FINITE RANK PERTURBATIONS OF LINEAR RELATIONS AND SINGULAR MATRIX PENCILS

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Abstract. We elaborate on the deviation of the Jordan structures of two linear relations that are finite-dimensional perturbations of each other. We compare the number of Jordan chains of length at least n corresponding to some eigenvalue to each other. In the operator case, it was recently proved that the difference of these numbers is independent of n and is at most the defect between the operators. One of the main results of this paper shows that in the case of linear relations this number has to be multiplied by $n+1$ and that this bound is sharp. The reason for this behaviour is the existence of singular chains.

We apply our results to one-dimensional perturbations of singular and regular matrix pencils. This is done by representing matrix pencils via linear relations. This technique allows for both proving known results for regular pencils as well as new results for singular ones.

1. Introduction

Given a pair of matrices $E, F \in \mathbb{C}^{d\times d}$, the associated matrix pencil is defined by

$$\mathcal{A}(s) := sE - F. \tag{1.1}$$

The theory of matrix pencils occupies an increasingly important place in linear algebra, due to its numerous applications. For instance, they appear in a natural way in the study of differential-algebraic equations of the form:

$$E\dot{x} = Fx, \quad x(0) = x_0, \tag{1.2}$$

which are a generalization of the abstract Cauchy problem. Substituting $x(t) = x_0 e^{st}$ into (1.2) leads to

$$(sE - F)x_0 = 0.$$

Hence, solutions of the above eigenvalue equation for the matrix pencil (1.1) correspond to solutions of the Cauchy problem (1.2).

The matrix pencil $\mathcal{A}$ is called regular if $\det(sE - F)$ is not identically zero, and it is called singular otherwise. Perturbation theory for regular matrix pencils $\mathcal{A}(s) := sE-F$ is a well developed field, we mention here only [14, 17, 21, 25] which is a short list of

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papers devoted to this subject. As an example, we describe a well-known result. Recall that for a matrix pencil $\mathcal{A}$ as in (1.1), an ordered family of vectors $(x_n, \ldots, x_0)$ is a Jordan chain of length $n+1$ at $\lambda \in \mathbb{C}$ if $x_0 \neq 0$ and

$$(F - \lambda E)x_0 = 0, \quad (F - \lambda E)x_1 = Ex_0, \quad \ldots, \quad (F - \lambda E)x_n = Ex_{n-1}.$$

Denote by $\mathcal{L}_\lambda^l(\mathcal{A})$ the subspace spanned by the elements of all Jordan chains up to length l at the eigenvalue $\lambda \in \mathbb{C}$. If $\mathcal{A}(s)$ is regular and if $\mathcal{P}(s)$ is a rank one pencil such that $(\mathcal{A} + \mathcal{P})(s)$ is also regular then for $n \in \mathbb{N}$ the following inequality holds:

$$\left| \dim \frac{\mathcal{L}_\lambda^{n+1}(\mathcal{A}+\mathcal{P})}{\mathcal{L}_\lambda^{n}(\mathcal{A}+\mathcal{P})} - \dim \frac{\mathcal{L}_\lambda^{n+1}(\mathcal{A})}{\mathcal{L}_\lambda^{n}(\mathcal{A})} \right| \leq 1. \tag{1.3}$$

In this form it can be found in [17], but it is mainly due to [14] and [25]. The proof of this inequality, as many other results concerning perturbation theory for regular matrix pencils, is based on a detailed analysis of the determinant.

On the other hand, to the best of our knowledge there is no perturbation theory for finite or low rank perturbations of singular matrix pencils. This is mainly due to the fact that a singular matrix pencil, by definition, has an identically zero determinant. However, some results exist for (generic) perturbations of singular matrix pencils small in norm, see e.g. [15].

Here we develop a different approach to treat rank one perturbations of singular matrix pencils. This is done by representing matrix pencils via linear relations, see also [5, 6, 11]. Each matrix $E \in \mathbb{C}^{d\times d}$ is considered as a linear relation via its graph, i.e. the subspace of $\mathbb{C}^d \times \mathbb{C}^d$ consisting of pairs of the form $\{x, Ex\}$, $x \in \mathbb{C}^d$. Also, the inverse E^{-1} (in the sense of linear relations) of a non-necessarily invertible matrix E is the subspace of $\mathbb{C}^d \times \mathbb{C}^d$ consisting of pairs of the form $\{Ex, x\}$, $x \in \mathbb{C}^d$. Multiplication of linear relations is defined in analogy to multiplication of matrices, see Section 2 for the details. Then, to a matrix pencil $\mathcal{A}(s) = sE - F$ we associate the linear relation $E^{-1}F$.

There exists a well developed spectral theory for linear relations, see e.g. [1, 12, 22]. An eigenvector at $\lambda \in \mathbb{C}$ of $E^{-1}F$ is a tuple of the form $\{x, \lambda x\} \in E^{-1}F$, $x \neq 0$. Jordan chains are defined in a similar way, see Section 3 below.

In Section 7 we show that (point) spectrum and Jordan chains of $E^{-1}F$ coincide with (point) spectrum and Jordan chains of the matrix pencil $\mathcal{A}$ in (1.1), respectively. This is the key to translate spectral properties of a matrix pencil to its associated linear relation and vice versa. The advantage of this approach is that it is applicable not only to regular matrix pencils, but also to singular matrix pencils.

Given a matrix pencil $\mathcal{A}$ as in (1.1), we consider one-dimensional perturbations of the form

$$\mathcal{P}(s) = w(su^* + v^*), \tag{1.4}$$

where $u, v, w \in \mathbb{C}^d$, $(u, v) \neq (0, 0)$ and $w \neq 0$. Then $\mathcal{A}$ and $\mathcal{A} + \mathcal{P}$ are rank-one perturbations of each other, which means that they differ by at most a rank-one matrix polynomial. Recall that the rank of a matrix pencil $\mathcal{P}$ is the largest $r \in \mathbb{N}$ such that $\mathcal{P}$, viewed as a matrix with polynomial entries, has minors of size r that are not the zero polynomial [14, 16]. As described above, to the matrix pencils $\mathcal{A}$ and $\mathcal{A} + \mathcal{P}$ there corresponds the linear relations $E^{-1}F$ and $(E+wu^*)^{-1}(F+wv^*)$ which turn out to be one-dimensional perturbations of each other, see Section 4. Then the main result of this paper consists of the following perturbation estimates for singular (and regular) matrix pencils:

(i) If $\mathcal{A}$ is regular but $\mathcal{A}+\mathcal{P}$ is singular, then

$$-1-n \leq \dim \frac{\mathcal{L}_\lambda^{n+1}(\mathcal{A}+\mathcal{P})}{\mathcal{L}_\lambda^{n}(\mathcal{A}+\mathcal{P})} - \dim \frac{\mathcal{L}_\lambda^{n+1}(\mathcal{A})}{\mathcal{L}_\lambda^{n}(\mathcal{A})} \leq 1.$$

(ii) If $\mathcal{A}$ is singular and $\mathcal{A}+\mathcal{P}$ is regular, then

$$-1 \leq \dim \frac{\mathcal{L}_\lambda^{n+1}(\mathcal{A}+\mathcal{P})}{\mathcal{L}_\lambda^{n}(\mathcal{A}+\mathcal{P})} - \dim \frac{\mathcal{L}_\lambda^{n+1}(\mathcal{A})}{\mathcal{L}_\lambda^{n}(\mathcal{A})} \leq n+1.$$

(iii) If both $\mathcal{A}$ and $\mathcal{A}+\mathcal{P}$ are singular, then

$$\left|\dim \frac{\mathcal{L}_\lambda^{n+1}(\mathcal{A}+\mathcal{P})}{\mathcal{L}_\lambda^{n}(\mathcal{A}+\mathcal{P})} - \dim \frac{\mathcal{L}_\lambda^{n+1}(\mathcal{A})}{\mathcal{L}_\lambda^{n}(\mathcal{A})}\right| \leq n+1.$$

This result follows from the corresponding result for one-dimensional perturbations of linear relations. This is the content of Sections 3 and 4, which is of independent interest. More precisely, given linear relations A and B in a linear space X which are one-dimensional perturbations of each other, we show that $N(A^{n+1})/N(A^n)$ is finite-dimensional if and only if $N(B^{n+1})/N(B^n)$ is finite-dimensional and, in this case,

$$\left|\dim \frac{N(B^{n+1})}{N(B^n)} - \dim \frac{N(A^{n+1})}{N(A^n)}\right| \leq n+1. \tag{1.5}$$

Here $N(A)$ denotes the kernel of the linear relation A, that is, the set of all $x \in X$ such that $\{x, 0\} \in A$. If, in addition, $A \subset B$ or $B \subset A$, we show that the left-hand side in (1.5) is bounded by n. However, in Section 5 we show that the bound in (1.5) is sharp. It is worth mentioning that if A and B are linear operators in X the left-hand side in (1.5) is bounded by 1, see [4].

In Section 6 we extend the above result to p-dimensional perturbations. In this case, we show that the left-hand side in (1.5) is bounded by $(n+1)p$. Again, this estimate improves to np in case that $A \subset B$ or $B \subset A$, and to p if A and B are operators, cf. [4].

2. Preliminaries

Throughout this paper X denotes a linear space over $\mathbb{K}$, where $\mathbb{K}$ stands for the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. Elements (pairs) from $X \times X$ will be denoted by $\{x, y\}$, where $x, y \in X$. A *linear relation* in X is a linear subspace of $X \times X$. Linear operators can be treated as linear relations via their graphs: each linear operator $T : D(T) \to X$ in X, where $D(T)$ stands for the domain of T, is identified with its graph

$$\Gamma(T) := \{\{x, Tx\} : \ x \in D(T)\}.$$

For the basic notions and properties of linear relations we refer to [12, 19], see also [1, 13, 22, 23, 24]. Here, we denote the domain and the range of a linear relation A in

X by $D(A)$ and $R(A)$, respectively,

$$D(A) = \{x \in X \ : \ \exists\, y : \ \{x, y\} \in A\} \quad \text{and} \quad R(A) = \{y \in X \ : \ \exists\, x : \ \{x, y\} \in A\}\,.$$

Furthermore, $N(A)$ and $M(A)$ denote the *kernel* and the *multivalued* part of A,

$$N(A) = \{x \in X \ : \ \{x, 0\} \in A\} \quad \text{and} \quad M(A) = \{y \in X \ : \ \{0, y\} \in A\}\,.$$

Obviously, a linear relation A is the graph of an operator if and only if $M(A) = \{0\}$. The inverse A^{-1} of a linear relation A always exists and is given by

$$A^{-1} = \{\{y, x\} \in X \times X \ : \ \{x, y\} \in A\}\,. \tag{2.1}$$

We recall that the product of two linear relations A and B in X is defined as

$$AB = \{\{x, z\} \ : \ \{y, z\} \in A \text{ and } \{x, y\} \in B \ \text{ for some } y \in X\}\,. \tag{2.2}$$

As for operators the product of linear relations is an associative operation. We denote $A^0 := I$, where I denotes the identity operator in X, and for $n = 1, 2, \ldots$ the n-th power of A is defined recursively by

$$A^n := AA^{n-1}. \tag{2.3}$$

Thus, we have $\{x_n, x_0\} \in A^n$ if and only if there exist $x_1, \ldots, x_{n-1} \in X$ such that

$$\{x_n, x_{n-1}\}, \{x_{n-1}, x_{n-2}\}, \ldots, \{x_1, x_0\} \in A. \tag{2.4}$$

In this case, (2.4) is called a *chain of A*. For this we also use the shorter notation $(x_n, \ldots, x_0)$.

For a linear relation T in X and $m \in \mathbb{N}$, consider the vector space of m-tuples of elements in T:

$$T^{(m)} := \underbrace{T \times T \times \cdots \times T}_{m \text{ times}}\,,$$

and also the space of m-tuples of elements in T which are chains of T:

$$\mathcal{S}_m^T := \big\{\, (\{x_m, x_{m-1}\}, \ldots, \{x_1, x_0\}) : (x_m, x_{m-1}, \ldots, x_0) \text{ is a chain of } T \big\}. \tag{2.5}$$

Clearly, $\mathcal{S}_m^T$ is a subspace of $T^{(m)}$.

Lemma 2.1. *Let A and C be linear relations in X such that $C \subset A$ and $\dim(A/C) = 1$. Then for each $m \in \mathbb{N}$ the following inequality holds:*

$$\dim(\mathcal{S}_m^A / \mathcal{S}_m^C) \ \le \ m. \tag{2.6}$$

Proof. We make use of Lemma 2.2 in [3] which states that whenever M_0, N_0, M_1, N_1 are subspaces of a linear space $\mathcal{X}$ such that $M_0 \subset M_1$ and $N_0 \subset N_1$, then

$$\dim \frac{M_1 \cap N_1}{M_0 \cap N_0} \ \le \ \dim \frac{M_1}{M_0} + \dim \frac{N_1}{N_0}.$$

With this lemma the proof of (2.6) is straightforward. Indeed, since $\mathcal{S}_m^C = \mathcal{S}_m^A \cap C^{(m)}$, we obtain from the lemma and from $\dim(A/C) = 1$ that

$$\dim(\mathcal{S}_m^A / \mathcal{S}_m^C) = \dim \frac{\mathcal{S}_m^A \cap A^{(m)}}{\mathcal{S}_m^A \cap C^{(m)}} \ \le \ \dim(A^{(m)} / C^{(m)}) = m,$$

which is (2.6). $\square$

The chain $(x_n, \dots, x_0)$ is called a *quasi-Jordan chain of* A (or a *quasi-Jordan chain of* A *at zero*) if $x_0 \in N(A)$, cf. [8]. If $(x_n, \dots, x_0)$ is a quasi-Jordan chain of A, then $x_j \in N(A^{j+1})$ for $j = 0, \dots, n$. If, in addition, $x_n \in M(A)$ and $(x_n, \dots, x_0) \neq (0, \dots, 0)$, then the chain is called a *singular chain* of A. Note that we admit linear dependence (and even zeros) within the elements of a quasi-Jordan chain.

For relations A and B in X the *operator-like sum* $A + B$ is the relation defined by

$$A + B = \{\{x, y + z\} \ : \ \{x, y\} \in A, \{x, z\} \in B\},$$

and for $\lambda \in \mathbb{C}$ one defines $\lambda A = \{\{x, \lambda y\} \ : \ \{x, y\} \in A\}$. Hence, we have

$$A - \lambda = \{\{x, y - \lambda x\} \ : \ \{x, y\} \in A\}.$$

Finally, we call the tuple $(x_n, \dots, x_0)$ a *quasi-Jordan chain of* A *at* $\lambda \in \mathbb{C}$, if $(x_n, \dots, x_0)$ is a quasi-Jordan chain of the linear relation $A - \lambda$. The tuple $(x_n, \dots, x_0)$ is called a quasi-Jordan chain of A at ∞, if $(x_n, \dots, x_0)$ is a quasi-Jordan chain of A^{-1}.

We reserve the notion of Jordan chain of a linear relation for a particular situation which is discussed in the next section.

3. Linear independence of Quasi-Jordan chains

Assume that T is a linear operator in X and consider $x_0, \dots, x_n \in D(T)$ such that

$$Tx_0 = 0 \quad \text{and} \quad Tx_j = x_{j-1}, \text{ for all } 1 \leq j \leq n.$$

Then $\{x_n, x_{n-1}\}, \{x_{n-1}, x_{n-2}\}, \dots, \{x_0, 0\} \in \Gamma(T)$. So, if we consider T also as a linear relation via its graph, $(x_n, \dots, x_0)$ is a quasi-Jordan chain of T.

As T is a linear operator, it is well-known that the following facts are equivalent:

(i) $x_0 \neq 0$.
(ii) The set of vectors $\{x_n, \dots, x_0\}$ is linear independent in X.
(iii) $[x_n] \neq 0$, where $[x_n]$ is the equivalence class in $N(T^{n+1})/N(T^n)$.
(iv) $[x_j] \neq 0$ for all $1 \leq j \leq n$, where $[x_j]$ is the equivalence class in $N(T^{j+1})/N(T^j)$.

Therefore, if T is a linear operator and $x_0 \neq 0$, $(x_n, \dots, x_0)$ is a quasi-Jordan chain of the linear relation $\Gamma(T)$ if and only if it is a Jordan chain of the linear operator T in the usual sense.

However, the four statements above are no longer equivalent for linear relations which contain singular chains, see the following example.

Example 3.1. Let x_0 and x_1 be two linear independent elements of X and let

$$A := \text{span } \{\{0, x_0\}, \{x_0, 0\}, \{x_1, x_0\}\}.$$

Then $x_0 \neq 0$ but $(0, x_0)$ is a quasi-Jordan chain with linear dependent entries, hence the equivalence of (i) and (ii) from above does not hold.

Moreover, (x_1, x_0) is a quasi-Jordan chain with linear independent entries. But, as $\{x_1, x_0\}$ and $\{0, x_0\}$ are both elements of A, due to linearity, also $\{x_1, 0\}$ is an element of A and, hence, $[x_1] = 0$, i.e. (iii) is not satisfied. Therefore, conditions (ii) and (iii) are neither equivalent for linear relations.

As it was mentioned before, the situation shown in the example is a consequence of the existence of singular chains in the relation A, or equivalently, the presence of vectors in the intersection of the kernel of A and the multivalued part of A^n for some $n \in \mathbb{N}$. For arbitrary linear relations we have the following equivalence.

Proposition 3.2. *Let A be a linear relation in X and $(x_n, \ldots, x_0)$ be a quasi-Jordan chain of A. Then the following statements are equivalent:*

(i) $x_0 \notin M(A^n)$.
(ii) $[x_n] \neq 0$, *where $[x_n]$ is the equivalence class in $N(A^{n+1})/N(A^n)$.*
(iii) $[x_j] \neq 0$ *for all $1 \leq j \leq n$, where $[x_j]$ is the equivalence class in $N(A^{j+1})/N(A^j)$.*

In particular, if any of the three equivalent statements holds, then the vectors $x_0, \ldots, x_n$ are linear independent in X.

Proof. Since $(x_n, \ldots, x_0)$ is a quasi-Jordan chain of A, we have that

$$\{x_n, x_{n-1}\}, \ldots, \{x_1, x_0\}, \{x_0, 0\} \in A. \tag{3.1}$$

We show that (i) and (ii) are equivalent. If $x_0 \in M(A^n)$, then there exist $y_1, \ldots, y_{n-1} \in X$ such that

$$\{0, y_{n-1}\}, \ldots, \{y_2, y_1\}, \{y_1, x_0\} \in A.$$

Subtracting this chain from the one in (3.1) we end with

$$\{x_n, x_{n-1} - y_{n-1}\}, \ldots, \{x_2 - y_2, x_1 - y_1\}, \{x_1 - y_1, 0\} \in A.$$

Thus, $x_n \in N(A^n)$, or equivalently, $[x_n] = 0$. Conversely, if $[x_n] = 0$ then $x_n \in N(A^n)$. Hence, there exist $u_1, \ldots, u_{n-1} \in X$ such that

$$\{x_n, u_{n-1}\}, \ldots, \{u_2, u_1\}, \{u_1, 0\} \in A.$$

Taking the difference of (3.1) and the chain above we obtain

$$\{0, x_{n-1} - u_{n-1}\}, \ldots, \{x_2 - u_2, x_1 - u_1\}, \{x_1 - u_1, x_0\} \in A,$$

i.e. $x_0 \in M(A^n)$.

Now we show that (ii) and (iii) are equivalent. Obviously (iii) implies (ii). Hence, assume $[x_n] \neq 0$. Then, by (i), $x_0 \notin M(A^n)$. But as $M(A^j) \subset M(A^n)$ for all $1 \leq j \leq n$, we have $x_0 \notin M(A^j)$ for all $1 \leq j \leq n$. Applying (ii) to every $[x_j]$ we obtain (iii).

It remains to show the additional statement concerning the linear independence of the vectors $x_0, \ldots, x_n$. This is the case if the equation $\sum_{j=0}^{n} \alpha_j x_j = 0$ implies that all α_j, $j = 0, \ldots, n$, are equal to 0. By (iii) we see that all x_j are non-zero. If not all α_j are equal to 0, let n_0 be the largest index j with $\alpha_j \neq 0$. It follows that

$$x_{n_0} = -\alpha_{n_0}^{-1} \sum_{j=0}^{n_0 - 1} \alpha_j x_j \in N(A^{n_0}),$$

hence $[x_{n_0}] = 0$, in contradiction to (iii). $\square$

The above considerations lead to the following definition of a Jordan chain for a linear relation.

Definition 3.3. Let $(x_n, \ldots, x_0)$ be a quasi-Jordan chain of a linear relation A in X. We call it a *Jordan chain of length $n+1$ in A* if

$$[x_n] \neq 0 \text{ in } N(A^{n+1})/N(A^n).$$

We remark that our Definition 3.3 is equivalent to the definition formulated in [22] but different from the one used in [11], where the term Jordan chain was used for an object which is here called quasi-Jordan chain together with the assumption that all elements of the quasi-Jordan chain are linear independent.

In the sequel we will make use of the following lemma.

Lemma 3.4. *Let A be a linear relation in X and let $(x_{k,n}, \ldots, x_{k,0})$, $k = 1, \ldots, m$, be m quasi-Jordan chains of A. Then*

$$\dim \operatorname{span}\{[x_{1,n}], \ldots, [x_{m,n}]\} = \dim \frac{\mathcal{L}}{\mathcal{L} \cap M(A^n)}, \tag{3.2}$$

where $\mathcal{L} := \operatorname{span}\{x_{1,0}, \ldots, x_{m,0}\}$.

Proof. In the case $m = 1$, the assertion follows from Proposition 3.2. Assume (3.2) holds for some m. We show it holds for $m+1$. So, let $(x_{k,n}, \ldots, x_{k,0})$, $k = 1, \ldots, m+1$, be $m+1$ quasi-Jordan chains of A and define

$$\mathcal{L}_m := \operatorname{span}\{x_{1,0}, \ldots, x_{m,0}\} \quad \text{and} \quad \mathcal{L}_{m+1} := \operatorname{span}\{x_{1,0}, \ldots, x_{m,0}, x_{m+1,0}\}.$$

We consider two cases, $x_{m+1,0} \in \mathcal{L}_m$ and $x_{m+1,0} \notin \mathcal{L}_m$. If $x_{m+1,0} \in \mathcal{L}_m$ then there exists numbers α_j, $j = 1, \ldots, m$, with $x_{m+1,0} = \sum_{j=1}^m \alpha_j x_{j,0}$ such that

$$\left\{ x_{m+1,n} - \sum_{j=1}^m \alpha_j x_{j,n}, x_{m+1,n-1} - \sum_{j=1}^m \alpha_j x_{j,n-1} \right\}, \ldots, \left\{ x_{m+1,1} - \sum_{j=1}^m \alpha_j x_{j,1}, 0 \right\} \in A.$$

Hence, $x_{m+1,n} - \sum_{j=1}^m \alpha_j x_{j,n} \in N(A^n)$. Then $[x_{m+1,n}] = \sum_{j=1}^m \alpha_j [x_{j,n}]$ and

$$\begin{aligned} \dim \operatorname{span}\{[x_{1,n}], \ldots, [x_{m+1,n}]\} &= \dim \operatorname{span}\{[x_{1,n}], \ldots, [x_{m,n}]\} && (3.3)\\ &= \dim \frac{\mathcal{L}_m}{\mathcal{L}_m \cap M(A^n)} = \dim \frac{\mathcal{L}_{m+1}}{\mathcal{L}_{m+1} \cap M(A^n)}, && (3.4) \end{aligned}$$

because in this case $\mathcal{L}_m = \mathcal{L}_{m+1}$.

On the other hand, if $x_{m+1,0} \notin \mathcal{L}_m$ then $\mathcal{L}_{m+1} = \mathcal{L}_m + \operatorname{span}\{x_{m+1,0}\}$. Now, we consider two subcases: $[x_{m+1,n}] \notin \operatorname{span}\{[x_{1,n}], \ldots, [x_{m,n}]\}$ or $[x_{m+1,n}] \in \operatorname{span}\{[x_{1,n}], \ldots, [x_{m,n}]\}$. Assume that $[x_{m+1,n}] \notin \operatorname{span}\{[x_{1,n}], \ldots, [x_{m,n}]\}$. In particular, $[x_{m+1,n}] \neq 0$. Then by Proposition 3.2, $x_{m+1,0} \notin M(A^n)$ and $\mathcal{L}_{m+1} \cap M(A^n) = \mathcal{L}_m \cap M(A^n)$. Therefore,

$$\begin{aligned} \dim \operatorname{span}\{[x_{1,n}], \ldots, [x_{m+1,n}]\} &= \dim \operatorname{span}\{[x_{1,n}], \ldots, [x_{m,n}]\} + 1 \\ &= \dim \frac{\mathcal{L}_m}{\mathcal{L}_m \cap M(A^n))} + 1 \\ &= \dim \frac{\mathcal{L}_{m+1}}{\mathcal{L}_{m+1} \cap M(A^n))}. \end{aligned}$$

Assume now that $[x_{m+1,n}] \in \operatorname{span}\{[x_{1,n}], \ldots, [x_{m,n}]\}$. Then there exists $\alpha_j \in \mathbb{K}$, $j = 1, \ldots, m$ such that $[x_{m+1,n}] = \sum_{j=1}^m \alpha_j [x_{j,n}]$ and (3.3) also holds. By Proposition 3.2,

$w_0 := x_{m+1,0} - \sum_{j=1}^{m} \alpha_j x_{j,0} \in M(A^n)$ and it follows that

$$\mathcal{L}_{m+1} \cap M(A^n) = \mathcal{L}_m \cap M(A^n) + \operatorname{span}\{w_0\}.$$

Moreover, it it easy to see that $\mathcal{L}_{m+1} = \mathcal{L}_m + \operatorname{span}\{w_0\}$. Hence, (3.4) also holds. Therefore, $\dim \operatorname{span}\{[x_{1,n}], \dots, [x_{m+1,n}]\} = \dim \frac{\mathcal{L}_{m+1}}{\mathcal{L}_{m+1} \cap M(A^n))}$. □

In the following we will study linear independence of quasi-Jordan chains.

Lemma 3.5. *Let $(x_{k,n}, \dots, x_{k,0})$, $k = 1, \dots, m$, be m quasi-Jordan chains of a linear relation A in X. Consider the following statements:*

(i) *The set $\{[x_{1,n}], \dots, [x_{m,n}]\}$ is linearly independent in $N(A^{n+1})/N(A^n)$.*
(ii) *The set $\{x_{k,j} : k = 1, \dots, m, j = 0, \dots, n\}$ is linearly independent in X.*
(iii) *The set of pairs*

$$\{\{x_{k,j}, x_{k,j-1}\} : k = 1, \dots, m, j = 1, \dots, n\} \cup \{\{x_{k,0}, 0\} : k = 1, \dots, m\}$$

is linearly independent in A.

Then the following implications hold: (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii). *If, in addition,*

$$\operatorname{span}\{x_{1,0}, \dots, x_{m,0}\} \cap M(A^n) = \{0\}, \tag{3.5}$$

holds, then the three conditions (i), (ii), *and* (iii) *are equivalent.*

Proof. The implication (ii)⇒(iii) is straightforward by use of the linear independence of the first components of the pairs in (iii). Let us prove the implication (i)⇒(ii). Assume that $\{[x_{1,n}], \dots, [x_{m,n}]\}$ is linearly independent. Let $\alpha_{k,j} \in \mathbb{K}$, $j = 0, \dots, n, k = 1, \dots, m$, such that

$$\sum_{j=0}^{n} \sum_{k=1}^{m} \alpha_{k,j} x_{k,j} = 0. \tag{3.6}$$

It is easily seen that the following tuple is a quasi-Jordan chain of A:

$$\left(\sum_{j=0}^{n} \sum_{k=1}^{m} \alpha_{k,j} x_{k,j}, \sum_{j=1}^{n} \sum_{k=1}^{m} \alpha_{k,j} x_{k,j-1}, \dots, \sum_{j=n-1}^{n} \sum_{k=1}^{m} \alpha_{k,j} x_{k,j-n+1}, \sum_{k=1}^{m} \alpha_{k,n} x_{k,0} \right). \tag{3.7}$$

From this and (3.6) it follows that $\sum_{k=1}^{m} \alpha_{k,n} x_{k,0} \in M(A^n)$, which, by Proposition 3.2, implies for equivalence classes in $N(A^{n+1})/N(A^n)$

$$\left[\sum_{j=0}^{n} \sum_{k=1}^{m} \alpha_{k,j} x_{k,j} \right] = \sum_{k=1}^{m} \alpha_{k,n} [x_{k,n}] = 0.$$

Hence, $\alpha_{k,n} = 0$ for $k = 1, \dots, m$ and (3.6) reads as

$$\sum_{j=0}^{n-1} \sum_{k=1}^{m} \alpha_{k,j} x_{k,j} = 0. \tag{3.8}$$

Now one can construct a quasi-Jordan chain as above starting with the sum in (3.8). Repeating the above argument shows $\alpha_{k,n-1} = 0$ for $k = 1, \dots, m$. Proceeding further in this manner yields (ii), since all $\alpha_{k,j}$ in (3.6) are equal to zero.

Now assume that $\operatorname{span}\{x_{1,0},\dots,x_{m,0}\}\cap M(A^n)=\{0\}$. We have to show that in this case (iii) implies (i). Let $\sum_{k=1}^m \alpha_k[x_{k,n}]=0$. Then it follows by Proposition 3.2 that $\sum_{k=1}^m \alpha_k x_{k,0}\in M(A^n)$, hence $\sum_{k=1}^m \alpha_k x_{k,0}=0$. Therefore

$$\sum_{k=1}^m \alpha_k\{x_{k,0},0\}=\{0,0\}$$

and (iii) implies that $\alpha_k=0$ for $k=1,\dots,m$, which shows (i).

□

4. One-dimensional perturbations

The following definition, taken from [2], specifies the idea of a rank one perturbation for linear relations.

Definition 4.1. Let A and B be linear relations in X. Then B is called an *one-dimensional perturbation* of A (and vice versa) if

$$\max\left\{\dim\frac{A}{A\cap B},\ \dim\frac{B}{A\cap B}\right\}=1. \tag{4.1}$$

In particular, A is called a *one-dimensional extension* of B if $B\subset A$ and $\dim(A/B)=1$.

The next lemma describes in which way (quasi-)Jordan chains of a one-dimensional extension A of a linear relation C can be linearly combined to become (quasi-)Jordan chains of C. The proof is based on the following simple principle: If M is a subspace of N and $\dim(N/M)=1$, then whenever $x,y\in N$, $y\notin M$, there exists some $\lambda\in\mathbb{K}$ such that $x-\lambda y\in M$.

Lemma 4.2. *Let A and C be linear relations in X such that $C\subset A$ and $\dim(A/C)=1$. If $(x_{k,n},\dots,x_{k,0})$, $k=1,\dots,m$, are m quasi-Jordan chains of A, then after a possible reordering, there exist $m-1$ quasi-Jordan chains $(y_{k,n},\dots,y_{k,0})$, $k=1,\dots,m-1$, of C such that*

$$y_{k,j}\in x_{k,j}+\operatorname{span}\{x_{m,\ell}:\ell=0,\dots,j\},\qquad k=1,\dots,m-1,\ j=0,\dots,n. \tag{4.2}$$

Moreover, if $\{[x_{1,n}],\dots,[x_{m,n}]\}$ is linearly independent in $N(A^{n+1})/N(A^n)$ then the set $\{[y_{1,n}],\dots,[y_{m-1,n}]\}$ is linearly independent in $N(C^{n+1})/N(C^n)$.

On the other hand, if the set $\{x_{k,j}:k=1,\dots,m,j=0,\dots,n\}$ is linearly independent in X then the set $\{y_{k,j}:k=1,\dots,m-1,j=0,\dots,n\}$ is linearly independent in X.

Proof. For any quasi-Jordan chain $(z_n,z_{n-1},\dots,z_0)$ of A we agree to write $\hat z_j=\{z_j,z_{j-1}\}$ for $j=1,\dots,n$ and $\hat z_0=\{z_0,0\}$. Consider the set

$$J:=\{(k,j)\in\{1,\dots,m\}\times\{0,\dots,n\}:\hat x_{k,j}\notin C\}.$$

If $J=\varnothing$ then all m quasi-Jordan chains are in C and the proof is completed. Therefore, assume $J\neq\varnothing$. Set

$$h:=\min\big\{j\in\{0,\dots,n\}:(k,j)\in J\text{ for some }k\in\{1,\dots,m\}\big\}.$$

Choose some $\kappa\in\{1,\dots,m\}$ such that $(\kappa,h)\in J$. After a reordering of the indices we can assume that $\kappa=m$.

Since $\hat{x}_{m,h} \notin C$, there exist $\alpha_{k,h} \in \mathbb{K}$, $k = 1, \dots m-1$, such that

$$\hat{x}_{k,h} - \alpha_{k,h}\hat{x}_{m,h} \in C$$

for $k = 1, \dots m-1$. If $h = n$, we stop here. Otherwise, there exist $\alpha_{k,h+1} \in \mathbb{K}$, $k = 1, \dots m-1$, such that

$$\hat{x}_{k,h+1} - \alpha_{k,h}\hat{x}_{m,h+1} - \alpha_{k,h+1}\hat{x}_{m,h} \in C$$

for $k = 1, \dots m-1$. If $h = n-1$, the process terminates. Otherwise, there exist $\alpha_{k,h+2} \in \mathbb{K}$ such that

$$\hat{x}_{k,h+2} - \alpha_{k,h}\hat{x}_{m,h+2} - \alpha_{k,h+1}\hat{x}_{m,h+1} - \alpha_{k,h+2}\hat{x}_{m,h} \in C$$

for $k = 1, \dots m-1$. We continue with this procedure up to n, where in the last step we find $\alpha_{k,n} \in \mathbb{K}$ such that

$$\hat{x}_{k,n} - \alpha_{k,h}\hat{x}_{m,n} - \alpha_{k,h+1}\hat{x}_{m,n-1} - \dots - \alpha_{k,n-1}\hat{x}_{m,h+1} - \alpha_{k,n}\hat{x}_{m,h} \in C$$

for $k = 1, \dots m-1$. Summarizing, we obtain numbers $\alpha_{k,j} \in \mathbb{K}$, $k = 1, \dots, m-1$, $j = h, \dots, n$, such that

$$\hat{u}_{k,j} := \hat{x}_{k,j} - \sum_{i=h}^{j} \alpha_{k,i}\, \hat{x}_{m,j+h-i} \;\in\; C$$

for all $k = 1, \dots, m-1$, $j = h, \dots, n$. We now define

$$y_{k,j} := x_{k,j} - \sum_{i=h}^{\min\{j+h,n\}} \alpha_{k,i}\, x_{m,j+h-i},$$

for $k = 1, \dots m-1$ and $j = 0, \dots, n$. For $0 \le j < h$ (if possible, i.e., $h > 0$),

$$\hat{y}_{k,j} = \hat{x}_{k,j} - \sum_{i=h}^{\min\{j+h,n\}} \alpha_{k,i}\, \hat{x}_{m,j+h-i} \;\in\; C$$

is a consequence of the definition of h, whereas for $j \ge h$ we also have

$$\hat{y}_{k,j} = \hat{u}_{k,j} - \sum_{i=j+1}^{\min\{j+h,n\}} \alpha_{k,i}\, \hat{x}_{m,j+h-i} \;\in\; C.$$

This shows that $(y_{k,n}, \dots, y_{k,0})$ is a quasi-Jordan chain of A for each $k = 1, \dots, m-1$. From the definition of $y_{k,j}$ we also see that $y_{k,j} \in x_{k,j} + \operatorname{span}\{x_{m,j}, \dots, x_{m,0}\}$ for all $j = 0, \dots, n$ and $k = 1, \dots, m-1$.

Now, assuming the linear independence of $\{[x_{1,n}], \dots, [x_{m,n}]\}$ in $N(A^{n+1})/N(A^n)$, we prove the linear independence of $\{[y_{1,n}], \dots, [y_{m-1,n}]\}$ in $N(C^{n+1})/N(C^n)$. Since $y_{k,0} = x_{k,0} - \alpha_{k,h}x_{m,0}$ for $k = 1, \dots, m-1$, the linear independence of $\{y_{1,0}, \dots, y_{m-1,0}\}$ in X easily follows from that of $\{x_{1,0}, \dots, x_{m,0}\}$. Furthermore,

$$\operatorname{span}\{y_{1,0}, \dots, y_{m-1,0}\} \cap M(C^n) \;\subset\; \operatorname{span}\{x_{1,0}, \dots, x_{m,0}\} \cap M(A^n),$$

and the claim follows from Lemma 3.4.

Finally, assume that the set $\{x_{k,j} : k = 1, \dots, m, j = 0, \dots, n\}$ is linearly independent. Also, let $\beta_{k,j} \in \mathbb{K}$, $k = 1, \dots, m-1$, $j = 0, \dots, n$, such that $\sum_{k=1}^{m-1} \sum_{j=0}^{n} \beta_{k,j} y_{k,j} = 0$. Then

$$\begin{aligned} 0 &= \sum_{k=1}^{m-1} \sum_{j=0}^{n} \beta_{k,j} \left(x_{k,j} - \sum_{i=h}^{\min\{j+h,n\}} \alpha_{k,i}\, x_{m,j+h-i} \right) \\ &= \sum_{k=1}^{m-1} \sum_{j=0}^{n} \beta_{k,j} x_{k,j} - \sum_{j=0}^{n} \sum_{i=h}^{\min\{j+h,n\}} \left(\sum_{k=1}^{m-1} \beta_{k,j} \alpha_{k,i} \right) x_{m,j+h-i} \end{aligned}$$

From this, we see that $\beta_{k,j} = 0$ for $k = 1, \dots, m-1$ and $j = 0, \dots, n$. Therefore, the set $\{y_{k,j} : k = 1, \dots, m-1, j = 0, \dots, n\}$ is linearly independent in X. $\square$

In the main result of this section, Theorem 4.5 below, we will compare the dimensions of $N(A^{n+1})/N(A^n)$ and $N(B^{n+1})/N(B^n)$ for two linear relations A and B that are one-dimensional perturbations of each other. To formulate it, we define the following value for two linear relations A and B in X and $n \in \mathbb{N} \cup \{0\}$:

$$\begin{aligned} s_n(A,B) := \max \big\{ \dim(\mathcal{L} \cap M(A^n)) : \mathcal{L} &\text{ is a subspace of } N(A \cap B) \cap R((A \cap B)^n), \\ &\mathcal{L} \cap M((A \cap B)^n) = \{0\} \big\}. \end{aligned} \tag{4.3}$$

The quantity $s_n(A,B)$ can be interpreted as the number of (linearly independent) singular chains of A of length n which are not singular chains of $A \cap B$.

Note that we always have $s_0(A,B) = s_0(B,A) = 0$. On the other hand, for $n \in \mathbb{N}$ usually we have $s_n(A,B) \neq s_n(B,A)$. For example, if $B \subset A$ then $s_n(B,A) = 0$, while $s_n(A,B)$ might be positive. Therefore, we also introduce the number

$$s_n[A,B] := \max\{s_n(A,B), s_n(B,A)\}.$$

The next proposition shows that this number is bounded by n.

Proposition 4.3. *Let A and B be linear relations in X such that B is a one-dimensional perturbation of A. Then for $n \in \mathbb{N} \cup \{0\}$ we have*

$$s_n[A,B] \,\leq\, n.$$

Proof. The claim is clear for $n = 0$. Let $n \geq 1$. It obviously suffices to prove that $s_n(A,B) \leq n$. If $A \subset B$ then $s_n(A,B) = 0$ and the desired inequality holds. Hence, let us assume that $\dim(A/A \cap B) = 1$ and set $C := A \cap B$.

Let $\mathcal{L}$ be a subspace of $N(C) \cap R(C^n)$ such that $\mathcal{L} \cap M(C^n) = \{0\}$. Towards a contradiction, suppose that $\dim(\mathcal{L} \cap M(A^n)) > n$. So, there exist linearly independent vectors $x_{1,0}, \dots, x_{n+1,0} \in \mathcal{L} \cap M(A^n)$. Then there exist $n+1$ singular chains of A of the form

$$X_k = (0, x_{k,n-1}, \dots, x_{k,0}), \quad k = 1, \dots, n+1,$$

and $\{X_1, \dots, X_{n+1}\}$ is linearly independent in $\mathcal{S}_n^A$, c.f. (2.5).

By Lemma 2.1, $\dim(\mathcal{S}_n^A/\mathcal{S}_n^C) \leq n$. Thus, there exists a non-trivial $Y \in \mathcal{S}_n^C$ such that $Y \in \operatorname{span}\{X_1, \dots, X_{n+1}\}$, i.e. there exist $\alpha_1, \dots, \alpha_{n+1} \in \mathbb{K}$ (not all zero) such that $Y = \sum_{k=1}^{n+1} \alpha_k X_k$.

So, Y is a non-trivial singular chain of C of the form $Y = (0, y_{n-1}, \dots, y_0)$, where

$$y_j = \sum_{k=1}^{n+1} \alpha_k x_{k,j}, \quad j = 0, 1, \dots, n-1.$$

In particular, $y_0 = \sum_{k=1}^{n+1} \alpha_k x_{k,0} \neq 0$ because $\{x_{1,0}, \dots, x_{n+1,0}\}$ is linearly independent. Now, since $x_{1,0}, \dots, x_{n+1,0} \in \mathcal{L}$, also $y_0 \in \mathcal{L}$ and hence $y_0 \in \mathcal{L} \cap M(C^n)$, which is the desired contradiction. □

We now present our first generalization of Theorem 2.2 in [4].In this case we assume that one of the two relations is a one-dimensional restriction of the other.

Theorem 4.4. *Let A and B be linear relations in X such that $A \subset B$ and $\dim(B/A) = 1$ and let $n \in \mathbb{N} \cup \{0\}$. Then the following holds:*

(i) *$N(A^{n+1})/N(A^n)$ is finite-dimensional if and only if $N(B^{n+1})/N(B^n)$ is finite-dimensional. Moreover,*

$$-\, s_n(B, A) \;\le\; \dim \frac{N(B^{n+1})}{N(B^n)} - \dim \frac{N(A^{n+1})}{N(A^n)} \;\le\; 1. \tag{4.4}$$

In particular, for $n \ge 1$ we have

$$\left| \dim \frac{N(B^{n+1})}{N(B^n)} - \dim \frac{N(A^{n+1})}{N(A^n)} \right| \;\le\; \max\{1, s_n(B, A)\} \;\le\; n. \tag{4.5}$$

(ii) *$N(A^n)$ is finite-dimensional if and only if $N(B^n)$ is finite-dimensional. Moreover, for $n \ge 1$,*

$$|\dim N(B^n) - \dim N(A^n)| \;\le\; \sum_{k=0}^{n-1} s_k(B, A) \;\le\; \frac{(n-1)n}{2}.$$

Proof. To prove the lower bound in item (i), suppose that there are

$$m := \dim \frac{N(B^{n+1})}{N(B^n)} + s_n(B, A) + 1$$

linearly independent vectors $[x_{1,n}], \dots, [x_{m,n}]$ in $N(A^{n+1})/N(A^n)$ and consider corresponding Jordan chains $(x_{k,n}, \dots, x_{k,0})$ of length $n+1$ of A, $k = 1, \dots, m$. By Lemma 3.4, the vectors $x_{1,0}, \dots, x_{m,0}$ are linearly independent and, if $\mathcal{L}_0 := \operatorname{span}\{x_{1,0}, \dots, x_{m,0}\}$ then

$$\mathcal{L}_0 \cap M(A^n) = \{0\}.$$

Denote the cosets of the vectors $x_{k,n}$ in $N(B^{n+1})/N(B^n)$ by $[x_{k,n}]_B$, $k = 1, \dots, m$. Since

$$s_n(B, A) = \max\left\{\dim(\mathcal{L} \cap M(B^n)) : \mathcal{L} \subset N(A) \cap R(A^n) \text{ subspace}, \mathcal{L} \cap M(A^n) = \{0\}\right\},$$

Lemma 3.4 implies that

$$\begin{aligned} \dim \operatorname{span}\{[x_{1,n}]_B, \dots, [x_{m,n}]_B\} &= m - \dim(\mathcal{L}_0 \cap M(B^n)) \\ &\ge m - s_n(B, A) = \dim \frac{N(B^{n+1})}{N(B^n)} + 1, \end{aligned}$$

which is a contradiction.

On the other hand, assume that there are

$$p := \dim \frac{N(A^{n+1})}{N(A^n)} + 2$$

linearly independent vectors $[y_{1,n}]_B, \dots, [y_{p,n}]_B$ in $\frac{N(B^{n+1})}{N(B^n)}$ and consider corresponding Jordan chains $(y_{k,n}, \dots, y_{k,0})$ of length $n+1$ of B, for $k = 1, \dots, p$. By Lemma 3.4, the vectors $y_{1,0}, \dots, y_{p,0}$ are linearly independent and, if $\mathcal{L}_Y := \operatorname{span}\{y_{1,0}, \dots, y_{p,0}\}$ then

$$\mathcal{L}_Y \cap M(B^n) = \{0\}.$$

Now, applying Lemma 4.2, we obtain $p-1$ Jordan chains $(z_{k,n}, \dots, z_{k,0})$ of length $n+1$ of A, $k = 1, \dots, p-1$, such that (after a possible reordering)

$$z_{k,j} \in y_{k,j} + \operatorname{span}\{y_{p,l} : l = 0, \dots, j\} \quad \text{for } k = 1, \dots, p-1,\ j = 0, \dots, n.$$

In particular, for each $k = 1, \dots, p-1$ there exists $\alpha_k \in \mathbb{K}$ such that $z_{k,0} = y_{k,0} + \alpha_k y_{p,0}$.

Hence, if $\mathcal{L}_Z := \operatorname{span}\{z_{1,0}, \dots, z_{p-1,0}\}$ it is easy to see that

$$\mathcal{L}_Z \cap M(A^n) = \{0\},$$

because $\mathcal{L}_Z \subseteq \mathcal{L}_Y$, $M(A^n) \subseteq M(B^n)$ and $\mathcal{L}_Y \cap M(B^n) = \{0\}$. Thus, by Lemma 3.5,

$$\dim \operatorname{span}\{[z_{1,n}], \dots, [z_{p-1}, n]\} = \dim \mathcal{L}_Z = p - 1 = \dim \frac{N(A^{n+1})}{N(A^n)} + 1,$$

which is a contradiction.

In order to prove item (ii), note that for a linear relation T we have

$$N(T^n) \cong N(T) \times \frac{N(T^2)}{N(T)} \times \cdots \times \frac{N(T^n)}{N(T^{n-1})}. \tag{4.6}$$

Hence, from item (i) we infer that $\dim N(A^n) < \infty$ if and only if $\dim N(B^n) < \infty$. Also, as a consequence of (4.5) and Proposition 4.3,

$$\begin{aligned}
|\dim N(B^n) - \dim N(A^n)| &= \left| \sum_{k=0}^{n-1} \dim \frac{N(A^{k+1})}{N(A^k)} - \sum_{k=0}^{n-1} \dim \frac{N(B^{k+1})}{N(B^k)} \right| \\
&\le \sum_{k=0}^{n-1} \left| \dim \frac{N(A^{k+1})}{N(A^k)} - \dim \frac{N(B^{k+1})}{N(B^k)} \right| \\
&\le \sum_{k=0}^{n-1} s_k(B, A) \ \le\ \frac{(n-1)n}{2}.
\end{aligned}$$

This concludes the proof of the theorem. $\square$

The next theorem is the main result of this section. It states that the estimate obtained in [4, Theorem 2.2] for operators have to be adjusted when considering arbitrary linear relations. Note that $s_n[A, B] = 0$ for operators A and B.

Theorem 4.5. *Let A and B be linear relations in X such that B is a one-dimensional perturbation of A and $n \in \mathbb{N} \cup \{0\}$. Then the following hold:*

(i) *$N(A^{n+1})/N(A^n)$ is finite-dimensional if and only if $N(B^{n+1})/N(B^n)$ is finite-dimensional. Moreover,*

$$-1 - s_n(B,A) \;\le\; \dim \frac{N(B^{n+1})}{N(B^n)} - \dim \frac{N(A^{n+1})}{N(A^n)} \;\le\; 1 + s_n(A,B). \tag{4.7}$$

In particular,

$$\left|\dim \frac{N(B^{n+1})}{N(B^n)} - \dim \frac{N(A^{n+1})}{N(A^n)}\right| \;\le\; 1 + s_n[A,B] \;\le\; n+1. \tag{4.8}$$

(ii) *$N(A^n)$ is finite-dimensional if and only if $N(B^n)$ is finite-dimensional. Moreover,*

$$|\dim N(B^n) - \dim N(A^n)| \;\le\; n + \sum_{k=0}^{n-1} s_k[A,B] \;\le\; \frac{n(n+1)}{2}.$$

Proof. Define $C := A \cap B$. Then $C \subset A$ and $C \subset B$ as well as $\dim(A/C) \le 1$ and $\dim(B/C) \le 1$. Moreover, note that

$$s_n(A,B) = s_n(A,C) \quad \text{and} \quad s_n(B,A) = s_n(B,C).$$

Therefore, using the notation $D_n(T) = \dim \frac{N(T^{n+1})}{N(T^n)}$ for a relation T in X, from Theorem 4.4 we obtain

$$D_n(B) - D_n(A) = (D_n(B) - D_n(C)) - (D_n(A) - D_n(C)) \;\le\; 1 + s_n(A,B)$$

Exchanging the roles of A and B leads to $D_n(A) - D_n(B) \le 1 + s_n(B,A)$. This proves (i). The proof of statement (ii) is analogous to the proof of its counterpart in Theorem 4.4. In this case, as a consequence of (4.8),

$$|\dim N(B^n) - \dim N(A^n)| \le \sum_{k=0}^{n-1} |D_k(A) - D_k(B)| \;\le\; \sum_{k=0}^{n-1} (1 + s_k[A,B]) \;\le\; \frac{n(n+1)}{2},$$

and the theorem is proved. □

In Section 5 below we prove that the bound $n+1$ in (4.8) of Theorem 4.5 is in fact sharp, meaning that there are examples of linear relations A and B which are one-dimensional perturbations of each other where the quantity on the left hand side of (4.8) coincides with $n+1$.

The following corollary deals with linear relations without singular chains. If neither A nor B has singular chains then we recover the bounds from the operator case, see Theorem 2.2 in [4].

Corollary 4.6. *Let A and B be linear relations in X without singular chains such that B is a one-dimensional perturbation of A. Then the following statements hold:*

(i) *$N(A^{n+1})/N(A^n)$ is finite dimensional if and only if $N(B^{n+1})/N(B^n)$ is finite dimensional. Moreover,*

$$\left|\dim \frac{N(A^{n+1})}{N(A^n)} - \dim \frac{N(B^{n+1})}{N(B^n)}\right| \;\le\; 1.$$

(ii) $N(A^n)$ *is finite dimensional if and only if* $N(B^n)$ *is finite dimensional. Moreover,*

$$|\dim N(A^n) - \dim N(B^n)| \,\leq\, n. \tag{4.9}$$

(iii) $N(A) \cap R(A^n)$ *is finite dimensional if and only if* $N(B) \cap R(B^n)$ *is finite dimensional. Moreover,*

$$|\dim(N(A) \cap R(A^n)) - \dim(N(B) \cap R(B^n))| \,\leq\, 1.$$

Proof. If A and B are linear relations in X without singular chains, then $s_n[A, B] = 0$ for each $n \in \mathbb{N}$. Therefore, items (i) and (ii) follow directly from items (i) and (ii) in Theorem 4.5. Finally, recall that for a linear relation T in X without singular chains we have $N(T^{n+1})/N(T^n) \cong N(T) \cap R(T^n)$, c.f. [23, Lemma 4.4]. Hence, (iii) follows from (i). □

5. Sharpness of the bound in Theorem 4.5

In this section we present an example which shows that the bound $n + 1$ in Theorem 4.5 can indeed be achieved and is therefore sharp. This is easy to see in the cases $n = 0$ and $n = 1$.

Example 5.1. (a) Let $n = 2$, and let $x_0, x_1, x_2, z_0, z_1, z_2, y_1, y_2, y_3$ be linearly independent vectors in X. Define the linear relations

$$\begin{aligned} A = \operatorname{span}\big\{&\{x_2, x_1\}, \{x_1, x_0\}, \{x_0, 0\},\\ &\{z_2, z_1\}, \{z_1, z_0\}, \{z_0, 0\},\\ &\boldsymbol{\{y_3, x_2 - y_2\}}, \{x_2 - y_2, y_1\}, \{y_1, 0\},\\ &\{z_2, y_2\}\big\} \end{aligned}$$

and

$$\begin{aligned} B = \operatorname{span}\big\{&\{x_2, x_1\}, \{x_1, x_0\}, \{x_0, 0\},\\ &\{z_2, z_1\}, \{z_1, z_0\}, \{z_0, 0\},\\ &\{x_2 - y_2, y_1\}, \{y_1, 0\},\\ &\{z_2, y_2\}, \boldsymbol{\{y_2, 0\}}\big\}. \end{aligned}$$

All pairs are contained in both A and B except for the two pairs $\boldsymbol{\{y_3, x_2 - y_2\}}$ and $\boldsymbol{\{y_2, 0\}}$ which are printed here in bold face. Therefore, A and B are one-dimensional perturbations of each other. It is easy to see that $M(A^2) = \operatorname{span}\{y_2 - z_1, x_1 - y_1 - z_0\}$ and thus $M(A^2) \cap \operatorname{span}\{x_0, z_0, y_1\} = \{0\}$. By Lemma 3.5, it follows that $[x_2]_A, [z_2]_A, [y_3]_A$ are linearly independent in $N(A^3)/N(A^2)$. As $N(B^2) = \operatorname{span}\{x_0, x_1, x_2, z_0, z_1, z_2, y_1, y_2\}$ it is clear that $N(B^3) = N(B^2)$, hence

$$\dim \frac{N(A^3)}{N(A^2)} - \dim \frac{N(B^3)}{N(B^2)} = 3 - 0 = 3 = n + 1.$$

(b) Let $n \in \mathbb{N}$, $n > 2$. For our example we need $(n + 1)^2$ linear independent vectors in the linear space X, say $x_{i,j}$ for $i = 1, \ldots, n$ and $j = 0, \ldots, n$ as well as $y_1, \ldots, y_{n+1}$. Let us consider the linear relation

$$A = \operatorname{span}\left[\{\{x_{k,n}, x_{k,n-1}\}, \ldots, \{x_{k,1}, x_{k,0}\}, \{x_{k,0}, 0\} : k = 1, \ldots, n\} \cup\right.$$

$$\cup\ \{y_{n+1}, x_{1,n} - y_n\} \cup \big\{\{x_{k,n} - y_{n-k+1}, x_{k+1,n} - y_{n-k}\} : k = 1, \dots, n-2\big\}$$
$$\cup \{x_{n-1,n} - y_2, y_1\} \cup \{y_1, 0\} \cup \{x_{n,n}, y_n\} \cup \big\{\{y_l, y_{l-1}\} : l = 3, \dots, n\big\}\big].$$

Notice that

$$N(A) = \operatorname{span}\{x_{1,0}, \dots, x_{n,0}, y_1\}.$$

In the following we compute the multivalued part of A^k for $k = 1, \dots, n$. Assume that $x \in M(A) \subset R(A)$. Then $\{0, x\} \in A$ and there exist scalars $\alpha_{i,j}, \beta_k, \gamma_l \in \mathbb{K}$ such that

$$x = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{i,j} x_{i,j-1} + \sum_{k=1}^{n-2} \gamma_k (x_{k+1,n} - y_{n-k}) + \gamma_{n-1} y_1 + \gamma_n y_n + \beta_n (x_{1,n} - y_n) + \sum_{l=2}^{n-1} \beta_l y_l$$

and

$$\begin{aligned} 0 &= \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{i,j} x_{i,j} + \sum_{k=1}^{n-2} \gamma_k (x_{k,n} - y_{n-k+1}) + \gamma_{n-1}(x_{n-1,n} - y_2) + \gamma_n x_{n,n} + \sum_{l=2}^{n} \beta_l y_{l+1} \\ &= \sum_{i=1}^{n} (\alpha_{i,n} + \gamma_i) x_{i,n} + \sum_{i=1}^{n} \sum_{j=1}^{n-1} \alpha_{i,j} x_{i,j} + \beta_n y_{n+1} + \sum_{k=1}^{n-2} (\beta_{n-k} - \gamma_k) y_{n-k+1} - \gamma_{n-1} y_2. \end{aligned}$$

Therefore,

$$\begin{cases} \alpha_{i,n} + \gamma_i = 0 & \text{for } i = 1, \dots, n, \\ \alpha_{i,j} = 0 & \text{for } i = 1, \dots, n,\ j = 1, \dots, n-1, \\ \beta_n = 0 & , \\ \gamma_k - \beta_{n-k} = 0 & \text{for } k = 1, \dots, n-2, \\ \gamma_{n-1} = 0 & . \end{cases}$$

Hence, we can rewrite the vector x as

$$\begin{aligned} x &= \sum_{i=1}^{n-2} \alpha_{i,n} x_{i,n-1} + \alpha_{n,n} x_{n,n-1} + \sum_{k=1}^{n-2} \gamma_k (x_{k+1,n} - y_{n-k}) + \gamma_n y_n + \sum_{l=2}^{n-1} \beta_l y_l \\ &= \sum_{k=1}^{n-2} \gamma_k (x_{k+1,n} - x_{k,n-1}) + \gamma_n (y_n - x_{n,n-1}). \end{aligned}$$

Thus,

$$M(A) = \operatorname{span}\left(\{y_n - x_{n,n-1}\} \cup \left\{x_{k+1,n} - x_{k,n-1} :\ k = 1, \dots, n-2\right\}\right). \tag{5.1}$$

If $x \in M(A^2)$, then there exists $y \in M(A)$ such that $\{y, x\} \in A$. Hence, if $y = \sum_{k=1}^{n-2} \alpha_k (x_{k+1,n} - x_{k,n-1}) + \alpha_{n-1}(y_n - x_{n,n-1})$ then

$$x - \sum_{k=1}^{n-2} \alpha_k (x_{k+1,n-1} - x_{k,n-2}) - \alpha_{n-1}(y_{n-1} - x_{n,n-2}) \in M(A).$$

Therefore,

$$\begin{aligned} M(A^2) = \operatorname{span}\big(&\{y_n - x_{n,n-1}\} \cup \left\{x_{k+1,n} - x_{k,n-1} :\ k = 1, \dots, n-2\right\} \cup \\ &\cup \{y_{n-1} - x_{n,n-2}\} \cup \left\{x_{k+1,n-1} - x_{k,n-2} :\ k = 1, \dots, n-2\right\}\big). \end{aligned}$$

Following the same arguments it can be shown that

$$M(A^{n-1}) = \operatorname{span}\left(\left\{x_{k+1,n-j} - x_{k,n-j-1} :\ k = 1,\dots,n-2,\ j = 0,\dots,n-2\right\}\cup \right.$$
$$\left. \cup \left\{y_{n-j} - x_{n,n-j-1} :\ j = 0,\dots,n-2\right\}\right).$$

and

$$M(A^{n}) = \operatorname{span}\left(\left\{x_{k+1,n-j} - x_{k,n-j-1} :\ k = 1,\dots,n-2,\ j = 0,\dots,n-1\right\}\cup \right.$$
$$\left. \cup \left\{y_{n-j} - x_{n,n-j-1} :\ j = 0,\dots,n-2\right\} \cup \left\{x_{n-1,n-1} - y_1 - x_{n,0}\right\}\right),$$

where the last vector above is a consequence of $\{y_2 - x_{n,1}, x_{n-1,n-1} - y_1 - x_{n,0}\} \in A$. From this it follows that

$$\operatorname{span}\{x_{1,0},\dots,x_{n,0},y_1\} \cap M(A^n) = \{0\}. \tag{5.2}$$

Indeed, if x is a vector contained in the set on the left hand side of (5.2), then

$$x = \alpha_1 x_{1,0} + \cdots + \alpha_n x_{n,0} + \alpha_{n+1} y_1 = \sum_{k=1}^{n-2} \beta_k (x_{k+1,1} - x_{k,0}) + \gamma(x_{n-1,n-1} - y_1 - x_{n,0}),$$

where $\alpha_j, \beta_k, \gamma \in \mathbb{K}$ for $j = 1,\dots,n+1$ and $k = 1,\dots,n-2$. This implies

$$\sum_{k=1}^{n-2}(\alpha_k+\beta_k)x_{k,0}+\alpha_{n-1}x_{n-1,0}+(\alpha_n+\gamma)x_{n,0}+(\alpha_{n+1}+\gamma)y_1-\sum_{k=1}^{n-2}\beta_k x_{k+1,1}-\gamma x_{n-1,n-1} = 0.$$

Since all the vectors involved are by assumption linearly independent, it follows that $\gamma = 0$ and also $\beta_k = 0$ for $k = 1,\dots,n-2$ and thus also $\alpha_j = 0$ for all $j = 1,\dots,n+1$. That is, $x = 0$.

Now, it follows from (5.2) and Lemma 3.5 that $[x_{1,n}]_A,\dots,[x_{n,n}]_A,[y_{n+1}]_A$ are linearly independent in $N(A^{n+1})/N(A^n)$. On the other hand, if we consider the linear relation

$$B = \operatorname{span}\left(\left\{\{x_{k,j}, x_{k,j-1}\} : k = 1,\dots,n,\ j = 1,\dots,n\right\} \cup \left\{\{x_{k,0}, 0\} : k = 1,\dots,n\right\}\right.$$
$$\cup \left\{\{x_{k,n} - y_{n-k+1}, x_{k+1,n} - y_{n-k}\} : k = 1,\dots,n-2\right\} \cup \{x_{n-1,n} - y_2, y_1\} \cup \{y_1, 0\}$$
$$\left.\cup\left\{\{x_{n,n}, y_n\}, \{y_n, y_{n-1}\},\dots,\{y_3, y_2\} \cup \{y_2, 0\}\right\}\right),$$

A and B are one-dimensional perturbations of each other. Also, it is straightforward to verify that $D(B) = N(B^n)$. In particular, $N(B^{n+1}) = N(B^n)$ so that

$$\dim \frac{N(A^{n+1})}{N(A^n)} - \dim \frac{N(B^{n+1})}{N(B^n)} = n + 1 - 0 = n + 1,$$

which shows that the worst possible bound is indeed achieved in this example.

6. Finite-dimensional perturbations

A linear relation B is a finite rank perturbation of another linear relation A if both differ by finitely many dimensions from each other. Following [2], we formalize this idea as follows.

Definition 6.1. Let A and B be linear relations in X and $n \in \mathbb{N}$, $n \geq 1$. Then B is called an *n-dimensional perturbation* of A (and vice versa) if

$$\max\left\{\dim \frac{A}{A\cap B},\, \dim \frac{B}{A\cap B}\right\} = n. \tag{6.1}$$

If X is a Hilbert space and A, B are closed linear relations in X then both quantities $\dim \frac{A}{A\cap B}$ and $\dim \frac{B}{A\cap B}$ are finite if and only if $P_A - P_B$ is a finite rank operator, where P_A and P_B are the orthogonal projections onto A and B, respectively, cf. [2].

Remark 6.2. Let A and B be linear relations in X which are p-dimensional perturbations of each other, $p > 1$. Then it is possible to construct a sequence of one-dimensional perturbations, starting in A and ending in B. Indeed, choose $\{\widehat{f}_1, \ldots, \widehat{f}_p\}$ and $\{\widehat{g}_1, \ldots, \widehat{g}_p\}$ in $X \times X$ such that

$$A = (A\cap B) \dotplus \operatorname{span}\{\widehat{f}_1, \ldots, \widehat{f}_p\} \quad \text{and} \quad B = (A\cap B) \dotplus \operatorname{span}\{\widehat{g}_1, \ldots, \widehat{g}_p\}.$$

Observe that $\{\widehat{f}_1, \ldots, \widehat{f}_p\}$ is linearly independent if and only if $\dim \frac{A}{A\cap B} = p$. Otherwise, some of the elements of $\{\widehat{f}_1, \ldots, \widehat{f}_p\}$ can be chosen as zero. An analogous statement holds for $\{\widehat{g}_1, \ldots, \widehat{g}_p\}$. Define $C_0 := A$, $C_p := B$, and

$$C_k := (A\cap B) \dotplus \operatorname{span}\{\widehat{f}_1, \ldots, \widehat{f}_{p-k}, \widehat{g}_{p-k+1}, \ldots, \widehat{g}_p\}, \quad k = 1, \ldots, p-1.$$

Obviously, C_{k+1} is a one-dimensional perturbation of C_k, $k = 0, \ldots, p-1$. If, in addition, $A \subset B$ is satisfied, then $\widehat{f}_j = 0$ for all $j = 1, \ldots, p$ holds and we obtain

$$A \subset C_j \subset C_{j+1} \subset B \quad \text{for } j = 1, \ldots, p-1.$$

Theorem 6.3. *Let A and B be linear relations in X such that B is a p-dimensional perturbation of A, $p \geq 1$, and $n \in \mathbb{N} \cup \{0\}$. Then the following conditions hold:*

(i) *$N(A^{n+1})/N(A^n)$ is finite-dimensional if and only if $N(B^{n+1})/N(B^n)$ is finite-dimensional. Moreover,*

$$\left|\dim \frac{N(A^{n+1})}{N(A^n)} - \dim \frac{N(B^{n+1})}{N(B^n)}\right| \leq (n+1)p.$$

(ii) *If, in addition in item* (i)*, $A \subset B$ is satisfied, then we have for $n \geq 1$*

$$\left|\dim \frac{N(A^{n+1})}{N(A^n)} - \dim \frac{N(B^{n+1})}{N(B^n)}\right| \leq np.$$

(iii) *$N(A^n)$ is finite-dimensional if and only if $N(B^n)$ is finite-dimensional. Moreover,*

$$|\dim N(A^n) - \dim N(B^n)| \leq \frac{n(n+1)}{2}p.$$

(iv) *If, in addition in item* (iii)*, $A \subset B$ is satisfied, then we have for $n \geq 1$*

$$|\dim N(A^n) - \dim N(B^n)| \leq \frac{n(n-1)}{2}p.$$

Proof. By Remark 6.2 there exist linear relations $C_0, \ldots, C_p$ in X with $C_0 = A$ and $C_p = B$ such that C_{k+1} is a one-dimensional perturbation of C_k, $k = 0, \ldots, p-1$. Hence, applying item (i) in Theorem 4.5 repeatedly, we obtain

$$\left|\dim \frac{N(B^{n+1})}{N(B^n)} - \dim \frac{N(A^{n+1})}{N(A^n)}\right| \leq \sum_{k=0}^{p-1} \left|\dim \frac{N(C_{k+1}^{n+1})}{N(C_{k+1}^n)} - \dim \frac{N(C_k^{n+1})}{N(C_k^n)}\right| \leq (n+1)p.$$

Also, applying item (ii) in Theorem 4.5 repeatedly,

$$|\dim N(A^n) - \dim N(B^n)| \leq \sum_{k=0}^{p-1} \left|\dim N(C_{k+1}^n) - \dim N(C_k^n)\right| \leq \frac{n(n+1)}{2}p,$$

which shows (iii). The statements (ii) and (iv) in the case $A \subset B$ follows in the same way from Remark 6.2 and Theorem 4.4. □

For linear relations A and B without singular chains we obtain the same (sharp) estimates as for operators, see [4].

Corollary 6.4. *Let A and B be linear relations in X without singular chains such that B is a p-dimensional perturbation of A, $p \geq 1$. Then the following conditions hold:*

(i) *$N(A^{n+1})/N(A^n)$ is finite-dimensional if and only if $N(B^{n+1})/N(B^n)$ is finite-dimensional. Moreover,*

$$\left|\dim \frac{N(A^{n+1})}{N(A^n)} - \dim \frac{N(B^{n+1})}{N(B^n)}\right| \leq p.$$

(ii) *$N(A^n)$ is finite-dimensional if and only if $N(B^n)$ is finite-dimensional. Moreover,*

$$|\dim N(A^n) - \dim N(B^n)| \leq np. \tag{6.2}$$

(iii) *$N(A) \cap R(A^n)$ is finite-dimensional if and only if $N(B) \cap R(B^n)$ is finite-dimensional. Moreover,*

$$|\dim(N(A) \cap R(A^n)) - \dim(N(B) \cap R(B^n))| \leq p.$$

Proof. The claims follow immediately applying repeatedly the results in Corollary 4.6 to the finite sequence of one-dimensional prturbations $A = C_0, C_1, \ldots, C_p = B$. □

7. Rank one perturbations of matrix pencils

In this section we apply our results to matrix pencils $\mathcal{A}$ of the form

$$\mathcal{A}(s) := sE - F, \tag{7.1}$$

where $s \in \mathbb{C}$ and E, F are square matrices in $\mathbb{C}^{d \times d}$. We will estimate the change of the number of Jordan chains of $\mathcal{A}$ under a perturbation with a rank-one matrix pencil.

Matrix pencils of the form (7.1) appear in a natural way in the study of differential algebraic equations which are a generalization of the abstract Cauchy problem:

$$E\dot{x} = Fx, \quad x(0) = x_0, \tag{7.2}$$

where $x_0 \in \mathbb{C}^d$ is the initial value. We do not assume E to be invertible. Nevertheless if we identify E with the linear relation given by the graph of E, then we have an inverse E^{-1} of E in the sense of linear relations, see (2.1). Moreover, if F is identified with the linear relation given by the graph of F, then one easily sees that (7.2) is equivalent to

$$\{x, \dot{x}\} \in E^{-1}F, \quad x(0) = x_0. \tag{7.3}$$

Also, we have that

$$E^{-1}F = \Big\{ \{x, y\} \in \mathbb{C}^d \times \mathbb{C}^d : Fx = Ey \Big\} = N[F;\ -E]. \tag{7.4}$$

Recall that $\lambda \in \mathbb{C}$ is an eigenvalue of $\mathcal{A}(s) = sE - F$ if zero is an eigenvalue of $\mathcal{A}(\lambda)$, and ∞ is an eigenvalue of $\mathcal{A}$ if zero is an eigenvalue of E. We denote the set of all eigenvalues of the pencil $\mathcal{A}$ with $\sigma_p(\mathcal{A})$. In the following we recall the notion of Jordan chains for matrix pencils, see e.g. [18, Section 1.4], [20, §11.2].

Definition 7.1. An ordered set $(x_n, \dots, x_0)$ in $\mathbb{C}^d$ is a *Jordan chain of length* $n+1$ *at* $\lambda \in \overline{\mathbb{C}}$ (for the matrix pencil $\mathcal{A}$) if $x_0 \neq 0$ and

$$\begin{array}{llll} \lambda \in \mathbb{C}: & (F - \lambda E)x_0 = 0, & (F - \lambda E)x_1 = Ex_0, \quad \dots, & (F - \lambda E)x_n = Ex_{n-1}, \\ \lambda = \infty: & Ex_0 = 0, & Ex_1 = Fx_0, \quad \dots, & Ex_n = Fx_{n-1}. \end{array} \tag{7.5}$$

Moreover, we denote by $\mathcal{L}_\lambda^l(\mathcal{A})$ the subspace spanned by the vectors of all Jordan chains up to length $l \geq 1$ at $\lambda \in \overline{\mathbb{C}}$.

Given a matrix pencil $\mathcal{A}$, the aim of this section is to obtain lower and upper bounds for the difference

$$\dim \frac{\mathcal{L}_\lambda^{n+1}(\mathcal{A} + \mathcal{P})}{\mathcal{L}_\lambda^{n}(\mathcal{A} + \mathcal{P})} - \dim \frac{\mathcal{L}_\lambda^{n+1}(\mathcal{A})}{\mathcal{L}_\lambda^{n}(\mathcal{A})},$$

where $\mathcal{P}$ is a rank-one matrix pencil, $n \in \mathbb{N}$ and $\lambda \in \overline{\mathbb{C}}$.

We start with a simple lemma, which follows directly from the definitions. It allow us to reduce the study of Jordan chains at some $\lambda \in \overline{\mathbb{C}}$ to Jordan chains at zero.

Lemma 7.2. *Given a matrix pencil $\mathcal{A}(s) = sE - F$, the following statements hold:*

(i) *$(x_n, \dots, x_0)$ is a Jordan chain of $\mathcal{A}$ at $\lambda \in \mathbb{C}$ if and only if it is a Jordan chain of the matrix pencil $\mathcal{B}(s) := sE - (F - \lambda E)$ at zero.*
(ii) *$(x_n, \dots, x_0)$ is a Jordan chain of $\mathcal{A}(s)$ at ∞ if and only if it is a Jordan chain of the dual matrix pencil $\mathcal{A}'(s) := sF - E$ at zero.*

The following proposition shows that the Jordan chains of the matrix pencil $\mathcal{A}$ coincide with the Jordan chains of the linear relation $E^{-1}F$. As the proof is simple and straightforward, we omit it.

Proposition 7.3. *For $n \in \mathbb{N}$ and $\lambda \in \overline{\mathbb{C}}$ the following two statements are equivalent.*

(i) *$(x_n, \dots, x_0)$ is a Jordan chain of $\mathcal{A}$ at λ.*
(ii) *$(x_n, \dots, x_0)$ is a quasi-Jordan chain of $E^{-1}F$ at λ.*

In particular, for $\lambda \in \mathbb{C}$ we have

$$\mathcal{L}_\lambda^n(\mathcal{A}) = N((E^{-1}F - \lambda)^n).$$

Note that the quasi-Jordan chains of a linear relation A at ∞ are the same as the quasi-Jordan chains of the inverse linear relation A^{-1} at zero. Therefore,

Corollary 7.4. *$(x_n, \dots, x_0)$ is a Jordan chain of $\mathcal{A}(s) = sE - F$ at ∞ if and only if $(x_n, \dots, x_0)$ is a quasi-Jordan chain of $F^{-1}E$ at zero. In particular,*

$$\mathcal{L}_\infty^n(\mathcal{A}) = N((F^{-1}E)^n).$$

Due to Proposition 7.3, for $n \in \mathbb{N}$ and $\lambda \in \mathbb{C}$ we have

$$\dim \frac{\mathcal{L}_\lambda^{n+1}(\mathcal{A})}{\mathcal{L}_\lambda^n(\mathcal{A})} = \dim \frac{N((E^{-1}F - \lambda)^{n+1})}{N((E^{-1}F - \lambda)^n)}.$$

On the other hand, Corollary 7.4 implies that

$$\dim \frac{\mathcal{L}_\infty^{n+1}(\mathcal{A})}{\mathcal{L}_\infty^n(\mathcal{A})} = \dim \frac{N((F^{-1}E)^{n+1})}{N((F^{-1}E)^n)}.$$

Given a matrix pencil $\mathcal{A}(s) = sE - F$, now we consider perturbations of the form:

$$\mathcal{P}(s) = w(su^* + v^*), \tag{7.6}$$

where $u, v, w \in \mathbb{C}^d$ and $w \neq 0$. These are rank-one matrix pencils. Recall that the rank of a matrix pencil $\mathcal{P}$ is the largest $r \in \mathbb{N}$ such that $\mathcal{P}$, viewed as a matrix with polynomial entries, has minors of size r that are not the zero polynomial [14, 16]. Then, $\mathcal{A}$ and $\mathcal{A} + \mathcal{P}$ are rank-one perturbations of each other, in the sense that they differ by (at most) a rank-one matrix pencil.

Lemma 7.5. *Given $\mathcal{A}(s) = sE - F$, let $\mathcal{P}$ be a rank-one matrix pencil as in* (7.6)*. Then, the linear relations*

$$E^{-1}F \quad \text{and} \quad (E + wu^*)^{-1}(F + wv^*)$$

are one-dimensional perturbations of each other in the sense of Definition 6.1*.*

Proof. Obviously, for $\mathcal{M} := E^{-1}F \cap (E + wu^*)^{-1}(F + wv^*)$ we have

$$\mathcal{M} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{C}^d \times \mathbb{C}^d : Fy = Ex \ \text{ and } \ (F + wv^*)y = (E + wu^*)x \right\}.$$

That is,

$$\mathcal{M} = E^{-1}F \cap \begin{pmatrix} -u \\ v \end{pmatrix}^\perp = (E + wu^*)^{-1}(F + wv^*) \cap \begin{pmatrix} -u \\ v \end{pmatrix}^\perp.$$

This implies

$$\dim \frac{E^{-1}F}{\mathcal{M}} \leq 1 \qquad \text{and} \qquad \dim \frac{(E + wu^*)^{-1}(F + wv^*)}{\mathcal{M}} \leq 1,$$

which proves the claim. ∎

Remark 7.6. Applying Lemma 7.5 to the dual matrix pencils $\mathcal{A}'$ and $\mathcal{P}'$, follows that

$$F^{-1}E \quad \text{and} \quad (F + wv^*)^{-1}(E + wu^*)$$

are one-dimensional perturbations of each other in the sense of Definition 6.1.

The following theorem is the second main result of this work. We consider here all possible situations of regular/singular matrix pencils $\mathcal{A}$ and regular/singular $\mathcal{A}+\mathcal{P}$. Recall that a matrix pencil $\mathcal{A}(s) = sE - F$ is called *regular* if its *characteristic polynomial* $\det(sE - A)$ is not the zero polynomial. Otherwise it is called $\mathcal{A}$ *singular*.

Theorem 7.7. *Given $\mathcal{A}(s) = sE - F$, let $\mathcal{P}$ be a rank-one matrix pencil as in* (7.6). *For $\lambda \in \overline{\mathbb{C}}$ and $n \in \mathbb{N}$, the following statements hold:*

(i) *If both pencils $\mathcal{A}$ and $\mathcal{A}+\mathcal{P}$ are regular, then*

$$\left| \dim \frac{\mathcal{L}_\lambda^{n+1}(\mathcal{A}+\mathcal{P})}{\mathcal{L}_\lambda^{n}(\mathcal{A}+\mathcal{P})} - \dim \frac{\mathcal{L}_\lambda^{n+1}(\mathcal{A})}{\mathcal{L}_\lambda^{n}(\mathcal{A})} \right| \leq 1. \tag{7.7}$$

(ii) *If $\mathcal{A}$ is regular but $\mathcal{A}+\mathcal{P}$ is singular, then*

$$-1 - n \leq \dim \frac{\mathcal{L}_\lambda^{n+1}(\mathcal{A}+\mathcal{P})}{\mathcal{L}_\lambda^{n}(\mathcal{A}+\mathcal{P})} - \dim \frac{\mathcal{L}_\lambda^{n+1}(\mathcal{A})}{\mathcal{L}_\lambda^{n}(\mathcal{A})} \leq 1.$$

(iii) *If $\mathcal{A}$ is singular and $\mathcal{A}+\mathcal{P}$ is regular, then*

$$-1 \leq \dim \frac{\mathcal{L}_\lambda^{n+1}(\mathcal{A}+\mathcal{P})}{\mathcal{L}_\lambda^{n}(\mathcal{A}+\mathcal{P})} - \dim \frac{\mathcal{L}_\lambda^{n+1}(\mathcal{A})}{\mathcal{L}_\lambda^{n}(\mathcal{A})} \leq n+1.$$

(iv) *If both $\mathcal{A}$ and $\mathcal{A}+\mathcal{P}$ are singular, then*

$$\left| \dim \frac{\mathcal{L}_\lambda^{n+1}(\mathcal{A}+\mathcal{P})}{\mathcal{L}_\lambda^{n}(\mathcal{A}+\mathcal{P})} - \dim \frac{\mathcal{L}_\lambda^{n+1}(\mathcal{A})}{\mathcal{L}_\lambda^{n}(\mathcal{A})} \right| \leq n+1.$$

Proof. According to Lemma 7.2, if $\lambda \in \mathbb{C}$ we may assume $\lambda = 0$. By Proposition 7.3, for $n \in \mathbb{N}$ we have that

$$\mathcal{L}_0^n(\mathcal{A}) = N\big((E^{-1}F)^n\big) \quad \text{and} \quad \mathcal{L}_0^n(\mathcal{A}+\mathcal{P}) = N(B^n),$$

where $B := (E + wu^*)^{-1}(F + wv^*)$. Due to Lemma 7.5 the linear relations $E^{-1}F$ and B are one-dimensional perturbations of each other and, by Theorem 4.5,

$$-1 - s_n(B, E^{-1}F) \leq \dim \frac{\mathcal{L}_0^{n+1}(\mathcal{A}+\mathcal{P})}{\mathcal{L}_0^{n}(\mathcal{A}+\mathcal{P})} - \dim \frac{\mathcal{L}_0^{n+1}(\mathcal{A})}{\mathcal{L}_0^{n}(\mathcal{A})} \leq 1 + s_n(E^{-1}F, B).$$

Then, Proposition 4.3 implies statement (iv). If the pencil $\mathcal{A}$ is regular then, by definition, not every complex number is an eigenvalue of $\mathcal{A}$. Hence, by Proposition 7.3, those numbers are neither eigenvalues of $E^{-1}F$. From [22] it follows that, in this case, $E^{-1}F$ has no singular chains and we conclude that

$$s_n(E^{-1}F, B) = 0,$$

see (4.3). Similarly, if $\mathcal{A}+\mathcal{P}$ is regular we obtain $s_n(B, E^{-1}F) = 0$, which shows the remaining statements (i)–(iii).

For $\lambda = \infty$ similar arguments can be used using $F^{-1}E$ and $C := (F+wv^*)^{-1}(E+wu^*)$ instead of $E^{-1}F$ and B, see Corollary 7.4 and Remark 7.6. □

Note that the estimate in item (i) of Theorem 7.7 was already known. The same result was shown in [14, Lemma 2.1] with the help of a result for polynomials, see also [25, Theorem 1]. On the other hand, the remaining estimates in Theorem 7.7 are completely new.

Under an additional assumption, which implies that one of the corresponding linear relations is contained in the other, we are able to improve the estimates from Theorem 7.7 in the cases in which one or both pencils are singular.

Theorem 7.8. *Given $\mathcal{A}(s) = sE - F$, let $\mathcal{P}$ be a rank-one matrix pencil as in* (7.6) *and assume that*

$$N[F; -E] \subseteq N[v^*; -u^*]. \tag{7.8}$$

Then, for $\lambda \in \overline{\mathbb{C}}$ and $n \in \mathbb{N}$ the following statements hold:

(i) *If $\mathcal{A}$ is regular but $\mathcal{A}+\mathcal{P}$ is singular, then*

$$-n \leq \dim \frac{\mathcal{L}_\lambda^{n+1}(\mathcal{A}+\mathcal{P})}{\mathcal{L}_\lambda^{n}(\mathcal{A}+\mathcal{P})} - \dim \frac{\mathcal{L}_\lambda^{n+1}(\mathcal{A})}{\mathcal{L}_\lambda^{n}(\mathcal{A})} \leq 1.$$

(ii) *If $\mathcal{A}$ is singular and $\mathcal{A}+\mathcal{P}$ is regular, then*

$$-1 \leq \dim \frac{\mathcal{L}_\lambda^{n+1}(\mathcal{A}+\mathcal{P})}{\mathcal{L}_\lambda^{n}(\mathcal{A}+\mathcal{P})} - \dim \frac{\mathcal{L}_\lambda^{n+1}(\mathcal{A})}{\mathcal{L}_\lambda^{n}(\mathcal{A})} \leq n.$$

(iii) *If both $\mathcal{A}$ and $\mathcal{A}+\mathcal{P}$ are singular, then*

$$\left| \dim \frac{\mathcal{L}_\lambda^{n+1}(\mathcal{A}+\mathcal{P})}{\mathcal{L}_\lambda^{n}(\mathcal{A}+\mathcal{P})} - \dim \frac{\mathcal{L}_\lambda^{n+1}(\mathcal{A})}{\mathcal{L}_\lambda^{n}(\mathcal{A})} \right| \leq n.$$

Proof. Observe that the assumption in (7.8) implies that $wv^*x = wu^*y$ for every $\{x, y\} \in N[F; -E]$. Therefore,

$$(F + wv^*)x = (E + wu^*)y \quad \text{for all } \{x, y\} \in E^{-1}F,$$

which means

$$E^{-1}F \subset (E + wv^*)^{-1}(F + wv^*).$$

The statements in Theorem 7.8 now follow from Theorem 4.4. □

Remark 7.9. In the following we present estimates for the so-called *Wong sequences*, which have their origin in [26]. Recently, Wong sequences have been used to prove the Kronecker canonical form, see [7, 9, 10]. For $E, F \in \mathbb{C}^{d\times d}$ the Wong sequence of second kind of the pencil $\mathcal{A}(s) := sE - F$ is defined as the sequence of subspaces $(\mathcal{W}_i(\mathcal{A}))_{i\in\mathbb{N}}$ given by

$$\mathcal{W}_0(\mathcal{A}) = \{0\}, \qquad \mathcal{W}_{i+1}(\mathcal{A}) = \left\{ x \in \mathbb{C}^d : Ex \in F\mathcal{W}_i(\mathcal{A}) \right\}, \quad i \in \mathbb{N}.$$

It is easily seen by induction that for $k \in \mathbb{N}$ we have

$$\mathcal{W}_n(\mathcal{A}) = N\big((F^{-1}E)^n\big).$$

Theorem 4.5 now yields the following statements on the behaviour of the Wong sequences of second kind under rank-one perturbations of the type (7.6):

(i) If both pencils $\mathcal{A}$ and $\mathcal{A}+\mathcal{P}$ are regular, then

$$\left|\dim \frac{\mathcal{W}_{n+1}(\mathcal{A}+\mathcal{P})}{\mathcal{W}_n(\mathcal{A}+\mathcal{P})} - \dim \frac{\mathcal{W}_{n+1}(\mathcal{A})}{\mathcal{W}_n(\mathcal{A})}\right| \leq 1.$$

(ii) If $\mathcal{A}$ is regular but $\mathcal{A}+\mathcal{P}$ is singular, then

$$-1-n \leq \dim \frac{\mathcal{W}_{n+1}(\mathcal{A}+\mathcal{P})}{\mathcal{W}_n(\mathcal{A}+\mathcal{P})} - \dim \frac{\mathcal{W}_{n+1}(\mathcal{A})}{\mathcal{W}_n(\mathcal{A})} \leq 1.$$

(iii) If $\mathcal{A}$ is singular and $\mathcal{A}+\mathcal{P}$ is regular, then

$$-1 \leq \dim \frac{\mathcal{W}_{n+1}(\mathcal{A}+\mathcal{P})}{\mathcal{W}_n(\mathcal{A}+\mathcal{P})} - \dim \frac{\mathcal{W}_{n+1}(\mathcal{A})}{\mathcal{W}_n(\mathcal{A})} \leq n+1.$$

(iv) If both $\mathcal{A}$ and $\mathcal{A}+\mathcal{P}$ are singular, then

$$\left|\dim \frac{\mathcal{W}_{n+1}(\mathcal{A}+\mathcal{P})}{\mathcal{W}_n(\mathcal{A}+\mathcal{P})} - \dim \frac{\mathcal{W}_{n+1}(\mathcal{A})}{\mathcal{W}_n(\mathcal{A})}\right| \leq n+1.$$

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