Operator least squares problems and Moore-Penrose inverses in Krein spaces

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Abstract

Given a Krein space $\mathcal{H}$ and B, C in $L(\mathcal{H})$, the bounded linear operators on $\mathcal{H}$, the minimization / maximization of expressions of the form $(BX - C)^{\#}(BX - C)$ as X runs over $L(\mathcal{H})$ is studied. Complete solutions are found for the problems posed, including solvability criteria and a characterization of the solutions when they exist. Min-max problems associated to Krein space decompositions of B are also considered, leading to a characterization of the Moore-Penrose inverse as the unique solution of a variational problem. Other generalized inverses are similarly described.

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1. Introduction

Indefinite metric spaces, and least squares problems on such spaces, naturally appear in various applications, especially in the search for alternative H^{∞} algorithms in system and control theory. Commonly, those least squares estimations are formulated and solved in Minkowski spaces (i.e., finite dimensional Krein spaces) [16, 17, 18]. Here, we discuss instead least-squares type problems for operators in Krein spaces of arbitrary countable dimension, adopting our approach from [12, 13], along with some arguments from [6, 7].

A principal goal of the paper is the study of the following operator approximation problem: Given two bounded operators B and C such that B has closed range, determine whether there exists a bounded operator X_0 for which the minimum

$$\min_{X \in L(\mathcal{H})} (BX - C)^{\#}(BX - C) \tag{1.1}$$

is attained. Here, $L(\mathcal{H})$ stands for the space of all the bounded linear operators from $\mathcal{H}$ to $\mathcal{H}$. The corresponding maximization problem can be likewise formulated. In Hilbert spaces where the norm is associated to the inner product, the operator norm (or any other unitarily invariant norm) is a natural choice for measuring distances. Instead, we opt for minimizing or maximizing the expression in (1.1) with respect to the operator order associated to the indefinite product in $\mathcal{H}$, where this is no longer the case.

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In solving problem (1.1) the concept of *indefinite inverses* is a fundamental tool, extending the concept of weighted inverses for matrices first presented by S.K. Mitra in [23] and later generalized to Hilbert space operators in [8, 6]. Over Krein spaces, an indefinite inverse of a closed range operator B is defined as any solution X_0 of the equation $B^{\#}(BX - I) = 0$, where $B^{\#}$ is the adjoint of B with respect to the Krein space inner product. When the range of B is definite, the weighted inverse is recovered. For instance, if the range of B is non-negative, X_0x is an (indefinite) least squares solution of the equation $Bz = x$; i.e.,

$$[\,x - X_0x, x - X_0x\,] \leq [\,x - z, x - z\,], \text{ for every } x,\ z \in \mathcal{H}.$$

Problem (1.1) admits a solution for any choice of C (and not just $C = I$) if and only if the range of B is regular and non-negative. In this case, the set of solutions is characterized as the set of indefinite inverses of B in the range of C, which is an affine manifold.

Given the indefinite nature of the subspaces involved, it is also natural to consider min-max problems by writing the operator B as the sum of two operators with non-negative and non-positive ranges, respectively. The results obtained when solving (1.1) can be applied to study this problem, in which case the set of solutions not only includes the set of indefinite inverses but also those operators with B-neutral range; i.e., those operators Z such that the range of BZ is in the cone of neutral vectors of $\mathcal{H}$.

In the Krein space framework, the analysis of the existence of an analog of the Moore-Penrose inverse of B; that is, a generalized inverse $B^{\dagger}$ such that $BB^{\dagger}$ and $B^{\dagger}B$ are selfadjoint (with respect to the indefinite inner product), was first carried out by X. Mary [22]. He proved that a bounded linear operator B admits a unique bounded Moore-Penrose inverse if and only if both the range and null space of B are regular subspaces. When the range and null space of B are definite and regular, we give a variational characterization of the Moore-Penrose inverse of B. Indeed, $B^{\dagger}$ is a distinguished element of the set of solutions of problem (1.1), in the sense that ${B^{\dagger}}^{\#}B^{\dagger} \leq Y^{\#}Y$ for any other solution Y of (1.1).

Following ideas from [12] we generalize the concept of Moore-Penrose inverse to operators with *pseudo-regular* range and nullspace. A (closed) subspace $\mathcal{S}$ of a Krein space $\mathcal{H}$ is called pseudo-regular if the algebraic sum $\mathcal{S} + \mathcal{S}^{[\perp]}$ is closed. Every closed subspace of a Pontryagin space is pseudo-regular. On the other hand, every pseudo-regular subspace of a Krein space is the range of a normal projection; see [14] and [21]. When the nullspace and range of a bounded operator are pseudo-regular, there exists a family of *generalized inverses*. The projections associated to these inverses are normal (but not necessarily selfadjoint). We characterize these generalized inverses in a variational way as well.

The paper comprises five sections, six if this introductory section is included. Section 2 is a brief expository overview of Krein spaces and operators on them, and serves to fix notation. It also presents some results that are needed further on. In Section 3 the notion of an indefinite inverse of an operator is defined. Section 4 deals with problem (1.1), first for $C = I$, and then for C an arbitrary bounded operator. The solutions to these problems are characterized as indefinite inverses of B. This indefinite minimization problem and its counterpart for the symmetric maximization problem are applied in Section 5, where the min-max problem is addressed. In Section 6, the main results about the Moore-Penrose inverse and other generalized inverses are presented.

2. Preliminaries

In the following, all Hilbert spaces are complex and separable. If $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces, $L(\mathcal{H}, \mathcal{K})$ stands for the space of all the bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$ and $CR(\mathcal{H}, \mathcal{K})$ for the subset of $L(\mathcal{H}, \mathcal{K})$ comprising all the operators with closed ranges. When $\mathcal{H} = \mathcal{K}$ we write, instead, $L(\mathcal{H})$ and $CR(\mathcal{H})$. The range and null space of $A \in L(\mathcal{H}, \mathcal{K})$ are denoted by $R(A)$ and $N(A)$, respectively. Given a subset $\mathcal{T} \subseteq \mathcal{K}$, the preimage of $\mathcal{T}$ under A is denoted by $A^{-1}(\mathcal{T})$ and $A^{-1}(\mathcal{T}) = \{x \in \mathcal{H} :\ Ax \in \mathcal{T}\}$.

The direct sum of two closed subspaces $\mathcal{M}$ and $\mathcal{N}$ of $\mathcal{H}$ is represented by $\mathcal{M} \dot{+} \mathcal{N}$. If $\mathcal{H}$ is decomposed as $\mathcal{H} = \mathcal{M} \dot{+} \mathcal{N}$, the oblique projection onto $\mathcal{M}$ with null space $\mathcal{N}$ is denoted $P_{\mathcal{M}//\mathcal{N}}$ and abbreviated $P_{\mathcal{M}}$ when $\mathcal{N} = \mathcal{M}^{\perp}$. The set of oblique projections is $\mathcal{Q} := \{Q \in L(\mathcal{H}) : Q^2 = Q\}$.

The following is a well-known result on range inclusion and factorizations of operators.

Lemma 2.1 (Douglas' Lemma [9])**.** *Let $Y, Z \in L(\mathcal{H})$. Then $R(Z) \subseteq R(Y)$ if and only if there exists $D \in L(\mathcal{H})$ such that $Z = YD$.*

Krein Spaces

Although familiarity with operator theory on Krein spaces is presumed, to fix notation, we include some basic definitions and results. Standard references on Krein spaces and operators on them are [1, 4, 5], as well as [10, 11].

Consider a linear space $\mathcal{H}$ with an indefinite metric, i.e., a sesquilinear Hermitian form $[\ ,\ ]$. A vector $x \in \mathcal{H}$ is said to be *positive* if $[\,x, x\,] > 0$. A subspace $\mathcal{S}$ of $\mathcal{H}$ is *positive* if every $x \in \mathcal{S}$, $x \neq 0$, is a positive vector. *Negative, nonnegative, nonpositive* and *neutral* vectors and subspaces are defined likewise.

We say that two closed subspaces $\mathcal{M}$ and $\mathcal{N}$ are *orthogonal*, and write $\mathcal{M}\ [\perp]\ \mathcal{N}$, if $[\,m, n\,] = 0$ for every $m \in \mathcal{M}$ and $n \in \mathcal{N}$. We denote the orthogonal direct sum of two closed subspaces $\mathcal{M}$ and $\mathcal{N}$ by $\mathcal{M}\ [\dot{+}]\ \mathcal{N}$.

Given any subspace $\mathcal{S}$ of $\mathcal{H}$, the *orthogonal companion* of $\mathcal{S}$ in $\mathcal{H}$, say $\mathcal{S}^{[\perp]}$, is defined as

$$\mathcal{S}^{[\perp]} := \{x \in \mathcal{H} : [\,x, s\,] = 0 \text{ for every } s \in \mathcal{S}\}.$$

The isotropic part $\mathcal{S}^o := \mathcal{S} \cap \mathcal{S}^{[\perp]}$ can be non-trivial.

An indefinite metric space $(\mathcal{H}, [\ ,\ ])$ is a *Krein space* if $\mathcal{H}$ admits a decomposition into an orthogonal direct sum in the form

$$\mathcal{H} = \mathcal{H}_+\ [\dot{+}]\ \mathcal{H}_- \tag{2.1}$$

where $(\mathcal{H}_+, [\ ,\ ])$ and $(\mathcal{H}_-, -[\ ,\ ])$ are Hilbert spaces. Any decomposition with these properties is called a *fundamental decomposition* of $\mathcal{H}$.

Given a Krein space $(\mathcal{H}, [\ ,\ ])$ with a fundamental descomposition $\mathcal{H} = \mathcal{H}_+\ [\dot{+}]\ \mathcal{H}_-$, the (orthogonal) direct sum of the Hilbert spaces $(\mathcal{H}_+, [\ ,\ ])$ and $(\mathcal{H}_-, -[\ ,\ ])$ is a Hilbert space. It is denoted by $(\mathcal{H}, \langle\ ,\ \rangle)$. Notice that the inner product $\langle\ ,\ \rangle$ and the corresponding quadratic norm $\|\ \|$ depend on the fundamental decomposition. A subspace $\mathcal{S}$ of $\mathcal{H}$ is called *uniformly positive* if, for some Hilbert space inner product $\langle\ ,\ \rangle$ on $\mathcal{H}$, there exists $\varepsilon > 0$ such that $[\,s, s\,] \geq \varepsilon \|s\|^2$ for every $s \in \mathcal{S}$. *Uniformly negative* subspaces are defined in a similar fashion.

Every fundamental decomposition of a Krein space $\mathcal{H}$ has an associated *signature operator*, $J := P_+ - P_-$ where $P_\pm := P_{\mathcal{H}_\pm // \mathcal{H}_\mp}$. The indefinite metric and the inner product corresponding to a fundamental decomposition of $\mathcal{H}$ with signature operator J are related to each other by

$$\langle\, x, y\,\rangle = [\,Jx, y\,] \text{ for every } x, y \in \mathcal{H}.$$

If $\mathcal{H}$ is a Krein space, $L(\mathcal{H})$ stands for the vector space of all the linear operators on $\mathcal{H}$ which are bounded in an associated Hilbert space $(\mathcal{H}, \langle\ ,\ \rangle)$. Since the norms generated by different fundamental decompositions of a Krein space $\mathcal{H}$ are equivalent (see, for instance, [4, Theorem 7.19]), $L(\mathcal{H})$ does not depend on the chosen underlying Hilbert space. Given $T \in L(\mathcal{H})$, $T^\#$ is the unique operator satisfying

$$[\,Tx, y\,] = [\,x, T^\# y\,] \text{ for every } x, y \in \mathcal{H}.$$

An operator $T \in L(\mathcal{H})$ is said to be *selfadjoint* if $T = T^\#$.

A *positive* operator $T \in L(\mathcal{H})$ satisfies $[\,Tx, x\,] \geq 0$ for every $x \in \mathcal{H}$. The notation $S \leq T$ signifies that $T - S$ is positive.

A (closed) subspace $\mathcal{S}$ of a Krein space $\mathcal{H}$ is a *regular subspace* if $\mathcal{H} = \mathcal{S}\ [\dot{+}]\ \mathcal{S}^{[\perp]}$. Equivalently, $\mathcal{S}$ is a regular subspace if it is the range of a selfadjoint projection, i.e., there exists $Q \in \mathcal{Q}$ such that $Q = Q^\#$ and $R(Q) = \mathcal{S}$ (see [4, Proposition 1.4.19]). In [2, Theorem 2.3], T. Ando proved that any selfadjoint projection on a Krein space can be decomposed as the sum of two selfadjoint projections with uniformly definite ranges. See also [15, 20].

Theorem 2.2. *Let $(\mathcal{H}, [\ ,\ ])$ be a Krein space and let Q be a selfadjoint projection. Then Q can be written as*

$$Q = Q_+ + Q_-,$$

where Q_+ and Q_- are selfadjoint projections such that $R(Q_+)$ is uniformly positive, $R(Q_-)$ is uniformly negative and $Q_+Q_- = Q_-Q_+ = 0$. Moreover, each fundamental decomposition of $\mathcal{H}$ provides a unique such decomposition of Q.

It happens that every closed subspace of a Krein space can be decomposed as the orthogonal direct sum of a closed positive subspace and a closed nonpositive subspace (see [4, Theorem 6.4], [5, Chapter V, Theorem 3.1]).

Lemma 2.3. *Let $(\mathcal{H}, [\ ,\ ])$ be a Krein space with fundamental descomposition $\mathcal{H} = \mathcal{H}_+ \ [\dot{+}]\ \mathcal{H}_-$ and corresponding Hilbert space inner product $\langle\ ,\ \rangle$. Let $\mathcal{S}$ be a closed subspace of $\mathcal{H}$. Then $\mathcal{S}$ can be represented uniquely as the orthogonal direct sum of a closed positive subspace $\mathcal{S}_+$ and a closed nonpositive subspace $\mathcal{S}_-$, i.e.,*

$$\mathcal{S} = \mathcal{S}_+ \ [\dot{+}]\ \mathcal{S}_-.$$

Furthermore, $\langle \mathcal{S}_+, \mathcal{S}_- \rangle = \{0\}$.

In [12, 13] least squares problems in the indefinite metric setting were studied. From those references we recall the definition of indefinite least squares solution.

Definition. Let $(\mathcal{H}, [\ ,\ ])$ be a Krein space and let $B \in CR(\mathcal{H})$. We say that $u \in \mathcal{H}$ is an *indefinite least squares solution* (ILSS) of $Bz = x$ if

$$[\,x - Bu, x - Bu\,] \leq [\,x - Bz, x - Bz\,] \ \text{ for every } x, z \in \mathcal{H}.$$

We conclude this section by stating necessary and sufficient conditions for the existence of an ILSS of the equation $Bz = x$, as proved in [5, Chapter I, Theorem 8.4] and in [13, Lemma 3.1].

Lemma 2.4. *Let $(\mathcal{H}, [\ ,\ ])$ be a Krein space and let $B \in CR(\mathcal{H})$. Then $u \in \mathcal{H}$ is an ILSS of the equation $Bz = x$ if and only if $R(B)$ is nonnegative and $x - Bu \in R(B)^{[\perp]}$.*

3. Indefinite inverses in Krein spaces

In [23] S.K. Mitra and C.R. Rao introduced the notion of the W-inverse of a matrix for a given positive weight W. Later, in [8] and [6], the concept was extended to Hilbert space operators. Specifically, given a Hilbert space $(\mathcal{H}, \langle\ ,\ \rangle)$, a positive operator $W \in L(\mathcal{H})$ and an operator $B \in CR(\mathcal{H})$, a *W-inverse* of B is defined to be an operator $X_0 \in L(\mathcal{H})$ such that, for each $x \in \mathcal{H}$, $X_0 x$ is a weighted least squares solution of $Bz = x$, i.e., so that

$$\langle\, W(BX_0x - x), BX_0x - x\,\rangle \leq \langle\, W(Bz - x), Bz - x\,\rangle \ \text{ for every } z \in \mathcal{H}.$$

In [8] it was proved that X_0 is a W-inverse of B if and only if $B^*W(BX_0 - I) = 0$ or, equivalently, X_0 satisfies the identities $W(BX_0)^2 = WBX_0 = (BX_0)^*W$. The first equality means that BX_0 is a *W-projection* while the second says that BX_0 is *W-selfadjoint*, see [8].

We extend the definition to a Krein space $(\mathcal{H}, [\ ,\ ])$, and all spaces are henceforth assumed to be Krein spaces unless otherwise stated.

Definition. Let $B \in CR(\mathcal{H})$. An operator $X_0 \in L(\mathcal{H})$ is an *indefinite inverse* of B if X_0 is a solution of

$$B^{\#}(BX - I) = 0.$$

Proposition 3.1. *Let $B \in CR(\mathcal{H})$. Then B admits an indefinite inverse if and only if $R(B)$ is regular.*

Proof. Suppose that X_0 is an indefinite inverse of B so that $B^{\#}(BX_0 - I) = 0$. Then, for every $x \in \mathcal{H}$, $BX_0x - x \in N(B^{\#}) = R(B)^{[\perp]}$ and therefore, $x \in R(B) + R(B)^{[\perp]}$. Thus $\mathcal{H} = R(B) + R(B)^{[\perp]}$. As, $\{0\} = R(B) \cap R(B)^{[\perp]}$, it follows that $\mathcal{H} = R(B)\ [\dot{+}]\ R(B)^{[\perp]}$ or, accordingly, $R(B)$ is regular.

Conversely, if $R(B)$ is regular then $\mathcal{H} = R(B)[\dot{+}]R(B)^{[\perp]}$, and so by applying $B^{\#}$, it results $R(B^{\#}) = R(B^{\#}B)$. By Douglas' Lemma (Lemma 2.1), it follows that $B^{\#}(BX - I) = 0$ admits a solution. □

By the proof of Proposition 3.1, for any $B \in CR(\mathcal{H})$, $R(B)$ is regular if, and only if, $R(B^{\#}) = R(B^{\#}B)$. In this case, $N(B) = N(B^{\#}B)$.

The next proposition characterizes the indefinite inverses of $B \in L(\mathcal{H})$ when $R(B)$ is regular.

Proposition 3.2. *Let $B \in L(\mathcal{H})$. Assume that $R(B)$ is regular. Then the following conditions are equivalent:*

i) X_0 *is an indefinite inverse of* B*,*

ii) X_0 *is a solution of the equation* $BX = Q$*, where* Q *is the selfadjoint projection onto* $R(B)$*,*

iii) X_0 *is an inner inverse of* B*, i.e.,* $BX_0B = B$*, and* $(BX_0)^{\#} = BX_0$*.*

Moreover, if $R(B)$ *is also uniformly positive, conditions* $i), ii), iii)$ *are also equivalent to:*

iv) For every $x \in \mathcal{H}$*,* X_0x *is an ILSS of* $Bz = x$*.*

A similar statement holds if $R(B)$ is uniformly negative.

Proof of Proposition 3.2. $i) \Rightarrow ii)$: Notice first that $B^{\#} = B^{\#}Q$, since $B = QB$ and $Q = Q^{\#}$. Then $B^{\#}(BX_0 - I) = 0$ is equivalent to $B^{\#}(BX_0 - Q) = 0$. Therefore, $R(BX_0 - Q) \subseteq N(B^{\#}) \cap R(Q) = N(Q) \cap R(Q) = \{0\}$. Thus $BX_0 = Q$.

$ii) \Rightarrow iii)$: If $BX_0 = Q$ then $BX_0B = QB = B$ and $(BX_0)^{\#} = Q^{\#} = Q = BX_0$.

$iii) \Rightarrow i)$: Suppose that $BX_0B = B$ and $(BX_0)^{\#} = BX_0$. Then $B^{\#}(BX_0 - I) = B^{\#}(X_0^{\#}B^{\#} - I) = B^{\#}X_0^{\#}B^{\#} - B^{\#} = 0$.

$i) \Leftrightarrow iv)$: X_0 is an indefinite inverse of B if, and only if, $B^{\#}(BX_0 - I) = 0$ if, and only if, for every $x \in \mathcal{H}$, $BX_0x - x \in R(B)^{[\perp]}$. Since $R(B)$ is nonnegative, Lemma 2.4 gives the equivalence. □

When $R(B)$ is closed and uniformly definite, the original definition of W-inverse for the indefinite metric is retrieved directly from item $iv)$ of the last proposition.

The more general concept of the *indefinite inverse* of B *in the range of* C is given next.

Definition. Let $B \in CR(\mathcal{H})$ and $C \in L(\mathcal{H})$. An operator $X_0 \in L(\mathcal{H})$ is an *indefinite inverse* of B *in* $R(C)$ if X_0 is a solution of

$$B^{\#}(BX - C) = 0.$$

Proposition 3.3. *Let* $B \in CR(\mathcal{H})$ *and* $C \in L(\mathcal{H})$*.* B *has an indefinite inverse in* $R(C)$ *if and only if* $R(C) \subseteq R(B) + R(B)^{[\perp]}$*.*

Proof. Suppose that X_0 is an indefinite inverse of B in $R(C)$. Then $B^{\#}(BX_0 - C) = 0$. So if $x \in \mathcal{H}$ then $BX_0x - Cx \in N(B^{\#}) = R(B)^{[\perp]}$ and therefore, $Cx \in R(B) + R(B)^{[\perp]}$. Thus, $R(C) \subseteq R(B) + R(B)^{[\perp]}$.

Conversely, if $R(C) \subseteq R(B) + R(B)^{[\perp]}$ then

$$R(B^{\#}C) \subseteq R(B^{\#}B).$$

As in the proof of Proposition 3.1, by Douglas' Lemma $B^{\#}(BX - C) = 0$ admits a solution. □

Corollary 3.4. *Let* $B \in CR(\mathcal{H})$ *and* $C \in L(\mathcal{H})$*. If* $R(B)$ *is regular then* X_0 *is an indefinite inverse of* B *in* $R(C)$ *if and only if* X_0 *is a solution of the equation* $BX = QC$*, with* Q *the selfadjoint projection onto* $R(B)$*.*

Proof. Suppose that $R(B)$ is regular. Then $R(B^{\#}) = R(B^{\#}B)$ or, equivalently, $N(B) = N(B^{\#}B)$.

Since $B = QB$, $Q = Q^{\#}$, if $B^{\#}(BX_0 - C) = 0$, then $B^{\#}(BX_0 - QC) = 0$ holds. Hence $R(BX_0 - QC) \subseteq N(B^{\#}) \cap R(Q) = N(Q) \cap R(Q) = \{0\}$, and $BX_0 = QC$. □

From the last corollary we have that, when $R(B)$ is regular, the set of indefinite inverses of B in $R(C)$ is the affine manifold

$$X_0 + L(\mathcal{H}, N(B)),$$

with X_0 any solution of the equation $BX = QC$.

Proposition 3.5. *Let* $B \in CR(\mathcal{H})$ *and* $C \in L(\mathcal{H})$ *satisfy that* $R(B)$ *is nonnegative and* $R(C) \subseteq R(B) + R(B)^{[\perp]}$*. Then* X_0 *is an indefinite inverse of* B *in* $R(C)$ *if and only if, for every* $x \in \mathcal{H}$*,* X_0x *is an ILSS of* $Bz = Cx$*, i.e.,*

$$[\,Cx - BX_0x, Cx - BX_0x\,] \leq [\,Cx - Bz, Cx - Bz\,] \quad \text{for every } z \in \mathcal{H}.$$

Proof. X_0 is an indefinite inverse of B in $R(C)$ if, and only if, $B^{\#}(BX_0 - C) = 0$ if, and only if, $BX_0x - Cx \in R(B)^{[\perp]}$ for every $x \in \mathcal{H}$. Since $R(B)$ is assumed to be nonnegative, Lemma 2.4 can be applied to get the equivalence. □

4. Indefinite least squares problems

To state the next problems let us recall that the order is the one induced by the positive operators in $(\mathcal{H}, [\ ,\ ])$.

Definition. Let $B \in CR(\mathcal{H})$ and $C \in L(\mathcal{H})$. An operator $X_0 \in L(\mathcal{H})$ is an *indefinite minimum solution* of $BX - C = 0$ if

$$(BX_0 - C)^{\#}(BX_0 - C) = \min_{X \in L(\mathcal{H})} (BX - C)^{\#}(BX - C). \tag{4.1}$$

Denote by ImS_C the set of indefinite minimum solutions of $BX - C = 0$ and simply ImS for $C = I$.

Consider the following problem: given two operators $B \in CR(\mathcal{H})$ and $C \in L(\mathcal{H})$, determine the existence of

$$\min_{X \in L(\mathcal{H})} (BX - C)^{\#}(BX - C); \tag{4.2}$$

i.e., analyze whether the set ImS_C is non-empty.

In a similar fashion, the analogous maximization problem can be considered. From now on, we only address the problem related to the existence of (4.2). The arguments we present in dealing with problem (4.2) can be adapted to the maximum problem.

Theorem 4.1. *Let $B \in CR(\mathcal{H})$ and $C \in L(\mathcal{H})$. Then the set ImS_C is non-empty if and only if $R(C) \subseteq R(B) + R(B)^{[\perp]}$ and $R(B)$ is nonnegative.*

Proof. Suppose that $X_0 \in ImS_C$, so that

$$[\,(BX_0 - C)x, (BX_0 - C)x\,] \le [\,(BX - C)x, (BX - C)x\,]$$

for every $x \in \mathcal{H}$ and every $X \in L(\mathcal{H})$. Let $z \in \mathcal{H}$ be arbitrary. Then, for every $x \in \mathcal{H} \setminus \{0\}$, there exists $X \in L(\mathcal{H})$ such that $z = Xx$. Therefore

$$[\,(BX_0 - C)x, (BX_0 - C)x\,] \le [\,Bz - Cx, Bz - Cx\,] \text{ for every } x, z \in \mathcal{H}.$$

So, for every $x \in \mathcal{H}$, $X_0 x$ is an ILSS of $Bz = Cx$. By Lemma 2.4, we get that $R(C) \subseteq R(B) + R(B)^{[\perp]}$ and $R(B)$ is nonnegative. Furthemore, by Proposition 3.5, we have that X_0 is an indefinite inverse of B in $R(C)$.

Conversely, if $R(C) \subseteq R(B) + R(B)^{[\perp]}$ and $R(B)$ is nonnegative then, by Proposition 3.3, B admits an indefinite inverse in $R(C)$. Now, if X_0 is an indefinite inverse of B in $R(C)$ then, by Proposition 3.5,

$$[\,(BX_0 - C)x, (BX_0 - C)x\,] \le [\,Bz - Cx, Bz - Cx\,] \text{ for every } x, z \in \mathcal{H}.$$

Given $x \in \mathcal{H}$ and $X \in L(\mathcal{H})$, set $z = Xx$, so that

$$[\,(BX_0 - C)x, (BX_0 - C)x\,] \le [\,(BX - C)x, (BX - C)x\,]$$

for every $x \in \mathcal{H}$ and every $X \in L(\mathcal{H})$. Thus $X_0 \in ImS_C$, as required to complete the proof. □

The proof of Theorem 4.1 gives a characterization of the set ImS_C :

Corollary 4.2. *Let $B \in CR(\mathcal{H})$ and $C \in L(\mathcal{H})$ satisfy that $R(B)$ is nonnegative and $R(C) \subseteq R(B) + R(B)^{[\perp]}$. Then $X_0 \in ImS_C$ if and only if X_0 is an indefinite inverse of B in $R(C)$; i.e., X_0 is solution of the normal equation*

$$B^{\#}(BX - C) = 0.$$

Consequently, ImS_C is the affine manifold

$$X_0 + L(\mathcal{H}, N(B^{\#}B)),$$

for any $X_0 \in ImS_C$. It follows from the structure of ImS_C that it is a closed subset of $L(\mathcal{H})$.

The next two corollaries also follow from Theorem 4.1.

Corollary 4.3. *Let $B \in CR(\mathcal{H})$. Then the set ImS_C is non-empty for every $C \in L(\mathcal{H})$ if and only if $R(B)$ is uniformly positive.*

In this case,

$$\min_{X \in L(\mathcal{H})} (BX - C)^{\#}(BX - C) = C^{\#}(I - Q)C$$

where Q is the selfadjoint projection onto $R(B)$.

Proof. Assume that, for every $C \in L(\mathcal{H})$, the set ImS_C is non-empty. In particular, the set ImS is non-empty. Then, by Theorem 4.1, $R(B)$ is regular and nonnegative. Whence $R(B)$ is uniformly positive.

Conversely, if $R(B)$ is uniformly positive then, for every $C \in L(\mathcal{H})$, we have that $R(C) \subseteq \mathcal{H} = R(B) + R(B)^{[\perp]}$. Hence, by Theorem 4.1, the set ImS_C is non-empty for every $C \in L(\mathcal{H})$; i.e., there exists an indefinite minimum solution $X_0 \in L(\mathcal{H})$ of $BX - C = 0$ or, equivalently, X_0 is a solution of the normal equation $B^{\#}(BX - C) = 0$ by Corollary 4.2. In this case, since $R(B)$ is regular, Corollary 3.4 gives that $BX_0 = QC$. Therefore,

$$\min_{X \in L(\mathcal{H})} (BX - C)^{\#}(BX - C) = (BX_0 - C)^{\#}(BX_0 - C) = C^{\#}(I - Q)C.$$

□

Corollary 4.4. *Let $B \in CR(\mathcal{H})$. Then the set ImS is non-empty if and only if $R(B)$ is uniformly positive.*

In this case, any element of ImS is an indefinite inverse of B and

$$\min_{X \in L(\mathcal{H})} (BX - I)^{\#}(BX - I) = I - Q$$

where Q is the selfadjoint projection onto $R(B)$.

Remark. By mimicking the arguments in the proof of Theorem 4.1, a similar result can be proved for operators acting between different Krein spaces. More precisely, let $(\mathcal{H}, [\ ,\ ]_{\mathcal{H}})$, $(\mathcal{K}, [\ ,\ ]_{\mathcal{K}})$ and $(\mathcal{F}, [\ ,\ ]_{\mathcal{F}})$ be Krein spaces. Let $B \in CR(\mathcal{H}, \mathcal{K})$ and $C \in L(\mathcal{F}, \mathcal{K})$. Then there exists $X_0 \in L(\mathcal{F}, \mathcal{H})$ such that

$$\min_{X \in L(\mathcal{F}, \mathcal{H})} (BX - C)^{\#}(BX - C) = (BX_0 - C)^{\#}(BX_0 - C)$$

if and only if $R(C) \subseteq R(B) + R(B)^{[\perp]_{\mathcal{K}}}$ and $R(B)$ is nonnegative.

4.1. Indefinite least squares problems: the pseudo-regular case

A (closed) subspace $\mathcal{S}$ of a Krein space $\mathcal{H}$ is called a *pseudo-regular subspace* if the algebraic sum $\mathcal{S} + \mathcal{S}^{[\perp]}$ is closed. This is equivalent to the equality $(\mathcal{S}^o)^{[\perp]} = \mathcal{S} + \mathcal{S}^{[\perp]}$; see [14]. In what follows $\mathcal{Q}_{\mathcal{S}}$ stands for the set of normal projections onto the pseudo-regular subspace $\mathcal{S}$, i.e.,

$$\mathcal{Q}_{\mathcal{S}} := \{Q \in L(\mathcal{H}) : Q^2 = Q,\ QQ^{\#} = Q^{\#}Q,\ R(Q) = \mathcal{S}\}.$$

Also, $\mathcal{S}$ is a pseudo-regular subspace if and only if $\mathcal{S}$ is the range of a normal projection; $Q \in \mathcal{Q}_{\mathcal{S}}$ (see [21, Theorem 4.3]). Unlike normal projections in Hilbert spaces, a normal projection in a Krein space need not be selfadjoint (moreover, a selfadjoint projection onto $\mathcal{S}$ need not necessarily exist).

The set $\mathcal{Q}_{\mathcal{S}}$ has infinitely many elements, unless $\mathcal{S}$ is regular. See [21] for further details.

Let $B \in CR(\mathcal{H})$. The next results relate the pseudo-regularity of $R(B)$, the indefinite inverse of B in $R(C)$ and the indefinite minimum solution of $BX - C = 0$.

The next lemma, stated in [12, Remark 2.1], is useful when dealing with pseudo-regular ranges.

Lemma 4.5. *Let $\mathcal{S}$ be a pseudo-regular subspace of $\mathcal{H}$. If $Q \in \mathcal{Q}_{\mathcal{S}}$ then*

$$Q^{\#}(I - Q)y = 0 \text{ if and only if } y \in \mathcal{S} + \mathcal{S}^{[\perp]}.$$

Proof. Let $Q \in \mathcal{Q}_{\mathcal{S}}$. If $P = Q(I-Q)^{\#}$ then $R(P) = \mathcal{S} \cap N(Q^{\#}) = \mathcal{S} \cap \mathcal{S}^{[\perp]} = \mathcal{S}^o$ and $N(P^{\#}) = R(P)^{[\perp]} = (\mathcal{S}^o)^{[\perp]} = \mathcal{S} + \mathcal{S}^{[\perp]}$.

Therefore, $0 = Q^{\#}(I-Q)y = P^{\#}y$ if and only if $y \in N(P^{\#}) = \mathcal{S} + \mathcal{S}^{[\perp]}$. □

Proposition 4.6. *Let $B \in CR(\mathcal{H})$ and $C \in L(\mathcal{H})$. If $R(B)$ is pseudo-regular and $R(C) \subseteq R(B)+R(B)^{[\perp]}$, then X_0 is an indefinite inverse of B in $R(C)$ if and only if $R(BX_0 - QC) \subseteq R(B)^o$, for any $Q \in \mathcal{Q}_{R(B)}$.*

Proof. Suppose that $R(B)$ is pseudo-regular and pick any $Q \in \mathcal{Q}_{R(B)}$. By Lemma 4.5, $(I-Q)y \in N(Q^{\#}) = N(B^{\#})$ for every $y \in R(B) + R(B)^{[\perp]}$. Since $R(C) \subseteq R(B) + R(B)^{[\perp]}$, we have that $B^{\#}(I-Q)C = 0$. So X_0 is a solution of $B^{\#}(BX - C) = 0$ if and only if $B^{\#}(BX_0 - QC) = 0$ or $R(BX_0 - QC) \subseteq R(B)^o$. □

Corollary 4.7. *Let $B \in CR(\mathcal{H})$. Then the set ImS_C is non-empty for every $C \in L(\mathcal{H})$ such that $R(C) \subseteq (R(B)^o)^{[\perp]}$ if and only if $R(B)$ is a pseudo-regular, nonnegative subspace of $\mathcal{H}$.*

In this case,

$$\min_{X \in L(\mathcal{H})} (BX - C)^{\#}(BX - C) = C^{\#}(I-Q)C,$$

for any $Q \in \mathcal{Q}_{R(B)}$.

Proof. Let $C \in L(\mathcal{H})$. Then the set ImS_C is non-empty if and only if $R(C) \subseteq R(B) + R(B)^{[\perp]}$ and $R(B)$ is nonnegative by Theorem 4.1.

Suppose that $R(B)$ is pseudo-regular and nonnegative. Then

$$(R(B)^o)^{[\perp]} = R(B) + R(B)^{[\perp]},$$

and, therefore, the set ImS_C is non-empty for every $C \in L(\mathcal{H})$ such that $R(C) \subseteq (R(B)^o)^{[\perp]}$.

Conversely, suppose that the set ImS_C is non-empty for every $C \in L(\mathcal{H})$ such that $R(C) \subseteq (R(B)^o)^{[\perp]}$. Then pick a C such that $R(C) = (R(B)^o)^{[\perp]} = \overline{R(B) + R(B)^{[\perp]}}$. By Theorem 4.1, it must happen that $R(C) \subseteq R(B) + R(B)^{[\perp]}$ and $R(B)$ is nonnegative. That is, $R(B)$ is pseudo-regular and nonnegative. In this case, let X_0 be an indefinite inverse of B in $R(C)$. By Corollary 4.2, X_0 is an indefinite minimum solution of $BX - C = 0$. By Proposition 4.6, $R(BX_0 - QC) \subseteq R(B)^o$. Then Lemma 4.5, along with $\mathcal{S} = R(B)$ and $R(BX_0 - QC) \subseteq R(B)^o$, yields the result. □

5. Min-Max least squares problems

In this section a min-max problem is studied for operators with range which is not necessary definite. In order to pose the problem, choose a fundamental decomposition $\mathcal{H} = \mathcal{H}_+ \, [\dot{+}] \, \mathcal{H}_-$ and fix the corresponding Hilbert space $(\mathcal{H}, \langle\ ,\ \rangle)$, so that, for all $x, y \in \mathcal{H}$, $\langle x, y \rangle = [\, Jx, y\,]$ with J the signature operator associated with the decomposition.

Let $B \in CR(\mathcal{H})$. By Lemma 2.3, $R(B)$ can be represented uniquely as

$$R(B) = \mathcal{S}_+ \, [\dot{+}] \, \mathcal{S}_- \tag{5.1}$$

with $\mathcal{S}_+$ a positive closed subspace of $\mathcal{H}$, $\mathcal{S}_-$ a nonpositive closed subspace of $\mathcal{H}$ and $\langle \mathcal{S}_+, \mathcal{S}_- \rangle = \{0\}$.

Consider $P_+ = P_{\mathcal{S}_+}$ and $P_- = P_{\mathcal{S}_-}$, the orthogonal projections from the Hilbert space $(\mathcal{H}, \langle\ ,\ \rangle)$ onto $\mathcal{S}_+$ and $\mathcal{S}_-$, respectively. It readily follows that $P_+ + P_- = P_{R(B)}$. Therefore, if $B_+ := P_+B$ and $B_- := P_-B$, then

$$B = B_+ + B_-, \quad R(B_+) = \mathcal{S}_+ \quad \text{and} \quad R(B_-) = S_-. \tag{5.2}$$

Since $N(P_{\pm}^{\#}) = \mathcal{S}_{\pm}^{[\perp]}$, it holds that $B_+^{\#}B_- = B^{\#}P_+^{\#}B_- = 0$ and $B_-^{\#}B_+ = 0$.

Observe that if $R(B)$ is regular then P_+ and P_- are the projections given by Theorem 2.2.

Definition. Let $C \in L(\mathcal{H})$. Let B in $CR(\mathcal{H})$ be represented as in (5.2). An operator $Z_0 \in L(\mathcal{H})$ is said to be an *indefinite min-max solution* of $BX - C = 0$ (corresponding to the decomposition given by J) if

$$(BZ_0 - C)^{\#}(BZ_0 - C) = \\ = \max_{Y \in L(\mathcal{H})} \; \min_{X \in L(\mathcal{H})} (B_+X + B_-Y - C)^{\#}(B_+X + B_-Y - C). \quad (5.3)$$

Denote by $ImMS_C$ the set of indefinite min-max solutions of $BX - C = 0$ for that decomposition.

The following result shows that an indefinite min-max solution of $BX - C = 0$ is independent of the selected fundamental decomposition of $\mathcal{H}$. In what follows, $\mathcal{C}$ denotes the cone of neutral vectors in $\mathcal{H}$.

Theorem 5.1. *Let $C \in L(\mathcal{H})$ and $B \in CR(\mathcal{H})$. An operator $Z_0 \in ImMS_C$, for some (and, hence, any) fundamental decomposition of $\mathcal{H}$, if and only if*

$$Z_0 = Z_1 + Z_2$$

where Z_1 is an indefinite inverse of B in $R(C)$ and $R(BZ_2) \subseteq \mathcal{C}$, where $\mathcal{C}$ denotes the cone of neutral vectors in $\mathcal{H}$.

Proof. Fix a fundamental decomposition $\mathcal{H} = \mathcal{H}_+ \,[\dot{+}]\, \mathcal{H}_-$, and consider $B = B_+ + B_-$ as in (5.2). Suppose that $Z_0 \in ImMS_C$ for that decomposition. Then Z_0 verifies (5.3). So, for every fixed $Y \in L(\mathcal{H})$, there exists $\min_{X \in L(\mathcal{H})} (B_+X + B_-Y - C)^{\#}(B_+X + B_-Y - C)$.

From Corollary 4.2 and by using that $B_+^{\#}B_- = 0$, the minimum is attained at $X_0 (= X_0(Y))$ if and only if

$$0 = B_+^{\#}(B_+X_0 - (C - B_-Y)) = B_+^{\#}(B_+X_0 - C). \quad (5.4)$$

That is, X_0 is an indefinite inverse of B_+ in $R(C)$ and, in particular, that X_0 does not depend on Y. Hence, for every $Y \in L(\mathcal{H})$,

$$(B_+X_0 + B_-Y - C)^{\#}(B_+X_0 + B_-Y - C) = \\ = \min_{X \in L(\mathcal{H})} (B_+X + B_-Y - C)^{\#}(B_+X + B_-Y - C)$$

and, since Z_0 satisfies (5.3),

$$(BZ_0 - C)^{\#}(BZ_0 - C) = \max_{Y \in L(\mathcal{H})} (B_+X_0 + B_-Y - C)^{\#}(B_+X_0 + B_-Y - C).$$

By Corollary 4.2 and using that $B_-^{\#}B_+ = 0$, we get that the maximum is attained at $Y_0 \in L(\mathcal{H})$ if and only if

$$0 = B_-^{\#}(B_-Y_0 - (C - B_+X_0)) = B_-^{\#}(B_-Y_0 - C). \quad (5.5)$$

Consequently,

$$(BZ_0 - C)^{\#}(BZ_0 - C) = (B_+X_0 + B_-Y_0 - C)^{\#}(B_+X_0 + B_-Y_0 - C). \quad (5.6)$$

By Douglas' Lemma, there exists $Z_1 \in L(\mathcal{H})$ satisfying $BZ_1 = B_+X_0 + B_-Y_0$. Thus by (5.6),

$$(BZ_0 - C)^{\#}(BZ_0 - C) = (BZ_1 - C)^{\#}(BZ_1 - C). \quad (5.7)$$

Since $B_-^{\#}B_+ = B_+^{\#}B_- = 0$, by (5.4) and (5.5), $B^{\#}(BZ_1 - C) = 0$. Consequently, Z_1 is an indefinite inverse of B in $R(C)$. Now, as Z_1 is an indefinite inverse of B in $R(C)$,

$$(BZ_0 - C)^{\#}(BZ_0 - C) = (BZ_1 - C + BZ_0 - BZ_1)^{\#}(BZ_1 - C + BZ_0 - BZ_1) =$$

$$= (BZ_1 - C)^{\#}(BZ_1 - C) + (B(Z_0 - Z_1))^{\#}B(Z_0 - Z_1).$$

Set $Z_2 := Z_0 - Z_1$. From equation (5.7), we conclude that $(BZ_2)^{\#}BZ_2 = 0$ or, equivalently, that $R(BZ_2) \subseteq \mathcal{C}$. Clearly, $Z_0 = Z_1 + Z_2$, with Z_1 and Z_2 as required.

Conversely, let $Z_1 \in L(\mathcal{H})$ be an indefinite inverse of B in $R(C)$, and $R(BZ_2) \subseteq \mathcal{C}$. If $Z_0 = Z_1 + Z_2$ then

$$(BZ_0 - C)^{\#}(BZ_0 - C) = (BZ_1 - C)^{\#}(BZ_1 - C).$$

Write $B = B_+ + B_-$ as in (5.2). Since $B^{\#}(BZ_1 - C) = 0$, we have that $R(BZ_1 - C) \subseteq N(B^{\#}) = (R(B_+)\ [\dot{+}]\ R(B_-))^{[\perp]} = N(B_+^{\#}) \cap N(B_-^{\#})$. Therefore, $B_+^{\#}(BZ_1 - C) = B_-^{\#}(BZ_1 - C) = 0$. Then, as $B_-^{\#}B_+ = B_+^{\#}B_- = 0$, it readily follows that, for every $X, Y \in L(\mathcal{H})$,

$$B_+^{\#}(B_+Z_1 - (C - B_-Y)) = B_+^{\#}(BZ_1 - C) = 0$$

and

$$B_-^{\#}(B_-Z_1 - (C - B_+X)) = B_-^{\#}(BZ_1 - C) = 0.$$

So, by Corollary 4.2, we obtain that

$$\begin{aligned}
&(BZ_0 - C)^{\#}(BZ_0 - C) = (BZ_1 - C)^{\#}(BZ_1 - C) = \\
&= (B_+Z_1 + B_-Z_1 - C)^{\#}(B_+Z_1 + B_-Z_1 - C) \\
&= \max_{Y \in L(\mathcal{H})} (B_-Y - (C - B_+Z_1))^{\#}(B_-Y - (C - B_+Z_1)) \\
&= \max_{Y \in L(\mathcal{H})} \min_{X \in L(\mathcal{H})} (B_+X + B_-Y - C)^{\#}(B_+X + B_-Y - C).
\end{aligned}$$

Therefore, $Z_0 \in ImMS_C$. □

Remark. Let $C \in L(\mathcal{H})$ and $B \in CR(\mathcal{H})$ such that B is represented as in (5.2). Then

$$\max_{Y \in L(\mathcal{H})} \min_{X \in L(\mathcal{H})} (B_+X + B_-Y - C)^{\#}(B_+X + B_-Y - C) =$$

$$= \min_{X \in L(\mathcal{H})} \max_{Y \in L(\mathcal{H})} (B_+X + B_-Y - C)^{\#}(B_+X + B_-Y - C).$$

Indeed, if $Z_0 \in ImMS_C$ then, as the last theorem asserts, $Z_0 = Z_1 + Z_2$ where Z_1 is an indefinite inverse of B in $R(C)$ and $R(BZ_2) \subseteq \mathcal{C}$. By the proof of the last theorem, on the other hand, for every $X, Y \in L(\mathcal{H})$,

$$B_+^{\#}(B_+Z_1 - (C - B_-Y)) = B_+^{\#}(BZ_1 - C) = 0$$

and

$$B_-^{\#}(B_-Z_1 - (C - B_+X)) = B_-^{\#}(BZ_1 - C) = 0.$$

A direct application of Corollary 4.2 gives

$$\begin{aligned}
&(BZ_0 - C)^{\#}(BZ_0 - C) = (BZ_1 - C)^{\#}(BZ_1 - C) = \\
&= (B_+Z_1 + B_-Z_1 - C)^{\#}(B_+Z_1 + B_-Z_1 - C) \\
&= \min_{X \in L(\mathcal{H})} (B_+X - (C - B_-Z_1))^{\#}(B_+X - (C - B_-Z_1)) \\
&= \min_{X \in L(\mathcal{H})} \max_{Y \in L(\mathcal{H})} (B_+X + B_-Y - C)^{\#}(B_+X + B_-Y - C).
\end{aligned}$$

Corollary 5.2. *Let $B \in CR(\mathcal{H})$ and $C \in L(\mathcal{H})$. Then, the set $ImMS_C$ is non-empty if and only if $R(C) \subseteq R(B) + R(B)^{[\perp]}$.*

Proof. Suppose that $Z_0 \in ImMS_C$. Then, by Theorem 5.1, $Z_0 = Z_1 + Z_2$ where $B^{\#}(BZ_1 - C) = 0$ and $R(BZ_2) \subseteq \mathcal{C}$. Therefore

$$R(C) \subseteq R(B) + R(B)^{[\perp]}.$$

Conversely, if $R(C) \subseteq R(B) + R(B)^{[\perp]}$ then by Proposition 3.3, there exists a solution of the normal equation $B^{\#}(BX - C) = 0$, say $Z_1 \in L(\mathcal{H})$. Put $Z_2 = 0$ and apply Theorem 5.1 to get that $Z_1 \in ImMS_C$. □

Corollary 5.3. *Let $B \in CR(\mathcal{H})$. Then the set $ImMS_C$ is non-empty for every $C \in L(\mathcal{H})$ if and only if $R(B)$ is regular.*

In this case, if B is represented with respect to any fixed fundamental decomposition of $\mathcal{H}$ as in (5.2), *then*

$$\max_{Y \in L(\mathcal{H})} \min_{X \in L(\mathcal{H})} (B_+X + B_-Y - C)^\#(B_+X + B_-Y - C) =$$
$$= C^\# \left[\max_{Y \in L(\mathcal{H})} (B_-Y - I)^\#(B_-Y - I)\right] \left[\min_{X \in L(\mathcal{H})} (B_+X - I)^\#(B_+X - I)\right] C =$$
$$= C^\#(I - Q)C,$$

where Q is the selfadjoint projection onto $R(B)$.

Proof. If $R(B)$ is regular then, for every $C \in L(\mathcal{H})$, $R(C) \subseteq R(B)[\dot{+}]R(B)^{[\perp]}$ and, by Corollary 5.2, the set $ImMS_C$ is non-empty.

Conversely, assume that, for every $C \in L(\mathcal{H})$ the set $ImMS_C$ is non-empty. Set $C = I$ and apply the corollary once again to get $\mathcal{H} = R(I) \subseteq R(B) + R(B)^{[\perp]}$ and $R(B)$ regular.

Suppose that $R(B)$ is regular. Given a fundamental decomposition of $\mathcal{H}$, Theorem 2.2 provides unique selfadjoint projections $Q_+, Q_- \in L(\mathcal{H})$ such that $Q = Q_+ + Q_-$ with $R(Q_+)$ uniformly positive and $R(Q_-)$ uniformly negative. Then, the subspaces $\mathcal{S}_\pm$ in the decomposition (5.1) of $R(B)$ and the operators $B_\pm$ in (5.2) are given by $\mathcal{S}_\pm = R(Q_\pm)$ and $B_\pm = Q_\pm B$.

Let $Z_0 \in ImMS_C$. Then by Theorem 5.1, $Z_0 = Z_1 + Z_2$ where Z_1 is an indefinite inverse in $R(C)$ and $R(BZ_2) \subseteq \mathcal{C}$. On the one hand,

$$\max_{Y \in L(\mathcal{H})} \min_{X \in L(\mathcal{H})} (B_+X + B_-Y - C)^\#(B_+X + B_-Y - C)$$
$$= (BZ_0 - C)^\#(BZ_0 - C) = (BZ_1 - C)^\#(BZ_1 - C) = C^\#(I - Q)C,$$

for $R(BZ_2) \subseteq \mathcal{C}$ and, by Corollary 3.4, $BZ_1 = QC$. On the other hand, Corollary 4.4 yields

$$C^\#(I - Q)C = C^\#(I - Q_-)(I - Q_+)C =$$
$$= C^\# \left[\max_{Y \in L(\mathcal{H})} (B_-Y - I)^\#(B_-Y - I)\right] \left[\min_{X \in L(\mathcal{H})} (B_+X - I)^\#(B_+X - I)\right] C.$$

By merging the above equations, the required identities are obtained and the proof is complete. □

6. The Moore-Penrose inverse in Krein spaces

In [22, Theorem 2.16] X. Mary proved that, given $B \in L(\mathcal{H})$, the range and nullspace of B are regular subspaces of $\mathcal{H}$ if and only if B admits a (unique) "Moore-Penrose inverse", in the sense that, there exists an operator $B^\dagger \in L(\mathcal{H})$ such that $BB^\dagger B = B$, $B^\dagger BB^\dagger = B^\dagger$, $(BB^\dagger)^\# = BB^\dagger$, $(B^\dagger B)^\# = B^\dagger B$. Moreover, by [22, Corollary 2.13], if Q is the selfadjoint projection onto $R(B)$ and P is the selfadjoint projection onto $N(B)^{[\perp]}$, then $BB^\dagger = Q$ and $B^\dagger B = P$.

In this section, we are interested in characterizing the Moore-Penrose inverse in a variational way. To this end, we consider $B \in CR(\mathcal{H})$ and $C \in L(\mathcal{H})$ and analyze the following problem: find conditions for the existence of an $X_0 \in ImS_C$ such that $X_0^\# X_0 \leq Y^\# Y$, for every $Y \in ImS_C$.

By Theorem 4.1, the set ImS_C is non-empty, or equivalently the equation $BX - C = 0$ admits an indefinite minimum solution if and only if $R(C) \subseteq R(B) + R(B)^{[\perp]}$ and $R(B)$ is nonnegative. In this case, the above problem becomes: determine whether there exists

$$\min_{X \in ImS_C} X^\# X \tag{6.1}$$

when $ImS_C \neq \emptyset$. Also, we analyze the existence of an operator $X_0 \in ImS_C$ where the minimum is achieved.

While we only formulate and address this particular problem, by altering the signatures of the subspaces involved, it is clear that this is not the only possibility. The methods used here work equally well in these other cases.

Theorem 6.1. *Let $B \in CR(\mathcal{H})$ and $C \in L(\mathcal{H})$. Then there exists a solution of problem* (6.1) *if and only if $R(B)$ and $N(B^\# B)$ are nonnegative and*

$$R(C) \subseteq B(N(B^\# B)^{[\perp]}) + R(B)^{[\perp]}.$$

Proof. Suppose that there exists a solution of problem (6.1). By Corollary 4.2, the set ImS_C can be described as

$$ImS_C = \{X = X_0 + Y : Y \in L(\mathcal{H}),\ R(Y) \subseteq N(B^\# B)\},$$

where X_0 is any solution of the equation $B^\#(BX - C) = 0$.

Therefore, problem (6.1) can be rephrased as: analyze the existence of

$$\min_{Z \in L(\mathcal{H})} (TZ + X_0)^\#(TZ + X_0), \tag{6.2}$$

where $T \in L(\mathcal{H})$ is such that $R(T) = N(B^\# B)$.

By Theorem 4.1, problem (6.2) has a solution if and only if $N(B^\# B)$ is nonnegative and $R(X_0) \subseteq N(B^\# B) + N(B^\# B)^{[\perp]}$. Applying $B^\# B$ to both sides of the inclusion, we have that

$$R(B^\# C) = R(B^\# B X_0) \subseteq B^\# B(N(B^\# B)^{[\perp]}).$$

Finally, applying $(B^\#)^{-1}$ to both sides of the inclusion, we get

$$R(C) \subseteq B(N(B^\# B)^{[\perp]}) + N(B^\#).$$

Conversely, suppose that $R(B)$ and $N(B^\# B)$ are nonnegative and $R(C) \subseteq B(N(B^\# B)^{[\perp]}) + R(B)^{[\perp]}$. Clearly, $R(C) \subseteq R(B) + R(B)^{[\perp]}$, so, by Theorem 4.1, there exists an indefinite minimum solution X_0 of $BX - C = 0$, or equivalently, $B^\#(BX_0 - C) = 0$. On the other hand, since $N(B^\# B)$ is nonnegative and $R(C) \subseteq B(N(B^\# B)^{[\perp]}) + R(B)^{[\perp]}$, it holds that

$$R(B^\# B X_0) = R(B^\# C) \subseteq B^\# B(N(B^\# B)^{[\perp]}).$$

Applying $(B^\# B)^{-1}$ to both sides of the inclusion yields

$$R(X_0) \subseteq N(B^\# B) + N(B^\# B)^{[\perp]}.$$

Therefore, by Theorem 4.1, there exists a solution of (6.2) and hence, there exists a solution of problem (6.1). □

It follows from the last theorem that, if $B \in CR(\mathcal{H})$ and $C \in L(\mathcal{H})$ are such that $R(B)$ and $N(B^\# B)$ are nonnegative, then there exists a solution of problem (6.1) if and only if $ImS_C \neq \emptyset$, and for every $X_0 \in ImS_C$, $R(X_0) \subseteq N(B^\# B) + N(B^\# B)^{[\perp]}$. Moreover:

Lemma 6.2. *Let $B \in CR(\mathcal{H})$ and $C \in L(\mathcal{H})$ such that $R(B)$ and $N(B^\# B)$ are nonnegative and $R(C) \subseteq B(N(B^\# B)^{[\perp]}) + R(B)^{[\perp]}$. Then X_1 is a solution of* (6.1) *if and only if $B^\#(BX_1 - C) = 0$ and $R(X_1) \subseteq N(B^\# B)^{[\perp]}$.*

Proof. Recall that X_1 is a solution of problem (6.1) if and only if $X_1 = TZ_1 + X_0$, with $T \in L(\mathcal{H})$ such that $R(T) = N(B^\# B)$, X_0 a solution of $B^\#(BX - C) = 0$ and Z_1 a solution of (6.2).

Since $N(B^\# B)$ is nonnegative, by Theorem 4.1 and Corollary 4.3, $Z_1 \in L(\mathcal{H})$ is a solution of (6.2) if and only if Z_1 is such that

$$T^\#(TZ_1 + X_0) = 0$$

that is,

$$T^\# X_1 = 0$$

or, equivalently,

$$R(X_1) \subseteq N(T^\#) = R(T)^{[\perp]} = N(B^\# B)^{[\perp]}.$$

□

As a corollary of Theorem 6.1, we have the following result.

Proposition 6.3. *Let $B \in CR(\mathcal{H})$. Then the following assertions are equivalent:*

i) There exists a solution of problem (6.1) *for $C = I$,*

ii) $R(B)$ and $N(B)$ are uniformly positive,

iii) there exists the Moore-Penrose inverse of B, $B^\dagger$, and $R(B)$ and $N(B)$ are nonnegative.

Proof. $i) \Leftrightarrow ii)$: If there exists a solution of problem (6.1) for $C = I$, by Theorem 6.1, $R(B)$ and $N(B^\# B)$ are nonnegative, and

$$\mathcal{H} \subseteq B(N(B^\# B)^{[\perp]}) + R(B)^{[\perp]}. \tag{6.3}$$

Then, clearly, $R(B)$ is regular and nonnegative; i.e., $R(B)$ is uniformly positive. Since $R(B)$ is regular then $R(B^\#) = R(B^\# B)$ or equivalently, $N(B) = N(B^\# B)$. Applying $B^\#$ to both sides of (6.3), we have that $R(B^\#) \subseteq B^\# B(N(B^\# B)^{[\perp]})$. Then,

$$R(B^\# B) = R(B^\#) \subseteq B^\# B(N(B^\# B)^{[\perp]}) \subseteq R(B^\# B).$$

Therefore

$$R(B^\# B) = B^\# B(N(B^\# B)^{[\perp]}),$$

and so,

$$\mathcal{H} = N(B^\# B) \;[\dotplus]\; N(B^\# B)^{[\perp]} = N(B) \;[\dotplus]\; N(B)^{[\perp]}.$$

Thus, $N(B)$ is regular and nonnegative and therefore uniformly positive.

Conversely, if $R(B)$ and $N(B)$ are uniformly positive, then

$$\mathcal{H} = R(B) \;[\dotplus]\; R(B)^{[\perp]} = N(B^\# B) \;[\dotplus]\; N(B^\# B)^{[\perp]},$$

where we used the fact that $N(B) = N(B^\# B)$ since $R(B)$ is regular. Applying $B^\# B$ to both sides of the second equality, we get that $R(B^\# B) = B^\# B(N(B^\# B)^{[\perp]})$. Then, applying $(B^\#)^{-1}$ to the left and right sides of the last equality, the inclusion $R(B) + R(B)^{[\perp]} \subseteq B(N(B^\# B)^{[\perp]}) + R(B)^{[\perp]}$ holds. Therefore we get

$$\mathcal{H} = B(N(B^\# B)^{[\perp]}) + R(B)^{[\perp]},$$

and, by Theorem 6.1, there exists a solution of problem (6.1) for $C = I$.

$ii) \Leftrightarrow iii)$: If $R(B)$ and $N(B)$ are regular subspaces, choose a fundamental decomposition $\mathcal{H} = \mathcal{H}_+ \;[\dotplus]\; \mathcal{H}_-$ and fix the corresponding Hilbert space $(\mathcal{H}, \langle\ ,\ \rangle)$, so that, for all $x, y \in \mathcal{H}$, $\langle x, y \rangle = [\,Jx, y\,]$ with J the signature operator associated with the decomposition. Consider $B_J^\dagger$ the Moore-Penrose inverse of B in the associated Hilbert space. Set

$$V = PB_J^\dagger Q,$$

where Q is the selfadjoint projection onto $R(B)$ and P is the selfadjoint projection onto $N(B)^{[\perp]}$. It is straightforward to see that $V = B^\dagger$, the Moore-Penrose inverse of B in the Krein space sense.

Conversely, suppose that B admits a Moore-Penrose inverse $B^\dagger$. From the definition of the Moore-Penrose inverse of B in the Krein space sense, it follows that $BB^\dagger$ and $B^\dagger B$ are selfadjoint projections. Moreover, $R(B) = R(BB^\dagger B) \subseteq R(BB^\dagger) \subseteq R(B)$, then $R(BB^\dagger) = R(B)$ so that $R(B)$ is regular. In the same way $N(B^\dagger B) = N(B)$ and then $N(B)$ is regular. See also [22, Theorem 2.6]. □

Remark. If $R(B)$ and $N(B)$ are regular subspaces, from the proof of $ii) \Rightarrow iii)$ of the last theorem, it follows that

$$B^\dagger = PB_J^\dagger Q,$$

where Q is the selfadjoint projection onto $R(B)$, P is the selfadjoint projection onto $N(B)^{[\perp]}$ and $B_J^\dagger$ is the Moore-Penrose inverse of B in any of the underlying Hilbert spaces $(\mathcal{H}, \langle\ ,\ \rangle)$ with $\langle x, y \rangle = [\,Jx, y\,]$, for every $x, y \in \mathcal{H}$.

Theorem 6.4. *Let $B \in CR(\mathcal{H})$ and suppose that $N(B)$ and $R(B)$ are uniformly positive. Then, the Moore-Penrose inverse $B^\dagger$ of B is the unique element X_0 of ImS such that $X_0^\# X_0 \leq Y^\# Y$, for every $Y \in ImS$.*

Proof. Since $N(B)$ and $R(B)$ are regular, the Moore-Penrose inverse of B, $B^\dagger$, exists. Consider Q the selfadjoint projection onto $R(B)$ and P the selfadjoint projection onto $N(B)^{[\perp]}$, then

$$B^{\#}(BB^\dagger - I) = B^{\#}(Q - I) = 0.$$

On the other hand,

$$R(B^\dagger) = R(B^\dagger BB^\dagger) \subseteq R(B^\dagger B) = R(P) = N(B)^{[\perp]}.$$

Hence, by Lemma 6.2, $B^\dagger$ is a solution of problem (6.1), with $C = I$.

Let $X_1 \in L(\mathcal{H})$ be any other solution of problem (6.1), with $C = I$. By Lemma 6.2, $X_1 \in ImS$ and $R(X_1) \subseteq N(B)^{[\perp]}$. Then, by Corollary 3.4, $BX_1 = Q = BB^\dagger$. Therefore,

$$X_1 = PX_1 = B^\dagger BX_1 = B^\dagger BB^\dagger = B^\dagger.$$

□

Remark. Let $B \in CR(\mathcal{H})$ and $C \in L(\mathcal{H})$, and suppose that $N(B)$ and $R(B)$ are uniformly positive. From the proofs of Proposition 6.3 and Theorem 6.4, problem (6.1) admits a unique solution; namely, $B^\dagger C$.

6.1. The Moore-Penrose inverse: the pseudo-regular case

In [12, Proposition 5.1], a family of generalized inverses of a closed range operator with pseudo-regular range and nullspace was given, and the associated projections turn out to be normal. In this section, we prove the equivalence between the existence of such a family of generalized inverses and the pseudo-regularity of the range and nullspace of an operator $B \in CR(\mathcal{H})$. We also give a formula for these generalized inverses and characterize them in a variational way, as we did in the last section with the Moore-Penrose inverse.

Given $B \in CR(\mathcal{H})$, recall that $\tilde{B}$ is a $\{1,2\}$-inverse of B if $\tilde{B}$ is a solution of the system

$$BXB = B,\ XBX = X.$$

If $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ is a Hilbert space, every $B \in CR(\mathcal{H})$ admits $\{1,2\}$-inverses, see [3, Theorem 3.1]. Using any of the underlying Hilbert structures, the same is then true in the Krein space $\mathcal{H}$. Observe that, if $\tilde{B}$ is a $\{1,2\}$-inverse of B, then $B\tilde{B}$ is a projection onto $R(B)$ and $\tilde{B}B$ is a projection with $N(\tilde{B}B) = N(B)$.

Proposition 6.5. *Let $B \in CR(\mathcal{H})$. Then, there exists a solution of the system*

$$\begin{cases} BXB = B,\ XBX = X, \\ (BX)^{\#}(BX) = (BX)(BX)^{\#}, \\ (XB)^{\#}(XB) = (XB)(XB)^{\#}, \end{cases} \tag{6.4}$$

if and only if $R(B)$ and $N(B)$ are pseudo-regular subspaces of $\mathcal{H}$.

In this case, $D \in L(\mathcal{H})$ is a solution of (6.4) *if and only if there exist $Q \in \mathcal{Q}_{R(B)}$ and $P \in \mathcal{Q}_{N(B)}$ such that*

$$D = (I - P)\tilde{B}Q, \tag{6.5}$$

where $\tilde{B}$ is any $\{1,2\}$-inverse of B.

Proof. Suppose that $R(B)$ and $N(B)$ are pseudo-regular subspaces. Let $Q \in \mathcal{Q}_{R(B)}$ and $P \in \mathcal{Q}_{N(B)}$. Let $\tilde{B}$ be any $\{1,2\}$-inverse of B. Let D be defined as in (6.5). From $BP = 0$ it follows immediately that

$$BD = Q, \text{ and } DB = I - P.$$

So the last two equations of the system are satisfied. Also,

$$BDB = QB = B \text{ and } DBD = (I - P)D = D.$$

Conversely, suppose that (6.4) admits a solution D. Let $Q = BD$ and $P = I - DB$, then P and Q are normal projections in $L(\mathcal{H})$. Moreover, $R(Q) = R(BD) \subseteq R(B)$. On the other hand, $R(BD) \supseteq$

$R(BDB) = R(B)$. Therefore, $R(Q) = R(B)$ and $R(B)$ is pseudo-regular. Also, $N(B) \subseteq N(DB) = N(P) \subseteq N(BDB) = N(B)$. So $N(B) = N(P)$ and $N(B)$ is pseudo-regular.

In this case, we have already proven that if D is as in (6.5), then D is a solution of (6.4). Conversely, let $D \in L(\mathcal{H})$ be any solution of (6.4) and $Q := BD \in \mathcal{Q}_{R(B)}$, $P := I - DB \in \mathcal{Q}_{N(B)}$. By definition, $\tilde{B} = D$ is a $\{1,2\}$-inverse of B. Hence $(I-P)\tilde{B}Q = DBD(BD) = DBD = D$. □

Proposition 6.6. *Let $B \in CR(\mathcal{H})$ such that $R(B)$ is pseudo-regular. Then there exists a solution of problem (6.1) for every $C \in L(\mathcal{H})$ such that $R(C) \subseteq \overline{B(N(B^{\#}B)^{[\perp]}) + R(B)^{[\perp]}}$ if and only if $N(B^{\#}B)$ is pseudo-regular and $N(B^{\#}B)$ and $R(B)$ are nonnegative.*

Proof. Suppose that $R(B)$ and $N(B^{\#}B)$ are nonnegative and pseudo-regular. Then, by [12, Lemma 3.4], $R(B^{\#}B)$ is closed. Since $N(B^{\#}B)$ is pseudo-regular, [19, Corollary 2.5] gives that $R(B^{\#}BB^{\#}B)$ is closed too.

Suppose that $R(C) \subseteq \overline{B(N(B^{\#}B)^{[\perp]}) + R(B)^{[\perp]}}$. Then

$$R(B^{\#}C) \subseteq B^{\#}[(\overline{B(R(B^{\#}B)) + R(B)^{[\perp]}})]$$

$$\subseteq B^{\#}[(R(B) \cap R(BB^{\#}B)^{[\perp]})^{[\perp]}] \subseteq [B^{-1}(R(B) \cap R(BB^{\#}B)^{[\perp]})]^{[\perp]} =$$
$$= [B^{-1}(R(BB^{\#}B)^{[\perp]})]^{[\perp]} = [B^{\#}R(BB^{\#}B)]^{[\perp]\,[\perp]} = R(B^{\#}BB^{\#}B),$$

where we used the fact that $B^{\#}(\mathcal{S}^{[\perp]}) \subseteq (B^{-1}(\mathcal{S}))^{[\perp]}$, for any closed subspace $\mathcal{S} \subseteq \mathcal{H}$ and that $R(B^{\#}BB^{\#}B)$ is closed. Then, applying $(B^{\#})^{-1}$ to both sides of the inclusion, we have that

$$R(C) \subseteq (B^{\#})^{-1}(R(B^{\#}BB^{\#}B)) = N(B^{\#}) + B(N(B^{\#}B)^{[\perp]}).$$

So, by Theorem 6.1, problem (6.1) admits a solution.

Conversely, suppose that there exists a solution of problem (6.1) for every $C \in L(\mathcal{H})$ such that $R(C) \subseteq \overline{B(N(B^{\#}B)^{[\perp]}) + R(B)^{[\perp]}}$. Pick C such that

$$R(C) = \overline{B(N(B^{\#}B)^{[\perp]}) + R(B)^{[\perp]}}.$$

By Theorem 6.1, $N(B^{\#}B)$ and $R(B)$ are nonnegative and

$$R(C) = \overline{B(N(B^{\#}B)^{[\perp]}) + R(B)^{[\perp]}} \subseteq B(N(B^{\#}B)^{[\perp]}) + R(B)^{[\perp]}.$$

Thus the subspace $B(N(B^{\#}B)^{[\perp]}) + R(B)^{[\perp]}$ is closed. Hence,

$$B^{-1}(B(N(B^{\#}B)^{[\perp]}) + R(B)^{[\perp]}) = N(B) + N(B^{\#}B)^{[\perp]} + N(B^{\#}B) =$$
$$= N(B^{\#}B)^{[\perp]} + N(B^{\#}B)$$

is closed, and so $N(B^{\#}B)$ is pseudo-regular. □

The next result is a corollary of Proposition 6.5. We will use it in the proof of Theorem 6.8 in order to characterized the solutions of (6.1) in term of pseudo-inverses when $R(B)$ and $N(B^{\#}B)$ are pseudo-regular.

Lemma 6.7. *Let $B \in CR(\mathcal{H})$ be such that $R(B)$ is a pseudo-regular subspace of $\mathcal{H}$. Given $Q \in \mathcal{Q}_{R(B)}$, let $B' = Q^{\#}B$. Then there exists a solution of the system*

$$\begin{cases} B'XB' = B',\ XB'X = X, \\ B'X = Q^{\#}Q, \\ (XB')^{\#}(XB') = (XB')(XB')^{\#}, \end{cases} \tag{6.6}$$

if and only if $N(B^{\#}B)$ is a pseudo-regular subspace of $\mathcal{H}$.

In this case, $D \in L(\mathcal{H})$ is a solution of (6.6) if and only if there exists $P \in \mathcal{Q}_{N(B^{\#}B)}$ such that

$$D = (I-P)\tilde{B}'Q^{\#}Q,$$

where $\tilde{B}'$ is a $\{1,2\}$-inverse of B'.

Proof. Note that

$$R(B') = R(Q^{\#}Q) \text{ and } N(B') = N(B^{\#}B).$$

Then apply Proposition 6.5 to B'. $\square$

Theorem 6.8. *Let $B \in CR(\mathcal{H})$ and $C \in L(\mathcal{H})$. For $R(B)$ and $N(B^{\#}B)$ nonnegative pseudo-regular subspaces and $R(C) \subseteq \overline{B(N(B^{\#}B)^{[\perp]}) + R(B)^{[\perp]}}$, set $X_1 = DC$, where $D \in L(\mathcal{H})$ is a solution of* (6.6), *then X_1 is a solution of problem* (6.1).

Proof. By the proof of Proposition 6.6, the set $B(N(B^{\#}B)^{[\perp]}) + R(B)^{[\perp]}$ is closed.

Given a solution D of (6.6), let $P \in \mathcal{Q}_{N(B^{\#}B)}$, $Q \in \mathcal{Q}_{R(B)}$ and $\tilde{B}'$ a $\{1,2\}$-inverse of $B' = Q^{\#}B$ satisfying

$$D = (I - P)\tilde{B}'Q^{\#}Q.$$

Observe that

$$Q^{\#}BDC = B'DC = Q^{\#}QC.$$

Since $R(C) \subseteq R(B) + R(B)^{[\perp]}$ and, by Lemma 4.5, $Q^{\#}Qy = Q^{\#}y$ for every $y \in R(B) + R(B)^{[\perp]}$, it follows that $Q^{\#}QC = Q^{\#}C$. Then

$$Q^{\#}BDC = Q^{\#}C.$$

Therefore $R(BDC - C) \subseteq N(Q^{\#}) = N(B^{\#})$ or, equivalently,

$$B^{\#}(BDC - C) = 0.$$

Then by Proposition 4.6, $DC \in ImS_C$.

On the other hand,

$$R(B^{\#}BDC) = R(B^{\#}C) \subseteq B^{\#}(B(N(B^{\#}B)^{[\perp]})),$$

so applying $(B^{\#}B)^{-1}$ to both sides of the inclusion, we have that

$$R(DC) \subseteq N(B^{\#}B) + N(B^{\#}B)^{[\perp]} = N(P^{\#}(I - P)).$$

Then

$$P^{\#}(I - P)DC = P^{\#}DC = 0.$$

Thus $R(DC) \subseteq N(B^{\#}B)^{[\perp]}$ and, by Lemma 6.2, $X_1 = DC$ is a solution of problem (6.1). $\square$

Remark. By Proposition 6.6, under the assumptions of the last theorem there exists a solution of problem (6.1). Furthermore, if $R(C) \not\subseteq R(B)^{[\perp]}$, a converse of Theorem 6.8 holds: if X_1 is a solution of problem (6.1) then $X_1 = DC$, where $D \in L(\mathcal{H})$ is a solution of (6.6).

For let X_1 be a solution of problem (6.1). Then by similar arguments as those in [12, Theorem 3.5], there exists $P' \in \mathcal{Q}_{N(B^{\#}B)}$ such that

$$X_1 = (I - P')X_0,$$

where $X_0 \in ImS_C$. Now let $Q \in \mathcal{Q}_{R(B)}$ and $\tilde{B}'$ be any $\{1,2\}$-inverse of B'. Set

$$D = (I - P')\tilde{B}'Q^{\#}Q.$$

Then by Lemma 6.7, D is a solution of (6.6). Proceeding as in the proof of the last theorem, we get that $DC \in ImS_C$. Then by Corollary 4.2, $X_0 = DC + Y$, with $R(Y) \subseteq N(B^{\#}B)$. Hence,

$$X_1 = (I - P')X_0 = (I - P')DC = DC.$$

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