$C^{1,\alpha}$ REGULARITY FOR THE FAR FIELD AND NEAR FIELD REFRACTOR PROBLEM

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Abstract. For index of refraction $\kappa < 1$, we prove $C^{1+\alpha}$ regularity for the Far Field Refractor and for the Near Field Refractor using Loepers method.

1. $C^{1,\alpha}$ for the Far Field Refractor

Let $\langle \bar{m}, x_0\rangle \geq \kappa$ and $\langle \hat{m}, x_0\rangle \geq \kappa$, and let $m_\lambda = (1-\lambda)\bar{m} + \lambda\hat{m}$, with $0 \leq \lambda \leq 1$. We parametrize the segment $[\bar{m}, \hat{m}]_{x_0}$ which is the intersection of the triangle with vertices $\bar{m}$, $\hat{m}$, x_0/κ with the sphere S^{n-1}. A point $m \in [\bar{m}, \hat{m}]_{x_0}$ can be obtained as the intersection of the line $x_0/\kappa + \beta\,\xi$ where $\xi = m_\lambda - \dfrac{1}{\kappa}x_0$, $\beta \in \mathbb{R}$, with the sphere S^{n-1}. Solving for β yields $\beta(\lambda) = \dfrac{-\langle x_0, \xi\rangle - \sqrt{\langle x_0, \xi\rangle^2 - (1-\kappa^2)\,|\xi|^2}}{\kappa|\xi|^2}$. Therefore, we obtain the parametrization

$$[\bar{m}, \hat{m}]_{x_0} = \left\{ m(\lambda) = \frac{1}{\kappa}x_0 + \beta(\lambda)\left(m_\lambda - \frac{1}{\kappa}x_0\right),\ \lambda \in [0,1] \right\}. \tag{1.1}$$

Notice that if $m \in [\bar{m}, \hat{m}]_{x_0}$, then we can write

$$m = \frac{1}{\kappa}x_0 + s\left(\bar{m} - \frac{1}{\kappa}x_0\right) + t\left(\hat{m} - \frac{1}{\kappa}x_0\right)$$

with $s, t \geq 0$ and $s + t \leq 1$; $s = (1-\lambda)\beta(\lambda)$, $t = \lambda\,\beta(\lambda)$.

Lemma 1.1. *Let $\bar{m}, \hat{m} \in \Omega^\star$ and $x_0 \in \Omega$, $(\Omega \cdot \Omega^\star \geq \kappa)$. Then for $m(\lambda) \in [\bar{m}, \hat{m}]_{x_0}$ and for all $x \in \Omega$,*

$$\max\left\{ \frac{1-\kappa\langle x_0, \bar{m}\rangle}{1-\kappa\langle x, \bar{m}\rangle}; \frac{1-\kappa\langle x_0, \hat{m}\rangle}{1-\kappa\langle x, \hat{m}\rangle} \right\} \geq \frac{1-\kappa\langle x_0, m(\lambda)\rangle}{1-\kappa\langle x, m(\lambda)\rangle} + C\lambda(1-\lambda)|x-x_0|^2|\bar{m}-\hat{m}|^2$$

where C depends only on κ.

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Proof. Assume

$$\frac{1-\kappa\langle x_0,\bar m\rangle}{1-\kappa\langle x,\bar m\rangle}\geq\frac{1-\kappa\langle x_0,\hat m\rangle}{1-\kappa\langle x,\hat m\rangle}, \tag{1.2}$$

and let $m=m(\lambda)\in[\bar m,\hat m]_{x_0}$. We will show

$$\frac{1-\kappa\langle x_0,\bar m\rangle}{1-\kappa\langle x,\bar m\rangle}\geq\frac{1-\kappa\langle x_0,m\rangle}{1-\kappa\langle x,m\rangle}+C\lambda(1-\lambda)|x-x_0|^2|\bar m-\hat m|^2.$$

A calculation shows that

$$\frac{1-\kappa\langle x_0,\bar m\rangle}{1-\kappa\langle x,\bar m\rangle}-\frac{1-\kappa\langle x_0,m\rangle}{1-\kappa\langle x,m\rangle}=\frac{\kappa\,\langle x-x_0,\bar m(1-\kappa\langle x_0,m\rangle)-m(1-\kappa\langle x_0,\bar m\rangle)\rangle}{(1-\kappa\langle x,\bar m\rangle)(1-\kappa\langle x,m\rangle)}$$

Since $m=\frac{1}{\kappa}x_0+s(\bar m-\frac{1}{\kappa}x_0)+t(\hat m-\frac{1}{\kappa}x_0)$ with $s=(1-\lambda)\beta(\lambda)$ and $t=\lambda\beta(\lambda)$, we obtain that $1-\kappa\langle x_0,m(\lambda)\rangle=s(1-\kappa\langle\bar m,x_0\rangle)+t(1-\kappa\langle\hat m,x_0\rangle)$. Hence

$$\begin{aligned}&\langle x-x_0,\bar m(1-\kappa\langle x_0,m\rangle)-m(1-\kappa\langle x_0,\bar m\rangle)\rangle\\&=\langle x-x_0,\bar m\,(s(1-\kappa x_0\cdot\bar m)+t(1-\kappa x_0\cdot\hat m))-\left(\frac{1}{\kappa}x_0+s(\bar m-\frac{1}{\kappa}x_0)+t(\hat m-\frac{1}{\kappa}x_0)\right)(1-\kappa\langle x_0,\bar m\rangle)\rangle\\&=\langle x-x_0,t(\bar m(1-\kappa\langle x_0,\hat m\rangle)-\hat m(1-\kappa\langle x_0,\bar m\rangle))\rangle+\frac{1}{\kappa}\langle x-x_0,x_0\rangle(s+t-1)(1-\kappa\langle x_0,\bar m\rangle).\end{aligned}$$

From (1.2), $\langle x-x_0,\bar m(1-\kappa\langle x_0,\hat m\rangle)-\hat m(1-\kappa\langle x_0,\bar m\rangle)\rangle\geq 0$ and since $|x|,|x_0|=1$, we have $-2\langle x-x_0,x_0\rangle=|x-x_0|^2$, and so

$$\begin{aligned}\frac{1-\kappa\langle x_0,\bar m\rangle}{1-\kappa\langle x,\bar m\rangle}-\frac{1-\kappa\langle x_0,m\rangle}{1-\kappa\langle x,m\rangle}&\geq\frac{1}{2\kappa}\,(1-(s+t))\,|x-x_0|^2\,\frac{1-\kappa\langle x_0,\bar m\rangle}{(1-\kappa\langle x,\bar m\rangle)(1-\kappa\langle x,m\rangle)}\\&\geq C_\kappa(1-(s+t))\,|x-x_0|^2.\end{aligned}$$

To complete the proof of the desired estimate, we shall prove that $1-(s+t)\geq C'_\kappa\,\lambda\,(1-\lambda)\,|\bar m-\hat m|^2$. In fact, notice that $s+t=\beta(\lambda)$ and

$$\begin{aligned}1-\beta(\lambda)&=\frac{\kappa|\xi|^2+\langle x_0,\xi\rangle+\sqrt{\langle x_0,\xi\rangle^2-(1-\kappa^2)|\xi|^2}}{\kappa|\xi|^2}\\&=\frac{\left(\kappa|\xi|^2+\langle x_0,\xi\rangle\right)^2-(\langle x_0,\xi\rangle^2-(1-\kappa^2)|\xi|^2)}{\kappa|\xi|^2\left(\kappa|\xi|^2+\langle x_0,\xi\rangle-\sqrt{\langle x_0,\xi\rangle^2-(1-\kappa^2)|\xi|^2}\right)}.\end{aligned}$$

Next, we have $(\kappa|\xi|^2+\langle x_0,\xi\rangle)^2-(\langle x_0,\xi\rangle^2-(1-\kappa^2)|\xi|^2)=|\xi|^2(\kappa^2|\xi|^2+2\kappa\langle x_0,\xi\rangle+1-\kappa^2)=|\xi|^2(|\kappa\xi+x_0|^2-\kappa^2)=|\xi|^2\kappa^2(|m_\lambda|^2-1)$. Therefore

$$1-\beta(\lambda)=\frac{\kappa(1-|m_\lambda|^2)}{-\kappa|\xi|^2-\langle x_0,\xi\rangle+\sqrt{\langle x_0,\xi\rangle^2-(1-\kappa^2)|\xi|^2}}.$$

Since $1-\beta(\lambda) > 0$ and $|m_\lambda| < 1$, for $0 < \lambda < 1$, it follows that $-\kappa|\xi|^2 - \langle x_0, \xi\rangle + \sqrt{\langle x_0,\xi\rangle^2 - (1-\kappa^2)|\xi|^2)} > 0$ and since $|\xi| \leq 1 + (1/\kappa)$, it is bounded above by a constant depending only on κ.

Finally, since $1-|m_\lambda|^2 = \lambda\,(1-\lambda)\,|\bar{m}-\hat{m}|^2$, the proof of the lemma is complete.

□

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Lemma 1.2. *There exists a constant M_κ depending only on κ such that for all x, y with $|x|, |y| \leq 1$ and for all $|m| = 1$ we have*

$$\begin{aligned} &- M_\kappa|y-x|^2 + \langle \frac{\kappa m}{(1-\kappa\langle x,m\rangle)^2}, y-x\rangle + \frac{1}{1-\kappa\langle x,m\rangle} \\ &\leq \frac{1}{1-\kappa\langle y,m\rangle} \\ &\leq \frac{1}{1-\kappa\langle x,m\rangle} + \langle \frac{\kappa m}{(1-\kappa\langle x,m\rangle)^2}, y-x\rangle + M_\kappa|y-x|^2. \end{aligned}$$

Set $q(x,m) := \dfrac{\kappa m}{(1-\kappa\langle x,m\rangle)^2}$.

Proof. It follows from Taylor's formula for $\dfrac{1}{1-\kappa\langle y,m\rangle}$ about x. □

The following is the main lemma of this section.

Lemma 1.3. *Let $B_{2\delta}$ be a geodesic ball in S^{n-1} with $B_{2\delta} \subseteq \Omega$. Suppose $x_i \in B_\delta$, $m_i \in N_\rho(x_i)$, $i = 1, 2$, are such that ($m_i \in \Omega^\star$) $|m_1 - m_2| \geq |x_1 - x_2|$. Then there exists $x_0 \in [x_1, x_2]$, the geodesic segment in S^{n-1} joining x_1, x_2 and contained in B_δ, such that*

$$\rho(x) \geq \frac{\rho(x_0)(1-\kappa\langle x_0, m(\lambda)\rangle)}{1-\kappa\langle x, m(\lambda)\rangle} + C_1\lambda(1-\lambda)|x-x_0|^2\,|m_1-m_2|^2 - C_2|x_1-x_2|\,|m_1-m_2| \tag{1.3}$$

for all $m(\lambda) \in [m_1, m_2]_{x_0}$ and for all $x \in \Omega$, with C_1 and C_2 positive constants depending only on κ and Λ.

Proof. We have $\rho(x) \geq \dfrac{\rho(x_i)\,(1-\kappa\langle x_i, m_i\rangle)}{1-\kappa\langle x, m_i\rangle}$ for all $x \in \Omega$, $i = 1, 2$. It follows by continuity that there exists $x_0 \in [x_1, x_2]$ such that

$$\frac{\rho(x_1)(1-\kappa\langle x_1, m_1\rangle}{1-\kappa\langle x_0, m_1\rangle} = \frac{\rho(x_2)(1-\kappa\langle x_2, m_2\rangle}{1-\kappa\langle x_0, m_2\rangle} := a_0.$$

Hence

$$\rho(x) \geq \max_{i=1,2}\left\{\frac{a_0(1-\kappa\langle x_0, m_i\rangle)}{1-\kappa\langle x, m_i\rangle}\right\} \tag{1.4}$$

for all $x \in \Omega$. Note in particular that $\rho(x_0) \geq a_0$.

We will prove the estimate

$$0 \leq \rho(x_0) - a_0 \leq C\,|x_1 - x_2|\,|m_1 - m_2|, \tag{1.5}$$

with C depending only on κ and Λ. Let $m_0 \in N_\rho(x_0)$, so

$$\begin{aligned}
\rho(x) &\geq \frac{\rho(x_0)(1-\kappa\langle x_0, m_0\rangle)}{1-\kappa\langle x, m_0\rangle} \\
&\geq \rho(x_0)(1-\kappa\langle x_0, m_0\rangle)\left(\frac{1}{1-\kappa\langle x_0, m_0\rangle} + \langle q(x_0, m_0), x - x_0\rangle - M_\kappa |x - x_0|^2\right) \\
&= \rho(x_0) + \rho(x_0)(1-\kappa\langle x_0, m_0\rangle)\left(\langle q(x_0, m_0), x - x_0\rangle - M_\kappa |x - x_0|^2\right) \qquad (1.6)
\end{aligned}$$

where we have used Lemma 1.2 about x_0.

For $0 \leq \mu \leq 1$, set $x_\mu = (1-\mu)\,x_1 + \mu\,x_2$. Since $x \cdot m \geq \kappa$ for all $x \in \Omega$ and $m \in \Omega^*$, $\kappa \leq x_\mu \cdot m \leq |x_\mu|$. Since $x_0 \in [x_1, x_2]$, there exists $0 \leq \mu \leq 1$ such that $x_0 = \dfrac{x_\mu}{|x_\mu|}$.

Pluging in $x = x_1$ in (1.6) and multiplying by $1 - \mu$, next pluging in $x = x_2$ in (1.6) and multiplying by μ, by adding the resulting inequalities and moving terms around we obtain

$$\begin{aligned}
\rho(x_0) &\leq (1-\mu)\rho(x_1) + \mu\rho(x_2) - \rho(x_0)(1-\kappa\langle x_0, m_0\rangle)\langle q(x_0, m_0), x_\mu - x_0\rangle \\
&\quad + M_\kappa \rho(x_0)(1-\kappa\langle x_0, m_0\rangle)\left((1-\mu)|x_1 - x_0|^2 + \mu|x_2 - x_0|^2\right). \qquad (1.7)
\end{aligned}$$

By direct computation

$$x_1 - x_0 = \frac{(|x_\mu| - 1)\,x_1 - \mu(x_2 - x_1)}{|x_\mu|} \tag{1.8}$$

$$x_2 - x_0 = \frac{(|x_\mu| - 1)\,x_2 + (1-\mu)(x_2 - x_1)}{|x_\mu|}. \tag{1.9}$$

Consequently $x_\mu - x_0 = \dfrac{(|x_\mu| - 1)x_\mu}{|x_\mu|}$. In addition, $(1-\mu)\,|x_1 - x_0|^2 + \mu\,|x_2 - x_0|^2 = 2\,(1 - |x_\mu|)$, and $0 \leq 1 - |x_\mu| \leq \dfrac{1 - |x_\mu|^2}{1+\kappa} = \dfrac{1}{1+\kappa}\,|x_1 - x_2|^2$. Using these estimates in (1.7) yields

$$\rho(x_0) \leq (1-\mu)\,\rho(x_1) + \mu\,\rho(x_2) + C\,|x_1 - x_2|^2 \tag{1.10}$$

with C depending only on κ and Λ.

On the other hand, from Lemma 1.2

$$\frac{1}{1-\kappa\langle x_0,m_1\rangle}\geq\frac{1}{1-\kappa\langle x_1,m_1\rangle}+\langle q(x_1,m_1),x_0-x_1\rangle-M_\kappa|x_0-x_1|^2,$$

so

$$\frac{1}{1-\kappa\langle x_1,m_1\rangle}\leq\frac{1}{1-\kappa\langle x_0,m_1\rangle}+\langle q(x_1,m_1),x_1-x_0\rangle+M_\kappa|x_0-x_1|^2.$$

Therefore,

$$\rho(x_i)=\frac{a_0\,(1-\kappa\langle x_0,m_i\rangle)}{1-\kappa\langle x_i,m_i\rangle}\leq a_0+a_0(1-\kappa\langle x_0,m_i\rangle)\langle q(x_i,m_i),x_1-x_0\rangle+a_0(1-\kappa\langle x_0,m_i\rangle)M_\kappa|x_0-x_i|^2,$$

for $i=1,2$. Multiplying the last inequality when $i=1$ by $1-\mu$, multiplying the last inequality when $i=2$ by μ, and adding them up yields

$$\begin{aligned}
&(1-\mu)\,\rho(x_1)+\mu\,\rho(x_2)\\
&\leq a_0\\
&\quad+a_0\left[(1-\kappa\langle x_0,m_1\rangle)\langle q(x_1,m_1),(1-\mu)(x_1-x_0)\rangle+(1-\kappa\langle x_0,m_2\rangle)\langle q(x_2,m_2),\mu(x_2-x_0)\rangle\right]\\
&\quad+a_0\,M_\kappa\left((1-\kappa\langle x_0,m_1\rangle)(1-\mu)|x_1-x_0|^2+(1-\kappa\langle x_0,m_2\rangle)\mu|x_2-x_0|^2\right)\\
&\quad=a_0+a_0\,K+L.
\end{aligned}$$

We have $|L|\leq C|x_1-x_2|^2$ with C depending only on κ and Λ. From (1.8) and (1.9)

$$\begin{aligned}
K&=\frac{\mu(1-\mu)}{|x_\mu|}\,\langle x_2-x_1,(1-\kappa\langle x_0,m_2\rangle)q(x_2,m_2)-(1-\kappa\langle x_0,m_1\rangle)q(x_1,m_1)\rangle\\
&\quad+\frac{|x_\mu|-1}{|x_\mu|}\left[(1-\kappa\langle x_0,m_1\rangle)(1-\mu)\langle q(x_1,m_1),x_1\rangle+(1-\kappa\langle x_0,m_2\rangle)\mu\langle q(x_2,m_2),x_2\rangle\right]\\
&=A+B.
\end{aligned}$$

Since $0\leq 1-|x_\mu|\leq|x_1-x_2|^2$, we have $|B|\leq C\,|x_1-x_2|^2$. To estimate $|A|$, write

$$\begin{aligned}
&(1-\kappa\langle x_0,m_2\rangle)q(x_2,m_2)-(1-\kappa\langle x_0,m_1\rangle)q(x_1,m_1))\\
&=\frac{(1-\kappa\langle x_1,m_1\rangle)^2(1-\kappa\langle x_0,m_2\rangle)m_2-(1-\kappa\langle x_2,m_2\rangle)^2(1-\kappa\langle x_0,m_1\rangle)m_1}{(1-\kappa\langle x_1,m_1\rangle)^2(1-\kappa\langle x_2,m_2\rangle)^2}.
\end{aligned}\tag{1.11}$$

The numerator of the last fraction equals

$$\begin{aligned}
&(1-\kappa\langle x_1,m_1\rangle)^2\,((1-\kappa\langle x_0,m_2\rangle)m_2-(1-\kappa\langle x_0,m_1\rangle)m_1)\\
&\qquad+x_2(1-\kappa\langle x_0,m_1\rangle x_1)m_1\left((1-\kappa\langle x_1,m_1\rangle)^2-(1-\kappa\langle x_2,m_2\rangle)^2\right).
\end{aligned}$$

It is easy to see that this expression is bounded in absolute value by $C\,(|m_1-m_2|+|x_1-x_2|)$. By assumption $|x_1-x_2|\leq|m_1-m_2|$ and since the denominator of (1.11) is positive and bounded below by a constant depending only on κ, we obtain that $|A| \leq C|x_1 - x_2|\,|m_1 - m_2|$ with C depending only on κ. Therefore, we have shown that

$$(1-\mu)\rho(x_1) + \mu\rho(x_2) \leq a_0 + C|x_1 - x_2|\,|m_1 - m_2|,$$

which combined with equation (1.10) and the assumption $|x_1 - x_2| \leq |m_1 - m_2|$ yields (1.5).

Now from (1.5) we have

$$\begin{aligned}\frac{\rho(x_0)(1-\kappa\langle x_0, m_i\rangle)}{1-\kappa\langle x, m_i\rangle} &= \frac{a_0(1-\kappa\langle x_0, m_i\rangle)}{1-\kappa\langle x, m_i\rangle} + \frac{(\rho(x_0)-a_0)(1-\kappa\langle x_0, m_i\rangle)}{1-\kappa\langle x, m_i\rangle}\\ &\leq \frac{a_0(1-\kappa\langle x_0, m_i\rangle)}{1-\kappa\langle x, m_i\rangle} + C|x_1 - x_2||m_1 - m_2|,\end{aligned}$$

$i = 1, 2$. Therefore

$$\max_{i=1,2}\left\{\frac{a_0(1-\kappa\langle x_0, m_i\rangle)}{1-\kappa\langle x, m_i\rangle}\right\} \geq \max_{i=1,2}\left\{\frac{\rho(x_0)(1-\kappa\langle x_0, m_i\rangle)}{1-\kappa\langle x, m_i\rangle}\right\} - C|x_1 - x_2||m_1 - m_2|$$

and so from (1.4) we obtain

$$\rho(x) \geq \max\left\{\frac{\rho(x_0)(1-\kappa\langle x_0, m_i\rangle)}{1-\kappa\langle x, m_i\rangle}\right\} - C|x_1 - x_2||m_1 - m_2|. \tag{1.12}$$

We can now apply Lemma 1.1 to conclude (1.3) for $m(\lambda) \in [m_1, m_2]_{x_0}$ and the proof of the lemma is complete. □

Let $|m| = 1$ and $m(\lambda)$ from (1.1). Writing

$$\begin{aligned}&\frac{1-\kappa\langle x_0, m(\lambda)\rangle}{1-\kappa\langle x, m(\lambda)\rangle} - \frac{1-\kappa\langle x_0, m\rangle}{1-\kappa\langle x, m\rangle}\\ &= \frac{\kappa\langle x - x_0, m(\lambda)(1-\kappa\langle x_0, m\rangle) - m(1-\kappa\langle x_0, m(\lambda)\rangle)\rangle}{(1-\kappa\langle x, m(\lambda)\rangle)(1-\kappa\langle x, m\rangle)}\end{aligned}$$

it follows that

$$\left|\frac{1-\kappa\langle x_0, m(\lambda)\rangle}{1-\kappa\langle x, m(\lambda)\rangle} - \frac{1-\kappa\langle x_0, m\rangle}{1-\kappa\langle x, m\rangle}\right| \leq C|x - x_0||m - m(\lambda)|.$$

with C depending only on κ. This estimate together with (1.3) yields the following lemma.

Lemma 1.4. *Assume ρ is a refractor defined in Ω with $\frac{1}{\Lambda} \leq \rho \leq \Lambda$ and let $B_{2\delta}$ be a geodesic ball with $B_{2\delta} \subseteq \Omega$. Let $x_i \in B_\delta$, $m_i \in N_\rho(x_i)$ with $m_i \in \Omega^\star$, $i = 1, 2$ and*

$|m_1 - m_2| \geq |x_1 - x_2|$. *Then there exists $x_0 \in [x_1, x_2]$, the geodesic segment in S^{n-1} joining x_1, x_2 and contained in B_δ, such that*

$$\rho(x) \geq \frac{\rho(x_0)(1-\kappa\langle x_0, m\rangle)}{1-\kappa\langle x, m\rangle} + C_1\lambda(1-\lambda)|x-x_0|^2|m_1-m_2|^2 - C_2|x_1-x_2|\,|m_1-m_2| - C_3|x-x_0|\,|m-m(\lambda)|$$

for all $m(\lambda) \in [m_1, m_2]_{x_0}$, for all $x \in \Omega$ and for all $m \in \Omega^\star$, where C_1, C_2 and C_3 are positive constants depending only on κ and Λ.

We now prove the main theorem.

Theorem 1.5. *Assume ρ is a refractor from Ω to Ω^* with $\frac{1}{\Lambda} \leq \rho \leq \Lambda$ and let $B_{2\delta}$ be a geodesic ball with $B_{2\delta} \subseteq \Omega$. Let $\bar{x}, \hat{x} \in B_\delta$, $\bar{m} \in N_\rho(\bar{x})$ and $\hat{m} \in N_\rho(\hat{x})$ with $\bar{m}, \hat{m} \in \Omega^\star$. There exists a constant C depending only on κ, Λ and δ, and a constant K depending only on κ and Λ, such that if $|\bar{m}-\hat{m}| \geq C|\bar{x}-\hat{x}|$, then*

$$|\bar{m}-\hat{m}| \leq K|\bar{x}-\hat{x}|^\alpha$$

with $\alpha = \frac{1}{2(4n-5)}$.

Proof. If we choose $C \geq 1$, then we can apply Lemma 1.4. Therefore there exist $x_0 \in [\bar{x}, \hat{x}]$, the geodesic segment, such that

$$\rho(x) \geq \frac{\rho(x_0)(1-\kappa\langle x_0, m\rangle)}{1-\kappa\langle x, m\rangle} + C_1|x-x_0|^2|\bar{m}-\hat{m}|^2 - C_2|\bar{x}-\hat{x}||\bar{m}-\hat{m}| - C_3|x-x_0||m-m(\lambda)|$$

for $m(\lambda) \in [\bar{m}, \hat{x}]_{x_0}$,with $\lambda \in [\frac{1}{4}, \frac{3}{4}]$, for all $x \in \Omega$ and for all $m \in \Omega^\star$

The positive constants C_1, C_2 and C_3 depend only on κ and Λ.

There exists a constant μ_0 depending on δ such that the μ_0-neighborhood of $[\bar{m}, \hat{m}]_{x_0}$ is contained in $\Omega^\star$.

Set $t_0 = \dfrac{C_3|m-m(\lambda)| + \sqrt{C_3^2|m-m(\lambda)|^2 + 4C_1C_2|m-m(\lambda)|^3|\bar{x}-\hat{x}|}}{2C_1|\bar{m}-\hat{m}|}$.

Note that if $|x-x_0| \geq t_0$ then $C_1|x-x_0|^2|\bar{m}-\hat{m}|^2 - C_2|\bar{x}-\hat{x}||\bar{m}-\hat{m}| - C_3|x-x_0||m-m(\lambda)| \geq 0$.

Set $\mu = \sqrt{|\bar{m}-\hat{m}|^3|\bar{x}-\hat{x}|}$ and assume $|m-m(\lambda)| \leq \mu$, then

$$t_0 \leq \frac{C_3 + \sqrt{C_3^2 + 4C_1C_2}}{2C_1}\sqrt{\frac{|\bar{x}-\hat{x}|}{|\bar{m}-\hat{m}|}}$$

.

Let $K = \dfrac{C_3 + \sqrt{C_3^2 + 4C_1C_2}}{2C_1}$ and take $C \geq 1$ and such that $K \leq C^{\frac{\delta}{2}}$. Note C depends only on κ, Λ and δ.

Assume $|\bar{m} - \hat{m}| \geq C|\bar{x} - \hat{x}|$

Set $\sigma = K\sqrt{\dfrac{|\bar{x} - \hat{x}|}{|\bar{m} - \hat{m}|}}$. Note that $t_0 \leq \sigma \leq \frac{\delta}{2}$.

Also note that $\mu \leq \dfrac{|\bar{m} - \hat{m}|^2}{\sqrt{C}} \leq \dfrac{4}{\sqrt{C}} \leq \mu_0$, provided $C \geq \dfrac{16}{\mu_0^2}$

Notice that if m is in the μ neighborhood of $\left\{[\bar{m}, \hat{m}]_{x_0} : \lambda \in [\frac{1}{4}, \frac{3}{4}]\right\}$ and $|x - x_0| \geq \sigma$, then

$\rho(x) \geq \dfrac{\rho(x_0)(1 - \kappa\langle x_0, m\rangle)}{1 - \kappa\langle x, m\rangle}$

Since $x_0 \in B_\delta$ and $B_\sigma(x_0) \subseteq B_{2\delta} \subseteq \Omega$, we have that there exists $\tilde{x} \in B_\sigma(x_0)$ such that

$$\rho(x) \geq \frac{\rho(\tilde{x})(1 - \kappa\langle \tilde{x}, m\rangle)}{1 - \kappa\langle x, m\rangle}$$

for all $x \in \Omega$. This implies that $m \in N_\rho(\tilde{x})$. Here it is important to know that $m \in \Omega^\star$

Therefore we have shown that the μ neighborhood of $\left\{[\bar{m}, \hat{m}]_{x_0} : \lambda \in [\frac{1}{4}, \frac{3}{4}]\right\}$ is contained in $N_\rho(B_\sigma(x_0))$.

Taking surface measure on the sphere, and using that the refractor measure is dominated by surface measure, we get $|\bar{m} - \hat{m}|\mu^{n-2} \leq C\sigma^{n-1}$.

This yields the result. □

2. $C^{1,\alpha}$ for the Near Field Refractor

In this section we will prove $C^{1,\alpha}$ regularity for the near field refractor using Loeper method.

Recall that the oval is $O(Y, b) = \{X \in R^n : |X| + \kappa|X - Y| = b\}$, with $\kappa|Y| < b < |Y|$.

A ray emanating from the origin in direction x is refracted at the point $X \in O(Y, b)$ to the point Y provided that $\langle \dfrac{X}{|X|}, \dfrac{Y - X}{|Y - X|}\rangle \geq \kappa$ which by the equation of the oval is equivalent to $\langle x, Y\rangle \geq b$.

The polar equation of the oval is $O(Y, b) = \{\rho(x)x : x \in S^{n-1}\}$ where

$$\rho(x; y, b) = \frac{b - \kappa\langle x, Y\rangle - \sqrt{(b - \kappa\langle x, Y\rangle)^2 - (1 - \kappa^2)(b^2 - \kappa^2|Y|^2)}}{1 - \kappa^2}$$

In order to specify a point X_0 on the oval, let $b = |X_0| + \kappa|X_0 - Y|$ and define the function $h(x, Y, X_0)$ by

$$h(x, Y, X_0) = \rho(x, Y, b)$$

The point X_0 will always be taken satisfying $\langle \frac{X_0}{|X_0|}, \frac{Y - X_0}{|Y - X_0|} \rangle \geq \kappa$.

Let $\Omega \subseteq S^{n-1}$ be open and $C_1 < C_2$ be constants.

Let $\Gamma_{C_1,C_2} = \{rx : x \in \Omega,\ C_1 \leq r \leq C_2\}$.

2.1. **Hypothesis on Σ.**

We now list the hypothesis we put on the target set Σ

Hypothesis A:

We assume that for each point $X \in \Gamma_{C_1,C_2}$, the target can be parametrized as

$$\Sigma = \{Y : Y = X + s_X(m)m,\ m \in S^{n-1},\ \langle m, x\rangle \geq \kappa\}$$

Notice carefully that $\langle \frac{X}{|X|}, \frac{Y - X}{|Y - X|} \rangle \geq \kappa$ for all $X \in \Gamma_{C_1,C_2}$ and for all $Y \in \Sigma$.

We also assume $0 < s_X \leq C$ for all $X \in \Gamma_{C_1,C_2}$

Hypothesis B:

We assume that for each $X \in \Gamma_{C_1,C_2}$, the function s_X is Lipschits. That is $|s_X(m) - s_X(\bar{m})| \leq C_X|m - \bar{m}|$ for all $m, \bar{m} \in S^{n-1}$ with $\langle m, x\rangle \geq \kappa$ and $\langle \bar{m}, x\rangle \geq \kappa$.

In particular, if $\bar{Y}, \hat{Y} \in \Sigma$ are given by $\bar{Y} = X + s_X(\bar{m})\bar{m}$ and $\hat{Y} = X + s_X(\hat{m})\hat{m}$, then $|\bar{Y} - \hat{Y}| \leq (C_X + C)|\bar{m} - \hat{m}|$. The last constant we will assume is uniform in X.

Notice that we also have the reverse inequality

$$|\hat{m} - \bar{m}| \leq 2\min\{\frac{1}{|\bar{Y} - X_0|}, \frac{1}{\hat{Y} - X_0|}\}|\bar{Y} - \hat{Y}| \leq C|\bar{Y} - \hat{Y}|$$

where in the last inequality we have used the next Hypothesis.

Hypothesis C:

Let $C(\kappa) = \frac{\kappa^2}{(1 + 2\kappa)(1 + \kappa^2)}$.

We assume

$$\frac{|X|}{|Y - X|} \leq C(\kappa)$$

for all $X \in \Gamma_{C_1,C_2}$ and for all $Y \in \Sigma$.

Notice that $|Y - X| \geq \frac{C_1}{C(\kappa)}$. Therefore $s_X \geq \frac{C_1}{C(\kappa)}$ for all $X \in \Gamma_{C_1,C_2}$

Hypothesis D:

Fix $X_0 \in \Gamma_{C_1,C_2}$, and $\bar{Y}, \hat{Y} \in \Sigma$. To simplify notation we will write s instead s_{X_0}.

Let $\bar{m} = \dfrac{\bar{Y} - X_0}{|\bar{Y} - X_0|}$ and $\hat{m} = \dfrac{\hat{Y} - X_0}{|\hat{Y} - X_0|}$

Consider the curve $[\bar{Y}, \hat{Y}]_{X_0}$ on Σ given by

$$[\bar{Y}, \hat{Y}]_{X_0} = \{Y(\lambda) = X_0 + s(m(\lambda))m(\lambda) : \lambda \in [0, 1]\}$$

where $m(\lambda)$ is the parametrization of $[\bar{m}, \hat{m}]_{x_0}$ as defined in (1.1).

Since $\langle m(\lambda), x_0\rangle \geq \kappa$ for all $\lambda \in [0, 1]$, the curve is well defined according to Hypothesis A.

Recall that if $m \in [\bar{m}, \hat{m}]_{x_0}$, then we can write

$$x_0 - \kappa m = \bar{\beta}(x_0 - \kappa\bar{m}) + \hat{\beta}(x_0 - \kappa\hat{m})$$

, where $\bar{\beta} = (1 - \lambda)\beta(\lambda)$ and $\hat{\beta} = \lambda\beta(\lambda)$.

We will assume that

$$\frac{1}{s} \geq \frac{\bar{\beta}}{\bar{s}} + \frac{\hat{\beta}}{\hat{s}}$$

Notice that this is the same as saying that $\dfrac{1}{s(m(\lambda))\beta(\lambda)}$ is a concave function of λ

Hypothesis E:

We assume that there exists μ_0 and C such that for any $X_0 \in \Gamma_{C_1,C_2}$, and $\bar{Y}, \hat{Y} \in \Sigma$,

$$H^{n-1}\Big(N_\mu(\{[\bar{Y}, \hat{Y}]_{X_0} : \frac{1}{4} \leq \lambda \leq \frac{3}{4}\}) \cap \Sigma\Big) \geq C\mu^{n-2}|\bar{Y} - \hat{Y}|$$

, for any $\mu \leq \mu_0$ and where H^{n-1} stands for $n - 1$ dimensional Hausdorff measure in R^n and N_μ is the μ neighborhood in R^n.

This finishes the list of hypothesis on Σ.

A word about each Hypothesis:

Hypothesis A and B are to ensure that each point $X \in \Gamma_{C_1,C_2}$ has the chance to being refracted to each point of Σ and to ensure visibility, that is, the ray refracted at $X \in \Gamma_{C_1,C_2}$ intersects Σ at only one point.

Hypothesis C imposes a positive and controlled distance between the refractor and the target and implies that the ovals to be used in the definition of the refractor are smooth with controlled derivatives.

Hypothesis D is the crucial AW hypothesis necessary for regularity.

Hypothesis E is a weak form of convexity of Σ with respect to points $X \in \Gamma_{C_1,C_2}$

2.2. **example.** Before we continue we give an example that shows that a horizontal plane (a part of it properly situated with respect to Γ_{C_1,C_2}) satisfies Hypothesis D.

We show that if $\Sigma = \{Y : Y_n = M\}$ and $\langle \frac{X}{|X|}, \frac{Y-X}{|Y-X|} \geq \kappa$ for all $X \in \Gamma_{C_1,C_2}$ and all $Y \in \Sigma$, then Σ satisfies Hypothesis D.

To see this, fix $X \in \Gamma_{C_1,C_2}$ with $0 < X_n < M$ and let $\bar{Y} = X + \bar{s}\bar{m}$ and $\hat{Y} = X + \hat{s}\hat{m}$ be in Σ.

For $Y = X + sm$ and $Y \in \Sigma$ we have $M = X_n + sm_n$, and hence

$$\frac{1}{s} = \frac{m_n}{M - X_n}.$$

For $m \in [\bar{m}, \hat{m}]_x$, we have $m = \frac{1}{\kappa}x + \bar{\beta}(\bar{m} - \frac{1}{\kappa}x) + \hat{\beta}(\hat{m} - \frac{1}{\kappa}x)$ and hence,
$m_n = \frac{1}{\kappa}x_n + \bar{\beta}(\bar{m}_n - \frac{1}{\kappa}x_n) + \hat{\beta}(\hat{m}_n - \frac{1}{\kappa}x_n)$, which gives

$$\frac{1}{s} = \frac{\frac{1}{\kappa}x_n(1 - (\bar{\beta} + \hat{\beta})) + \bar{\beta}\bar{m}_n + \hat{\beta}\hat{m}_n}{M - x_n} \geq$$

$$\frac{\bar{\beta}\bar{m}_n}{M - x_n} + \frac{\hat{\beta}\hat{m}_n}{M - x_n} = \frac{\bar{\beta}}{\bar{s}} + \frac{\hat{\beta}}{\hat{s}}$$

and we are done.

Let us now study the function $h(x, Y, X_0)$ for $Y \in \Sigma$ and $X_0 \in \Gamma_{C_1,C_2}$.

Write $Y = X_0 + sm$ and recall $b = |X_0| + \kappa|Y - X_0| = |X_0| + \kappa s$. It follows that

$$b - \kappa^2\langle x, Y\rangle = |X_0|(1 - \kappa^2\langle x, x_0\rangle) + \kappa s(1 - \kappa\langle x, m\rangle)$$

and

$$b^2 - \kappa^2|Y|^2 = (1 - \kappa^2)|X_0|^2 + 2\kappa s(1 - \kappa\langle x_0, m\rangle)$$

. Set $B = \dfrac{b - \kappa^2\langle x, Y\rangle}{1 - \kappa^2}$ and $C = \dfrac{b^2 - \kappa^2|Y|^2}{1 - \kappa^2}$.
We then have

$$h(x, Y, X_0) = B - \sqrt{B^2 - C}$$

In order to get to our crucial lemma, first we need three auxiliary lemmas.

Lemma 2.1. *Let $\bar{Y}, \hat{Y} \in \Sigma$ and $X_0 \in \Gamma_{C_1,C_2}$. Then, with the notation as above, we have* $\hat{B} \geq \bar{B} - \sqrt{\bar{B}^2 - \bar{C}}$

Proof. We have, $\bar{B}-\sqrt{\bar{B}^2-\bar{C}}=\dfrac{\bar{C}}{\sqrt{\bar{B}^2-\bar{C}}}\leq\dfrac{\bar{C}}{\bar{B}}$.

So, it is enough to show $\bar{C}\leq\bar{B}\hat{B}$ and this amounts to

$$\frac{(1-\kappa^2)|X_0|^2+2\kappa\bar{s}(1-\kappa\langle x_0,\bar{m}\rangle)}{1-\kappa^2}\leq$$
$$\frac{\left(|X_0|(1-\kappa^2\langle x,x_0\rangle)+\kappa\bar{s}(1-\kappa\langle x,\bar{m}\rangle)\right)\left(|X_0|(1-\kappa^2\langle x,x_0\rangle)+\kappa\hat{s}(1-\kappa\langle x,\hat{m}\rangle)\right)}{(1-\kappa^2)^2}$$

The above inequality is equivalent to

$$|X_0|^2\left(1-\frac{(1-\kappa^2\langle x,x_0\rangle)^2}{(1-\kappa^2)^2}\right)+\frac{2\kappa|X_0|\bar{s}(1-\kappa\langle x_0,\bar{m})}{1-\kappa^2}\leq$$
$$\frac{\kappa|X_0|(1-\kappa^2\langle x,x_0\rangle)}{(1-\kappa^2)^2}\Big(\bar{s}(1-\langle x,\bar{m}\rangle)+\hat{s}(1-\langle x,\hat{m}\rangle)\Big)+\frac{\kappa^2}{(1-\kappa^2)^2}\bar{s}\hat{s}(1-\langle x,\bar{m}\rangle)(1-\langle x,\hat{m}\rangle)$$

The LHS is $\leq|X_0|^2+2\kappa|X_0|\bar{s}$ and the RHS is $\geq\frac{\kappa^2}{(1-\kappa^2)^2}\bar{s}\hat{s}(1-\kappa)^2=\frac{\kappa^2}{(1+\kappa)^2}\bar{s}\hat{s}$.

Therefore,we need $|X_0|^2+2\kappa|X_0|\bar{s}\leq\frac{\kappa^2}{(1+\kappa)^2}\bar{s}\hat{s}$.

Equivalently,

$$\frac{|X_0|}{\bar{s}}\frac{|X_0|}{\hat{s}}+2\kappa\frac{|X_0|}{\hat{s}}\leq\frac{\kappa^2}{(1+\kappa)^2}$$

and this follows from Hypothesis C.

□

The second auxiliary lemma is as follows,

Lemma 2.2. *Consider the function* $f(B,C)=B-\sqrt{B^2-C}$ *on the set* $0\leq C\leq B^2$ *and* $B\geq 0$ *Fix* $(\bar{B},\bar{C})$ *and assume* $f(\bar{B},\bar{C})\leq B$*. Then* $f(B,C)\leq f(\bar{B},\bar{C})$ *if and only if* $C-\bar{C}\leq 2(B-\bar{B})f(\bar{B},\bar{C})$*. In addition, if* $C-\bar{C}\leq 2(B-\bar{B})f(\bar{B},\bar{C})-E$ *for some* $E\geq 0$ *then* $f(B,C)\leq f(\bar{B},\bar{C})-\dfrac{E}{B+\sqrt{B^2-C}-f(\bar{B},\bar{C})}$

Proof. Assume that $C-\bar{C}\leq 2(B-\bar{B})f(\bar{B},\bar{C})-E$, for some $E\geq 0$

We have

$$f(B,C)-f(\bar{B},\bar{C})=\frac{C-\bar{C}-(f(B,C)+f(\bar{B},\bar{C}))(B-\bar{B})}{\sqrt{B^2-C}+\sqrt{\bar{B}^2-\bar{C}}}\leq$$
$$\frac{2(B-\bar{B})f(\bar{B},\bar{C})-E-(f(B,C)+f(\bar{B},\bar{C}))(B-\bar{B})}{\sqrt{B^2-C}+\sqrt{\bar{B}^2-\bar{C}}}=$$
$$\frac{(f(\bar{B},\bar{C})-f(B,C))(B-\bar{B})-E}{\sqrt{B^2-C}+\sqrt{\bar{B}^2-\bar{C}}}$$

Therefore,

$$(f(B,C) - f(\bar{B},\bar{C}))\Big(1 + \frac{B-\bar{B}}{\sqrt{B^2-C} + \sqrt{\bar{B}^2-\bar{C}}}\Big) \leq \frac{-E}{\sqrt{B^2-C} + \sqrt{\bar{B}^2-\bar{C}}}$$

which implies

$$f(B,C) \leq f(\bar{B},\bar{C}) - \frac{E}{B + \sqrt{B^2-C} - f(\bar{B},\bar{C})}$$

Conversely, assume $f(B,C) \leq f(\bar{B},\bar{C})$, that is $B - \sqrt{B^2-C} \leq f(\bar{B},\bar{C})$ and this implies

$$0 \leq B - f(\bar{B},\bar{C}) \leq \sqrt{B^2-C}$$

where the first inequality is by assumption. Hence,

$$C \leq 2Bf(\bar{B},\bar{C}) - f(\bar{B},\bar{C})^2 = 2(B-\bar{B})f(\bar{B},\bar{C}) + \bar{C}$$

□

The third auxiliary lemma says that the oval passing thru X_0 is inside the ellipsoid passing thru X_0

Lemma 2.3. *Assume* $\langle x_0, m\rangle \geq \kappa$ *and let* $Y = X_0 + sm \;\; s > 0$, *then* $\{X : |X| + \kappa|X-Y| \leq |X_0| + \kappa|X_0 - Y|\} \subseteq \{X : |X| - \kappa\langle X, m\rangle \leq |X_0| - \kappa\langle X_0, m\rangle\}$.

In particular

$$h(x, Y, X_0) \leq \frac{|X_0|(1-\kappa\langle x_0, m\rangle)}{1-\kappa\langle x, m\rangle}$$

for all $x \in S^{n-1}$

Proof. Assume $|X| + \kappa|X-Y| \leq |X_0| + \kappa|X_0 - Y|$, then

$$|X| - \kappa\langle X, m\rangle = |X| + \kappa|X-Y| - \kappa\langle X, m\rangle + \kappa|X-Y| \leq |X_0| + \kappa|X_0-Y| - \kappa\Big(\langle X, m\rangle + |X-Y|\Big) =$$

$$|X_0| + \kappa|X_0 - Y| - \kappa\Big(\langle X-Y, m\rangle + |X-Y|\Big) - \kappa\langle Y, m\rangle \leq$$

$$|X_0| + \kappa|X_0 - Y| - \kappa\langle Y, m\rangle = |X_0| + \kappa|X_0 - Y| - \kappa\langle Y - X_0, m\rangle - \kappa\langle X_0, m\rangle =$$

$$|X_0| + \kappa s - \kappa\langle sm, m\rangle - \kappa\langle X_0, m\rangle =$$

$$|X_0| - \kappa\langle X_0, m\rangle$$

□

We are now ready for the crucial lemma

Lemma 2.4. *There exists a universal constant C such that if $\bar{Y}, \hat{Y} \in \Sigma$ and $X_0 \in \Gamma_{C_1,C_2}$ and $Y = X_0 + sm$ with $x_0 - \kappa m = \bar{\beta}(x_0 - \kappa\bar{m}) + \hat{\beta}(x_0 - \kappa\hat{m})$ and $\frac{1}{s} \geq \frac{\bar{\beta}}{\bar{s}} + \frac{\hat{\beta}}{\hat{s}}$. Then*

$$C\lambda(1-\lambda)|\bar{Y} - \hat{Y}|^2|x - x_0|^2 + h(x, Y, X_0) \leq \max\{h(x, \bar{Y}, X_0), h(x, \hat{Y}, X_0)\}$$

for all $x \in S^{n-1}$

Proof. Fix $x \in S^{n-1}$ and assume without lost of generality that $h(x, \bar{Y}, X_0) \geq h(x, \hat{Y}, X_0)$.

We will show

$$C\lambda(1-\lambda)|\bar{Y} - \hat{Y}|^2|x - x_0|^2 + h(x, Y, X_0) \leq h(x, \bar{Y}, X_0).$$

By lemma (2.1), we have $\hat{B} \geq \bar{B} - \sqrt{\bar{B}^2 - \bar{C}}$ and hence by lemma (2.2), we have

$$\hat{C} - \bar{C} \leq 2f(\bar{B}, \bar{C})(\hat{B} - \bar{B}).$$

The above means

$$2\kappa|X_0|\Big(\hat{s}(1 - \kappa\langle x_0, \hat{m}\rangle) - \bar{s}(1 - \kappa\langle x_0, \bar{m}\rangle)\Big) \leq$$
$$2\kappa f(\bar{B}, \bar{C})\Big(\hat{s}(1 - \kappa\langle x, \hat{m}\rangle) - \bar{s}(1 - \kappa\langle x, \bar{m}\rangle)\Big)$$

and equivalently

$$\hat{s}\Big(|X_0|(1 - \kappa\langle x_0, \hat{m}\rangle) - f(\bar{B}, \bar{C})(1 - \kappa\langle x, \hat{m}\rangle)\Big) \leq$$
$$\bar{s}\Big(|X_0|(1 - \kappa\langle x_0, \bar{m}\rangle) - f(\bar{B}, \bar{C})(1 - \kappa\langle x, \bar{m}\rangle)\Big)$$

. We will show that

$$C - \bar{C} \leq 2f(\bar{B}, \bar{C})(B - \bar{B}) - E$$

for E to be specify at the end. We need to prove that

$$|X_0|\Big(s(1 - \kappa\langle x_0, m\rangle) - \bar{s}(1 - \kappa\langle x_0, \bar{m}\rangle)\Big) \leq$$
$$f(\bar{B}, \bar{C})\Big(s(1 - \kappa\langle x, m\rangle) - \bar{s}(1 - \kappa\langle x, \bar{m}\rangle)\Big) - \frac{(1-\kappa^2)E}{2\kappa}$$

Equivalently we will show

$$s\Big(|X_0|(1 - \kappa\langle x_0, m\rangle) - f(\bar{B}, \bar{C})(1 - \kappa\langle x, m\rangle)\Big) \leq$$
$$\bar{s}\Big(|X_0|(1 - \kappa\langle x_0, \bar{m}\rangle) - f(\bar{B}, \bar{C})(1 - \kappa\langle x, \bar{m}\rangle)\Big) - \frac{(1-\kappa^2)E}{2\kappa}$$

. We have

$$s\Big(|X_0|(1 - \kappa\langle x_0, m\rangle) - f(\bar{B}, \bar{C})(1 - \kappa\langle x, m\rangle)\Big) =$$
$$s\Big(|X_0|\Big(\bar{\beta}(1 - \kappa\langle x_0, \bar{m}\rangle) + \hat{\beta}(1 - \kappa\langle x_0, \hat{m}\rangle)\Big) - f(\bar{B}, \bar{C})(1 - \kappa\langle x, m\rangle)\Big) =$$

$$s\bar{\beta}\Big(|X_0|(1-\kappa\langle x_0,\bar{m}\rangle)-f(\bar{B},\bar{C})(1-\kappa\langle x,\bar{m}\rangle)\Big)+$$
$$s\hat{\beta}\Big(|X_0|(1-\kappa\langle x_0,\hat{m}\rangle)-f(\bar{B},\bar{C})(1-\kappa\langle x,\hat{m}\rangle)\Big)+$$
$$sf(\bar{B},\bar{C})\Big(\bar{\beta}(1-\kappa\langle x,\bar{m}\rangle)+\hat{\beta}(1-\kappa\langle x,\hat{m}\rangle)-(1-\kappa\langle x,m\rangle)\Big)$$

Now,

$$\bar{\beta}(1-\kappa\langle x,\bar{m}\rangle)+\hat{\beta}(1-\kappa\langle x,\hat{m}\rangle)-(1-\kappa\langle x,m\rangle)=$$
$$(\bar{\beta}+\hat{\beta}-1)(1-\langle x,x_0\rangle)=\frac{1}{2}(\bar{\beta}+\hat{\beta}-1)|x-x_0|^2$$

And from above we have

$$s\hat{\beta}\Big(|X_0|(1-\kappa\langle x_0,\hat{m}\rangle)-f(\bar{B},\bar{C})(1-\kappa\langle x,\hat{m}\rangle)\Big)\leq$$
$$\frac{s\hat{\beta}\bar{s}}{\hat{s}}\Big(|X_0|(1-\kappa\langle x_0,\bar{m}\rangle)-f(\bar{B},\bar{C})(1-\kappa\langle x,\bar{m}\rangle)\Big)$$

. Therefore, we have

$$s\Big(|X_0|(1-\kappa\langle x_0,m\rangle)-f(\bar{B},\bar{C})(1-\kappa\langle x,m\rangle)\Big)\leq$$
$$\frac{s(\bar{\beta}\hat{s}+\hat{\beta}\bar{s})}{\hat{s}}\Big(|X_0|(1-\kappa\langle x_0,\bar{m}\rangle)-f(\bar{B},\bar{C})(1-\kappa\langle x,\bar{m}\rangle)\Big)-sf(\bar{B},\bar{C})\frac{(1-(\bar{\beta}+\hat{\beta}))}{2}|x-x_0|^2\leq$$

$$\bar{s}\Big(|X_0|(1-\kappa\langle x_0,\bar{m}\rangle)-f(\bar{B},\bar{C})(1-\kappa\langle x,\bar{m}\rangle)\Big)-sf(\bar{B},\bar{C})\frac{(1-(\bar{\beta}+\hat{\beta}))}{2}|x-x_0|^2 \tag{2.13}$$

where in the last inequality we recall that $f(\bar{B},\bar{C})=h(x,\bar{Y},X_0)$ and hence by lemma (2.3) we have

$$|X_0|(1-\kappa\langle x_0,\bar{m}\rangle)-f(\bar{B},\bar{C})(1-\kappa\langle x,\bar{m}\rangle)\geq 0$$

and by hypothesis we have

$$\bar{\beta}\hat{s}+\hat{\beta}\bar{s}\leq\frac{\bar{s}\hat{s}}{s} \tag{2.14}$$

Now,define $E=\dfrac{\kappa sf(\bar{B},\bar{C})(1-(\bar{\beta}+\hat{\beta}))|x-x_0|^2}{1-\kappa^2}$.
We have proved that

$$C-\bar{C}\leq 2f(\bar{B},\bar{C})(B-\bar{B})-E$$

. Since by lemma (2.1), we have $B\geq f(\bar{B},\bar{C})$, applying lemma (2.2) we get

$$f(B,C)+\frac{E}{B+\sqrt{B^2-C}-f(\bar{B},\bar{C})}\leq f(\bar{B},\bar{C})$$

That is

$$h(x,Y,X_0) + \frac{E}{B + \sqrt{B^2 - C} - f(\bar{B},\bar{C})} \leq h(x,\bar{Y},X_0)$$

.

Finally, we estimate $\dfrac{E}{B + \sqrt{B^2 - C} - f(\bar{B},\bar{C})}$ from below.

We have

$$sf(\bar{B},\bar{C})(1-(\bar{\beta}+\hat{\beta}))|x-x_0|^2 \geq Cs\lambda(1-\lambda)|\bar{m}-\hat{m}|^2|x-x_0|^2 \geq$$

$$Cs\lambda(1-\lambda)|\bar{Y}-\hat{Y}|^2|x-x_0|^2$$

where we have used the constants in Hypothesis A and B.

Also, $B+\sqrt{B^2-C}-f(\bar{B},\bar{C}) \leq B+\sqrt{B^2-C} \leq 2B \leq |X_0|+\kappa s \leq Cs$ using Hypothesis C.

Hence,

$$\frac{E}{B + \sqrt{B^2 - C} - f(\bar{B},\bar{C})} \geq C\lambda(1-\lambda)|\bar{Y}-\hat{Y}|^2|x-x_0|^2$$

finishing the proof. The constant C is universal. □

We continue with the analysis of the function $h(x,Y,X_0)$ for $Y \in \Sigma$ and $X_0 \in \Gamma_{C_1,C_2}$.

First we need to bound from below the quantity inside the square root.

Lemma 2.5. *There exist a universal constant C such that if $Y \in \Sigma$ and $X_0 \in \Gamma_{C_1,C_2}$ and $b = |X_0| + \kappa|Y - X_0|$ then*

$$(b-\kappa^2\langle x,Y\rangle)^2 - (1-\kappa^2)(b^2-\kappa^2|Y|^2) \geq C$$

for all x with $|x| \leq 1$

Proof. We can write $Y = X_0 + sm$ with $\langle x_0, m\rangle \geq \kappa$ and $s \geq C$. This is by Hypothesis A and C. Hence $|Y|^2 - b^2 = |X_0 + sm| - (|X_0| + \kappa s)^2 = 2s|X_0|(\langle x_0,m\rangle - \kappa) + s^2(1-\kappa^2) \geq s^2(1-\kappa^2)$, and hence

$$|Y| - b = \frac{|Y|^2 - b^2}{|Y| + b} \geq \frac{s^2(1-\kappa^2)}{2|X_0| + (1+\kappa)s} \geq C$$

Next note that

$$(b-\kappa^2\langle x,Y\rangle)^2 - (1-\kappa^2)(b^2-\kappa^2|Y|^2) = \kappa^2\Big((b-\langle x,Y\rangle)^2 + (1-\kappa^2)(|Y|^2 - (\langle x,Y\rangle)^2\Big)$$

We minimize the above quantity with $|x| \leq 1$.

If the minimum occurs at x with $|x| < 1$ then $b = \kappa^2\langle x,Y\rangle$ which is impossible since $b \geq \kappa|Y|$

And if the minimum occurs at x with $|x| = 1$ then $x = \frac{Y}{|Y|}$ and $\Big((b - \langle x, Y\rangle)^2 + (1 - \kappa^2)(|Y|^2 - (\langle x, Y\rangle)^2\Big) \geq (|Y| - b)^2 \geq C$ by the above □

Lemma 2.6. *There exists C universal such that if* $Y \in \Sigma$, $t > 0$ *and* $(1 + t)X_0 \in \Gamma_{C_1,C_2}$, *then* $0 \leq h(x, Y, (1 + t)X_0) - h(x, Y, X_0) \leq Ct|X_0|$

Proof. Note $b(t) = (1 + t)|X_0| + \kappa|Y - (1 + t)X_0|$ is increasing in t and $b(t) - b(0) \leq (1 + \kappa)t|X_0|$

Let $Q(t) = (b(t) - \kappa^2\langle x, Y\rangle)^2 - (1 - \kappa^2(b(t)^2 - \kappa|Y|^2)$

We can write

$$h(x, Y, (1{+}t)X_0) - h(x, Y, X_0) = \frac{(b(t) - b(0))\Big(\sqrt{Q(t)} + \sqrt{Q(0)} + \kappa^2(b(t) - \langle x, Y\rangle + b(0) - \langle x, Y\rangle)}{\sqrt{Q(t)} + \sqrt{Q(0)}}$$

and hence

$$0 \leq h(x, Y, (1 + t)X_0) - h(x, Y, X_0) \leq C(b(t) - b(0)) \leq Ct|X_0|$$

□

Lemma 2.7. *There exist C universal such that if* $\bar{Y}, Y \in \Sigma$ *and* $X_0 \in \Gamma_{C_1,C_2}$, *then* $|\nabla_x h(x_0, Y, X_0) - \nabla_x h(x_0, \bar{Y}, X_0)| \leq C|Y - \bar{Y}|$

Proof. Let $Y = X_0 + sm$ and $\bar{Y} = X_0 + \bar{s}\bar{m}$ and $b = |X_0| + \kappa s$ A calculation shows that

$$\frac{\partial h}{\partial x_i}(x, Y, X_0) = \frac{\kappa^2 h(x, Y, X_0) Y_i}{\sqrt{(b - \kappa^2\langle x, Y\rangle)^2 - (1 - \kappa^2(b^2 - \kappa|Y|^2)}}.$$

In particular, at $x = x_0$, we get

$$\frac{\partial h}{\partial x_i}(x_0, Y, X_0) = \frac{\kappa^2|X_0|Y_i}{\sqrt{(b - \kappa^2\langle x_0, Y\rangle)^2 - (1 - \kappa^2(b^2 - \kappa|Y|^2)}} =$$

$$\frac{\kappa^2|X_0|Y_i}{\kappa s(1 - \kappa\langle x_0, m\rangle)}$$

, where in the last equality we have used that

$$\sqrt{(b - \kappa^2\langle x_0, Y\rangle)^2 - (1 - \kappa^2(b^2 - \kappa|Y|^2)} = \kappa s(1 - \kappa\langle x_0, m\rangle)$$

Therefore,

$$\nabla_x h(x_0, Y, X_0) - \nabla_x h(x_0, \bar{Y}, X_0) = \kappa|X_0|\Big(\frac{Y}{|Y - X_0|(1 - \kappa\langle x_0, m\rangle)} - \frac{Y}{|\bar{Y} - X_0|(1 - \kappa\langle x_0, \bar{m}\rangle)}\Big)$$

. The estimate thus follows from the estimate $|m - \bar{m}| \leq C|Y - \bar{Y}|$. □

Lemma 2.8. *There exists a universal constant M such that if $X_0 \in \Gamma_{C_1,C_2}$, $Y \in \Sigma$ and $x \in S^{n-1}$, then*

$$|h(x,Y,X_0) - h(x_0,Y,X_0) - \langle \nabla_x h(x_0,Y,X_0), x - x_0\rangle| \le M|x-x_0|^2$$

Proof. This follows from Taylor theorem and the estimate in lemma (2.5) □

Lemma 2.9. *There exists a universal constant C such that if $X_0 \in \Gamma_{C_1,C_2}$ and $\bar{Y}, Y \in \Sigma$ and $x \in S^{n-1}$,then*

$$|h(x,Y,X_0) - h(x,\bar{Y},X_0)| \le C|Y-\bar{Y}||x-x_0|$$

Proof. We have, for some $\tilde{Y} \in [\bar{Y}, Y]$ the straight segment, and for some $\tilde{x} \in [x_0, x]$, the straight segment

$$h(x,Y,X_0) - h(x,\bar{Y},X_0) = \sum_{k=1}^{n} \frac{\partial h}{\partial y_k}(x,\tilde{Y},X_0)(Y_k - \bar{Y}_k) =$$

$$\sum_{k=1}^{n} \Big(\frac{\partial h}{\partial y_k}(x,\tilde{Y},X_0) - \frac{\partial h}{\partial y_k}(x_0,\tilde{Y},X_0)\Big)(Y_k - \bar{Y}_k)$$

$$\sum_{k,l=1}^{n} \frac{\partial^2 h}{\partial y_k \partial x_l}(\tilde{x},\tilde{Y},X_0)(x_l - x_i^0)(Y_k - \bar{Y}_k)$$

where we have used that $h(x_0,Y,X_0) = |X_0|$, for all Y and hence,$\frac{\partial h}{\partial y_k}(x_0,\tilde{Y},X_0) = 0$

It remains to notice that writing $Y = X_0 + sm$ and $\bar{Y} = X_0 + \bar{s}\bar{m}$, then $\tilde{Y} = (1-\lambda)\bar{Y} + \lambda Y$ for some $\lambda \in [0,1]$, and hence $Y = X_0 + (1-\lambda)\bar{s}\bar{m} + \lambda s m = X_0 + w$

$$|Y|^2 - b^2 = |X_0 + w|^2 - (|X_0| + \kappa|w|)^2 = |w|^2(1-\kappa^2) + 2\langle X_0, w\rangle - 2\kappa|X_0||w|$$

And note

$$\langle X_0, w\rangle = (1-\lambda)\bar{s}\langle X_0, \bar{m}\rangle + \lambda s\langle X_0, m\rangle \ge (1-\lambda)\bar{s}\kappa|X_0| + \lambda s\kappa|X_0| \ge \kappa|X_0||w|$$

Thus,

$$|Y|^2 - b^2 \ge (1-\kappa^2)|w|^2 \ge C\min\{\bar{s}^2, s^2\} \ge C$$

and the estimate follows again by lemma (2.5) □

This ends the study of the function $h(x,Y,X_0)$.

We now turn to the definition of refractor and prove our main theorem.

We say $u : \Omega \to [C_1, C_2]$ is a refractor from Ω to Σ if for each $x_0 \in \Omega$, there exists $Y \in \Sigma$ such that

$$u(x) \geq h(x, Y, X_0)$$

for all $x \in \Omega$, where $X_0 = u(x_0)x_0$.

If the above holds, we say $Y\partial u(x_0)$

Note that we are assuming $X = u(x)x \in \Gamma_{C_1,C_2}$ for all $x \in \Omega$.

Assume that there is a constant C such that for all balls B_σ such that $B_\sigma \cap S^{n-1} \subseteq \Omega$, we have

$$H^{n-1}(\partial u(B_\sigma) \leq C\sigma^{n-1}. \tag{2.15}$$

where H^{n-1} is the Hausdorff $n-1$ dimensional measure in R^n.

We will show $u \in C^{1,\alpha}(\Omega)$.

The proof will follow from two lemmas.

Lemma 2.10. *There exist universal constants K_1, K_2 such that if $B_{2\delta} \cap S^{n-1} \subseteq \Omega$, $\bar{x}, \hat{x} \in B_\delta \cap S^{n-1}$, and $\bar{Y} \in \partial u(\bar{x})$, $\hat{Y} \in \partial u(\hat{x})$, with $|\bar{Y} - \hat{Y}| \geq |\bar{x} - \hat{x}|$. Then, there exists $x_0 \in B_\delta \cap S^{n-1}$ such that, setting $X_0 = u(x_0)x_0$,if $Y(\lambda) \in [\bar{Y}, \hat{Y}]_{X_0}$,then*

$$u(x) \geq h(x, Y(\lambda), X_0) + K_1\lambda(1-\lambda)|\bar{Y} - \hat{Y}|^2|x - x_0|^2 - K_2|\bar{x} - \hat{x}||\bar{Y} - \hat{Y}|^2$$

for all $x \in \Omega$

Proof. Let $\bar{X} = u(\bar{x})\bar{x}$ and $\hat{X} = u(\hat{x})\hat{x}$ We have $u(x) \geq h(x, \bar{Y}, \bar{X})$ and $u(x) \geq h(x, \hat{Y}, \hat{X})$, for all $x \in \Omega$.

There exists $x_0 \in [\bar{x}, \hat{x}]$, the geodesic segment, such that $h(x_0, \bar{Y}, \bar{X}) = h(x_0, \hat{Y}, \hat{X}) = \rho_0$

Let $\tilde{X}_0 = \rho_0 x_0$ and $X_0 = u(x_0)x_0$. Note $\rho_0 \leq u(x_0)$.

We claim

$$u(x_0) - \rho_0 \leq C|\bar{x} - \hat{x}||\bar{Y} - \hat{Y}|$$

for some structural constant C.

We will prove the claim at the end. Let us assume the claim.

Then, from lemma (2.6), we get,

$$h(x, \bar{Y}, \bar{X}) = h(x, \bar{Y}, \tilde{X}_0) \geq h(x, \bar{Y}, X_0) - C(u(x_0) - \rho_0) \geq h(x, \bar{Y}, X_0) - C|\bar{x} - \hat{x}||\bar{Y} - \hat{Y}|$$

and

$$h(x,\hat{Y},\bar{X}) = h(x,\hat{Y},\tilde{X}_0) \geq h(x,\hat{Y},X_0) - C(u(x_0)-\rho_0) \geq h(x,\hat{Y},X_0) - C|\bar{x}-\hat{x}||\bar{Y}-\hat{Y}|$$

for all $x \in \Omega$.

And, hence, we have

$$u(x) \geq \max\{h(x,\bar{Y},\tilde{X}_0), h(x,\hat{Y},\tilde{X}_0)\} \geq \max\{h(x,\bar{Y},X_0), h(x,\hat{Y},X_0)\} - C|\bar{x}-\hat{x}||\bar{Y}-\hat{Y}| \geq$$

$$h(x,Y(\lambda),X_0) + K_1\lambda(1-\lambda)|\bar{Y}-\hat{Y}|^2|x-x_0|^2 - K_2|\bar{x}-\hat{x}||\bar{Y}-\hat{Y}|$$

, where in the last inequality we have used lemma (2.4) and renamed the resulting constants.

It remains to prove the claim.

Since $x_0 \in [\bar{x},\hat{x}]$, we can write $x_0 = \dfrac{(1-t)\bar{x}+t\hat{x}}{|(1-t)\bar{x}+t\hat{x}|} := \dfrac{x_t}{|x_t|}$, for some $t \in [0,1]$.

Let $Y_0 \in \partial u(x_0)$, then

$$u(x) \geq h(x,Y_0,X_0) \geq h(x_0,Y_0,X_0) + \langle \nabla_x h(x_0,Y_0,X_0), x-x_0\rangle - M|x-x_0|^2 =$$

$$u(x_0) + \langle \nabla_x h(x_0,Y_0,X_0), x-x_0\rangle - M|x-x_0|^2.$$

where we have used lemma (2.8).Therefore,

$$(1-t)u(\bar{x})+tu(\hat{x}) \geq u(x_0)+\langle \nabla_x h(x_0,Y_0,X_0), (1-t)\bar{x}+t\hat{x}-x_0\rangle - M\Big((1-t)|\bar{x}-x_0|^2+t|\hat{x}-x_0|^2\Big)$$

Recall

$$\bar{x}-x_0 = \frac{\bar{x}(|x_t|-1)-t(\hat{x}-\bar{x})}{|x_t|}$$

and

$$\hat{x}-x_0 = \frac{\hat{x}(|x_t|-1)+(1-t)(\hat{x}-\bar{x})}{|x_t|}$$

and

$$(1-t)|\bar{x}-x_0|^2 + t|\hat{x}-x_0|^2 = 2(1-|x_t|) \leq |\bar{x}-\hat{x}|^2$$

and thus we get

$$u(x_0) \leq (1-t)u(\bar{x}) + tu(\hat{x}) + C|\bar{x}-\hat{x}|^2$$

Next, we have

$$u(\bar{x}) = h(\bar{x},\bar{Y},\tilde{X}_0) \leq h(x_0,\bar{Y},\tilde{X}_0) + \langle \nabla_x h(x_0,\bar{Y},\tilde{X}_0), \bar{x}-x_0\rangle + M|\bar{x}-x_0|^2 =$$

$$\rho_0 + \langle \nabla_x h(x_0,\bar{Y},\tilde{X}_0), \bar{x}-x_0\rangle + M|\bar{x}-x_0|^2$$

and similarly,

$$u(\hat{x}) \leq \rho_0 + \langle \nabla_x h(x_0,\hat{Y},\tilde{X}_0), \hat{x}-x_0\rangle + M|\hat{x}-x_0|^2$$

and hence,

$$(1-t)u(\bar{x})+tu(\hat{x}) \leq \rho_0+(1-t)\langle\nabla_x h(x_0,\bar{Y},\tilde{X}_0),\bar{x}-x_0\rangle+t\langle\nabla_x h(x_0,\hat{Y},\tilde{X}_0),\hat{x}-x_0\rangle+M\Big((1-t)|\bar{x}-x_0|^2+t|\hat{x}-x_0|^2\Big)$$

. The last term is $\leq C|\bar{x}-\hat{x}|^2$.

We estimate the middle term. Inserting the expressions above we get

$$(1-t)\langle\nabla_x h(x_0,\bar{Y},\tilde{X}_0),\bar{x}-x_0\rangle + t\langle\nabla_x h(x_0,\hat{Y},\tilde{X}_0),\hat{x}-x_0\rangle =$$

$$(1-t)t\langle\nabla_x h(x_0,\hat{Y},\tilde{X}_0)-\nabla_x h(x_0,\bar{Y},\tilde{X}_0),\hat{x}-\bar{x}\rangle+$$
$$\frac{|x_t|-1}{|x_t|}\Big(\langle\nabla_x h(x_0,\bar{Y},\tilde{X}_0),\bar{x}\rangle+\langle\nabla_x h(x_0,\hat{Y},\tilde{X}_0),\hat{x}\rangle\Big)$$

. The absolute value of the last term is $\leq C|\bar{x}-\hat{x}|^2$.

And the absolute value of the first term is $\leq C|\bar{Y}-\hat{Y}||\bar{x}-\hat{x}|$ by lemma (2.7). Since $|\bar{x}-\hat{x}| \leq |\bar{Y}-\hat{Y}|$, the claim is proved, and the lemma follows.

We now use lemma (2.9), to slightly modify the above.

Lemma 2.11. *Under the same hypothesis as in lemma* (2.10),*there exist universal constants K_1, K_2, K_3 and there exists $x_0 \in B_\sigma \cap S^{n-1}$ such that for all $Y(\lambda) \in [\bar{Y},\hat{Y}]_{X_0}$, for all $Y \in \Sigma$ and for all $x \in \Omega$,*

$$u(x) \geq h(x,Y,X_0) + K_1\lambda(1-\lambda)|\bar{Y}-\hat{Y}|^2|x-x_0|^2 - K_2|Y-Y(\lambda)||x-x_0| - K_3|\bar{Y}-\hat{Y}||\bar{x}-\hat{x}|$$

where $X_0 = u(x_0)x_0$.

Proof. The proof follows directly from lemmas (2.9) and (2.10) □

Now, we prove the main theorem,

Theorem 2.12. *Assume $B_{2\delta}\cap S^{n-1} \subseteq \Omega$. There exist constants $\tilde{C}_1, \tilde{C}_2$ depending on δ and structure, such that if $\bar{x},\hat{x} \in B_\delta \cap S^{n-1}$ and $\bar{Y} \in \partial u(\bar{x})$, $\hat{Y} \in \partial u(\hat{x})$, and $|\bar{Y}-\hat{Y}| \geq \tilde{C}_1|\bar{x}-\hat{x}|$. Then $|\bar{Y}-\hat{Y}| \leq \tilde{C}_2|\bar{x}-\hat{x}|^\alpha$. Where $\alpha = \frac{1}{4n-5}$*

Proof. By lemma (2.12), there exists $x_0 \in [\bar{x},\hat{x}] \subseteq B_\delta$, such that for all $Y(\lambda) \in [\bar{Y},\hat{Y}]_{X_0}$ with $\frac{1}{4} \leq \lambda \leq \frac{3}{4}$, for all $Y \in \Sigma$ and for all $x \in \Omega$, we have

$$u(x) \geq h(x,Y,X_0) + K_1|\bar{Y}-\hat{Y}|^2|x-x_0|^2 - K_2|Y-Y(\lambda)||x-x_0| - K_3|\bar{Y}-\hat{Y}||\bar{x}-\hat{x}|$$

where $X_0 = u(x_0)x_0$ and $K_i, i = 1,2,3$ are universal.

Let $t_0 = \dfrac{K_2|Y-Y(\lambda)| + \sqrt{K_2^2|Y-Y(\lambda)|^2 + 4K_1K_3|\bar{Y}-\hat{Y}|^3|\bar{x}-\hat{x}|}}{2K_1|\bar{Y}-\hat{Y}|^2}$.

Note that if $|x-x_0| \geq t_0$, then $K_1|\bar{Y}-\hat{Y}|^2|x-x_0|^2-K_2|Y-Y(\lambda)||x-x_0|-K_3|\bar{Y}-\hat{Y}||\bar{x}-\hat{x}| \geq 0$.

Let

$$\mu = \sqrt{|\bar{Y} - \hat{Y}|^3|\bar{x} - \hat{x}|}$$

and assume $|Y - Y(\lambda)| \leq \mu$, then

$$t_0 \leq \frac{K_2 + \sqrt{K_2^2 + 4K_1K_3}}{2K_1} \sqrt{\frac{|\bar{x} - \hat{x}|}{|\hat{Y} - \bar{Y}|}} := K\sqrt{\frac{|\bar{x} - \hat{x}|}{|\hat{Y} - \bar{Y}|}}$$

. Let

$$\sigma = K\sqrt{\frac{|\bar{x} - \hat{x}|}{|\hat{Y} - \bar{Y}|}}$$

. Let $C \geq 1$ be large enough depending on δ and structure such that $\dfrac{K}{\sqrt{C}} \leq \dfrac{\delta}{2}$ and $\dfrac{(\mathrm{diam}(\Sigma))^2}{\sqrt{C}} \leq \mu_0$ Set $\tilde{C}_1 := C$. Assume $|\bar{m} - \hat{m}| \geq |\bar{Y} - \hat{Y}|$, then

$$t_0 \leq \sigma \leq \frac{\delta}{2}$$

and

$$\mu \leq \frac{|\bar{Y} - \hat{Y}|^2}{\sqrt{\tilde{C}_1}} \leq \frac{(\mathrm{diam}(\Sigma))^2}{\sqrt{\tilde{C}_1}} \leq \mu_0$$

where μ_0 is the constant in Hypothesis E.

Let $Y \in \Sigma$ and $|Y - Y(\lambda)| \leq \mu$ for some $\frac{1}{4} \leq \lambda \leq \frac{3}{4}$.

We will show that

$$Y \in \partial u(B(x_0, \sigma) \cap S^{n-1})$$

Notice that $B(x_0, \sigma) \cap S^{n-1} \subseteq B_{2\delta} \cap S^{n-1} \subseteq \Omega$ and if $|x - x_0| \geq \sigma$ and $x \in \Omega$, then $u(x) \geq h(x, Y, X_0)$.

Therefore setting $X = u(x)x$, we have $|X|+\kappa|X-Y| \geq |X_0|+\kappa|X_0-Y|$ for $|x-x_0| \geq \sigma$ and $x \in \Omega$.

It follows that

$$\min\{|X| + \kappa|X - Y| : X = u(x)x\ x \in \Omega\} = |\tilde{X}| + \kappa|\tilde{X} - Y|$$

for some $\tilde{X} = u(\tilde{x})\tilde{x}$ with $\tilde{x} \in B(x_0, \sigma) \cap S^{n-1}$.

This implies that $u(x) \geq h(x, Y, \tilde{X})$, for all $x \in \Omega$.

That is $Y \in \partial u(\tilde{x})$

Hence, we have shown that

$$N_\mu(\{[\bar{Y},\hat{Y}]_{X_0} : \frac{1}{4} \leq \lambda \leq \frac{3}{4}\}) \cap \Sigma \subset \partial u(B(x_0,\sigma) \cap S^{n-1})$$

We now take H^{n-1} measure and use Hypothesis E on the left and the Hypothesis on the refractor on the right to get

$$C_\star \mu^{n-2} |\bar{Y} - \hat{Y}| \leq C^\star \sigma^{n-1}$$

. This gives,

$$|\bar{Y} - \hat{Y}| \leq \tilde{C}_2 |\bar{x} - \hat{x}|^\alpha$$

, with $\tilde{C}_2$ depending on $\tilde{C}_1$ and structure. □

We can now show $u \in C^{1,\alpha}$

Theorem 2.13. *If Σ satisfies hypothesis A,B,C,D and D and u is a refractor from Ω to Σ satisfying the measure condition* (2.15), *then $u \in C^{1,\alpha}(\Omega)$*

Proof. Let $x_0 \in \Omega$. First we show $\partial u(x_0)$ has only one element. Fix $\delta > 0$ such that $B(x_0, 2\delta) \cap S^{n-1} \subseteq \Omega$ and suppose Y_1 and Y_0 are in $\partial u(x_0)$, with $Y_1 \neq Y_0$. Let $\bar{x} \in B(x_0,\delta) \cap S^{n-1}$ and $\bar{Y} \in \partial u(\bar{x})$.

By theorem (2.12), we have $|\bar{Y} - Y_0| \leq C|\bar{x} - x_0|^\alpha$ and $|\bar{Y} - Y_1| \leq C|\bar{x} - x_0|^\alpha$ where the constant C depends on δ. Hence, $|Y_1 - Y_0| \leq 2C|\bar{x} - x_0|^\alpha$, so if we take $\bar{x}$ close enough to x_0 we will reach a contradiction.

Let $Y \in \partial u(x_0)$. First we show that for any $\eta \perp x_0$, we have $D_\eta u(x_0) = \langle \nabla h(x_0, Y, X_0), \eta \rangle$, where $X_0 = u(x_0)x_0$. To see this, let c be any curve such that $c(0) = x_0$ and $c'(0) = \eta$ and $c(t) \in B(x_0,\delta) \cap S^{n-1}$ for all t near 0.

We have

$$u(c(t)) - u(x_0) \geq h(c(t), Y, X_0) - h(x_0, Y, X_0)$$

for all t near 0.

Let $Y(t) \in \partial u(c(t))$ and let $X(t) = u(c(t))c(t)$, then since $u(x) \geq h(x, Y(t), X(t))$ for all $x \in \Omega$, we get

$$u(x_0) - u(c(t)) \geq h(x_0, Y(t), X(t)) - h(c(t), Y(t), X(t))$$

for all t. Therefore, we have for all $t > 0$, small

$$\frac{h(c(t), Y, X_0) - h(x_0, Y, X_0)}{t} \leq \frac{u(c(t)) - u(x_0)}{t} \leq \frac{h(c(t), Y(t), X(t)) - h(x_0, Y(t), X(t))}{t}$$

Note that for each t

$$\frac{h(c(t),Y(t),X(t)) - h(x_0,Y(t),X(t))}{t} = \langle \nabla h(\tilde{x},Y(t),X(t)), \frac{c(t)-c(0)}{t}\rangle$$

for some $\tilde{x} \in [x_0, c(t)]$

Letting $t \to 0$, proves the claim. We have used that $Y(t) \to Y$ as $t \to 0$ thanks to theorem (2.12) and that

$X(t) \to X_0$ by continuity of u.

Define $\tilde{u}(X) = u(\frac{X}{|X|})$. We will show that for $|x_0| = 1$, we have $\nabla\tilde{u}(x_0) = \nabla^T h(x_0, Y, X_0)$, where

$$\nabla^T h(x_0,Y,X_0) = \nabla h(x_0,Y,X_0) - \langle \nabla h(x_0,Y,X_0), x_0\rangle x_0$$

To prove the claim, let $c(t) = \dfrac{x_0 + te_i}{|x_0 + te_i|}$ and note that $c(0) = x_0$ and $c'(0) = e_i - \langle x_0, e_i\rangle x_0$.

And we have $\dfrac{\tilde{u}(x_0 + te_i) - \tilde{u}(x_0)}{t} = \dfrac{u(c(t)) - u(x_0)}{t}$, thus, letting $t \to 0$ and using the first part we get

$$\frac{\partial \tilde{u}}{\partial x_i}(x_0) = \langle \nabla h(x_0,Y,X_0), e_i - \langle x_0, e_i\rangle x_0\rangle$$

and this proves the claim.

Next, let $\bar{x}, \hat{x} \in B(x_0,\delta) \cap S^{n-1}$.

Let $\bar{Y} \in \partial u(\bar{x})$ and $\hat{Y} \in \partial u(\hat{x})$, then

$$|\nabla\tilde{u}(\bar{x}) - \nabla\tilde{u}(\hat{x})| = |\nabla^T h(\bar{x},\bar{Y},\bar{X}) - \nabla^T h(\hat{x},\hat{Y},\hat{X})| \leq$$
$$C\Big(|\bar{X} - \hat{X}| + |\bar{x} - \hat{x}| + |\bar{Y} - \hat{Y}|\Big) \leq$$
$$C|\bar{x} - \hat{x}|^\alpha.$$

We have used theorem (2.12) in the last inequality.

We prove the first inequality.

First note

$$|\nabla^T h(\bar{x},\bar{Y},\bar{X}) - \nabla^T h(\hat{x},\hat{Y},\hat{X})| \leq 2|\nabla h(\bar{x},\bar{Y},\bar{X}) - \nabla h(\hat{x},\hat{Y},\hat{X})| + C|\bar{x} - \hat{x}|$$

.

Next, we write

$$|\nabla h(\bar{x},\bar{Y},\bar{X}) - \nabla h(\hat{x},\hat{Y},\hat{X})| \leq |\nabla h(\bar{x},\bar{Y},\bar{X}) - \nabla h(\bar{x},\hat{Y},\bar{X})| +$$
$$|\nabla h(\bar{x},\hat{Y},\bar{X}) - \nabla h(\hat{x},\hat{Y},\bar{X})| +$$
$$|\nabla h(\hat{x},\hat{Y},\bar{X}) - \nabla h(\hat{x},\hat{Y},\hat{X})|$$

And we have $|\nabla h(\bar{x}, \bar{Y}, \bar{X}) - \nabla h(\bar{x}, \hat{Y}, \bar{X})| \leq C|\bar{Y} - \hat{Y}|$ by (2.7).

Also, $|\nabla h(\bar{x}, \hat{Y}, \bar{X}) - \nabla h(\hat{x}, \hat{Y}, \bar{X})| \leq C|\bar{x} - \hat{x}|$ using mean value in x and

$$\nabla h(\hat{x}, \hat{Y}, \bar{X}) - \nabla h(\hat{x}, \hat{Y}, \hat{X}) =$$
$$\frac{\kappa^2 h(\hat{x}, \hat{Y}, \bar{X})\hat{Y}}{\sqrt{(b - \kappa^2\langle x, \hat{Y}\rangle)^2 - (1-\kappa^2)(b^2 - \kappa^2|\hat{Y}|^2)}} - \frac{\kappa^2 h(\hat{x}, \hat{Y}, \hat{X})\hat{Y}}{\sqrt{(\hat{b} - \kappa^2\langle x, \hat{Y}\rangle)^2 - (1-\kappa^2)(\hat{b}^2 - \kappa^2|\hat{Y}|^2)}}$$

, where $b = |\bar{X}| + \kappa|\hat{Y} - \bar{X}|$ and $\hat{b} = |\hat{X}| + \kappa|\hat{Y} - \hat{X}|$. Since by assumption $\hat{Y} \in \Sigma$ and $\bar{X}, \hat{X} \in \Gamma_{C_1,C_2}$, it follows using lemma (2.5), that

$$|\nabla h(\hat{x}, \hat{Y}, \bar{X}) - \nabla h(\hat{x}, \hat{Y}, \hat{X})| \leq C|\bar{X} - \hat{X}|$$

□

2.3. **A property of refractors.** Let us assume that if Y_0 is a point of Σ, then the tangent plane to Σ at Y_0 does not intersect the graph of u.

Let

$$S^\star = \{Y \in \Sigma : Y \in \partial u(\bar{x}) \cap \partial u(\hat{x}),\ \bar{x} \neq \hat{x} \in \Omega\}.$$

We claim $H^{n-1}(S^\star) = 0$.

Define $u^\star : R^n \to R$ by

$$u^\star(Y) = \min\{|X| + \kappa|X - Y| : X = u(x)x,\ x \in \Omega\}.$$

It is easy to see that $u^\star$ is Lipschitz in R^n.

Note that if $\bar{Y} \in \partial u(\bar{x})$, then for $X = u(x)x$ and $\bar{X} = u(\bar{x})\bar{x}$, we have

$$|X| + \kappa|X - \bar{Y}| \geq |\bar{X}| + \kappa|\bar{X} - \bar{Y}|$$

and hence

$$u^\star(\bar{Y}) = |\bar{X}| + \kappa|\bar{X} - \bar{Y}|$$

and

$$u^\star(Y) \leq u^\star(\bar{Y}) + \kappa|\bar{X} - Y| - \kappa|\bar{X} - \bar{Y}|,$$

for all $Y \in R^n$. In particular, if $Y_0 \in S^\star$ and say $Y_0 \in \partial u(\bar{x}) \cap \partial u(\hat{x})$, then

$$u^\star(Y) \leq u^\star(Y_0) + \kappa|\bar{X} - Y| - \kappa|\bar{X} - Y_0|,$$

and

$$u^\star(Y) \leq u^\star(Y_0) + \kappa|\hat{X} - Y| - \kappa|\hat{X} - Y_0|,$$

for all $Y \in R^n$.

Assume $O \subseteq R^{n-1}$ is open and $\psi : R^{n-1} \to R^n$ is Lipschitz and such that $\Sigma = \psi(\bar{O})$ and ψ is one to one in $\bar{O}$.

Let $\tilde{S} = \psi^{-1}(S^\star)$. We show $H^{n-1}(\tilde{S}) = 0$.

Let $h(Y') = u^\star(\psi(Y'))$.

h is Lipschitz in R^{n-1} and we claim that if $Y' \in \tilde{S}$, the h not differentiable at Y'.

Let $Y_0' \in \tilde{S}$ and let $Y_0 = \psi(Y_0')$. Assume $Y_0 \in \partial u(\bar{x}) \cap \partial u(\hat{x})$.

Then,

$$h(Y') \leq h(Y_0') + |\bar{X} - \psi(Y')| - |\bar{X} - \psi(Y_0')|$$

and

$$h(Y') \leq h(Y_0') + |\hat{X} - \psi(Y')| - |\hat{X} - \psi(Y_0')|$$

If h is differentiable at Y_0' then

$$\nabla_{Y'}(|\bar{X} - \psi(Y')|) = \nabla_{Y'}(|\hat{X} - \psi(Y')|)$$

at $Y' = Y_0'$.

And therefore,

$$D\psi(Y_0')^T \frac{Y_0 - \bar{X}}{|Y_0 - \bar{X}|} = D\psi(Y_0')^T \frac{Y_0 - \hat{X}}{|Y_0 - \hat{X}|}$$

Set $w = \dfrac{Y_0 - \bar{X}}{|Y_0 - \bar{X}|} - \dfrac{Y_0 - \hat{X}}{|Y_0 - \hat{X}|}$.

So, $D\psi(Y_0')^T w = 0$. This says that $\langle v_k, w \rangle = 0$, for $k = 1, ..., n-1$, where v_k are the columns of $D\psi(Y_0')$ and this $n-1$ vectors span the tangent plane to Σ at Y_0.

Therefore w is normal to the tangent plane to Σ at Y_0.

In particular, the line $Y_0 + t\dfrac{1}{2}\Big(\dfrac{Y_0 - \bar{X}}{|Y_0 - \bar{X}|} + \dfrac{Y_0 - \hat{X}}{|Y_0 - \hat{X}|}\Big)$ is on the tangent plane to Σ at Y_0.

However, this line intersects the straight segment $[\bar{X}, \hat{X}]$, which implies that either both $\bar{X}$ and $\hat{X}$ are on the tangent plane or they are on opposite sides of the tangent plane. In either case, since they are points on the graph of u, the tangent plane intersects the graph of u, which is a contradiction to our assumption.

Therefore h is not differentiable at points in $\tilde{S}$ and hence $H^{n-1}(\tilde{S}) = 0$.

This implies, since ψ is Lipschitz, that $H^{n-1}(S^\star) = 0$ as we wanted to show.

2.4. **A pointwise condition.** In this subsection we will motivate Hypothesis D.

First, we need a new parametrization of the curve $[\bar{m}, \hat{m}]_{x_0}$.

Let

$$\nabla^T h(x, Y, X_0) = \nabla h(x, Y, X_0) - \langle \nabla h(x, Y, X_0), x \rangle x.$$

Note that for $Y = X_0 + sm$, we get

$$\nabla^T h(x_0, Y, X_0) = \kappa |X_0| \frac{m - \langle m, x_0 \rangle x_0}{1 - \kappa \langle m, x_0 \rangle}$$

.

Set $v = \kappa |X_0| \dfrac{m - \langle m, x_0 \rangle x_0}{1 - \kappa \langle m, x_0 \rangle}$ and solve for m, with $\langle m, x_0 \rangle \geq \kappa$ and $|m| = 1$.

First note $\langle m, x_0 \rangle = \dfrac{|v|^2 + |X_0| \sqrt{\kappa^2 |X_0|^2 - (1 - \kappa^2)|v|^2}}{\kappa(|v|^2 + |X_0|^2}$, and thus

$$\frac{1 - \kappa \langle m, x_0 \rangle}{\kappa |X_0|} = \frac{1 - \kappa^2}{\kappa} \frac{1}{|X_0| + \sqrt{\kappa^2 |X_0|^2 - (1 - \kappa^2)|v|^2}}$$

. Note that $v \perp x_0$ and $|v|^2 \leq \dfrac{\kappa^2 |X_0|^2}{1 - \kappa^2}$ Set

$$t(v) = \frac{1 - \kappa^2}{\kappa} \frac{1}{|X_0| + \sqrt{\kappa^2 |X_0|^2 - (1 - \kappa^2)|v|^2}}$$

.

We can write $m = \langle m, x_0 \rangle x_0 + t(v)v$ and hence

$$m = \frac{1}{\kappa} x_0 + t(v - X_0)$$

.

Given $\bar{m}, \hat{m} \in S^{n-1}$ such that $\langle \bar{m}, x_0 \rangle \geq \kappa$ and $\langle \hat{m}, x_0 \rangle \geq \kappa$, we let

$\bar{v} = \kappa |X_0| \dfrac{\bar{m} - \langle \bar{m}, x_0 \rangle x_0}{1 - \kappa \langle \bar{m}, x_0 \rangle}$ and

$\hat{v} = \kappa |X_0| \dfrac{\hat{m} - \langle \hat{m}, x_0 \rangle x_0}{1 - \kappa \langle \hat{m}, x_0 \rangle}$.

Let $v_\gamma = (1 - \gamma)\bar{v} + \gamma \hat{v}$. We parametrize the curve $[\bar{m}, \hat{m}]_{x_0}$ as follows:

$$m(\gamma) = \frac{1}{\kappa} x_0 + t(v_\gamma)(v_\gamma - X_0)$$

, $\gamma \in [0, 1]$.

It is important to note that setting (with obvious notation, $t := t(v_\gamma)$)

$$\bar{\beta} = \frac{t(1 - \gamma)}{\bar{t}}, \ \hat{\beta} = \frac{t\gamma}{\hat{t}},$$

then

$$x_0 - \kappa m = \bar{\beta}(x_0 - \kappa \bar{m}) + \hat{\beta}(x_0 - \kappa \hat{m})$$

. The Hypothesis D reads

$$\frac{1}{s} \geq \frac{\bar{\beta}}{\bar{s}} + \frac{\hat{\beta}}{\hat{s}},$$

which is equivalent to

$$\frac{1}{st} \geq (1-\gamma)\frac{1}{\bar{s}\bar{t}} + \gamma\frac{1}{\hat{s}\hat{t}}$$

. In other words, $\frac{1}{st}$ is a concave function of v for $v \perp x_0$.

Therefore Hypothesis D is equivalent to

$$\langle D_v^2(\frac{1}{st})\xi, \xi\rangle \leq 0,$$

for all $\xi \perp x_0$.

Let us motivate the above condition.

Set $(D^2)^T h(x, Y, X_0) = D^2 h(x, Y, X_0) - \langle \nabla h(x, Y, X_0), x\rangle\rangle I$

Fix $\bar{Y}, \hat{Y} \in \Sigma$ and $X_0 \in \Gamma_{C_1,C_2}$. Write $\bar{Y} = X_0 + \bar{s}\bar{m}$ and $\hat{Y} = X_0 + \hat{s}\hat{m}$. and let

$$Y(\gamma) = X_0 + s(v_\gamma)m(v_\gamma)$$

for $\gamma \in [0,1]$ be a parametrization of $[\bar{Y}, \hat{Y}]_{X_0}$

Let

$$S = \{x \in S^{n-1} : h(x, \bar{Y}, X_0) = h(x, \hat{Y}, X_0)\}$$

For $x \in S$, we want a necessary condition such that

$$(1-\gamma)h(x, \bar{Y}, X_0) + \gamma h(x, \hat{Y}, X_0) \geq h(x, Y(\gamma), X_0)$$

holds.

Set $\nabla^T h(x_0, \bar{Y}, X_0) = \bar{v}$ and $\nabla^T h(x_0, \hat{Y}, X_0) = \hat{v}$, so $\bar{v}, \hat{v} \perp x_0$.

Notice that

$$\nabla^T h(x_0, Y(\gamma), X_0) = v_\gamma$$

Let $\xi = \hat{v} - \bar{v}$.

Let c be any curve in S such that $c(0) = x_0$ and set $\eta = c'(0)$. Note that $\xi, \eta \perp x_0$ and $\xi \perp \eta$

Let $\phi(t) = (1-\gamma)h(c(t), \bar{Y}, X_0) + \gamma h(c(t), \hat{Y}, X_0) - h(c(t), Y(\gamma), X_0)$.

We have

$$\phi'(t) = \langle (1-\gamma)\nabla h(c(t), \bar{Y}, X_0) + \gamma \nabla h(c(t), \hat{Y}, X_0) - \nabla h(c(t), Y(\gamma), X_0), c'(t)\rangle =$$

$$\langle (1-\gamma)\nabla^T h(c(t), \bar{Y}, X_0) + \gamma \nabla^T h(c(t), \hat{Y}, X_0) - \nabla^T h(c(t), Y(\gamma), X_0), c'(t)\rangle.$$

Note that $\phi'(0) = 0$ and

$$\phi''(0) = \langle\Big((1-\gamma)(D^2)^T h(x_0, \bar{Y}, X_0) + \gamma(D^2)^T h(x_0, \hat{Y}, X_0) - (D^2)^T h(x_0, Y(\gamma), X_0)\Big)\eta, \eta\rangle.$$

We need that $\phi''(0) \geq 0$.

Let

$$H(\gamma) = \langle(D^2)^T h(x_0, Y(\gamma), X_0)\Big)\eta, \eta\rangle.$$

We need that $H''(\gamma) \geq 0$.

Note that

$$H''(\gamma) = \langle D_v^2\Big(\langle(D^2)^T h(x_0, Y(\bar{v} + \gamma\xi), X_0)\Big)\eta, \eta\rangle\Big)\xi, \xi\rangle.$$

We will compute this last quantity.

First we compute

$$\langle(D^2)^T h(x_0, Y(v), X_0)\Big)\eta, \eta\rangle$$

Set $Q = (b - \kappa^2\langle x, Y\rangle)^2 - (1-\kappa^2)(b^2 - \kappa^2|Y|^2)$.

We have

$$h_i = Q^{\frac{-1}{2}}\kappa^2 Y_i h$$

and

$$h_{i,j} = Q^{\frac{-3}{2}}\kappa^4 Y_i Y_j\Big(b - \kappa^2\langle x, Y\rangle + Q^{\frac{1}{2}}\Big)h$$

At $x = x_0$,and for $Y = X_0 + sm$ we have

$$Q^{\frac{1}{2}} = \kappa s(1 - \kappa\langle x_0, m\rangle)$$

and

$$(b - \kappa^2\langle x, Y\rangle + Q^{\frac{1}{2}} = (1-\kappa^2)|X_0| + 2\kappa s(1 - \kappa\langle x_0, m\rangle)$$

and hence at $x = x_0$, $Y = X_0 + sm$ and $b = |X_0| + ks$ we have

$$h_i = \frac{\kappa^2|X_0|Y_i}{\kappa s(1 - \kappa\langle x_0, m\rangle)}$$

and

$$h_{i,j} = \frac{\kappa^4 Y_i Y_j|X_0|\Big((1-\kappa^2)|X_0| + 2\kappa s(1 - \kappa\langle x_0, m\rangle)\Big)}{\Big(\kappa s(1 - \kappa\langle x_0, m\rangle)\Big)^3}$$

. Therefore, for $|\eta| = 1$, we get

$$\langle(D^2)^T h(x_0, Y(v), X_0)\eta, \eta\rangle = \frac{\kappa^4|X_0|\Big((1-\kappa^2)|X_0| + 2\kappa s(1 - \kappa\langle x_0, m\rangle)\Big)\langle Y, \eta\rangle^2}{\Big(\kappa s(1 - \kappa\langle x_0, m\rangle)\Big)^3} - \frac{\kappa^2|X_0|\langle Y, x_0\rangle}{\kappa s(1 - \kappa\langle x_0, m\rangle)}.$$

Now, $\langle Y, x_0\rangle = |X_0| + s\langle m, x_0\rangle$ and $\langle Y, \eta\rangle = s\langle m, \eta\rangle$, since $\eta \perp x_0$.

Also recall $1-\kappa\langle m,x_0\rangle=\kappa|X_0|t$, thus we get

$$\langle (D^2)^T h(x_0,Y(v),X_0)\eta,\eta\rangle=\frac{\Big((1-\kappa^2)+2\kappa^2 st)s^2\langle m,\eta\rangle^2}{\kappa^2|X_0|s^3t^3}-\frac{\kappa|X_0|+\kappa s\langle m,x_0\rangle}{\kappa st}$$

Finally note that since $m=\frac{1}{\kappa}x_0+t(v-X_0)$ with $v\perp x_0$, we have $\langle m,\eta\rangle=t\langle v,\eta\rangle$ and hence after simplification we get

$$\langle (D^2)^T h(x_0,Y(v),X_0)\eta,\eta\rangle=\frac{\Big((1-\kappa^2)\langle v,\eta\rangle^2-\kappa^2|X_0|^2}{\kappa^2|X_0|st}+\frac{2\langle v,\eta\rangle^2}{|X_0|}-\frac{1}{\kappa t}+|X_0|$$

Now we can finish the computation of

$$\langle D_v^2\Big(\langle (D^2)^T h(x_0,Y(\bar v+\gamma\xi),X_0)\Big)\eta,\eta\rangle\Big)\xi,\xi\rangle.$$

Using that $\eta\perp\xi$, we get

$$\langle D_v^2\Big(\langle (D^2)^T h(x_0,Y(\bar v+\gamma\xi),X_0)\Big)\eta,\eta\rangle\Big)\xi,\xi\rangle=\frac{\Big((1-\kappa^2)\langle v,\eta\rangle^2-\kappa^2|X_0|\Big)}{\kappa^2|X_0|}\langle D_v^2(\frac{1}{st})\xi,\xi\rangle-\frac{1}{\kappa}\langle D_v^2(\frac{1}{t})\xi,\xi\rangle$$

Recall $\frac{1}{t}=\dfrac{\kappa\Big(|X_0|+\sqrt{\kappa^2|X_0|^2-(1-\kappa^2)|v|^2}}{1-\kappa^2}$ to compute

$$\langle D_v^2(\frac{1}{t})\xi,\xi\rangle=-(\kappa^2|X_0|^2-(1-\kappa^2)|v|^2)^{-\frac{3}{2}}\Big((1-\kappa^2)\langle v,\xi\rangle^2+(\kappa^2|X_0|^2-(1-\kappa^2)|v|^2)|\xi|^2\Big).$$

Hence we get

$$\langle D_v^2\Big(\langle (D^2)^T h(x_0,Y(\bar v+\gamma\xi),X_0)\Big)\eta,\eta\rangle\Big)\xi,\xi\rangle=\frac{\Big((1-\kappa^2)\langle v,\eta\rangle^2-\kappa^2|X_0|\Big)}{\kappa^2|X_0|}\langle D_v^2(\frac{1}{st})\xi,\xi\rangle+$$

$$\frac{\kappa^2|X_0|^2-(1-\kappa^2)(|v|^2-\langle\xi,v\rangle^2)}{(\kappa^2|X_0|^2-(1-\kappa^2)|v|^2)^{\frac{3}{2}}}$$

Noting that $\Big((1-\kappa^2)\langle v,\eta\rangle^2-\kappa^2|X_0|^2\Big)\le 0$, the condition

$$\langle D_v^2\Big(\langle (D^2)^T h(x_0,Y(\bar v+\gamma\xi),X_0)\Big)\eta,\eta\rangle\Big)\xi,\xi\rangle\ge 0$$

is equivalent to

$$\langle D_v^2(\frac{1}{st})\xi,\xi\rangle\le\frac{\kappa^2|X_0|^2-(1-\kappa^2)(|v|^2-\langle\xi,v\rangle^2)}{\kappa^2|X_0|^2-(1-\kappa^2)\langle v,\eta\rangle^2}\frac{\kappa^2|X_0|^2}{(\kappa^2|X_0|^2-(1-\kappa^2)|v|^2)^{\frac{3}{2}}}$$

for all $\eta\perp\xi$, with $|\xi|,|\eta|=1$

Note that $\langle v,\xi\rangle^2+\langle v,\eta\rangle^2\le|v|^2$ (with equality in R^3) and hence

$$\frac{\kappa^2|X_0|^2-(1-\kappa^2)(|v|^2-\langle\xi,v\rangle^2)}{\kappa^2|X_0|^2-(1-\kappa^2)\langle v,\eta\rangle^2}\le 1$$

Therefore, we arrive at the necessary condition

$$\langle D_v^2(\frac{1}{st})\xi,\xi\rangle \leq \frac{\kappa^2|X_0|^2}{(\kappa^2|X_0|^2-(1-\kappa^2)|v|^2)^{\frac{3}{2}}}$$

.

Notice that we are assuming in Hypothesis D the stronger condition $\langle D_v^2(\frac{1}{st})\xi,\xi\rangle \leq 0$ and we have shown in lemma (2.4) that this sufficient for the estimate in this lemma.

Next, we will show that there exits a positive constant C depending on structure such that under the new hypothesis

$$\langle D_v^2(\frac{1}{st})\xi,\xi\rangle \leq 2C \tag{2.16}$$

for all $\xi \perp x_0$ with $|\xi|=1$ the inequality in lemma (2.4) still holds.

It is easy to prove that the inequality (2.16) implies that

$$\frac{1}{st} \geq (1-\gamma)\frac{1}{\bar{s}\bar{t}} + \gamma\frac{1}{\hat{s}\hat{t}} - C\gamma(1-\gamma)|\bar{v}-\hat{v}|^2$$

and hence, that

$$\frac{1}{s} \geq \frac{\bar{\beta}}{\bar{s}} + \frac{\hat{\beta}}{\hat{s}} - Ct\gamma(1-\gamma)|\bar{v}-\hat{v}|^2.$$

Set $K := Ct\gamma(1-\gamma)|\bar{v}-\hat{v}|^2$.

We have

$$\bar{\beta}\hat{s} + \hat{\beta}\bar{s} \leq \frac{\bar{s}\hat{s}}{s} + K\bar{s}\hat{s} \tag{2.17}$$

Using (2.17) in place of (2.14) in inequality (2.13), we see that in order for the estimate to continue holding we need that

$$K\bar{s}\Big(|X_0|(1-\kappa\langle x_0,\bar{m}\rangle - h(x,\bar{Y},X_0)(1-\kappa\langle x,\bar{m}\rangle)\Big) \leq \frac{1-(\bar{\beta}+\hat{\beta})}{4}|x-x_0|^2 h(x,\bar{Y},X_0) \tag{2.18}$$

In order to verify (2.18) we will prove that

$$0 \leq \bar{s}\Big(|X_0|(1-\kappa\langle x_0,\bar{m}\rangle - h(x,\bar{Y},X_0)(1-\kappa\langle x,\bar{m}\rangle)\Big) \leq C_\kappa|x-x_0|^2 \tag{2.19}$$

and that

$$K \leq 2C|X_0|(1-(\bar{\beta}+\hat{\beta})) \tag{2.20}$$

We prove (2.18):

Write $X = h(x,Y,X_0)x$ with $Y = X_0 + \bar{s}\bar{m}$.

We have $|X|+\kappa|X-Y| = |X_0|+\kappa|X_0-Y|$, which after simplification can be written as

$$|X_0|(1-\kappa\langle\bar{m},x_0\rangle) - |X|(1-\kappa\langle\bar{m},x\rangle) = \frac{\kappa^2|X-X_0|^2 - (|X|-|X_0|)^2}{2\kappa\bar{s}}$$

By lemma (2.3), the left hand side above is non negative and the right hand side is smaller than $\dfrac{\kappa|X-X_0|^2}{2\bar{s}} \leq C_\kappa \dfrac{|x-x_0|^2}{\bar{s}}$

and this proves the estimate.

And to prove (2.20), we have

$$1-(\bar{\beta}+\hat{\beta}) = t(\frac{1}{t} - \frac{1-\gamma}{\bar{t}} - \frac{\gamma}{\hat{t}}) \geq \frac{t\gamma(1-\gamma)|\bar{v}-\hat{v}|^2}{2|X_0|}$$

where in the last inequality we have used Taylors theorem. The estimate is proved.

Therefore, in order for (2.18) to hold, we need $C|X_0|$ to be bounded above by a constant depending on structure.

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