2 \leq \left\| \begin{bmatrix} \alpha_j^1(1) & \dots & \alpha_j^1(p) \\ \vdots & & \vdots \\ \alpha_j^m(1) & \dots & \alpha_j^m(p) \end{bmatrix} \begin{bmatrix} c_j(1) \\ \vdots \\ c_j(p) \end{bmatrix} \right\|_{C^{m\times 1}}^2, \qquad \forall \begin{bmatrix} c_j(1) \\ \vdots \\ c_j(p) \end{bmatrix} \in \mathbb{C}^p,$$

we obtain

$$\sum_{i=1}^{m}\Big\| \sum_{j\geq 1}\big(\sum_{\nu=1}^{p} c_j(\nu)\alpha_j^i(\nu)\big) k_{\lambda_j(1)}\Big\|^2 \geq D_1^2 D_0^2 \sum_{j\geq 1}\sum_{\nu=1}^{p}\left|c_j(\nu)\right|^2. \qquad (3.22)$$

Observe that, as with D_1, the constant D_0 does not depend on η_p, since only depends on the η in condition (2) of Theorem 2.4.

To estimate the second term of (3.21) we notice that by Theorem 2.7, there is a constant $C_S > 0$ depending only on the original sequence S (and then independent of η_p) such that

$$\begin{aligned}
\Big\| \sum_{j\geq 1} c_j(\nu)\alpha_j^i(\nu)\left(k_{\lambda_j(\nu)} - k_{\lambda_j(1)}\right)\Big\|^2 &\leq \mathcal{B}(k_{\lambda_j(\nu)} - k_{\lambda_j(1)})^2 \sum_{j\geq 1}\left|c_j(\nu)\alpha_j^i(\nu)\right|^2 \\
&\leq C_S^2\eta_p^2 \sum_{j\geq 1}\left|c_j(\nu)\alpha_j^i(\nu)\right|^2.
\end{aligned}$$

for all $2 \leq \nu \leq p$ and $1 \leq i \leq m$. Thus

$$\sum_{i=1}^{m}\sum_{\nu=2}^{p} \Big\| \sum_{j\geq 1} c_j(\nu)\alpha_j^i(\nu)\left(k_{\lambda_j(\nu)} - k_{\lambda_j(1)}\right)\Big\|^2 \leq C_S^2\eta_p^2 \sum_{\nu=2}^{p}\sum_{j\geq 1}\big|c_j(\nu)\big|^2. \tag{3.23}$$

If we insert (3.22) and (3.23) into (3.21), we get

$$\begin{aligned}
&\sum_{i=1}^{m} \Big\| \sum_{j\geq 1}\sum_{\nu=1}^{p} c_j(\nu)\alpha_j^i(\nu)k_{\lambda_j(\nu)}\Big\|^2 \geq \\
&\geq \frac{1}{2}D_1^2D_0^2\sum_{j\geq 1}\sum_{\nu=1}^{p}\big|c_j(\nu)\big|^2 - mC_S\eta_p^2\sum_{\nu=2}^{p}\sum_{j\geq 1}\big|c_j(\nu)\big|^2 \\
&\geq \left(\frac{1}{2}D_1^2D_0^2 - mC_S\eta_p^2\right)\sum_{j\geq 1}\sum_{\nu=1}^{p}\big|c_j(\nu)\big|^2.
\end{aligned}$$

We can take η_p small enough for the expression between brackets to be $\geq (1/4)D_1^2D_0^2$, which proves the theorem. □

We go back to our original problem. By [1, Thm. 5.6] any normal operator N on a Hilbert space that admits a finite union of orbits as a frame is diagonalizable. Therefore N is unitarily equivalent to an operator A as in:

Theorem 3.5. *Let A be a diagonal operator with respect to the standard base in $\ell^2(J)$, where $J = \mathbb{N}$ or it is finite, and let $a^1, \dots, a^m \in \ell^2(J)$. Then $\{A^n a^i : n \in \mathbb{N}_0,\ 1 \leq i \leq m\}$ is a frame if and only if*

- *Each a^i is given by (2.4), where $i = 1, \dots, m$.*
- *The sequence of eigenvalues $\{\lambda_j : j \in J\}$ of A and the double sequence $\{\alpha_j^i : j \in J,\ 1 \leq i \leq m\}$ appearing in (2.4) satisfy Theorem 2.4.*

We finish the paper with a comment on the tails of the orbits. Keep the previous notations and conventions. If $n_0 \geq 1$ is an integer, the calculation leading to (2.5) gives

$$\sum_{n\geq n_0} |\langle A^n\tilde{a}^i, c\rangle|^2 = \|\sum_j \alpha_j^i c_j \overline{\lambda}_j^{n_0} k_{\lambda_j}\|^2 \quad \text{for } 1 \leq i \leq m.$$

Therefore, if (2.8) holds and we write $\mathcal{J} = \{j : \lambda_j \neq 0\}$,

$$\sum_{i=1}^{m}\sum_{n\geq n_0} |\langle A^n\tilde{a}^i, c\rangle|^2 \geq D^2 \sum_j |\overline{\lambda}_j^{n_0} c_j|^2 \geq D^2 \min_{j\in\mathcal{J}}\{|\lambda_j|^{n_0}\} \sum_{j\in\mathcal{J}} |c_j|^2$$

$\forall c \in \ell^2$, meaning that $\{A^n\tilde{a}^i : n \geq n_0,\ 1 \leq i \leq m\}$ is a frame for $(\operatorname{Ker} A)^{\perp}$.

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