Schur complements of selfadjoint Krein space operators

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Abstract

Given a bounded selfadjoint operator W on a Krein space $\mathcal{H}$ and a closed subspace $\mathcal{S}$ of $\mathcal{H}$, the Schur complement of W to $\mathcal{S}$ is defined under the hypothesis of weak complementability. A variational characterization of the Schur complement is given and the set of selfadjoint operators W admitting a Schur complement with these variational properties is shown to coincide with the set of $\mathcal{S}$-weakly complementable selfadjoint operators.

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1. Introduction

The notion of Schur complement (or shorted operator) of B to $\mathcal{S}$ for a positive operator B on a Hilbert space $\mathcal{H}$ and $\mathcal{S} \subseteq \mathcal{H}$ a closed subspace, was introduced by M.G. Krein [16]. When $\leq_{\mathcal{H}}$ is the usual order in $L(\mathcal{H})$, he proved that the set $\{X \in L(\mathcal{H}) : 0 \leq_{\mathcal{H}} X \leq_{\mathcal{H}} B \text{ and } R(X) \subseteq \mathcal{S}^{\perp}\}$ has a maximum element, which he defined as the Schur complement $B_{/\mathcal{S}}$ of B to $\mathcal{S}$. This notion was later rediscovered by Anderson and Trapp [1]. If B is represented as the 2×2 block matrix $\begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$ with respect to the decomposition of $\mathcal{H} = \mathcal{S} \oplus \mathcal{S}^{\perp}$, they established the formula

$$B_{/\mathcal{S}} = \begin{pmatrix} 0 & 0 \\ 0 & c - y^*y \end{pmatrix}$$

where y is the unique solution of the equation $b = a^{1/2}x$ such that the range inclusion $R(y) \subseteq \overline{R(a)}$ holds. The solution always exists because B is positive, in which case a is also positive and the range inclusion $R(b) \subseteq R(a^{1/2})$ holds.

In [4] Antezana et al., extended the Schur complement to any bounded operator B satisfying a weak complementability condition with respect to a given pair of closed subspaces $\mathcal{S}$ and $\mathcal{T}$, by giving an Anderson-Trapp type formula. In particular, if B is a bounded salfadjoint operator, $\mathcal{S} = \mathcal{T}$ and $B = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$, this condition reads $R(b) \subseteq R(|a|^{1/2})$, which as noted, is automatic for positive operators.

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Later, Massey and Stojanoff [20] studied many properties of the Schur complement of an $\mathcal{S}$-weakly complementable selfadjoint operator B when $\mathcal{S}$ is B-positive.

In this paper we show that if B is a bounded selfadjoint operator which is $\mathcal{S}$-weakly complementable then $B_{/\mathcal{S}}$ can be characterized as the solution of a $min - max$ problem, extending the original approach of Krein. But, more importantly, the converse is true, in the sense that if the solution of this $min - max$ problem exists then B has to be $\mathcal{S}$-weakly complementable. In other words, the $\mathcal{S}$-weakly complementable operators are exactly those selfadjoint operators admitting a Schur complement that satisfies these variational properties.

A closed-form expression for the Schur complement $B_{/\mathcal{S}}$ of B to $\mathcal{S}$ is also established, in terms of a family of densely defined projections with prescribed nullspace $\mathcal{S}^{\perp}$ (Theorem 3.14). This formula is new even in the case of positive B.

We then turn to the consideration of a bounded selfadjoint operator W on a Krein space $(\mathcal{H}, [\ ,\ ])$. For a fixed signature operator J on $\mathcal{H}$, JW is selfadjoint in the Hilbert space inner product $\langle\ ,\ \rangle$ associated with J. If $\mathcal{S}$ is a given closed subspace of $\mathcal{H}$, JW is assumed to be $\mathcal{S}$-weakly complementable and J_α is any other signature operator on $\mathcal{H}$ then two key results are established: $J_\alpha W$ is $\mathcal{S}$-weakly complementable (Theorem 4.4) and $J\ (JW)_{/\mathcal{S}} = J_\alpha\ (J_\alpha W)_{/\mathcal{S}}$ (Theorem 4.5).

Based on these results we extend the notions of $\mathcal{S}$-weak complementability and Schur complement to the Krein space setting. A bounded selfadjoint operator W on a Krein space $\mathcal{H}$ is $\mathcal{S}$-weakly complementable if, for some (and, hence, any) signature operator J, JW is $\mathcal{S}$-weakly complementable in the corresponding Hilbert space. If this is the case then the Schur complement of W to $\mathcal{S}$ is $W_{/[\mathcal{S}]} := J\ (JW)_{/\mathcal{S}}$.

In this fashion we obtain a simple way of computing the Schur complement of $\mathcal{S}$-weakly complementable selfadjoint operators in Krein spaces. This definition allows us to "translate" the properties obtained in Hilbert spaces to the Krein space setting in a straightforward way.

If $\mathcal{S}$ is a regular subspace of $\mathcal{H}$ (meaning that $\mathcal{H} = \mathcal{S}\ [\dotplus]\ \mathcal{S}^{[\perp]}$) then it is possible to give a characterization of the $\mathcal{S}$-weak complementability of W in terms of the entries of the first row of the 2×2 block matrix representation of W with respect to $\mathcal{S}\ [\dotplus]\ \mathcal{S}^{[\perp]}$. Indeed if $W = (w_{ij})_{i,j=1,2}$ and $w_{11} = dd^{\#}$ is a polar factorization of w_{11} then W is $\mathcal{S}$-weakly complementable if, and only if, $R(w_{12}) \subseteq R(d)$. In this case, $W_{/[\mathcal{S}]} = \begin{pmatrix} 0 & 0 \\ 0 & w_{22} - y^{\#}y \end{pmatrix}$ with y the only solution of the equation $w_{12} = dx$. The result may be viewed as a neat Krein space counterpart of the Hilbert space results in [4].

Based on a formula given by Pekarev [22], Maestripieri and Martínez Pería [18] extended the concept of the Schur complement to bounded selfadjoint operators in Krein spaces with the so-called "unique factorization property". Another approach was given by Mary [19]. He defined the Schur complement of a bounded operator $W = (w_{ij})_{i,j=1,2}$ when the range $R(w_{11})$ and the nullspace $N(w_{11})$ of w_{11} are regular subspaces. The approach we adopt has greater scope and is less restrictive.

The paper has three additional sections. Section 2 is a brief expository introduction to Krein spaces and operators on them and serves to fix the notation and give some results that are needed in the following sections. Section 3 is entirely devoted to the study of complementability and the Schur complement of a selfadjoint operator on a Hilbert space. In Section 4 we present our main results concerning the Schur complement of a Krein space operator. This section includes three subsections: the first deals with the notion of weak complementability on Krein spaces; the second presents an application inspired on some completion problems previously considered in Hilbert and Krein spaces by Baidiuk and Hassi in [6] and [7]; in the last subsection our notion of Schur complement in the Krein space setting is compared to those in [18] and [19].

2. Preliminaries

We assume that all Hilbert spaces are complex and separable. If $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces, $L(\mathcal{H}, \mathcal{K})$ stands for the space of all the bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. When $\mathcal{H} = \mathcal{K}$ we write, for short, $L(\mathcal{H})$. The domain, range and nullspace of any given $A \in L(\mathcal{H}, \mathcal{K})$ are denoted by $Dom(A)$, $R(A)$ and $N(A)$, respectively. Given a subset $\mathcal{T} \subseteq \mathcal{K}$, the preimage of $\mathcal{T}$ under A is denoted by $A^{-1}(\mathcal{T})$ so $A^{-1}(\mathcal{T}) = \{x \in \mathcal{H} :\ Ax \in \mathcal{T}\}$.

The direct sum of two closed subspaces $\mathcal{M}$ and $\mathcal{N}$ of $\mathcal{H}$ is represented by $\mathcal{M} \dotplus \mathcal{N}$. If $\mathcal{H}$ is decomposed as $\mathcal{H} = \mathcal{M} \dotplus \mathcal{N}$, the projection onto $\mathcal{M}$ with nullspace $\mathcal{N}$ is denoted by $P_{\mathcal{M}//\mathcal{N}}$ and abbreviated $P_{\mathcal{M}}$

when $\mathcal{N} = \mathcal{M}^{\perp}$. In general, $\mathcal{Q}$ indicates the subset of all the oblique projections in $L(\mathcal{H})$, namely, $\mathcal{Q} := \{Q \in L(\mathcal{H}) : Q^2 = Q\}$.

Denote by $GL(\mathcal{H})$ the group of invertible operators in $L(\mathcal{H})$, $L(\mathcal{H})^+$ the cone of positive semidefinite operators in $L(\mathcal{H})$ and $GL(\mathcal{H})^+ := GL(\mathcal{H}) \cap L(\mathcal{H})^+$. Given two operators $S, T \in L(\mathcal{H})$, the notation $T \leq_{\mathcal{H}} S$ signifies that $S - T \in L(\mathcal{H})^+$. Given any $T \in L(\mathcal{H})$, $|T| := (T^*T)^{1/2}$ is the modulus of T and $T = U|T|$ is the polar decomposition of T, with U the partial isometry such that $N(U) = N(T)$.

The following is a well-known result about range inclusion and factorizations of operators.

Lemma 2.1 (Douglas' Lemma [12])**.** *Let $Y \in L(\mathcal{K}_1, \mathcal{H})$ and $Z \in L(\mathcal{K}_2, \mathcal{H})$. Then $R(Z) \subseteq R(Y)$ if and only if there exists $D \in L(\mathcal{K}_2, \mathcal{K}_1)$ such that $Z = YD$.*

Amongst the solutions of the equation $Z = YX$, there exists a unique operator $D_0 \in L(\mathcal{H})$ such that $N(Z) = N(D_0)$ and $R(D_0) \subseteq \overline{R(Y^*)}$.

The operator D_0 is called the *reduced solution* of $Z = YX$.

Given $B \in L(\mathcal{H})$ selfadjoint and $\mathcal{S}$ a closed subspace of $\mathcal{H}$, we say that $\mathcal{S}$ is *B-positive* if $\langle Bs, s \rangle > 0$ for every $s \in \mathcal{S},\ s \neq 0$. *B-nonnegative*, *B-neutral*, *B-negative* and *B-nonpositive* subspaces are defined analogously. If $\mathcal{S}$ and $\mathcal{T}$ are two closed subspaces of $\mathcal{H}$, the notation $\mathcal{S} \ \oplus_B\ \mathcal{T}$ is used to indicate the orthogonal direct sum of $\mathcal{S}$ and $\mathcal{T}$ when, in addition, $\langle Bs, t \rangle = 0$ for every $s \in \mathcal{S}$ and $t \in \mathcal{T}$.

The following is a consequence of the spectral theorem for Hilbert space selfadjoint operators.

Lemma 2.2. *Let $B \in L(\mathcal{H})$ selfadjoint and $\mathcal{S}$ a closed subspace of $\mathcal{H}$. Then $\mathcal{S}$ can be represented as*

$$\mathcal{S} = \mathcal{S}_+ \ \oplus_B \ \mathcal{S}_- \tag{2.1}$$

where $\mathcal{S}_+$ and $\mathcal{S}_-$ are closed, $\mathcal{S}_+$ is B-nonnegative, $\mathcal{S}_-$ is B-nonpositive.

Let

$$B = \begin{bmatrix} a & b \\ b^* & c \end{bmatrix} \begin{matrix} \mathcal{S} \\ \mathcal{S}^{\perp} \end{matrix} \tag{2.2}$$

be the matrix decomposition of B induced by $\mathcal{S}$ and consider $\mathcal{S} = \mathcal{S}_+ \ \oplus_B \ \mathcal{S}_-$ as in (2.1). Then, the matrix representations of $a, |a|, |a|^{1/2} \in L(\mathcal{S})$ induced by $\mathcal{S}_+$ are: $a = \begin{bmatrix} a_+ & 0 \\ 0 & -a_- \end{bmatrix}$, $|a| = \begin{bmatrix} a_+ & 0 \\ 0 & a_- \end{bmatrix}$, $|a|^{1/2} = \begin{bmatrix} a_+^{1/2} & 0 \\ 0 & a_-^{1/2} \end{bmatrix}$, respectively.

Let us write $b := \begin{bmatrix} b_+ \\ b_- \end{bmatrix} : \mathcal{S}^{\perp} \to \begin{bmatrix} \mathcal{S}_+ \\ \mathcal{S}_- \end{bmatrix}$, where $b_{\pm} = P_{\mathcal{S}_{\pm}} b$. Then B can be written as

$$B = \begin{bmatrix} a_+ & 0 & b_+ \\ 0 & -a_- & b_- \\ b_+^* & b_-^* & c \end{bmatrix} \begin{matrix} \mathcal{S}_+ \\ \mathcal{S}_- \\ \mathcal{S}^{\perp} \end{matrix} . \tag{2.3}$$

In [1, Theorem 3], the Schur complement $B_{/\mathcal{S}}$ of an operator $B \in L(\mathcal{H})^+$ was characterized in the following fashion: if the matrix representation of B is given by (2.2), then

$$B_{/\mathcal{S}} = \begin{bmatrix} 0 & 0 \\ 0 & c - f^*f \end{bmatrix},$$

where f is the reduced solution of $a^{1/2}x = b$ (which always exists for positive operators). The next lemma characterizes the positive operators in terms of its matrix decomposition. It follows easily from the fact that $B - B_{/\mathcal{S}} \geq_{\mathcal{H}} 0$.

Lemma 2.3. *Let $\mathcal{S} \subseteq \mathcal{H}$ be a closed subspace and $B \in L(\mathcal{H})$ a selfadjoint operator with matrix decomposition induced by $\mathcal{S}$ as in* (2.2)*. Then $B \in L(\mathcal{H})^+$ if and only if*

$$a \geq_{\mathcal{H}} 0,\ \ b = b^*,\ \ R(b) \subseteq R(a^{1/2}),\ \ \textit{and}\ c = f^*f + t,$$

with f the reduced solution of the equation $b = a^{1/2}x$ and $t \geq_{\mathcal{H}} 0$.

Krein Spaces

Although familiarity with operator theory on Krein spaces is presumed, we include some basic notions. Standard references on Krein spaces and operators on them are [3], [5] and [8]. We also refer to [13] and [14] and as authoritative accounts of the subject.

Consider a linear space $\mathcal{H}$ with an indefinite metric; i.e., a sesquilinear Hermitian form $[\ ,\ ]$. A vector $x \in \mathcal{H}$ is said to be *positive* if $[x, x] > 0$. A subspace $\mathcal{S}$ of $\mathcal{H}$ is *positive* if every $x \in \mathcal{S}$, $x \neq 0$, is a positive vector. *Negative, nonnegative, nonpositive* and *neutral* vectors and subspaces are defined likewise.

We say that two closed subspaces $\mathcal{M}$ and $\mathcal{N}$ are *orthogonal*, and write $\mathcal{M}\ [\perp]\ \mathcal{N}$, if $[m, n] = 0$ for every $m \in \mathcal{M}$ and $n \in \mathcal{N}$. Denote the orthogonal direct sum of two closed subspaces $\mathcal{M}$ and $\mathcal{N}$ by $\mathcal{M}\ [\dot{+}]\ \mathcal{N}$.

Given any subspace $\mathcal{S}$ of $\mathcal{H}$, the *orthogonal companion* of $\mathcal{S}$ in $\mathcal{H}$ is defined as

$$\mathcal{S}^{[\perp]} := \{x \in \mathcal{H} : [x, s] = 0 \text{ for every } s \in \mathcal{S}\}.$$

An indefinite metric space $(\mathcal{H}, [\ ,\ ])$ is a *Krein space* if it admits a decomposition as an orthogonal direct sum

$$\mathcal{H} = \mathcal{H}_+\ [\dot{+}]\ \mathcal{H}_-, \tag{2.4}$$

where $(\mathcal{H}_+, [\ ,\ ])$ and $(\mathcal{H}_-, -[\ ,\ ])$ are Hilbert spaces. Any decomposition with these properties is called a *fundamental decomposition* of $\mathcal{H}$.

Given a Krein space $(\mathcal{H}, [\ ,\ ])$ with a fundamental decomposition $\mathcal{H} = \mathcal{H}_+\ [\dot{+}]\ \mathcal{H}_-$, the (orthogonal) direct sum of the Hilbert spaces $(\mathcal{H}_+, [\ ,\ ])$ and $(\mathcal{H}_-, -[\ ,\ ])$ is a Hilbert space, $(\mathcal{H}, \langle\ ,\ \rangle)$. Notice that the inner product $\langle\ ,\ \rangle$ and the corresponding quadratic norm $\|\ \|$ depend on the fundamental decomposition.

Every fundamental decomposition of $\mathcal{H}$ has an associated *signature operator*: $J := P_+ - P_-$ with $P_\pm := P_{\mathcal{H}_\pm // \mathcal{H}_\mp}$. The indefinite metric and the inner product corresponding to a fundamental decomposition of $\mathcal{H}$ with signature operator J are related to each other by

$$\langle x, y \rangle = [Jx, y] \quad (x, y \in \mathcal{H}).$$

If $\mathcal{H}$ is a Krein space, $L(\mathcal{H})$ stands for the vector space of all the linear operators on $\mathcal{H}$ which are bounded in an associated Hilbert space $(\mathcal{H}, \langle\ ,\ \rangle)$. Since the norms generated by different fundamental decompositions of a Krein space $\mathcal{H}$ are equivalent (see, for instance, [5, Theorem 7.19]), $L(\mathcal{H})$ does not depend on the chosen underlying Hilbert space, all of which are equivalent.

Given $T \in L(\mathcal{H})$, $T^\#$ is the unique operator satisfying

$$[Tx, y] = [x, T^\# y] \text{ for every } x, y \in \mathcal{H}.$$

$L(\mathcal{H})^s$ denotes the set of the operators $T \in L(\mathcal{H})$ such that $T = T^\#$. The selfadjoint operator $T \in L(\mathcal{H})$ is *positive* if $[Tx, x] \geq 0$ for every $x \in \mathcal{H}$. The notation $S \leq T$ signifies that $T - S$ is positive.

A (closed) subspace $\mathcal{S}$ of a Krein space $\mathcal{H}$ is *regular* if it is itself a Krein space in the indefinite metric of $\mathcal{H}$. A subspace $\mathcal{S}$ is regular if and only if $\mathcal{H} = \mathcal{S}\ [\dot{+}]\ \mathcal{S}^{[\perp]}$ or, equivalently, if it is the range of a selfadjoint projection, i.e., there exists $Q \in \mathcal{Q}$ such that $Q = Q^\#$ and $R(Q) = \mathcal{S}$ (see [5, Proposition 1.4.19]). Clearly, $\mathcal{S}$ is regular if and only if $\mathcal{S}^{[\perp]}$ is regular.

Suppose that $\mathcal{S}$ is a regular subspace with fundamental decomposition $\mathcal{S} = \mathcal{S}_+\ [\dot{+}]\ \mathcal{S}_-$. Then, by [14, Theorem 1.6], there exists a fundamental decomposition of $\mathcal{H} = \mathcal{H}_+\ [\dot{+}]\ \mathcal{H}_-$ such that $\mathcal{S}_\pm \subseteq \mathcal{H}_\pm$. In this case,

$$\mathcal{H}_\pm = \mathcal{S}_\pm\ [\dot{+}]\ \mathcal{N}_\pm,$$

where $\mathcal{S}^{[\perp]} = \mathcal{N}_+\ [\dot{+}]\ \mathcal{N}_-$ is a fundamental decomposition of $\mathcal{S}^{[\perp]}$. Now, consider $J_1 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \begin{matrix} \mathcal{S}_+ \\ \mathcal{S}_- \end{matrix}$ and $J_2 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \begin{matrix} \mathcal{N}_+ \\ \mathcal{N}_- \end{matrix}$, signature operators of $\mathcal{S}$ and $\mathcal{S}^{[\perp]}$, respectively. Then

$$J = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix} \begin{matrix} \mathcal{S} \\ \mathcal{S}^{[\perp]} \end{matrix} \tag{2.5}$$

is a signature operator for $\mathcal{H}$.

Given $W \in L(\mathcal{H})^s$ and $\mathcal{S}$ a closed subspace of $\mathcal{H}$, we say that $\mathcal{S}$ is W-*positive* if $[\,Ws, s\,] > 0$ for every $s \in \mathcal{S},\ s \neq 0$. W-*nonnegative*, W-*neutral*, W-*negative* and W-*nonpositive* subspaces are defined likewise. If $\mathcal{S}$ and $\mathcal{T}$ are two closed subspaces of $\mathcal{H}$, the notation $\mathcal{S}\ [\dot{+}]_W\ \mathcal{T}$ is used to indicate the direct sum of $\mathcal{S}$ and $\mathcal{T}$ when, additionally, $[\,Ws, t\,] = 0$ for every $s \in \mathcal{S}$ and $t \in \mathcal{T}$.

3. Complementability and Schur complement for selfadjoint operators in Hilbert spaces

The notion of complementability of an operator $B \in L(\mathcal{H})$ with respect to two given closed subspaces $\mathcal{S}$ and $\mathcal{T}$ of $\mathcal{H}$ was studied for matrices by Ando [2] and extended to operators in Hilbert spaces by Carlson and Haynsworth [10]. In [4] Antezana et al. defined a weaker notion, that of *weak complementability*, and extended the notion of the Schur complement to this context. We use these ideas when $\mathcal{S} = \mathcal{T}$ and B is selfadjoint. In what follows, we recall both definitions for this particular case:

Definition. Let $B \in L(\mathcal{H})$ selfadjoint and $\mathcal{S} \subseteq \mathcal{H}$ a closed subspace. Then B is $\mathcal{S}$-*complementable* if

$$\mathcal{H} = \mathcal{S} + B^{-1}(\mathcal{S}^{\perp}).$$

In [11] it was shown that B is $\mathcal{S}$-complementable if and only if there exists a B-selfadjoint projection onto $\mathcal{S}$; i.e., the set

$$\mathcal{P}(B, \mathcal{S}) := \{Q \in \mathcal{Q} : R(Q) = \mathcal{S},\ BQ = Q^* B\}$$

is not empty. It was also proven that, if

$$B = \begin{bmatrix} a & b \\ b^* & c \end{bmatrix} \begin{matrix} \mathcal{S} \\ \mathcal{S}^{\perp} \end{matrix}\ , \tag{3.1}$$

then B is $\mathcal{S}$-complementable if and only if $R(b) \subseteq R(a)$.

This naturally leads to the following definition.

Definition. Let $\mathcal{S} \subseteq \mathcal{H}$ be a closed subspace, and $B \in L(\mathcal{H})$ selfadjoint with representation as in (3.1). Then B is $\mathcal{S}$-*weakly complementable* if

$$R(b) \subseteq R(|a|^{1/2}).$$

When $R(a)$ is closed both notions coincide and therefore the notion of weak complementability is distinct only in the infinite dimensional setting. Every positive operator B is $\mathcal{S}$-weakly complementable.

Proposition 3.1. *Let $B \in L(\mathcal{H})$ selfadjoint. B is $\mathcal{S}$-weakly complementable for every closed subspace $\mathcal{S} \subseteq \mathcal{H}$ if and only if B is semidefinite.*

Proof. If B is semidefinite then B is $\mathcal{S}$-weakly complementable for every closed subspace $\mathcal{S} \subseteq \mathcal{H}$, because in this case, if $B \in L(\mathcal{H})$ is represented as in (3.1) for any $\mathcal{S}$, by Lemma 2.3, $R(b) \subseteq R((\pm a)^{1/2})$.

Conversely, suppose that B is $\mathcal{S}$-weakly complementable for every closed subspace $\mathcal{S} \subseteq \mathcal{H}$ and that B is not definite. Then there exists $x_0 \in \mathcal{H} \setminus \{0\}$ such that x_0 is B-neutral and $x_0 \notin N(B)$. Let $\mathcal{S} = span\ \{x_0\}$ and suppose that $B \in L(\mathcal{H})$ is represented as in (3.1). Then $\langle\, By, y\,\rangle = \langle\, ay, y\,\rangle = 0$ for every $y \in \mathcal{S}$. Hence $a = 0$ and $b = 0 = b^*$, because B is $\mathcal{S}$-weakly complementable. Then, $\mathcal{S} \subseteq N(B)$ which is a contradiction. Therefore, B is semidefinite. □

Also, B is $\mathcal{S}$-weakly complementable and $\mathcal{S}$ is B-nonnegative if and only if $a \in L(\mathcal{S})^+$ and $R(b) \subseteq R(a^{1/2})$. In fact, if $\mathcal{S}$ is B-nonnegative then, for every $s \in \mathcal{S}$,

$$0 \leq \langle\, Bs, s\,\rangle = \langle\, as, s\,\rangle\,,$$

whence $a \in L(\mathcal{S})^+$ and, since B is $\mathcal{S}$-weakly complementable, $R(b) \subseteq R(a^{1/2})$. The converse is similar. Analogously, B is $\mathcal{S}$-weakly complementable and $\mathcal{S}$ is B-nonpositive if and only if $-a \in L(\mathcal{S})^+$ and $R(b) \subseteq R((-a)^{1/2})$.

We recall the definition of Schur complement for an $\mathcal{S}$-weakly complementable selfadjoint operator.

Definition. Let $\mathcal{S} \subseteq \mathcal{H}$ be a closed subspace and $B \in L(\mathcal{H})$ selfadjoint $\mathcal{S}$-weakly complementable. When B is as in (3.1), let f be the reduced solution of $b = |a|^{1/2}x$ and $a = u|a|$ the polar decomposition of a. The *Schur complement* of B to $\mathcal{S}$ is defined as

$$B_{/\mathcal{S}} := \begin{bmatrix} 0 & 0 \\ 0 & c - f^*uf \end{bmatrix}.$$

$B_{\mathcal{S}} := B - B_{/\mathcal{S}}$ is the $\mathcal{S}$-*compression* of B.

If B is positive, $B_{/\mathcal{S}}$ coincides with the usual Schur complement of B to $\mathcal{S}$.

Variational characterization of the Schur complement

For $B \in L(\mathcal{H})$ selfadjoint and $\mathcal{S}$ a closed subspace of $\mathcal{H}$, define

$$\mathcal{M}^-(B, \mathcal{S}^\perp) := \{X \in L(\mathcal{H}) : X = X^*, X \leq_{\mathcal{H}} B,\ R(X) \subseteq \mathcal{S}^\perp\},$$

$$\mathcal{M}^+(B, \mathcal{S}^\perp) := \{X \in L(\mathcal{H}) : X = X^*, B \leq_{\mathcal{H}} X,\ R(X) \subseteq \mathcal{S}^\perp\}.$$

The next proposition shows that B is $\mathcal{S}$-weakly complementable if and only if $\mathcal{M}^-(B, \mathcal{S}_+^\perp)$ and $\mathcal{M}^+(B, \mathcal{S}_-^\perp)$ are non-empty, where $\mathcal{S} = \mathcal{S}_+ \oplus_B \mathcal{S}_-$ is any decomposition as in (2.1).

Proposition 3.2. *Let $B \in L(\mathcal{H})$ selfadjoint and $\mathcal{S}$ a closed subspace of $\mathcal{H}$. Suppose that $\mathcal{S} = \mathcal{S}_+ \oplus_B \mathcal{S}_-$ is any decomposition as in* (2.1)*. Then, the following statements are equivalent:*

i) B is $\mathcal{S}$-weakly complementable;

ii) there exist $B_1, B_2, B_3 \in L(\mathcal{H})$ selfadjoint, $B_2, B_3 \geq_{\mathcal{H}} 0$ such that $B = B_1 + B_2 - B_3$ and $\mathcal{S} \subseteq N(B_1)$, $\mathcal{S}_- \subseteq N(B_2)$, $\mathcal{S}_+ \subseteq N(B_3)$;

iii) the sets $\mathcal{M}^-(B, \mathcal{S}_+^\perp)$ and $\mathcal{M}^+(B, \mathcal{S}_-^\perp)$ are non-empty;

iv) B is $\mathcal{S}_\pm$-weakly complementable.

Proof. *i)* $\Rightarrow$ *ii)* : Let $\mathcal{S} = \mathcal{S}_+ \oplus_B \mathcal{S}_-$ be any decomposition as in (2.1). Suppose that the matrix representation of B induced by $\mathcal{S}_+$ is as in (2.3) and B is $\mathcal{S}$-weakly complementable. Then $R(b_+) \subseteq R(a_+^{1/2})$ and $R(b_-) \subseteq R(a_-^{1/2})$. In fact, since $R(b) \subseteq R(|a|^{1/2})$, for every $y \in \mathcal{S}^\perp$, there exists $s \in \mathcal{S}$ such that $by = |a|^{1/2}s$. Therefore, for every $y \in \mathcal{S}^\perp$, $b_\pm y = P_{\mathcal{S}_\pm} by = P_{\mathcal{S}_\pm}|a|^{1/2}s = a_\pm^{1/2}s$. Let f be the reduced solution of $b_+ = a_+^{1/2}x$ and g the reduced solution of $b_- = -a_-^{1/2}x$.

Set

$$B_1 := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c - f^*f + g^*g \end{bmatrix} \begin{matrix} \mathcal{S}_+ \\ \mathcal{S}_- \\ \mathcal{S}^\perp \end{matrix}, B_2 := \begin{bmatrix} a_+ & 0 & b_+ \\ 0 & 0 & 0 \\ b_+^* & 0 & f^*f \end{bmatrix} \begin{matrix} \mathcal{S}_+ \\ \mathcal{S}_- \\ \mathcal{S}^\perp \end{matrix} \text{ and } B_3 := \begin{bmatrix} 0 & 0 & 0 \\ 0 & a_- & -b_- \\ 0 & -b_-^* & g^*g \end{bmatrix} \begin{matrix} \mathcal{S}_+ \\ \mathcal{S}_- \\ \mathcal{S}^\perp \end{matrix}.$$

Then $B = B_1 + B_2 - B_3$, $\mathcal{S} \subseteq N(B_1)$, $S_- \subseteq N(B_2)$, $S_+ \subseteq N(B_3)$ and, by Lemma 2.3, $B_2, B_3 \geq_{\mathcal{H}} 0$.

ii) $\Rightarrow$ *iii)* : Since $B_1 + B_2 = B + B_3 \geq_{\mathcal{H}} B$ and $R(B_1 + B_2) \subseteq \mathcal{S}_-^\perp$, $B_1 + B_2 \in \mathcal{M}^+(B, \mathcal{S}_-^\perp)$. Similarly, $B_1 - B_3 \in \mathcal{M}^-(B, \mathcal{S}_+^\perp)$.

iii) $\Rightarrow$ *iv)* : Suppose that $X_0 \in \mathcal{M}^-(B, \mathcal{S}_+^\perp)$. Since $R(X_0) \subseteq \mathcal{S}_+^\perp$, the matrix representation of X_0 induced by $\mathcal{S}_+$ is $X_0 = \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix} \begin{matrix} \mathcal{S}_+ \\ \mathcal{S}_+^\perp \end{matrix}$, for some $d \in L(\mathcal{S}_+^\perp)$. Suppose that the matrix representation of B induced by $\mathcal{S}_+$ is $B = \begin{bmatrix} a' & b' \\ b'^* & c' \end{bmatrix} \begin{matrix} \mathcal{S}_+ \\ \mathcal{S}_+^\perp \end{matrix}$, with $a' \in L(\mathcal{S}_+)^+$. Since

$$\begin{bmatrix} a' & b' \\ b'^* & c' - d \end{bmatrix} = B - X_0 \geq_{\mathcal{H}} 0,$$

by Lemma 2.3, $R(b') \subseteq R(a'^{1/2})$ and B is $\mathcal{S}_+$-weakly complementable. In a similar way, B is $\mathcal{S}_-$-weakly complementable.

iv) $\Rightarrow$ *i)* : Suppose that the matrix representation of B induced by $\mathcal{S}_+$ is as in (2.3), since B is $\mathcal{S}_\pm$-weakly complementable, $R(b_\pm) \subseteq R(a_\pm^{1/2})$. Thus,

$$R(b) \subseteq R(b_+) + R(b_-) \subseteq R(a_+^{1/2}) \oplus R(a_-^{1/2}) = R(|a|^{1/2}),$$

and B is $\mathcal{S}$-weakly complementable. □

The following result characterizes the weak $\mathcal{S}$-complementability of B when $\mathcal{S}$ is B-nonnegative. A similar result holds in the B-nonpositive case. Several of the equivalences were also proven in [20, Proposition 3.3]. Nonetheless, we include the proofs for the sake of completeness.

Proposition 3.3. *Let $B \in L(\mathcal{H})$ selfadjoint and $\mathcal{S}$ a closed subspace of $\mathcal{H}$. Then the following statements are equivalent:*

i) $\mathcal{S}$ is B-nonnegative and B is $\mathcal{S}$-weakly complementable;

ii) there exist $B_1, B_2 \in L(\mathcal{H})$ selfadjoint, $B_2 \geq_{\mathcal{H}} 0$ such that $B = B_1 + B_2$ and $\mathcal{S} \subseteq N(B_1)$;

iii) the set $\mathcal{M}^-(B, \mathcal{S}^\perp)$ is non-empty;

iv) the set $\mathcal{M}^-(B, \mathcal{S}^\perp)$ has a maximum element, namely,

$$B_{/\mathcal{S}} = max\ \mathcal{M}^-(B, \mathcal{S}^\perp).$$

Proof. If $\mathcal{S}$ is B-nonnegative then, in the decomposition of $\mathcal{S}$ as in (2.1), $\mathcal{S}_+ = \mathcal{S}$ and $\mathcal{S}_- = \{0\}$. Applying Proposition 3.2, the equivalence *i)* $\Leftrightarrow$ *ii)* and the implication *ii)* $\Rightarrow$ *iii)* follow.

iii) $\Rightarrow$ *i)* : Suppose that $X_0 \in \mathcal{M}^-(B, \mathcal{S}^\perp)$. For all $s \in \mathcal{S}$, $\langle Bs, s \rangle \geq \langle X_0 s, s \rangle = 0$, because $X_0 \leq_{\mathcal{H}} B$ and $R(X_0) \subseteq \mathcal{S}^\perp$. Then $\mathcal{S}$ is B-nonnegative. In this case, applying Proposition 3.2, we also have that B is $\mathcal{S}$-weakly complementable.

iii) $\Leftrightarrow$ *iv)* : Suppose that $X_0 \in \mathcal{M}^-(B, \mathcal{S}^\perp)$, then by *iii)* $\Rightarrow$ *i)*, $\mathcal{S}$ is B-nonnegative and B is $\mathcal{S}$-weakly complementable. Decompose X_0 as $X_0 = X_{0+} - X_{0-}$, with $X_{0\pm} \in L(\mathcal{S}^\perp)^+$. Since $X_{0+} - X_{0-} \leq_{\mathcal{H}} B$, it follows that $0 \leq_{\mathcal{H}} X_{0+} \leq_{\mathcal{H}} B + X_{0-}$. Thus, by [1, Theorem 1],

$$0 \leq_{\mathcal{H}} X_{0+} \leq (B + X_{0-})_{/\mathcal{S}} = B_{/\mathcal{S}} + X_{0-},$$

where the last equality is a result of the fact that if $Z \in L(\mathcal{H})$ is selfadjoint and $R(Z) \subseteq \mathcal{S}^\perp$ then $B+Z$ is $\mathcal{S}$-weakly complementable and $(B+Z)_{/\mathcal{S}} = B_{/\mathcal{S}} + Z$. Therefore $X_0 \leq_{\mathcal{H}} B_{/\mathcal{S}}$. Finally, as $B_{/\mathcal{S}}$ is selfadjoint, $R(B_{/\mathcal{S}}) \subseteq \mathcal{S}^\perp$ and, by Lemma 2.3, $B_{/\mathcal{S}} \leq_{\mathcal{H}} B$. Hence $B_{/\mathcal{S}} \in \mathcal{M}^-(B, \mathcal{S}^\perp)$. Thus $B_{/\mathcal{S}} = max\ \mathcal{M}^-(B, \mathcal{S}^\perp)$.

The converse is straightforward. □

The Schur complement of B to $\mathcal{S}$ satisfies a variational characterization as a min-max if and only if B is $\mathcal{S}$-weakly complementable, as the following theorem shows.

Theorem 3.4. *Let $B \in L(\mathcal{H})$ selfadjoint and $\mathcal{S}$ a closed subspace of $\mathcal{H}$. Suppose that $\mathcal{S} = \mathcal{S}_+ \oplus_B \mathcal{S}_-$ is any decomposition as in* (2.1). *Then B is $\mathcal{S}$-weakly complementable if and only if there exist $min\ \mathcal{M}^+(max\ \mathcal{M}^-(B, \mathcal{S}_+^\perp), \mathcal{S}_-^\perp)$ and $max\ \mathcal{M}^-(min\ \mathcal{M}^+(B, \mathcal{S}_-^\perp), \mathcal{S}_+^\perp)$. In this case,*

$$B_{/\mathcal{S}} = min\ \mathcal{M}^+(max\ \mathcal{M}^-(B, \mathcal{S}_+^\perp), \mathcal{S}_-^\perp) = max\ \mathcal{M}^-(min\ \mathcal{M}^+(B, \mathcal{S}_-^\perp), \mathcal{S}_+^\perp).$$

Proof. Suppose that B is $\mathcal{S}$-weakly complementable. If the matrix representation of B induced by $\mathcal{S}_+$ is as in (2.3), then $R(b_\pm) \subseteq R(a_\pm^{1/2})$. Let f be the reduced solution of $b_+ = a_+^{1/2} x$ and g the reduced solution of $b_- = -a_-^{1/2} x$. Then, by Proposition 3.3,

$$B_{/\mathcal{S}_+} = max\ \mathcal{M}^-(B, \mathcal{S}_+^\perp) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -a_- & b_- \\ 0 & b_-^* & c - f^* f \end{bmatrix} \begin{matrix} \mathcal{S}_+ \\ \mathcal{S}_- \\ \mathcal{S}^\perp \end{matrix}.$$

Thus $B_{/\mathcal{S}_+}$ is $\mathcal{S}_-$-weakly complementable and $\mathcal{S}_-$ is $B_{/\mathcal{S}_+}$-nonpositive. Again by Proposition 3.3,

$$(B_{/\mathcal{S}_+})_{/\mathcal{S}_-} = min\ \mathcal{M}^+\ (max\ \mathcal{M}^-(B, \mathcal{S}_+^\perp), \mathcal{S}_-^\perp) = \begin{bmatrix} 0 & 0 \\ 0 & c - f^*f + g^*g \end{bmatrix}.$$

In a similar way,

$$(B_{/\mathcal{S}_-})_{/\mathcal{S}_+} = max\ \mathcal{M}^-\ (min\ \mathcal{M}^+(B, \mathcal{S}_-^\perp), \mathcal{S}_+^\perp) = \begin{bmatrix} 0 & 0 \\ 0 & c - f^*f + g^*g \end{bmatrix}.$$

Conversely, if there exist $min\ \mathcal{M}^+(max\ \mathcal{M}^-(B, \mathcal{S}_+^\perp), \mathcal{S}_-^\perp)$ and $max\ \mathcal{M}^-(min\ \mathcal{M}^+(B, \mathcal{S}_-^\perp), \mathcal{S}_+^\perp)$, then the sets $\mathcal{M}^-(B, \mathcal{S}_+^\perp)$ and $\mathcal{M}^+(B, \mathcal{S}_-^\perp)$ are non-empty. So by Proposition 3.2, B is $\mathcal{S}$-weakly complementable.

In this case, notice that

$$b = \begin{bmatrix} b_+ \\ b_- \end{bmatrix} = \begin{bmatrix} a_+^{1/2} & 0 \\ 0 & a_-^{1/2} \end{bmatrix} \begin{bmatrix} f \\ -g \end{bmatrix} = |a|^{1/2}(f - g),$$

and since

$$R(f-g) \subseteq \overline{R(a_+^{1/2})} \oplus \overline{R(a_-^{1/2})} = \overline{R(|a|^{1/2})},$$

$y := f - g$ is the reduced solution of the equation $b = |a|^{1/2}x$. Also, if $u = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{matrix} \mathcal{S}_+ \\ \mathcal{S}_- \end{matrix}$, then $a = u|a| = |a|u$ is the polar decomposition of a. Therefore $y^*uy = \begin{bmatrix} f^* & -g^* \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} f \\ -g \end{bmatrix} = f^*f - g^*g$ and

$$B_{/\mathcal{S}} = min\ \mathcal{M}^+(max\ \mathcal{M}^-(B, \mathcal{S}_+^\perp), \mathcal{S}_-^\perp) = max\ \mathcal{M}^-(min\ \mathcal{M}^+(B, \mathcal{S}_-^\perp), \mathcal{S}_+^\perp).$$

□

Corollary 3.5. *Let $B \in L(\mathcal{H})$ selfadjoint and $\mathcal{S}$ a closed subspace of $\mathcal{H}$. Suppose that $\mathcal{S} = \mathcal{S}_+ \oplus_B \mathcal{S}_-$ is any decomposition as in* (2.1)*. If B is $\mathcal{S}$-weakly complementable then*

$$B_{/\mathcal{S}} = (B_{/\mathcal{S}_+})_{/\mathcal{S}_-} = (B_{/\mathcal{S}_-})_{/\mathcal{S}_+}.$$

In [1, Theorem 5], Anderson and Trapp proved that if $B \in L(\mathcal{H})^+$ and $\mathcal{S}$ is a closed subspace of $\mathcal{H}$ then

$$B_{/\mathcal{S}} = inf\ \{Q^*BQ : Q \in \mathcal{Q},\ N(Q) = \mathcal{S}\}.$$

More generally:

Theorem 3.6. *Let $B \in L(\mathcal{H})$ selfadjoint and $\mathcal{S}$ a closed subspace of $\mathcal{H}$. Suppose that $\mathcal{S} = \mathcal{S}_+ \oplus_B \mathcal{S}_-$ is any decomposition as in* (2.1)*. Then B is $\mathcal{S}$-weakly complementable if and only if there exist*

$$\underset{Q_- \in \mathcal{Q},\ N(Q_-) = \mathcal{S}_-}{sup} \left(\underset{Q_+ \in \mathcal{Q},\ N(Q_+) = \mathcal{S}_+}{inf} Q_-^*Q_+^*BQ_+Q_- \right)$$

and

$$\underset{Q_+ \in \mathcal{Q},\ N(Q_+) = \mathcal{S}_+}{inf} \left(\underset{Q_- \in \mathcal{Q},\ N(Q_-) = \mathcal{S}_-}{sup} Q_+^*Q_-^*BQ_-Q_+ \right).$$

In this case,

$$\begin{aligned} B_{/\mathcal{S}} &= \underset{Q_- \in \mathcal{Q},\ N(Q_-) = \mathcal{S}_-}{sup} \left(\underset{Q_+ \in \mathcal{Q},\ N(Q_+) = \mathcal{S}_+}{inf} Q_-^*Q_+^*BQ_+Q_- \right) \\ &= \underset{Q_+ \in \mathcal{Q},\ N(Q_+) = \mathcal{S}_+}{inf} \left(\underset{Q_- \in \mathcal{Q},\ N(Q_-) = \mathcal{S}_-}{sup} Q_+^*Q_-^*BQ_-Q_+ \right). \end{aligned}$$

In order to prove Theorem 3.6, we require the following lemmas.

Lemma 3.7. *Let $B \in L(\mathcal{H})$ selfadjoint and $\mathcal{S}$ a closed subspace of $\mathcal{H}$. Suppose that $\mathcal{S} = \mathcal{S}_+ \oplus_B \mathcal{S}_-$ is any decomposition as in* (2.1)*. If B is $\mathcal{S}$-weakly complementable and $B = B_1 + B_2 - B_3$ is any decomposition as in Proposition 3.2, then*

$$B_{/\mathcal{S}} = B_1 + B_{2/\mathcal{S}_+} - B_{3/\mathcal{S}_-}.$$

Proof. Since B is $\mathcal{S}$-weakly complementable, by Proposition 3.2, B is $\mathcal{S}_\pm$-weakly complementable. Then, proceeding as in the proof of Proposition 3.3, it can be checked that $B_1 + B_2 - B_{3/\mathcal{S}_-} = min\ \mathcal{M}^+(B, \mathcal{S}_-^\perp)$.

On the other hand,

$$B_1 + B_{2/\mathcal{S}_+} - B_{3/\mathcal{S}_-} = max\ \mathcal{M}^-(B_1 + B_2 - B_{3/\mathcal{S}_-}, \mathcal{S}_+^\perp).$$

In fact, $B_1 + B_{2/\mathcal{S}_+} - B_{3/\mathcal{S}_-} \in \mathcal{M}^-(B_1 + B_2 - B_{3/\mathcal{S}_-}, \mathcal{S}_+^\perp)$. This follows since $B_1 + B_{2/\mathcal{S}_+} - B_{3/\mathcal{S}_-}$ is selfadjoint, $B_1 + B_{2/\mathcal{S}_+} - B_{3/\mathcal{S}_-} \leq_{\mathcal{H}} B_1 + B_2 - B_{3/\mathcal{S}_-}$ and, by [1, Corollary 4], $R(B_{3/\mathcal{S}_-}) \subseteq \overline{R(B_3)} \subseteq \mathcal{S}_+^\perp$. Let $Y \in \mathcal{M}^-(B_1 + B_2 - B_{3/\mathcal{S}_-}, \mathcal{S}_+^\perp)$, and decompose Y as $Y = Y_+ - Y_-$, with $Y_\pm \in L(\mathcal{S}_+^\perp)^+$. Since $Y_+ - Y_- \leq_{\mathcal{H}} B_1 + B_2 - B_{3/\mathcal{S}_-}$,

$$0 \leq_{\mathcal{H}} Y_+ + B_{3/\mathcal{S}_-} \leq_{\mathcal{H}} B_1 + B_2 + Y_-.$$

Now, since $R(B_1 + Y_-) \subseteq \mathcal{S}_+^\perp$, [1, Theorem 1] gives that

$$0 \leq_{\mathcal{H}} Y_+ + B_{3/\mathcal{S}_-} \leq_{\mathcal{H}} (B_1 + B_2 + Y_-)_{/\mathcal{S}_+} = B_1 + B_{2/\mathcal{S}_+} + Y_-;$$

i.e., $Y \leq_{\mathcal{H}} B_1 + B_{2/\mathcal{S}_+} - B_{3/\mathcal{S}_-}$. Therefore,

$$B_1 + B_{2/\mathcal{S}_+} - B_{3/\mathcal{S}_-} = max\ \mathcal{M}^-(min\ \mathcal{M}^+(B, \mathcal{S}_-^\perp), \mathcal{S}_+^\perp) = B_{/\mathcal{S}}.$$

□

Lemma 3.8. *Let $B \in L(\mathcal{H})^+$ and $\mathcal{S}$ a closed subspace of $\mathcal{H}$ decomposed as in* (2.1)*. For any $Q_- \in \mathcal{Q},\ N(Q_-) = \mathcal{S}_-$,*

$$\underset{Q_+ \in \mathcal{Q},\ N(Q_+) = \mathcal{S}_+}{inf} Q_-^* Q_+^* B Q_+ Q_- = Q_-^* B_{/\mathcal{S}_+} Q_-.$$

Proof. Let $Q_- \in \mathcal{Q},\ N(Q_-) = \mathcal{S}_-$. By [1, Theorem 5], $B_{/\mathcal{S}_+} \leq Q_+^* B Q_+$, for every $Q_+ \in \mathcal{Q},\ N(Q_+) = \mathcal{S}_+$. Therefore $Q_-^* B_{/\mathcal{S}_+} Q_-$ is a lower bound of the set $\{Q_-^* Q_+^* B Q_+ Q_- : Q_+ \in \mathcal{Q},\ N(Q_+) = \mathcal{S}_+\}$.

If B is invertible then, by [11, Section 4], B is $\mathcal{S}_+$-complementable. So, by [11, Proposition 4.2], there exists $Q_+ \in \mathcal{Q},\ N(Q_+) = \mathcal{S}_+$ such that $B_{/\mathcal{S}_+} = Q_+^* B Q_+$. Then clearly in this case, the infimum is actually a minimum.

For a non invertible B, consider $\varepsilon > 0$. If F is any lower bound of the set $\{Q_-^* Q_+^* B Q_+ Q_- : Q_+ \in \mathcal{Q},\ N(Q_+) = \mathcal{S}_+\}$ then, for any $Q_+ \in \mathcal{Q},\ N(Q_+) = \mathcal{S}_+$,

$$F \leq Q_-^* Q_+^* B Q_+ Q_- \leq Q_-^* Q_+^* (B + \varepsilon I) Q_+ Q_-.$$

Since $B + \varepsilon I$ is invertible, it follows that $F \leq Q_-^* (B + \varepsilon I)_{/\mathcal{S}_+} Q_-$. As ε is arbitrary, [1, Corollary 2] yields $F \leq Q_-^* B_{/\mathcal{S}_+} Q_-$.

□

Proof of Theorem 3.6. Suppose that B is $\mathcal{S}$-weakly complementable and write $B = B_1 + B_2 - B_3$, with $\mathcal{S} \subseteq N(B_1)$, $\mathcal{S}_- \subseteq N(B_2)$, $\mathcal{S}_+ \subseteq N(B_3)$ and $B_2, B_3 \geq_{\mathcal{H}} 0$ (see Proposition 3.2). Let $Q_- \in \mathcal{Q},\ N(Q_-) = \mathcal{S}_-$. Then, for any $Q_+ \in \mathcal{Q},\ N(Q_+) = \mathcal{S}_+$,

$$Q_-^* Q_+^* B Q_+ Q_- = B_1 + Q_-^* Q_+^* B_2 Q_+ Q_- - Q_-^* B_3 Q_-.$$

By Lemma 3.8, $\underset{Q_+ \in \mathcal{Q},\ N(Q_+) = \mathcal{S}_+}{inf} Q_-^* Q_+^* B_2 Q_+ Q_- = Q_-^* B_{2/\mathcal{S}_+} Q_-$. Therefore,

$$\begin{aligned} \underset{Q_+ \in \mathcal{Q},\ N(Q_+) = \mathcal{S}_+}{inf} Q_-^* Q_+^* B Q_+ Q_- &= B_1 + Q_-^* B_{2/\mathcal{S}_+} Q_- - Q_-^* B_3 Q_- \\ &= B_1 + B_{2/\mathcal{S}_+} - Q_-^* B_3 Q_-, \end{aligned}$$

where we used the fact that $R(B_{2/\mathcal{S}_+}) \subseteq \overline{R(B_2)} \subseteq \mathcal{S}_-^{\perp}$ (see [1, Corollary 4]). Finally, by [20, Proposition 3.7] and Lemma 3.7,

$$\underset{Q_-\in\mathcal{Q},\ N(Q_-)=\mathcal{S}_-}{sup} \left(\underset{Q_+\in\mathcal{Q},\ N(Q_+)=\mathcal{S}_+}{inf} Q_-^* Q_+^* B Q_+ Q_- \right) =$$

$$= \underset{Q_-\in\mathcal{Q},\ N(Q_-)=\mathcal{S}_-}{sup} \left(B_1 + B_{2/\mathcal{S}_+} - Q_-^* B_3 Q_- \right) = B_1 + B_{2/\mathcal{S}_+} - B_{3/\mathcal{S}_-} = B_{/\mathcal{S}}.$$

The second equality follows in a similar way.

Conversely, suppose that

$$\underset{Q_-\in\mathcal{Q},\ N(Q_-)=\mathcal{S}_-}{sup} \left(\underset{Q_+\in\mathcal{Q},\ N(Q_+)=\mathcal{S}_+}{inf} Q_-^* Q_+^* B Q_+ Q_- \right)$$

and

$$\underset{Q_+\in\mathcal{Q},\ N(Q_+)=\mathcal{S}_+}{inf} \left(\underset{Q_-\in\mathcal{Q},\ N(Q_-)=\mathcal{S}_-}{sup} Q_+^* Q_-^* B Q_- Q_+ \right).$$

exist. Then, for every $Q_- \in \mathcal{Q}, N(Q_-) = \mathcal{S}_-$, there exists

$$\underset{Q_+\in\mathcal{Q},\ N(Q_+)=\mathcal{S}_+}{inf} Q_-^* Q_+^* B Q_+ Q_-.$$

In particular, for $Q_- = P_{\mathcal{S}_-^{\perp}}$, $T_0 := \underset{Q_+\in\mathcal{Q},\ N(Q_+)=\mathcal{S}_+}{inf} P_{\mathcal{S}_-^{\perp}} Q_+^* B Q_+ P_{\mathcal{S}_-^{\perp}}$. Then $P_{\mathcal{S}_-^{\perp}} Q_+^* B Q_+ P_{\mathcal{S}_-^{\perp}} - T_0 \geq_{\mathcal{H}} 0$ for every $Q_+ \in \mathcal{Q}$ such that $N(Q_+) = \mathcal{S}_+$. Thus $T_0 = P_{\mathcal{S}_-^{\perp}} Q_+^* B Q_+ P_{\mathcal{S}_-^{\perp}} - (P_{\mathcal{S}_-^{\perp}} Q_+^* B Q_+ P_{\mathcal{S}_-^{\perp}} - T_0)$ is selfadjoint.

Since $R(Q_+^* B Q) \subseteq \mathcal{S}_+^{\perp}$, then $P_{\mathcal{S}_-^{\perp}} Q_+^* B Q_+ P_{\mathcal{S}_-^{\perp}} = P_{\mathcal{S}^{\perp}} Q_+^* B Q_+ P_{\mathcal{S}^{\perp}}$ and

$$T_0 = \underset{Q_+\in\mathcal{Q},\ N(Q_+)=\mathcal{S}_+}{inf} P_{\mathcal{S}^{\perp}} Q_+^* B Q_+ P_{\mathcal{S}^{\perp}}.$$

Suppose that the matrix decomposition of T_0 is given by $T_0 = \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{12}^* & t_{22} & t_{23} \\ t_{13}^* & t_{23}^* & t_{33} \end{bmatrix} \begin{matrix} \mathcal{S}_+ \\ \mathcal{S}_- \\ \mathcal{S}^{\perp} \end{matrix}$. Then

$$P_{\mathcal{S}^{\perp}} Q_+^* B Q_+ P_{\mathcal{S}^{\perp}} - T_0 = \begin{bmatrix} -t_{11} & -t_{12} & -t_{13} \\ -t_{12}^* & -t_{22} & -t_{23} \\ -t_{13}^* & t_{23}^* & P_{\mathcal{S}^{\perp}} Q_+^* B Q_+ P_{\mathcal{S}^{\perp}} - t_{33} \end{bmatrix} \geq_{\mathcal{H}} 0.$$

Then $t_{11} \leq_{\mathcal{H}} 0$ and $t_{22} \leq_{\mathcal{H}} 0$. Also, since $P_{\mathcal{S}^{\perp}} T_0 P_{\mathcal{S}^{\perp}} \leq_{\mathcal{H}} P_{\mathcal{S}^{\perp}} Q_+^* B Q_+ P_{\mathcal{S}^{\perp}}$ for every $Q_+ \in \mathcal{Q}$ such that $N(Q_+) = \mathcal{S}_+$, then $P_{\mathcal{S}^{\perp}} T_0 P_{\mathcal{S}^{\perp}} \leq_{\mathcal{H}} T_0$. Therefore

$$T_0 - P_{\mathcal{S}^{\perp}} T_0 P_{\mathcal{S}^{\perp}} = \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{12}^* & t_{22} & t_{23} \\ t_{13}^* & t_{23}^* & 0 \end{bmatrix} \geq_{\mathcal{H}} 0.$$

Then, by Lemma 2.3, $t_{11} \geq_{\mathcal{H}} 0$ so $t_{11} = 0$, $t_{22} \geq_{\mathcal{H}} 0$ so $t_{22} = 0$, and also $t_{12} = t_{12}^* = t_{13} = t_{13}^* = t_{23} = t_{23}^* = 0$. Hence, $R(T_0) \subseteq \mathcal{S}^{\perp}$. Therefore

$$P_{\mathcal{S}^{\perp}} Q_+^* (B - T_0) Q_+ P_{\mathcal{S}^{\perp}} \geq_{\mathcal{H}} 0 \text{ for every } Q_+ \in \mathcal{Q} \text{ such that } N(Q_+) = \mathcal{S}_+.$$

Let us show that $\langle (B - T_0)x, x \rangle \geq 0$ for every $x \in \mathcal{S}_-^{\perp} = \mathcal{S}_+ \oplus \mathcal{S}^{\perp}$. Fix $x \in \mathcal{S}_-^{\perp}$; if $x \in \mathcal{S}_+$ then $\langle (B - T_0)x, x \rangle = \langle Bx, x \rangle \geq 0$, because $\mathcal{S}_+ \subseteq N(T_0)$ and $\mathcal{S}_+$ is B-nonnegative. If $x \notin \mathcal{S}_+$ then $P_{\mathcal{S}^{\perp}} x \neq 0$ and there exists a subspace $\mathcal{M}$ such that $x \in \mathcal{M}$ and $\mathcal{M} \dotplus \mathcal{S}_+ = \mathcal{H}$. Take $Q_+ = P_{\mathcal{M}\,//\,\mathcal{S}_+}$; then

$x = Q_+x = Q_+P_{\mathcal{S}^\perp}x$. Thus $\langle\,(B-T_0)x, x\,\rangle = \left\langle\, P_{\mathcal{S}^\perp}Q_+^*(B-T_0)Q_+P_{\mathcal{S}^\perp}x, x\,\right\rangle \geq 0$. Since $x \in \mathcal{S}_-^\perp$ is arbitrary, $\langle\,(B-T_0)x, x\,\rangle \geq 0$ for every $x \in \mathcal{S}_-^\perp$. If the matrix decomposition of B is as in (2.3),

$$P_{\mathcal{S}_-^\perp}(B-T_0)P_{\mathcal{S}_-^\perp} = \begin{bmatrix} a_+ & 0 & b_+ \\ 0 & 0 & 0 \\ b_+^* & 0 & c - t_{33} \end{bmatrix} \geq_{\mathcal{H}} 0.$$

Then, by Lemma 2.3, $R(b_+) \subseteq R(a_+^{1/2})$.

Analogously, since $\underset{Q_+\in\mathcal{Q},\ N(Q_+)=\mathcal{S}_+}{inf}\left(\underset{Q_-\in\mathcal{Q},\ N(Q_-)=\mathcal{S}_-}{sup} Q_+^*Q_-^*BQ_-Q_+\right)$ exists, it follows that $R(b_-) \subseteq R(a_-^{1/2})$. Therefore $R(b) \subseteq R(b_+) + R(b_-) \subseteq R(a_+^{1/2}) \oplus R(a_-^{1/2}) = R(|a|^{1/2})$ and B is $\mathcal{S}$-weakly complementable.

□

Corollary 3.9. *Let $B \in L(\mathcal{H})$ selfadjoint and $\mathcal{S}$ a closed subspace of $\mathcal{H}$. Suppose that $\mathcal{S}$ is B-nonnegative. Then B is $\mathcal{S}$-weakly complementable if and only if there exists $inf\ \{Q^*BQ : Q \in \mathcal{Q},\ N(Q) = \mathcal{S}\}$. In this case,*

$$B_{/\mathcal{S}} = inf\ \{Q^*BQ : Q \in \mathcal{Q},\ N(Q) = \mathcal{S}\}. \tag{3.2}$$

A similar result holds when $\mathcal{S}$ is B-nonpositive, replacing inf by sup.

The following proposition shows that the infimum in (3.2) is indeed a minimum if and only if B is $\mathcal{S}$-complementable and $\mathcal{S}$ is B-nonnegative.

Proposition 3.10. *Let $B \in L(\mathcal{H})$ selfadjoint and $\mathcal{S}$ a closed subspace of $\mathcal{H}$. Suppose that B is $\mathcal{S}$-weakly complementable. Then*

$$B_{/\mathcal{S}} = min\ \{Q^*BQ : Q \in \mathcal{Q},\ N(Q) = \mathcal{S}\}$$

if and only if B is $\mathcal{S}$-complementable and $\mathcal{S}$ is B-nonnegative. In this case,

$$B_{/\mathcal{S}} = B(I-Q),$$

for any $Q \in \mathcal{P}(B, \mathcal{S})$.

A similar result holds when $\mathcal{S}$ is B-nonpositive, replacing min by max.

Proof. Suppose that B is $\mathcal{S}$-complementable and $\mathcal{S}$ is B-nonnegative. Then, by [20, Proposition 4.6], $B_{/\mathcal{S}} = min\ \{Q^*BQ : Q \in \mathcal{Q},\ N(Q) = \mathcal{S}\}$. In this case $B_{/\mathcal{S}} = B(I-Q)$, for any $Q \in \mathcal{P}(B, \mathcal{S})$.

Conversely, suppose that $B_{/\mathcal{S}} = min\ \{Q^*BQ : Q \in \mathcal{Q},\ N(Q) = \mathcal{S}\}$. Let B be as in (3.1) and $Q := \begin{pmatrix} 0 & e \\ 0 & I \end{pmatrix}$ with $e \in L(\mathcal{S}^\perp, \mathcal{S})$. Then $Q \in \mathcal{Q}$ and $\mathrm{N}(Q) = \mathcal{S}$. Let f be the reduced solution of $b = |a|^{1/2}x$ and $a = u|a|$ the polar decomposition of a. From $B_{/\mathcal{S}} \leq_{\mathcal{H}} Q^*BQ$, it is easy to check that

$$0 \leq \left\langle\, u(f + |a|^{1/2}ue)y), (f + |a|^{1/2}ue)y\,\right\rangle \text{ for every } y \in \mathcal{S}^\perp \text{ and } e \in L(\mathcal{S}^\perp, \mathcal{S}).$$

Since $R(f) \subseteq \overline{R(|a|^{1/2})}$ then $\overline{R(|a|^{1/2})} = \overline{R(f)} \oplus \overline{R(|a|^{1/2})} \cap R(f)^\perp$. In particular, for every $s \in \mathcal{S}$, $|a|^{1/2}s = t + v$, with $t \in \overline{R(f)}$ and $v \in \overline{R(|a|^{1/2}) \cap R(f)^\perp}$. If $s \in \mathcal{S}$ and $\varepsilon > 0$, then there exist $y_\varepsilon \in \mathcal{S}^\perp$ and $e_\varepsilon \in L(\mathcal{S}^\perp, \mathcal{S})$ such that $\| |a|^{1/2}s - (fy_\varepsilon + |a|^{1/2}ue_\varepsilon y_\varepsilon)\| < \varepsilon$. Therefore

$$\begin{aligned}
\langle\, as, s\,\rangle &= \left\langle\, u|a|^{1/2}s, |a|^{1/2}s\,\right\rangle \\
&= \left\langle\, u\ \underset{\varepsilon\to 0}{lim}[(f + |a|^{1/2}ue_\varepsilon)y_\varepsilon], \underset{\varepsilon\to 0}{lim}[(f + |a|^{1/2}ue_\varepsilon)y_\varepsilon]\,\right\rangle \\
&= \underset{\varepsilon\to 0}{lim}\left\langle\, u(f + |a|^{1/2}ue_\varepsilon)y_\varepsilon, (f + |a|^{1/2}ue_\varepsilon)y_\varepsilon\,\right\rangle \geq 0
\end{aligned}$$

and $\mathcal{S}$ is B-nonnegative.

In this case, by Proposition 3.3, $B_{/\mathcal{S}} \in \mathcal{M}^-(B, \mathcal{S}^\perp)$, so that $B_{/\mathcal{S}} \leq_{\mathcal{H}} B$. Let $Q_0 \in \mathcal{Q}$ with $N(Q_0) = \mathcal{S}$ such that $B_{/\mathcal{S}} = Q_0^* B Q_0$. Then, $B - Q_0^* B Q_0 \geq_{\mathcal{H}} 0$. From $Q_0^*(B - Q_0^* B Q_0)Q_0 = 0$, it follows that $(B - Q_0^* B Q_0)Q_0 = 0$, which implies that $BQ_0 = Q_0^* B Q_0$. Thus, $E_0 := I - Q_0 \in \mathcal{P}(B, \mathcal{S})$ and B is $\mathcal{S}$-complementable. □

If $B \in L(\mathcal{H})$ is selfadjoint and $\mathcal{S}$ a closed subspace of $\mathcal{H}$ such that B is $\mathcal{S}$-complementable, then there exists a B-selfadjoint projection Q into $\mathcal{S}$ that can be decomposed as the sum of two B-selfadjoint projections Q_+, Q_- with B-nonnegative and B-nonpositive ranges, respectively.

Lemma 3.11. *Let $B \in L(\mathcal{H})$ selfadjoint and $\mathcal{S}$ a closed subspace of $\mathcal{H}$. Suppose that $\mathcal{S} = \mathcal{S}_+ \oplus_B \mathcal{S}_-$ is any decomposition as in* (2.1). *Then, the following statements are equivalent:*

i) B is $\mathcal{S}$-complementable;

ii) B is $\mathcal{S}_\pm$-complementable;

iii) B is $\mathcal{S}$-weakly complementable and $B_{/\mathcal{S}_\pm}$ is $\mathcal{S}_\mp$-complementable.

In this case, there exists $Q \in \mathcal{P}(B, \mathcal{S})$ that can be decomposed as $Q = Q_+ + Q_-$, where $Q_\pm \in \mathcal{P}(B, \mathcal{S}_\pm)$. Moreover, $R(Q_+) \perp R(Q_-)$ and $Q_+Q_- = Q_-Q_+ = 0$.

Proof. $i) \Leftrightarrow ii)$: Let $\mathcal{S} = \mathcal{S}_+ \oplus_B \mathcal{S}_-$ be any decomposition as in (2.1). Suppose that the matrix representation of B is as in (2.3) and B is $\mathcal{S}$-complementable. Then $R(b_+) \subseteq R(a_+)$ and $R(b_-) \subseteq R(a_-)$. In fact, since $R(b) \subseteq R(a)$, for every $y \in \mathcal{S}^\perp$, there exists $s \in \mathcal{S}$ such that $by = as$. Therefore, for every $y \in \mathcal{S}^\perp$, $b_\pm y = P_{\mathcal{S}_\pm} by = P_{\mathcal{S}_\pm} as = a_\pm s$ and B is $\mathcal{S}_\pm$-complementable. The converse follows in a similar way using that $R(a) = R(a_+) \oplus R(a_-)$.

$i) \Leftrightarrow iii)$: It can be proven in a similar way as in $i) \Leftrightarrow ii)$ using the decomposition of $B_{/\mathcal{S}_\pm}$ given in the proof of Theorem 3.4. In this case, let f be the reduced solution of $b_+ = a_+ x$ and g the reduced solution of $b_- = -a_- x$. Set $Q_+ := \begin{bmatrix} I & 0 & f \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \mathcal{S}_+ \\ \mathcal{S}_- \\ \mathcal{S}^\perp \end{matrix}$ and $Q_- := \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & -g \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \mathcal{S}_+ \\ \mathcal{S}_- \\ \mathcal{S}^\perp \end{matrix}$. Then $Q_\pm \in \mathcal{P}(B, \mathcal{S}_\pm)$, $R(Q_+) \perp R(Q_-)$ and $Q_+Q_- = Q_-Q_+ = 0$. Finally, since

$$R(f - g) \subseteq \overline{R(a_+)} \oplus \overline{R(a_-)} = \overline{R(a)},$$

$y := f - g$ is the reduced solution of the equation $b = ax$. Therefore $Q := Q_+ + Q_- = \begin{bmatrix} I & 0 & f \\ 0 & I & -g \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \mathcal{S}_+ \\ \mathcal{S}_- \\ \mathcal{S}^\perp \end{matrix} = \begin{bmatrix} I & y \\ 0 & 0 \end{bmatrix} \begin{matrix} \mathcal{S} \\ \mathcal{S}^\perp \end{matrix} \in \mathcal{P}(B, \mathcal{S})$. □

Corollary 3.12. *Let $B \in L(\mathcal{H})$ selfadjoint and $\mathcal{S}$ a closed subspace of $\mathcal{H}$ such that B is $\mathcal{S}$-complementable. Then*

$$B_{/\mathcal{S}} = B(I - Q) = (I - Q)^* B$$

for any $Q \in \mathcal{P}(B, \mathcal{S})$.

Proof. Let $Q^0 \in \mathcal{P}(B, \mathcal{S})$ such that there exist $Q^0_\pm \in \mathcal{P}(B, \mathcal{S}_\pm)$ with $Q^0 = Q^0_+ + Q^0_-$, $R(Q^0_+) \perp R(Q^0_-)$ and $Q^0_+ Q^0_- = Q^0_- Q^0_+ = 0$, as in Lemma 3.11. Set $\mathcal{S}_\pm := R(Q^0_\pm)$, then $\mathcal{S} = \mathcal{S}_+ \oplus_B \mathcal{S}_-$, as in Lemma 2.2. By Lemma 3.11, $B_{/\mathcal{S}_+}$ is $\mathcal{S}_-$-complementable and since $\mathcal{S}_-$ is $B_{/\mathcal{S}_+}$-nonpositive, Corollary 3.5 together with Proposition 3.10 give

$$B_{/\mathcal{S}} = (B_{/\mathcal{S}_+})_{/\mathcal{S}_-} = B_{/\mathcal{S}_+}(I - Q^0_-).$$

Then, once again by Lemma 3.11, B is $\mathcal{S}_+$-complementable and by Proposition 3.10,

$$B_{/\mathcal{S}} = B_{/\mathcal{S}_+}(I - Q^0_-) = B(I - Q^0_+)(I - Q^0_-) = B(I - Q^0_+ - Q^0_-) = B(I - Q^0).$$

Now take any $Q \in \mathcal{P}(B, \mathcal{S})$, then by [11, Theorem 3.5] and [17, Proposition 3.2], $Q = Q^0 + T$, for some $T \in L(\mathcal{H})$ with $R(T) \subseteq N(B) \cap \mathcal{S}$ and $\mathcal{S} \subseteq N(T)$. Therefore

$$B_{/\mathcal{S}} = B(I - Q^0) = B(I - (Q - T)) = B(I - Q).$$

□

Corollary 3.12 shows that $B_{/\mathcal{S}}$ coincides with the Schur complement defined in [15] for a bounded selfadjoint operator B and $\mathcal{S} \subseteq \mathcal{H}$ a closed subspace such that B is $\mathcal{S}$-complementable.

Corollary 3.13. *Let $B \in L(\mathcal{H})$ selfadjoint and $\mathcal{S}$ a closed subspace of $\mathcal{H}$ such that B is $\mathcal{S}$-weakly complementable. Suppose that $\mathcal{S} = \mathcal{S}_+ \oplus_B \mathcal{S}_-$ is any decomposition as in (2.1). Then, B is $\mathcal{S}$-complementable if and only if*

$$B_{/\mathcal{S}} = \underset{Q_-\in\mathcal{Q},\ N(Q_-)=\mathcal{S}_-}{max} \left(\underset{Q_+\in\mathcal{Q},\ N(Q_+)=\mathcal{S}_+}{min} Q_-^* Q_+^* B Q_+ Q_- \right)$$
$$= \underset{Q_+\in\mathcal{Q},\ N(Q_+)=\mathcal{S}_+}{min} \left(\underset{Q_-\in\mathcal{Q},\ N(Q_-)=\mathcal{S}_-}{max} Q_+^* Q_-^* B Q_- Q_+ \right).$$

Proof. Suppose that B is $\mathcal{S}$-complementable. Then, by Theorem 3.6, Lemma 3.11 and Corollary 3.12, the result follows.

Conversely, suppose that

$$B_{/\mathcal{S}} = \underset{Q_-\in\mathcal{Q},\ N(Q_-)=\mathcal{S}_-}{max} \left(\underset{Q_+\in\mathcal{Q},\ N(Q_+)=\mathcal{S}_+}{min} Q_-^* Q_+^* B Q_+ Q_- \right)$$
$$= \underset{Q_+\in\mathcal{Q},\ N(Q_+)=\mathcal{S}_+}{min} \left(\underset{Q_-\in\mathcal{Q},\ N(Q_-)=\mathcal{S}_-}{max} Q_+^* Q_-^* B Q_- Q_+ \right).$$

Suppose that the matrix representation of B is as in (2.3). Since B is $\mathcal{S}$-weakly complementable, take B_1, B_2 and B_3 as in the proof of Proposition 3.2. Let $Q_- \in \mathcal{Q}$ with $N(Q_-) = \mathcal{S}_-$. Then, by the proof of Theorem 3.6,

$$\underset{Q_+\in\mathcal{Q},\ N(Q_+)=\mathcal{S}_+}{min} Q_-^* Q_+^* B Q_+ Q_- = Q_-^*(B_1 + B_{2/\mathcal{S}_+} - B_3)Q_-.$$

Observe that

$$B_1 + B_{2/\mathcal{S}_+} - B_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -a_- & b_- \\ 0 & b_-^* & c - f^*f \end{bmatrix} \begin{matrix} \mathcal{S}_+ \\ \mathcal{S}_- \\ \mathcal{S}^\perp \end{matrix}.$$

Since there exists $\underset{Q_-\in\mathcal{Q},\ N(Q_-)=\mathcal{S}_-}{max} Q_-^*(B_1 + B_{2/\mathcal{S}_+} - B_3)Q_-$, by Proposition 3.10, $B_1 + B_{2/\mathcal{S}_+} - B_3$ is $\mathcal{S}_-$-complementable and then $R(b_-) \subseteq R(a_-)$. In a similar fashion, $B_1 + B_2 - B_{3/\mathcal{S}_-}$ is $\mathcal{S}_+$-complementable and $R(b_+) \subseteq R(a_+)$. Therefore, $R(b) \subseteq R(b_-) + R(b_+) \subseteq R(a_-) \oplus R(a_+) = R(a)$. Hence B is $\mathcal{S}$-complementable.

□

A formula for the Schur complement

When the operator B is $\mathcal{S}$-complementable, the Schur complement can be written as $B_{/\mathcal{S}} = (I - F)B$, for any bounded projection with $N(F) = \mathcal{S}^\perp$ such that $(FB)^* = FB$. In fact, from Corollary 3.12, it suffices to take $F = Q^*$, for any $Q \in \mathcal{P}(B, \mathcal{S})$.

In this section, we show that a similar formula for $B_{/\mathcal{S}}$ can be given when B is $\mathcal{S}$-weakly complementable. In this case the projection need not be bounded, but it is densely defined.

Theorem 3.14. *Let $B \in L(\mathcal{H})$ selfadjoint and $\mathcal{S}$ a closed subspace of $\mathcal{H}$. Then B is $\mathcal{S}$-weakly complementable if and only if there exists a densely defined projection E with $N(E) = \mathcal{S}^\perp$ such that $E|P_\mathcal{S} B P_\mathcal{S}|^{1/2}, EB \in L(\mathcal{H})$ and EB is selfadjoint. In this case,*

$$B_{/\mathcal{S}} = (I - E)B.$$

Remark. The densely defined projection E is closed if and only if the pair $(B, \mathcal{S})$ is quasi-compatible; i.e., $\mathcal{H} = \overline{\mathcal{S} + B^{-1}(\mathcal{S}^\perp)}$. Moreover, $E \in L(\mathcal{H})$ if and only if B is $\mathcal{S}$-complementable.

Proof. Suppose that B is $\mathcal{S}$-weakly complementable. If the matrix decomposition of B induced by $\mathcal{S}$ is as in (3.1), let f be the reduced solution of $b = |a|^{1/2}x$ and $a = u|a|$ the polar decomposition of a. Write $(|a|^{1/2})^\dagger$ for the Moore-Penrose inverse of $|a|^{1/2}$ and set

$$E = \begin{bmatrix} I & 0 \\ f^*u(|a|^{1/2})^\dagger & 0 \end{bmatrix}.$$

Then $Dom(E) = Dom(|a|^{1/2})^\dagger) \oplus \mathcal{S}^\perp$ and E is a densely defined projection with $N(E) = \mathcal{S}^\perp$. On the other hand, since $R(B) \subseteq R(|a|^{1/2}) \oplus \mathcal{S}^\perp$, the product $(I-E)B$ is well defined. Moreover

$$\begin{aligned}(I-E)B &= \begin{bmatrix} 0 & 0 \\ -f^*u(|a|^{1/2})^\dagger & I \end{bmatrix} \begin{bmatrix} |a|^{1/2}u|a|^{1/2} & |a|^{1/2}f \\ f^*|a|^{1/2} & c \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & c - f^*uf \end{bmatrix} = B_{/\mathcal{S}}\end{aligned}$$

is bounded and selfadjoint. Finally,

$$E|P_\mathcal{S}BP_\mathcal{S}|^{1/2} = \begin{bmatrix} |a|^{1/2} & 0 \\ f^*u & 0 \end{bmatrix} \in L(\mathcal{H}).$$

Conversely, suppose that there exists a densely defined projection E with $N(E) = \mathcal{S}^\perp$ such that $E|P_\mathcal{S}BP_\mathcal{S}|^{1/2}, EB \in L(\mathcal{H})$ and EB is selfadjoint. Then the matrix decomposition of $I-E$ is

$$I - E = \begin{bmatrix} 0 & 0 \\ y & I \end{bmatrix},$$

with $y : Dom(y) \subseteq \mathcal{S} \to \mathcal{S}^\perp$ and $\overline{Dom(y)} = \mathcal{S}$. If the matrix decomposition of B is as in (3.1), since $(I-E)B$ is selfadjoint, it follows that, $ya = -b^*$ and yb is bounded and selfadjoint. From the fact that $E|P_\mathcal{S}BP_\mathcal{S}|^{1/2} \in L(\mathcal{H})$, we have that $y|a|^{1/2} \in L(\mathcal{S}, \mathcal{S}^\perp)$ and, since $ya = -b^*$, we also have that $y|a|^{1/2}u|a|^{1/2} = -b^*$. Then $b = |a|^{1/2}(-y|a|^{1/2}u)^*$, $R(b) \subseteq R(|a|^{1/2})$ and B is $\mathcal{S}$-weakly complementable. □

4. Schur complement in Krein spaces

In this section we adapt the definitions of complementability and weak complementability given in Section 3 to a bounded selfadjoint operator W acting on a Krein space $(\mathcal{H}, [\ ,\ ])$. From now on all spaces are assumed to be Krein spaces unless otherwise stated.

Definition. Let $W \in L(\mathcal{H})^s$ and $\mathcal{S}$ a closed subspace of $\mathcal{H}$. The operator W is called $\mathcal{S}$-*complementable* if

$$\mathcal{H} = \mathcal{S} + W^{-1}(\mathcal{S}^{[\perp]}).$$

If W is $\mathcal{S}$-complementable then, for any fundamental decomposition $\mathcal{H} = \mathcal{H}_+ \ [\dot{+}]\ \mathcal{H}_-$ with signature operator J, we get that $\mathcal{H} = \mathcal{S} + (JW)^{-1}(\mathcal{S}^\perp)$. Therefore, W is $\mathcal{S}$-complementable if and only if the selfadjoint operator JW is $\mathcal{S}$-complementable in (the Hilbert space) $(\mathcal{H}, \langle\ ,\ \rangle)$ for any (and then for every) signature operator J. From this, it follows that W is $\mathcal{S}$-complementable if and only if there exists a projection Q onto $\mathcal{S}$ such that $WQ = Q^\# W$.

In this case, if the matrix representation of JW induced by $\mathcal{S}$ is

$$JW = \begin{bmatrix} a & b \\ b^* & c \end{bmatrix}, \tag{4.1}$$

the $\mathcal{S}$-complementability of W is equivalent to $R(b) \subseteq R(a)$ (see [11, Proposition 3.3]).

In a similar fashion we define the $\mathcal{S}$-weak complementability in Krein spaces, with respect to a fixed signature operator J.

Definition. Let $W \in L(\mathcal{H})^s$ and $\mathcal{S}$ a closed subspace of $\mathcal{H}$. The operator W is $\mathcal{S}$-*weakly complementable* with respect to a signature operator J if JW is $\mathcal{S}$-weakly complementable in $(\mathcal{H}, \langle\ ,\ \rangle)$.

Next, we show that the $\mathcal{S}$-weak complementability of W does not depend on the signature operator. In order to do so, we need to establish some technical lemmas. Some additional notation is also required: consider J and J_α two signature operators and set $\alpha = J_\alpha J$. Denote by $\mathcal{H} = (\mathcal{H}, \langle\ ,\ \rangle)$ the Hilbert space associated to J and by $\mathcal{H}_\alpha = (\mathcal{H}, \langle\ ,\ \rangle_{J_\alpha})$, where $\langle x, y \rangle_{J_\alpha} = [\, J_\alpha x, y\,] = \left\langle \alpha^{-1}x, y \right\rangle$, the Hilbert space associated to J_α. Then $\alpha \geq_{\mathcal{H}} 0$ and $\alpha \geq_{\mathcal{H}_\alpha} 0$. Notice that $\mathcal{S}^{\perp_\alpha} = \alpha(\mathcal{S}^\perp)$ is the orthogonal complement of $\mathcal{S}$ in $\mathcal{H}_\alpha$ and, for $T \in L(\mathcal{H})$, $T^{*_\alpha} = \alpha T^* \alpha^{-1}$ is the adjoint of T in $\mathcal{H}_\alpha$. Also, $T^{*_\alpha} = T$ if and only if $\alpha T^* = T\alpha$.

Denote by $|T|_\alpha$ the modulus of T in $\mathcal{H}_\alpha$ and by $P^\alpha_{\mathcal{S}} = P_{\mathcal{S}//\mathcal{S}^{\perp_\alpha}}$ the orthogonal projection onto $\mathcal{S}$ in $\mathcal{H}_\alpha$. If $T \geq_{\mathcal{H}_\alpha} 0$, we indicate by $T^{1/2_\alpha}$ the square root of T in $\mathcal{H}_\alpha$. Frequently, we will use that if $T \geq_{\mathcal{H}} 0$ or $T = T^*$ then $\alpha T \geq_{\mathcal{H}_\alpha} 0$ or $\alpha T = (\alpha T)^{*_\alpha}$, respectively.

Lemma 4.1. *Let $\mathcal{S}$ be a closed subspace of $\mathcal{H}$, J and J_α two signature operators and $\alpha = J_\alpha J$. Then*

$$\tilde{\alpha} := P^\alpha_{\mathcal{S}} \alpha|_{\mathcal{S}} = (P_{\mathcal{S}}\alpha^{-1}|_{\mathcal{S}})^{-1} \in GL(\mathcal{S})^+.$$

Proof. The projection $P^\alpha_{\mathcal{S}}$ can be expressed in terms of $P_{\mathcal{S}}$ and α as

$$P^\alpha_{\mathcal{S}} = P_{\mathcal{S}}(P_{\mathcal{S}}\alpha^{-1}P_{\mathcal{S}} + (I - P_{\mathcal{S}})\alpha^{-1}(I - P_{\mathcal{S}}))^{-1}\alpha^{-1};$$

see [11, Section 4]. Therefore,

$$P^\alpha_{\mathcal{S}}\alpha|_{\mathcal{S}} = P_{\mathcal{S}}(P_{\mathcal{S}}\alpha^{-1}P_{\mathcal{S}} + (I - P_{\mathcal{S}})\alpha^{-1}(I - P_{\mathcal{S}}))^{-1}|_{\mathcal{S}} =$$

$$= (P_{\mathcal{S}}\alpha^{-1}|_{\mathcal{S}})^{-1} \in GL(\mathcal{S})^+.$$

□

Lemma 4.2. *Let $\mathcal{S}$ be a closed subspace of $\mathcal{H}$, J and J_α two signature operators, $\alpha = J_\alpha J$ and $a \in L(\mathcal{S})$. Let $a' = P^\alpha_{\mathcal{S}}\alpha a$. Then*

$$R(|a'|_\alpha^{1/2_\alpha}) = R((P^\alpha_{\mathcal{S}}\alpha|a|P^\alpha_{\mathcal{S}})^{1/2_\alpha}) = \tilde{\alpha}R(|a|^{1/2}). \tag{4.2}$$

Proof. First observe that

$$(P^\alpha_{\mathcal{S}}\alpha|a|P^\alpha_{\mathcal{S}})^{1/2_\alpha} = (\tilde{\alpha}|a|P^\alpha_{\mathcal{S}})^{1/2_\alpha} = P^\alpha_{\mathcal{S}}\alpha X_0 P^\alpha_{\mathcal{S}},$$

with $X_0 = \tilde{\alpha}^{-1/2}(\tilde{\alpha}^{1/2}|a|\tilde{\alpha}^{1/2})^{1/2}\tilde{\alpha}^{-1/2}$. In fact, $X_0 \in L(\mathcal{S})^+$, because $\tilde{\alpha}^{1/2} \in L(\mathcal{S})^+$ by Lemma 4.1. Clearly, $X_0\tilde{\alpha}X_0 = |a|$. Therefore, $(\tilde{\alpha}X_0)^2 = \tilde{\alpha}|a|$. Also, $\tilde{\alpha}X_0P^\alpha_{\mathcal{S}} = P^\alpha_{\mathcal{S}}\alpha X_0 P^\alpha_{\mathcal{S}} \geq_{\mathcal{H}_\alpha} 0$.

Then $(\tilde{\alpha}X_0P^\alpha_{\mathcal{S}})^2 = \tilde{\alpha}X_0P^\alpha_{\mathcal{S}}\tilde{\alpha}X_0P^\alpha_{\mathcal{S}} = \tilde{\alpha}X_0\tilde{\alpha}X_0P^\alpha_{\mathcal{S}} = \tilde{\alpha}|a|P^\alpha_{\mathcal{S}}$. Thus,

$$(\tilde{\alpha}|a|P^\alpha_{\mathcal{S}})^{1/2_\alpha} = \tilde{\alpha}X_0P^\alpha_{\mathcal{S}} = P^\alpha_{\mathcal{S}}\alpha X_0 P_{\mathcal{S}}.$$

Now, since $X_0\tilde{\alpha}X_0 = |a|$, Douglas' Lemma yields $R(X_0\tilde{\alpha}^{1/2}) = R(|a|^{1/2})$. Therefore, because $\tilde{\alpha}^{1/2} \in GL(\mathcal{S})^+$, $R(X_0) = R(|a|^{1/2})$ (see Lemma 4.1). Then

$$R((P^\alpha_{\mathcal{S}}\alpha|a|P^\alpha_{\mathcal{S}})^{1/2_\alpha}) = R(P^\alpha_{\mathcal{S}}\alpha X_0 P_{\mathcal{S}}) = R(\tilde{\alpha}X_0P^\alpha_{\mathcal{S}}) = \tilde{\alpha}R(X_0) = \tilde{\alpha}R(|a|^{1/2})$$

and the second equality in (4.2) follows. To prove the first equality, note that $R(|a'|_\alpha) = R(a') = R(\tilde{\alpha}a) = R(\tilde{\alpha}|a|)$. Then, applying Douglas' Lemma and the operator monotonicity of the square root in $\mathcal{H}_\alpha$ (see [21]), we get that

$$R(|a'|_\alpha^{1/2_\alpha}) = R((\tilde{\alpha}|a|)^{1/2_\alpha}) = R((P^\alpha_{\mathcal{S}}\alpha|a|P^\alpha_{\mathcal{S}})^{1/2_\alpha}).$$

□

Lemma 4.3. *Let $\mathcal{S}$ be a closed subspace of $\mathcal{H}$, J and J_α two signature operators, $\alpha = J_\alpha J$ and $a \in L(\mathcal{S})$. Let $a' = P_{\mathcal{S}}^{\alpha}\alpha a$, $\Gamma := \begin{bmatrix} |a|^{1/2} & 0 \\ 0 & I \end{bmatrix} \begin{matrix} \mathcal{S} \\ \mathcal{S}^\perp \end{matrix}$, $\Gamma_\alpha := \begin{bmatrix} |a'|_\alpha^{1/2_\alpha} & 0 \\ 0 & I \end{bmatrix} \begin{matrix} \mathcal{S} \\ \mathcal{S}^{\perp_\alpha} \end{matrix}$ and E a densely defined projection with $N(E) = \mathcal{S}^\perp$ such that $E\Gamma \in L(\mathcal{H})$. Then $\alpha E \alpha^{-1}$ is a densely defined projection with $N(\alpha E\alpha^{-1}) = \mathcal{S}^{\perp_\alpha}$ such that $\alpha E \alpha^{-1}\Gamma_\alpha \in L(\mathcal{H})$.*

Proof. Clearly, $\alpha E\alpha^{-1}$ is a densely defined projection with $N(\alpha E\alpha^{-1}) = \mathcal{S}^{\perp_\alpha}$. Let us see that $\alpha E\alpha^{-1}\Gamma_\alpha \in L(\mathcal{H})$. Since E is a densely defined projection with $N(E) = \mathcal{S}^\perp$ such that $E\Gamma \in L(\mathcal{H})$, the matrix decomposition of E is

$$E = \begin{bmatrix} I & 0 \\ y & 0 \end{bmatrix} \begin{matrix} \mathcal{S} \\ \mathcal{S}^\perp \end{matrix} = (I + y)P_{\mathcal{S}},$$

with $y : Dom(y) \subseteq \mathcal{S} \to \mathcal{S}^\perp$, $\overline{Dom(y)} = \mathcal{S}$ and $y|a|^{1/2} \in L(\mathcal{S}, \mathcal{S}^\perp)$. Also, $\Gamma_\alpha = |a'|_\alpha^{1/2_\alpha} P_{\mathcal{S}}^{\alpha} + (I - P_{\mathcal{S}}^{\alpha})$. Since, by Lemma 4.2, $R(|a'|_\alpha^{1/2_\alpha}) = R(\tilde{\alpha}|a|^{1/2})$, $|a'|_\alpha^{1/2_\alpha} \geq_{\mathcal{H}_\alpha} 0$ and $\tilde{\alpha}|a|^{1/2} \geq_{\mathcal{H}_\alpha} 0$, there exists $g \in GL(\mathcal{S})$ such that

$$|a'|_\alpha^{1/2_\alpha} = \tilde{\alpha}|a|^{1/2} g.$$

Then $P_{\mathcal{S}}\alpha^{-1}|a'|_\alpha^{1/2_\alpha} = P_{\mathcal{S}}\alpha^{-1}\tilde{\alpha}|a|^{1/2}g = P_{\mathcal{S}}\alpha^{-1}P_{\mathcal{S}}\tilde{\alpha}|a|^{1/2}g = \tilde{\alpha}^{-1}\tilde{\alpha}|a|^{1/2}g = |a|^{1/2}g$.

Therefore, $R(P_{\mathcal{S}}\alpha^{-1}|a'|_\alpha^{1/2_\alpha}) = R(|a|^{1/2}) \subseteq Dom(y)$ and

$$yP_{\mathcal{S}}\alpha^{-1}|a'|_\alpha^{1/2_\alpha} = y|a|^{1/2}g \in L(\mathcal{S}, \mathcal{S}^\perp).$$

Thus

$$\alpha E\alpha^{-1}\Gamma_\alpha = \alpha P_{\mathcal{S}}\alpha^{-1}|a'|_\alpha^{1/2_\alpha} P_{\mathcal{S}}^{\alpha} + \alpha y P_{\mathcal{S}}\alpha^{-1}|a'|_\alpha^{1/2_\alpha} P_{\mathcal{S}}^{\alpha} \in L(\mathcal{H}).$$

□

Now we are ready to show that the weak $\mathcal{S}$-complementability of W does not depend on the fundamental decomposition of $\mathcal{H}$.

Theorem 4.4. *Let $W \in L(\mathcal{H})^s$ and $\mathcal{S}$ be a closed subspace of $\mathcal{H}$. Suppose that W is $\mathcal{S}$-weakly complementable for some signature operator J. Then W is $\mathcal{S}$-weakly complementable for any other signature operator J_α.*

Proof. Suppose that W is $\mathcal{S}$-weakly complementable for some signature operator J and the matrix decomposition of JW is as in (4.1). By Theorem 3.14, there exists a densely defined projection E with $N(E) = \mathcal{S}^\perp$ such that $EJW \in L(\mathcal{H})$ is selfadjoint. Also, if $\Gamma := \begin{bmatrix} |a|^{1/2} & 0 \\ 0 & I \end{bmatrix} \begin{matrix} \mathcal{S} \\ \mathcal{S}^\perp \end{matrix}$, then $E\Gamma \in L(\mathcal{H})$. Let J_α be another signature operator and $\alpha = J_\alpha J$. If $J_\alpha W = \alpha JW = \begin{bmatrix} a' & b' \\ b'^* & c' \end{bmatrix} \begin{matrix} \mathcal{S} \\ \mathcal{S}^{\perp_\alpha} \end{matrix}$ then

$$a' = P_{\mathcal{S}}^{\alpha}\alpha JW P_{\mathcal{S}}^{\alpha} = P_{\mathcal{S}}^{\alpha}\alpha a P_{\mathcal{S}}^{\alpha}.$$

Consider $\alpha E\alpha^{-1}$ and $\Gamma_\alpha := \begin{bmatrix} |a'|_\alpha^{1/2_\alpha} & 0 \\ 0 & I \end{bmatrix} \begin{matrix} \mathcal{S} \\ \mathcal{S}^{\perp_\alpha} \end{matrix}$. By Lemma 4.3, $\alpha E\alpha^{-1}$ is a densely defined projection with $N(\alpha E\alpha^{-1}) = \mathcal{S}^{\perp_\alpha}$ such that $\alpha E\alpha^{-1}\Gamma_\alpha \in L(\mathcal{H})$. Also, $(\alpha E\alpha^{-1})(J_\alpha W) = \alpha EJW \in L(\mathcal{H})$ and is selfadjoint. Therefore, again by Theorem 3.14, $J_\alpha W$ is $\mathcal{S}$-weakly complementable. □

From now on, since the $\mathcal{S}$-weak complementability does not depend on the fundamental decomposition of $\mathcal{H}$, we simply say that W is $\mathcal{S}$-weakly complementable, whenever W is $\mathcal{S}$-weakly complementable with respect to a signature operator J. In particular, if $W \geq 0$ then W is $\mathcal{S}$-weakly complementable.

Following the ideas of [4], we extend the notion of Schur complement to selfadjoint operators in Krein spaces:

Definition. Let $W \in L(\mathcal{H})^s$, $\mathcal{S}$ a closed subspace of $\mathcal{H}$ and J a signature operator. Suppose that W is $\mathcal{S}$-weakly complementable. The *Schur complement* of W to $\mathcal{S}$ corresponding to J is

$$W_{/[\mathcal{S}]}^{J} = J(JW)_{/\mathcal{S}},$$

and the $\mathcal{S}$-*compression* of W is $W_{[\mathcal{S}]}^{J} = W - W_{/[\mathcal{S}]}^{J} = J(JW)_{\mathcal{S}}$.

Theorem 4.5. *Let $W \in L(\mathcal{H})^s$ and $\mathcal{S}$ be a closed subspace of $\mathcal{H}$. Suppose that W is $\mathcal{S}$-weakly complementable and J is a signature operator, then*

$$W^{J}_{/[\mathcal{S}]} = W^{J_\alpha}_{/[\mathcal{S}]},$$

for any other signature operator J_α; i.e., the Schur complement does not depend on the fundamental decomposition of $\mathcal{H}$.

Henceforth we write $W_{/[\mathcal{S}]}$ for this operator.

Proof. Suppose that W is $\mathcal{S}$-weakly complementable. Then, by Theorem 4.4, JW is $\mathcal{S}$-weakly complementable for any signature operator J. By Theorem 3.14, there exists a densely defined projection E with $N(E) = \mathcal{S}^\perp$ such that EJW is selfadjoint, $E\Gamma \in L(\mathcal{H})$ and $W^J_{/[\mathcal{S}]} = J(I-E)JW$. Let J_α be another signature operator, $\alpha = J_\alpha J$ and consider $\alpha E \alpha^{-1}$. Then, by Theorem 4.4, $\alpha E\alpha^{-1}$ is a densely defined projection with $N(\alpha E \alpha^{-1}) = \mathcal{S}^{\perp_\alpha}$ such that $\alpha E \alpha^{-1}\Gamma_\alpha \in L(\mathcal{H})$. Therefore, by Theorem 3.14,

$$W^{J_\alpha}_{/[\mathcal{S}]} = J_\alpha \alpha (I-E)\alpha^{-1} J_\alpha W = J(I-E)JW = W^J_{/[\mathcal{S}]}.$$

□

Corollary 4.6. *Let $W \in L(\mathcal{H})^s$ and $\mathcal{S}$ be a closed subspace of $\mathcal{H}$. Suppose that W is $\mathcal{S}$-weakly complementable. Then there exists a densely defined projection E with $N(E) = \mathcal{S}^{[\perp]}$ such that*

$$W_{/[\mathcal{S}]} = (I-E)W.$$

Let $W \in L(\mathcal{H})^s$ and $\mathcal{S}$ a closed subspace of $\mathcal{H}$, define

$$\mathcal{N}^-(W, \mathcal{S}^{[\perp]}) := \{X \in L(\mathcal{H})^s : X \leq W,\ R(X) \subseteq \mathcal{S}^{[\perp]}\},$$

$$\mathcal{N}^+(W, \mathcal{S}^{[\perp]}) := \{X \in L(\mathcal{H})^s : W \leq X,\ R(X) \subseteq \mathcal{S}^{[\perp]}\}.$$

If J is any signature operator,

$$\mathcal{N}^\pm(W, \mathcal{S}^{[\perp]}) = J\mathcal{M}^\pm(JW, \mathcal{S}^\perp).$$

Let $W \in L(\mathcal{H})^s$ and $\mathcal{S}$ a closed subspace of $\mathcal{H}$. Then, applying Lemma 2.2 to $B = JW$, with J any signature operator, $\mathcal{S}$ can be decomposed as

$$\mathcal{S} = \mathcal{S}_+ \ [\dot{+}]_W \ \mathcal{S}_-, \tag{4.3}$$

where $\mathcal{S}_+$ and $\mathcal{S}_-$ are closed, $\mathcal{S}_+$ is W-nonnegative, $\mathcal{S}_-$ is W-nonpositive and $\mathcal{S}_+ \perp \mathcal{S}_-$.

Proposition 4.7. *Let $W \in L(\mathcal{H})^s$ selfadjoint and $\mathcal{S}$ a closed subspace of $\mathcal{H}$.*

Suppose that $\mathcal{S} = \mathcal{S}_+ \ [\dot{+}]_W \ \mathcal{S}_-$ is any decomposition as in (4.3) *for some signature operator J. Then the following statements are equivalent:*

i) W is $\mathcal{S}$-weakly complementable;

ii) there exist $W_1, W_2, W_3 \in L(\mathcal{H})^s$, $W_2, W_3 \geq 0$ such that $W = W_1 + W_2 - W_3$ and $\mathcal{S} \subseteq N(W_1)$, $\mathcal{S}_- \subseteq N(W_2)$, $\mathcal{S}_+ \subseteq N(W_3)$;

iii) the sets $\mathcal{N}^-(W, \mathcal{S}_+^{[\perp]})$ and $\mathcal{N}^+(W, \mathcal{S}_-^{[\perp]})$ are non-empty;

iv) W is $\mathcal{S}_\pm$-weakly complementable.

Proof. This follows from Proposition 3.2. □

The following theorem proves that the set $\mathcal{N}^-(W, \mathcal{S}^{[\perp]})$ has a maximum element if and only if $\mathcal{S}$ is W-nonnegative and W is $\mathcal{S}$-weakly complementable. A similar result can be proven if $\mathcal{S}$ is W-nonpositive and W is $\mathcal{S}$-weakly complementable.

Proposition 4.8. *Let $W \in L(\mathcal{H})^s$ and $\mathcal{S}$ a closed subspace of $\mathcal{H}$. Then $\mathcal{S}$ is W-nonnegative and W is $\mathcal{S}$-weakly complementable if and only if the set $\mathcal{N}^-(W, \mathcal{S}^{[\perp]})$ has a maximum element.*

In this case,

$$W_{/[\mathcal{S}]} = max\ \mathcal{N}^-(W, \mathcal{S}^{[\perp]}).$$

Proof. This follows from Proposition 3.3. □

Theorem 4.9. *Let $W \in L(\mathcal{H})^s$ and $\mathcal{S}$ a closed subspace of $\mathcal{H}$. Suppose that $\mathcal{S} = \mathcal{S}_+ \ [\dot{+}]_W \ \mathcal{S}_-$ is any decomposition as in (4.3) for some signature operator J. Then W is $\mathcal{S}$-weakly complementable if and only if there exists $min\ \mathcal{N}^+(max\ \mathcal{N}^-(W, \mathcal{S}_+^{[\perp]}), \mathcal{S}_-^{[\perp]})$ and $max\ \mathcal{N}^-(min\ \mathcal{N}^+(B, \mathcal{S}_-^{[\perp]}), \mathcal{S}_+^{[\perp]})$.*

In this case,

$$\begin{aligned} W_{/[\mathcal{S}]} &= min\ \mathcal{N}^+(max\ \mathcal{N}^-(W, \mathcal{S}_+^{[\perp]}), \mathcal{S}_-^{[\perp]}) \\ &= max\ \mathcal{N}^-(min\ \mathcal{N}^+(B, \mathcal{S}_-^{[\perp]}), \mathcal{S}_+^{[\perp]}). \end{aligned}$$

Proof. This follows from Theorem 3.4. □

Corollary 4.10. *Let $W \in L(\mathcal{H})^s$ and $\mathcal{S}$ a closed subspace of $\mathcal{H}$. Suppose that $\mathcal{S} = \mathcal{S}_+ \ [\dot{+}]_W \ \mathcal{S}_-$ is any decomposition as in (4.3) for some signature operator J. If W is $\mathcal{S}$-weakly complementable, then*

$$W_{/[\mathcal{S}]} = (W_{/[\mathcal{S}_+]})_{/[\mathcal{S}_-]} = (W_{/[\mathcal{S}_-]})_{/[\mathcal{S}_+]}.$$

Theorem 4.11. *Let $W \in L(\mathcal{H})^s$ and $\mathcal{S}$ a closed subspace of $\mathcal{H}$. Suppose that $\mathcal{S} = \mathcal{S}_+ \ [\dot{+}]_W \ \mathcal{S}_-$ is any decomposition as in (4.3) for some signature operator J. W is $\mathcal{S}$-weakly complementable if and only if there exist*

$$\underset{E_- \in \mathcal{Q},\ N(E_-) = \mathcal{S}_-}{sup} \left(\underset{E_+ \in \mathcal{Q},\ N(E_+) = \mathcal{S}_+}{inf} E_-^{\#} E_+^{\#} W E_+ E_- \right)$$

and

$$\underset{E_+ \in \mathcal{Q},\ N(E_+) = \mathcal{S}_+}{inf} \left(\underset{E_- \in \mathcal{Q},\ N(E_-) = \mathcal{S}_-}{sup} E_+^{\#} E_-^{\#} W E_- E_+ \right).$$

In this case,

$$\begin{aligned} W_{/[\mathcal{S}]} &= \underset{E_- \in \mathcal{Q},\ N(E_-) = \mathcal{S}_-}{sup} \left(\underset{E_+ \in \mathcal{Q},\ N(E_+) = \mathcal{S}_+}{inf} E_-^{\#} E_+^{\#} W E_+ E_- \right) \\ &= \underset{E_+ \in \mathcal{Q},\ N(E_+) = \mathcal{S}_+}{inf} \left(\underset{E_- \in \mathcal{Q},\ N(E_-) = \mathcal{S}_-}{sup} E_+^{\#} E_-^{\#} W E_- E_+ \right). \end{aligned}$$

Proof. For any signature operator J, if $(\mathcal{H}, \langle\ ,\ \rangle)$ is the associated Hilbert space,

$$J\{Q^* J W Q : Q \in \mathcal{Q},\ N(Q) = \mathcal{S}\} = \{E^{\#} W E : E \in \mathcal{Q},\ N(E) = \mathcal{S}\}. \tag{4.4}$$

Also, there exists $inf_{\leq}\ \{E^{\#} W E : E = E^2,\ N(E) = \mathcal{S}\}$ if and only if there exists $inf_{\leq_{\mathcal{H}}} \{Q^* J W Q : Q \in \mathcal{Q},\ N(Q) = \mathcal{S}\}$. Moreover

$$inf_{\leq}\ \{E^{\#} W E : E = E^2,\ N(E) = \mathcal{S}\} = J inf_{\leq_{\mathcal{H}}} \{Q^* J W Q : Q \in \mathcal{Q},\ N(Q) = \mathcal{S}\}. \tag{4.5}$$

Analogously, there exists $sup_{\leq}\ \{E^{\#} W E : E = E^2,\ N(E) = \mathcal{S}\}$ if and only if there exists $sup_{\leq_{\mathcal{H}}} \{Q^* J W Q : Q \in \mathcal{Q},\ N(Q) = \mathcal{S}\}$. Moreover

$$sup_{\leq}\ \{E^{\#} W E : E = E^2,\ N(E) = \mathcal{S}\} = J sup_{\leq_{\mathcal{H}}} \{Q^* J W Q : Q \in \mathcal{Q},\ N(Q) = \mathcal{S}\}. \tag{4.6}$$

The result follows from (4.4), (4.5), (4.6) and Theorem 3.6. □

Corollary 4.12. *Let $W \in L(\mathcal{H})^s$ and $\mathcal{S}$ a closed subspace of $\mathcal{H}$. Suppose that $\mathcal{S}$ is W-nonnegative. Then W is $\mathcal{S}$-weakly complementable if and only if there exists $inf\ \{E^{\#}WE : E = E^2,\ N(E) = \mathcal{S}\}$.*

In this case,

$$W_{/[\mathcal{S}]} = inf\ \{E^{\#}WE : E = E^2,\ N(E) = \mathcal{S}\}.$$

A similar result holds when $\mathcal{S}$ is W-nonpositive, replacing inf by sup.

Proposition 4.13. *Let $W \in L(\mathcal{H})^s$ and $\mathcal{S}$ a closed subspace of $\mathcal{H}$. Suppose that W is $\mathcal{S}$-weakly complementable. Then,*

$$W_{/[\mathcal{S}]} = min\ \{E^{\#}WE : E \in \mathcal{Q},\ N(E) = \mathcal{S}\}$$

if and only if W is $\mathcal{S}$-complementable and $\mathcal{S}$ is W-nonnegative.

In this case,

$$W_{/[\mathcal{S}]} = W(I - Q),$$

with Q any projection onto $\mathcal{S}$ such that $WQ = Q^{\#}W$.

A similar result holds when $\mathcal{S}$ is W-nonpositive, replacing min by max.

Proof. For any signature operator J, if $(\mathcal{H}, \langle\ ,\ \rangle)$ is the associated Hilbert space, by (4.4) and Proposition 3.10,

$$\begin{aligned} W_{/[\mathcal{S}]} &= J(JW)_{/\mathcal{S}} = J\ min_{\leq_{\mathcal{H}}}\{Q^*JWQ : Q \in \mathcal{Q},\ N(Q) = \mathcal{S}\} \\ &= min\ \{E^{\#}WE : E \in \mathcal{Q},\ N(E) = \mathcal{S}\} \end{aligned}$$

if and only if $\mathcal{H} = \mathcal{S} + (JW)^{-1}(\mathcal{S}^{\perp})$ and $\mathcal{S}$ is JW-nonnegative (in the Hilbert space $\mathcal{H}$) if and only if W is $\mathcal{S}$-complementable and $\mathcal{S}$ is W-nonnegative.

The operator Q is any projection onto $\mathcal{S}$ such that $WQ = Q^{\#}W$ if and only if $Q \in \mathcal{P}(JW, \mathcal{S})$ for any signature operator J. Therefore, in these cases, by Proposition 3.10, $W_{/[\mathcal{S}]} = J(JW)_{/\mathcal{S}} = J(JW)(I-Q) = W(I - Q)$, for any of these projections. □

Corollary 4.14. *Let $W \in L(\mathcal{H})^s$ and $\mathcal{S}$ a closed subspace of $\mathcal{H}$ such that W is $\mathcal{S}$-complementable. Then*

$$W_{/[\mathcal{S}]} = W(I - Q),$$

for Q any projection onto $\mathcal{S}$ such that $WQ = Q^{\#}W$.

Proof. This follows proceeding as in Corollary 3.12 and by Proposition 4.13. □

Theorem 4.15. *Let $W \in L(\mathcal{H})^s$ and $\mathcal{S}$ a closed subspace of $\mathcal{H}$ such that W is $\mathcal{S}$-weakly complementable. Suppose that $\mathcal{S} = \mathcal{S}_+ \ [\dot{+}]_W\ \mathcal{S}_-$ is any decomposition as in (4.3) for some signature operator J. Then, W is $\mathcal{S}$-complementable if and only if*

$$\begin{aligned} W_{/[\mathcal{S}]} &= \min_{E_+ \in \mathcal{Q},\ N(E_+) = \mathcal{S}_+} \left(\max_{E_- \in \mathcal{Q},\ N(E_-) = \mathcal{S}_-} E_-^{\#}E_+^{\#}WE_+E_- \right) \\ &= \max_{E_- \in \mathcal{Q},\ N(E_-) = \mathcal{S}_-} \left(\min_{E_+ \in \mathcal{Q},\ N(E_+) = \mathcal{S}_+} E_+^{\#}E_-^{\#}WE_-E_+ \right). \end{aligned}$$

Proof. This follows by (4.4) and Corollary 3.13. □

Weak complementability for regular subspaces

Any $W \in L(\mathcal{H})^s$ can be written in the form

$$W = DD^{\#}$$

where $D \in L(\mathcal{K}, \mathcal{H})$ for some Krein space $\mathcal{K}$ and $N(D) = \{0\}$. This factorization, in general, is not unique. Such factorizations are known as *Bognár-Krámli factorizations*, see [9].

Let J be any signature operator of $\mathcal{H}$. Then, JW is selfadjoint in the corresponding Hilbert space. If $JW = U|JW| = |JW|U$ is the polar factorization of JW, then $\mathcal{K} := \overline{R(|JW|)}$ is a Krein space with signature operator $J_{\mathcal{K}} := U|_{\mathcal{K}}$. Define $D : \mathcal{K} \to \mathcal{H}$ by

$$Dk := J|JW|^{1/2}k, \ k \in \mathcal{K}. \tag{4.7}$$

Then, $N(D) = \{0\}$, $D^{\#} = J_{\mathcal{K}}|JW|^{1/2} = U|JW|^{1/2}$ and $DD^{\#} = W$ (cf. [14, Theorem 1.1]).

Definition. A Bognár-Krámli factorization of an operator $W \in L(\mathcal{H})^s$ which is constructed by the method described above is called a *polar factorization* of W (see [14, Lecture 6]).

Lemma 4.16. *Let $W \in L(\mathcal{H})^s$ have polar factorizations $W = DD^{\#} = EE^{\#}$ with $D \in L(\mathcal{K}, \mathcal{H})$ and $E \in L(\mathcal{K}', \mathcal{H})$. Then*

$$R(D) = R(E).$$

Proof. In this case, following similar arguments as in [14, Theorem 6.1] and [14, Theorem 6.2], it can be shown that there exists a unique $L \in L(\mathcal{K}', \mathcal{K})$ such that $E = DL$ and $D = EL^{\#}$. Clearly, $R(D) = R(E)$. □

Let $\mathcal{S}$ be a regular subspace of $\mathcal{H}$, then $W \in L(\mathcal{H})^s$ can be represented as a 2×2 block matrix in the form

$$W = \begin{pmatrix} w_{11} & w_{12} \\ w_{12}^{\#} & w_{22} \end{pmatrix} \begin{matrix} \mathcal{S} \\ \mathcal{S}^{[\perp]} \end{matrix} . \tag{4.8}$$

Theorem 4.17. *Let $W \in L(\mathcal{H})^s$ and $\mathcal{S}$ a regular subspace of $\mathcal{H}$. Suppose that W is represented as in (4.8) and $w_{11} = dd^{\#}$ is any polar factorization of w_{11}. Then W is $\mathcal{S}$-weakly complementable if and only if $R(w_{12}) \subseteq R(d)$.*

In this case,

$$W_{/[\mathcal{S}]} = \begin{pmatrix} 0 & 0 \\ 0 & w_{22} - y^{\#}y \end{pmatrix} \begin{matrix} \mathcal{S} \\ \mathcal{S}^{[\perp]} \end{matrix} ,$$

with $y \in L(\mathcal{S}^{[\perp]}, \mathcal{K})$ the only solution of the equation $w_{12} = dx$.

Proof. Take $J = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix} \begin{matrix} \mathcal{S} \\ \mathcal{S}^{[\perp]} \end{matrix}$ a signature operator for $\mathcal{H}$ as in (2.5).

Then, $JW = \begin{pmatrix} J_1 w_{11} & J_1 w_{12} \\ J_2 w_{12}^{\#} & J_2 w_{22} \end{pmatrix} \begin{matrix} \mathcal{S} \\ \mathcal{S}^{\perp} \end{matrix}$ and W is $\mathcal{S}$-weakly complementable if and only if $R(J_1 w_{12}) \subseteq R(|J_1 w_{11}|^{1/2})$ or, equivalently, $R(w_{12}) \subseteq R(J_1|J_1 w_{11}|^{1/2}) = R(d)$. Indeed, if $e := J_1|J_1 w_{11}|^{1/2}$ then, by (4.7), $w_{11} = ee^{\#}$ is a polar factorization of w_{11} and, by Lemma 4.16, $R(e) = R(d)$.

In this case, let $y \in L(\mathcal{S}^{[\perp]}, \mathcal{K})$ be the only solution of the equation $w_{12} = dx$. Observe that

$$(JW)_{/\mathcal{S}} = \begin{pmatrix} 0 & 0 \\ 0 & J_2 w_{22} - f^* u f \end{pmatrix} \begin{matrix} \mathcal{S} \\ \mathcal{S}^{\perp} \end{matrix} ,$$

where f is the reduced solution of $J_1 w_{12} = |J_1 w_{11}|^{1/2} x$ and u is the partial isometry corresponding to the polar decomposition of $J_1 w_{11}$. Clearly, $w_{12} = J_1 |J_1 w_{11}|^{1/2} f = ef$, so that $ef = dy$. As in the proof of Lemma 4.16, a unique bounded operator l can be found such that $e = dl$ and $d = el^{\#}$. Therefore, $y^{\#} d^{\#} = f^{\#} e^{\#} = f^{\#} l^{\#} d^{\#}$ and, since $d^{\#}$ has a dense range, $y^{\#} = f^{\#} l^{\#}$. In a similar way, $l^{\#} y = f$. Therefore, $y^{\#} y = f^{\#} l^{\#} y = f^{\#} f$. Finally, since $J_2 f^* u = f^{\#}$, it follows that

$$W_{/[\mathcal{S}]} = J(JW)_{/\mathcal{S}} = \begin{pmatrix} 0 & 0 \\ 0 & w_{22} - J_2 f^* u f \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & w_{22} - y^{\#} y \end{pmatrix} \begin{matrix} \mathcal{S} \\ \mathcal{S}^{[\perp]} \end{matrix} .$$

□

An application to a completion problem

Let $\mathcal{S}$ be a regular subspace of $\mathcal{H}$ and consider a bounded incomplete block operator

$$W^0 = \begin{pmatrix} w_{11} & w_{12} \\ w_{12}^{\#} & * \end{pmatrix} \begin{matrix} \mathcal{S} \\ \mathcal{S}^{[\perp]} \end{matrix}, \tag{4.9}$$

with $w_{11} \in L(\mathcal{S})^s$.

Following the ideas of Baidiuk in [6, Theorem 2.1], the next theorem solves a completion problem for any bounded incomplete operator W^0 of the form (4.9).

Proposition 4.18. *Let $\mathcal{S}$ be a regular subspace of $\mathcal{H}$ and W^0 be an incomplete block operator of the form (4.9). Assume that the number of negative squares $\nu_-[w_{11}]$ of the quadratic form $[\,w_{11}f, f\,]$, $f \in \mathcal{S}$, is finite. Let $w_{11} = dd^{\#}$ be any polar factorization of w_{11}. Then, there exists a completion W of W^0 with some operator $w_{22} \in L(\mathcal{S}^{[\perp]})^s$ such that $\nu_-[W] = \nu_-[w_{11}]$ if and only if $R(w_{12}) \subseteq R(d)$.*

In this case, if y is the unique bounded solution of the equation $w_{12} = dx$, the operator $y^{\#}y \in L(\mathcal{S}^{[\perp]})$ is the minimum in the solution set

$$\mathcal{W} = \{w_{22} \in L(\mathcal{S}^{[\perp]})^s : W = \begin{pmatrix} w_{11} & w_{12} \\ w_{12}^{\#} & w_{22} \end{pmatrix} : \nu_-[W] = \nu_-[w_{11}]\},$$

and this solution set admits the description

$$\mathcal{W} = \{w_{22} \in L(\mathcal{S}^{[\perp]})^s : w_{22} = y^{\#}y + z, \text{ where } z = z^{\#} \geq 0\}.$$

Proof. Take $J = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix} \begin{matrix} \mathcal{S} \\ \mathcal{S}^{[\perp]} \end{matrix}$ a signature operator for $\mathcal{H}$ as in (2.5).

Then, $JW = \begin{pmatrix} J_1 w_{11} & J_1 w_{12} \\ J_2 w_{12}^{\#} & * \end{pmatrix} \begin{matrix} \mathcal{S} \\ \mathcal{S}^{\perp} \end{matrix}$. Then, by [6, Theorem 2.1], there exists a completion W of W^0 if and only if $R(J_1 w_{12}) \subseteq R(|J_1 w_{11}|^{1/2})$ or equivalently, proceeding as in Theorem 4.17, $R(w_{12}) \subseteq R(d)$.

In this case, by [6, Theorem 2.1], any selfadjoint operator completion of JW^0 admits the representation

$$JW = \begin{pmatrix} J_1 w_{11} & J_1 w_{12} \\ J_2 w_{12}^{\#} & J_2 w_{22} \end{pmatrix} \begin{matrix} \mathcal{S} \\ \mathcal{S}^{\perp} \end{matrix},$$

with $J_2 w_{22} = f^* u f + z$ and f the reduced solution of $J_1 w_{12} = |J_1 w_{11}|^{1/2} x$, u the partial isometry corresponding to the polar decomposition of $J_1 w_{11}$, and $z \in L(\mathcal{S}^{\perp})^+$. Then, as in the proof of Theorem 4.17, if y is the unique bounded solution of the equation $w_{12} = dx$, the set of completions of W^0 has the form $W = \begin{pmatrix} w_{11} & w_{12} \\ w_{12}^{\#} & w_{22} \end{pmatrix}$ with $w_{22} = y^{\#}y + z$, where $z = z^{\#} \geq 0$. □

Remark. Any completion W of an incomplete block operator W^0 of the form (4.9) has the same $\mathcal{S}$-compression: $W_{[\mathcal{S}]}$. Moreover, for any completion W, $W_{[\mathcal{S}]} \leq W$.

Comparison with other notions of Schur complement in Krein spaces

In [19], Mary proved that any weakly regular operator $B \in L(\mathcal{K}, \mathcal{H})$ (i.e., any operator such that $\overline{R(B)}$ and $N(B)$ are regular subspaces) admits a (unique) closed Moore-Penrose inverse. That is, there exists an operator $B^{\dagger} : Dom(B^{\dagger}) = R(B) \,[\dot{+}]\, R(B)^{[\perp]} \subseteq \mathcal{H} \to \mathcal{K}$ such that $BB^{\dagger}$ is a symmetric projection from $Dom(B^{\dagger})$ onto $R(B)$ with nullspace $N(B)$ and $B^{\dagger}B$ is a symmetric projection from $\mathcal{K}$ onto $R(B^{\dagger}) = N(B)^{[\perp]}$ with nullspace $N(B)$ (see [19, Corollary 2.9 and Lemma 2.2]).

Let $W \in L(\mathcal{H})^s$ and $\mathcal{S}$ a regular subspace of $\mathcal{H}$. Suppose that W is represented as in (4.8) and $w_{11} = dd^{\#}$ is a polar factorization of w_{11}. If $\overline{R(w_{11})}$ is regular, then $d, d^{\#}$ and w_{11} are weakly regular, therefore, there exist $d^{\dagger}, (d^{\#})^{\dagger}$ and $w_{11}^{\dagger}$, which are weakly regular and $(d^{\#})^{\dagger} = (d^{\dagger})^{\#}$ (see [19, Theorem 2.8 and Theorem 2.15]).

Suppose that W is $\mathcal{S}$-weakly complementable. Then $R(w_{12}) \subseteq R(d)$ (see Theorem 4.17) and $d(d^\dagger w_{12}) = w_{12}$. Therefore, $d^\dagger w_{12} \in L(\mathcal{S}^{[\perp]}, \mathcal{K})$ is the unique solution of the equation $dx = w_{12}$. Thus, by Theorem 4.17,

$$W_{/[\mathcal{S}]} = \begin{pmatrix} 0 & 0 \\ 0 & w_{22} - (d^\dagger w_{12})^\# d^\dagger w_{12} \end{pmatrix} \begin{matrix} \mathcal{S} \\ \mathcal{S}^{[\perp]} \end{matrix}.$$

If in addition $R(w_{11})$ is closed, then $R(d)$ is regular. Thus, $(d^\dagger)^\# d^\dagger w_{12}$ is well defined, $w_{11}^\dagger = (d^\#)^\dagger d^\dagger \in L(\mathcal{S})$ and

$$W_{/[\mathcal{S}]} = \begin{pmatrix} 0 & 0 \\ 0 & w_{22} - w_{12}^\# w_{11}^\dagger w_{12} \end{pmatrix} = W_{/[\mathcal{S}]}^{XM},$$

where $W_{/[\mathcal{S}]}^{XM}$ is the Schur complement of W to $\mathcal{S}$ as defined by Mary. See [19, Theorem 2.20].

For a positive operator W in a Hilbert space $\mathcal{H}$ and a closed subspace $\mathcal{S} \subseteq \mathcal{H}$, Pekarev [22] showed that the Schur complement $W_{/\mathcal{S}}$ of W to $\mathcal{S}$ can be expressed as $W_{/\mathcal{S}} = W^{1/2}(I - P_{\mathcal{M}})W^{1/2}$ where $\mathcal{M} = \overline{W^{1/2}(\mathcal{S})}$. In [18], Pekarev's result was taken as an inspiration to extend the concept to the more general Krein space setting. In that paper a (bounded) selfadjoint operator W is said to have the unique factorization property (UFP) if for any two Bognár-Krámli factorizations of $W = D_1 D_1^\# = D_2 D_2^\#$, there is an isomorphism U such that $D_1 = D_2 U$.

For W selfadjoint with the UFP and a closed subspace $\mathcal{S} \subseteq \mathcal{H}$, consider $\mathcal{M} = \overline{D^\#(\mathcal{S})}$ and suppose that $\mathcal{M}$ is a regular subspace of $\mathcal{K}$. The Schur complement of W to $\mathcal{S}$ is then defined in that paper as

$$W_{/[\mathcal{S}]}^{M-MP} = D(I - Q)D^\#,$$

where Q is the selfadjoint projection onto $\mathcal{M}$.

The next result show that, when W is $\mathcal{S}$-complementable the regularity of $\mathcal{M}$ can be omitted. If $P_{\mathcal{M}//\mathcal{M}^{[\perp]}}$ is the *projection-like* operator with domain $\mathcal{M} \, [\dotplus] \, \mathcal{M}^{[\perp]}$, range $\mathcal{M}$ and nullspace $\mathcal{M}^{[\perp]}$, then the operator $D(I - P_{\mathcal{M}//\mathcal{M}^{[\perp]}})D^\#$ is well defined and bounded. In this case,

$$D(I - P_{\mathcal{M}//\mathcal{M}^{[\perp]}})D^\# = W_{/[\mathcal{S}]}.$$

Since any bounded selfadjoint operator W can be written in the form $W = DD^\#$ with $D : \mathcal{K} \to \mathcal{H}$ injective, $\mathcal{K}$ a Krein space, it follows that W need not have the UFP.

Proposition 4.19. *Let $W \in L(\mathcal{H})^s$ and $\mathcal{S}$ a closed subspace of $\mathcal{H}$. Suppose that $W = DD^\#$ with $D \in L(\mathcal{K}, \mathcal{H})$, $N(D) = \{0\}$ and W is $\mathcal{S}$-complementable. Let $\mathcal{M} = \overline{D^\#(\mathcal{S})}$. Then*

$$W_{/[\mathcal{S}]} = D(I - P_{\mathcal{M}//\mathcal{M}^{[\perp]}})D^\#.$$

Proof. Since W is $\mathcal{S}$-complementable, we have that $\mathcal{H} = \mathcal{S} + W^{-1}(\mathcal{S}^{[\perp]})$. Suppose that $W = DD^\#$ with $D \in L(\mathcal{K}, \mathcal{H})$ and $N(D) = \{0\}$. Then

$$R(D^\#) = D^\#(\mathcal{S}) \; [+] \; R(D^\#) \cap D^\#(\mathcal{S})^{[\perp]}.$$

Furthermore, the sum is direct because $\{0\} = R(D^\#)^{[\perp]} \supseteq D^\#(\mathcal{S})^{[\perp]} \cap \overline{D^\#(\mathcal{S})}$. Therefore,

$$R(D^\#) = D^\#(\mathcal{S}) \; [\dotplus] \; R(D^\#) \cap D^\#(\mathcal{S})^{[\perp]} \subseteq \mathcal{M} \; [\dotplus] \; \mathcal{M}^{[\perp]}.$$

Let $T := P_{\mathcal{M}//\mathcal{M}^{[\perp]}} D^\#$; since $R(D^\#) \subseteq Dom(P_{\mathcal{M}//\mathcal{M}^{[\perp]}}) = \mathcal{M} \; [\dotplus] \; \mathcal{M}^{[\perp]}$, T is well defined. Let Q be any projection onto $\mathcal{S}$ such that $WQ = Q^\# W$. Then, for every $x \in \mathcal{H}$,

$$Tx = TQx + T(I - Q)x.$$

Since $Qx \in \mathcal{S}$, $TQx = P_{\mathcal{M}//\mathcal{M}^{[\perp]}} D^\# Qx = D^\# Qx$. Also, $T(I - Q)x = 0$ because $D^\#(I - Q)x \in R(D^\#(I - Q)) = D^\# N(Q) \subseteq D^\#(W^{-1}(\mathcal{S}^{[\perp]})) = R(D^\#) \cap D^\#(\mathcal{S})^{[\perp]} \subseteq \mathcal{M}^{[\perp]} = N(T)$. Therefore,

$$T = P_{\mathcal{M}//\mathcal{M}^{[\perp]}} D^\# = D^\# Q \in L(\mathcal{H}).$$

Thus, by Corollary 4.14,

$$W_{/[\mathcal{S}]} = W(I - Q) = DD^\#(I - Q) = D(I - P_{\mathcal{M}//\mathcal{M}^{[\perp]}})D^\#.$$

□

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