

Projective geometry of the Poincaré disk of a C^* -algebra

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Abstract

We study the Poincaré disk $\mathcal{D} = \{a \in \mathcal{A} : \|a\| < 1\}$ of a C^* -algebra \mathcal{A} from a projective point of view: \mathcal{D} is regarded as an open subset of the projective line \mathcal{AP}_1 , the space of complemented rank one submodules of \mathcal{A}^2 . We introduce the concept of cross ratio of four points in \mathcal{AP}_1 . Our main result establishes the relation between the exponential map $\text{Exp}_{z_0}(z_1)$ of \mathcal{D} ($z_0, z_1 \in \mathcal{D}$) and the cross ratio of the four-tuple

$$\delta(-\infty), \delta(0) = z_0, \delta(1) = z_1, \delta(+\infty),$$

where δ is the unique geodesic of \mathcal{D} joining z_0 and z_1 at times $t = 0$ and $t = 1$, respectively.

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1 Introduction

Let \mathcal{A} be a unital C^* -algebra and consider the Poincaré disk

$$\mathcal{D} = \{z \in \mathcal{A} : \|z\| < 1\}$$

and the Poincaré halfspace

$$\mathcal{H} = \{h \in \mathcal{A} : \frac{1}{2i}(h - h^*) \text{ is positive and invertible}\}.$$

This paper is the second part of the study began in [1], where we considered \mathcal{D} and \mathcal{H} as homogeneous spaces of the unitary groups $\mathcal{U}(\theta_{\mathcal{D}})$ and $\mathcal{U}(\theta_{\mathcal{H}})$ of certain quadratic forms $\theta_{\mathcal{D}}$ and $\theta_{\mathcal{H}}$ in \mathcal{A}^2 , respectively. These spaces have reductive connections and invariant Finsler metrics, which make them non positively curved length spaces (see [10]).

In the present paper we study the \mathcal{D} from the projective point of view. More precisely, we introduce the projective line \mathcal{AP}_1 of \mathcal{A} , and regard \mathcal{D} as an open subset of \mathcal{AP}_1 . In this way, the group $\mathcal{U}(\theta)$ appears as the group of projective transformations which preserve \mathcal{D} . This allows us to introduce the concept of *cross ratio* of four points in \mathcal{AP}_1 , in the sense considered by Zelikin [17], adapted to our situation. Geometrically, the construction of the cross ratio is illustrated by the following figure:

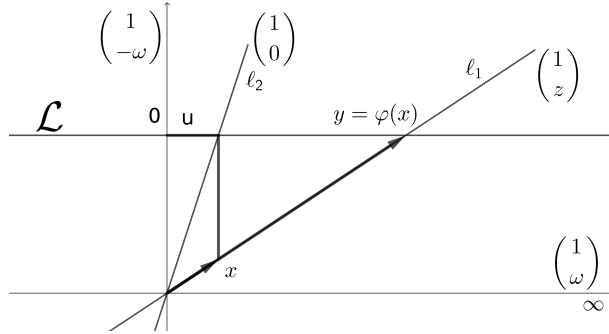


Figure 1

Here the plane represents \mathcal{A}^2 and the lines are elements of \mathcal{AP}_1 , i.e. complemented submodules of \mathcal{A} generated by a single element. The cross ratio of $\infty, 0, \ell_1$ and ℓ_2 (in this precise order) has the following geometric meaning: in the projective line \mathcal{AP}_1 we choose ∞ as the point of infinity, then $\mathcal{AP}_1 \setminus \{\infty\}$ turns into an *affine line* (\mathcal{L} in the figure above). In this affine line, the line ℓ_1 defines a point 0, and the affine line turns into a *vector line*. In this vector line we choose a point u , and the vector line \mathcal{L} turns into a *scalar line* (identified with \mathcal{A} by means of the basis u). Finally, if in this scalar line we choose a fourth element $\ell_2 \neq 0$, this point has a coordinate in the basis u . Classically, this scalar is the cross ratio of $\infty, 0, \ell_1, \ell_2$. These features can be observed in Figure 1, in the line \mathcal{L} . In this paper, the cross ratio of these four lines will be an endomorphism φ of ℓ_1 . The map φ associates to each $\mathbf{x} \in \ell_1$ the point $\varphi(\mathbf{x}) + \mathbf{y}$, obtained by two successive projections: first to 0, parallel to ∞ ; next to ℓ_1 , parallel to ℓ_2 . This is clear in the figure above.

We introduce a fibre bundle Γ whose fibers are C^* -algebras (isomorphic to \mathcal{A}), over the base space \mathcal{D} . In the tangent bundle $T\mathcal{D}$, we define an inner product with values in Γ (similarly as a Riemannian structure). If $z \in \mathcal{D}$ and $X, Y \in (T\mathcal{D})_z$, we denote this inner product by $\langle X, Y \rangle_z$. It is an element in Γ_z (the fiber of Γ over z). Then, by definition, the cross ratio of four points in \mathcal{D} is an element of Γ .

On the other hand, we compute explicitly the exponential map of the natural connection over $T\mathcal{D}$, which is bijective at every tangent space, and its inverse $\text{Log}_z : \mathcal{D} \rightarrow (T\mathcal{D})_z$. We consider the morphism $\langle \text{Log}_z(w), \text{Log}_z(w) \rangle_z^{1/2}$ as a *distance operator* from z to w in \mathcal{D} .

In Theorem 10.1 and Corollary 10.2 we state our main result, which clarifies the relationship between $\langle \text{Log}_z(w), \text{Log}_z(w) \rangle_z^{1/2}$ and the cross ratio of $-\infty, z, w, \infty$. Here $\pm\infty$ denote the limits

$$\text{SOT} - \lim_{t \rightarrow \pm\infty} \gamma(t),$$

where γ is the geodesic from z to w in time 1, and the algebra \mathcal{A} is considered in its universal representation. The formulas in Theorem 10.1 and Corollary 10.2 enable one to view the differential geometry of the Poincaré disk \mathcal{D} as projective geometry of spaces of operators.

The contents of the paper are the following. In Section 2 we introduce the *regular* elements of \mathcal{A}^2 , which will be used to define \mathcal{AP}_1 and \mathcal{AP}_1^θ .

In Section 3 we define the projective \mathcal{AP}_1 line of \mathcal{A} , and introduce the operator cross ratio of four submodules (following ideas introduced in [17] for closed subspaces): it is defined as a (possibly empty) set of module homomorphisms.

In Section 4 we introduce our object of study, the θ -hyperbolic part \mathcal{AP}_1^θ of \mathcal{AP}_1 . We state the identifications of \mathcal{AP}_1^θ with the disk \mathcal{D} and with the space of projections \mathcal{Q}_ρ (θ -orthogonal

projections whose ranges are the elements of \mathcal{AP}_1^θ). We introduce the action of the θ -unitary group $\mathcal{U}(\theta)$, as well as the Borel subgroup $\mathcal{B}(\theta)$.

In Section 5 we describe the tangent bundle of \mathcal{AP}_1^θ .

In Section 6 we recall the basic facts of the geometry of \mathcal{D} done in [1]. Also we compute the explicit form of the exponential map and its inverse Log.

In Section 7 we compute the limit points of geodesics at $t = \pm\infty$. In particular, we show that the partial isometry of $z \in \mathcal{D}$ coincides with the $\text{SOT} - \lim_{t \rightarrow \infty} \delta(t)$, where δ is the geodesic joining 0 and z .

In Section 8 we introduce the $\mathcal{U}(\theta)$ -invariant Finsler metric in \mathcal{AP}_1^θ .

In Section 9 we consider the cross ratio of the points $\ell_{-\infty}, \ell_0, \ell, \ell_{+\infty}$, where ℓ_0 is the submodule corresponding to the origin in \mathcal{D} , ℓ is an arbitrary point in $\ell \neq \ell_0$, and $\ell_{\mp\infty}$ are the $\mp\infty$ -limit points of the geodesic starting at ℓ_0 at $t = 0$ and reaching ℓ at $t = 1$. We show that this set is always non empty, and that there is a natural choice of a specific homomorphism $cr(\ell_0, \ell)$, or $cr(0, z)$ if $\ell \simeq z \in \mathcal{D}$ (which is the only possible choice for a strongly dense subset of $\ell \in \mathcal{AP}_1^\theta$, in the case when \mathcal{A} is a von Neumann algebra). We finish Section 9 by giving a first glimpse of our main result, which relates $cr(0, z)$ with the metric geometry of \mathcal{D} (or equivalently, \mathcal{AP}_1^θ). Namely,

$$\|cr(0, z)\|_{\mathcal{B}(\ell_z)} = \frac{1}{2}d(0, z),$$

where $\|cr(0, z)\|_{\mathcal{B}(\ell_z)}$ stands for the norm of the endomorphism $cr(0, z) : \ell_z \rightarrow \ell_z$.

In Section 10 we briefly introduce the coefficient bundle over \mathcal{AP}_1^θ (using the model \mathcal{Q}_ρ), and an Hermitian structure on \mathcal{AP}_1^θ , with values in the coefficient bundle. This is done in order to state our main results of this paper, which are the formulas given in Theorem 10.1 and Corollary 10.2, relating the cross ratio $cr(\ell_1, \ell_2)$ with the logarithm map $\text{Log}_{\ell_1} \ell_2$.

In Section 11 we consider a relevant example, namely, when there exists a C^* -subalgebra \mathcal{B} of the center of \mathcal{A} and a conditional expectation $\mathbf{tr} : \mathcal{A} \rightarrow \mathcal{B}$ satisfying $\mathbf{tr}(xy) = \mathbf{tr}(yx)$ for all $x, y \in \mathcal{A}$; an important particular case is when \mathcal{A} is commutative. In this case the computations are much simpler, and the formula relating the cross ratio and the logarithm is

$$e^{|\text{Log}_{\ell_1} \ell_2|} = cr(\ell_1, \ell_2).$$

2 Preliminaries

Let \mathcal{A} be a unital C^* -algebra, G the group of invertible elements of \mathcal{A} , G^+ the subset of G of positive elements. Consider $\mathcal{A}^2 = \mathcal{A} \times \mathcal{A}$ as a right \mathcal{A} - C^* -module, with \mathcal{A} -valued inner product given by

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1^* y_1 + x_2^* y_2,$$

where $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathcal{A}^2$. The space \mathcal{A}^2 is endowed with the usual C^* -module norm

$$\|\mathbf{x}\| = \|\langle \mathbf{x}, \mathbf{x} \rangle\|^{1/2} = \|x_1^* x_1 + x_2^* x_2\|^{1/2}.$$

We shall say that an element $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathcal{A}^2$ is *regular* if

$$\langle \mathbf{x}, \mathbf{x} \rangle \in G^+.$$

We shall denote by \mathcal{A}_{reg}^2 the set of regular elements. Clearly \mathcal{A}_{reg}^2 is an open subset of \mathcal{A}^2 . The C*-algebra of adjointable operators of \mathcal{A}^2 (bounded \mathcal{A} -linear operators acting in \mathcal{A}^2 which admit an adjoint for the \mathcal{A} -valued inner product) identifies with the algebra $M_2(\mathcal{A})$ of 2×2 matrices with entries in \mathcal{A} , acting by left multiplication. Let $Gl_2(\mathcal{A})$ be the group of invertible elements of $M_2(\mathcal{A})$. Let us state the following basic facts:

Proposition 2.1. *With the current notations, we have that*

1. \mathcal{A}_{reg}^2 is an open subset of \mathcal{A} .
2. $Gl_2(\mathcal{A})$ acts on \mathcal{A}_{reg}^2 by left multiplication.
3. G acts on \mathcal{A}_{reg}^2 by right multiplication.
4. Both actions commute.
5. For each $\mathbf{x} \in \mathcal{A}_{reg}^2$ the orbit $Gl_2(\mathcal{A}) \cdot \mathbf{x}$ is open (and closed) in \mathcal{A}_{reg}^2 .
6. For each $\mathbf{x} \in \mathcal{A}_{reg}^2$, the map $Gl_2(\mathcal{A}) \rightarrow Gl_2(\mathcal{A}) \cdot \mathbf{x} \subset \mathcal{A}_{reg}^2$, $\tilde{g} \mapsto \tilde{g} \cdot \mathbf{x}$ defines a principal fibre bundle with smooth local cross sections.

We shall omit the proof: the proposition holds for \mathcal{A}^n instead of \mathcal{A}^2 . We refer to [5] (Section 1) for complete proofs of these facts, noticing that

$$\mathcal{A}_{reg}^n = \{\mathbf{x} \in \mathcal{A}^n : \text{there exists } \mathbf{y} \in \mathcal{A}^n \text{ such that } y_1x_1 + \cdots + y_nx_n = 1\}.$$

Indeed, if $\mathbf{x} \in \mathcal{A}_{reg}^n$, then defining \mathbf{y} given by $y_k = \langle \mathbf{x}, \mathbf{x} \rangle^{-1} x_k^*$ provides an n -tuple such that $y_1x_1 + \cdots + y_nx_n = 1$. Conversely, if $y_1x_1 + \cdots + y_nx_n = 1$, and we set $\mathbf{y}^* = (y_1^*, \dots, y_n^*)$, by the Schwarz inequality for C*-modules (see for instance [13], Prop. 1.2.4), we have

$$1 = \langle \mathbf{y}^*, \mathbf{x} \rangle = |\langle \mathbf{y}^*, \mathbf{x} \rangle|^2 \leq \|\langle \mathbf{y}^*, \mathbf{y}^* \rangle\| \langle \mathbf{x}, \mathbf{x} \rangle,$$

which implies that $\langle \mathbf{x}, \mathbf{x} \rangle$ is invertible.

This set \mathcal{A}_{reg}^n is known, in the K -theoretic setting, as the set of n -unimodular rows in \mathcal{A} . The paper by M. Rieffel [16] contains a thorough treatment of this subject.

We are interested in the single orbit

$$\mathcal{O} := Gl_2(\mathcal{A}) \cdot \mathbf{e}_1,$$

where $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Here and throughout, denote $\hat{g} := (\tilde{g}^*)^{-1}$.

We consider the sphere

$$\mathbb{S} = \mathbb{S}(\mathcal{A}^2) = \{\mathbf{x} \in \mathcal{A}^2 : \langle \mathbf{x}, \mathbf{x} \rangle = 1\},$$

and the intersection

$$\mathbb{S}_{\mathcal{O}} = \mathbb{S} \cap \mathcal{O}.$$

The next result will not be needed in the rest of the paper, we state it here to describe the topology of the sphere \mathbb{S} in the case when $\mathcal{A} = \mathcal{B}(\mathcal{H})$.

Theorem 2.2. *If $\mathcal{A} = \mathcal{B}(\mathcal{H})$, for an infinite dimensional Hilbert space \mathcal{H} , then*

1. *Each $\mathbf{x} \in \mathbb{S}$ defines an isometry $\mathbf{x} : \mathcal{H} \rightarrow \mathcal{H}^2$ such that $R(\mathbf{x})^\perp$ is infinite dimensional.*
2. *\mathbb{S} is connected.*

Proof.

$$\|\mathbf{x}\xi\|^2 = \|x_1\xi\|^2 + \|x_2\xi\|^2 = \langle x_1^*x_1\xi, \xi \rangle + \langle x_2^*x_2\xi, \xi \rangle = \langle \xi, \xi \rangle = \|\xi\|^2.$$

Let us check that $R(\mathbf{x})^\perp$ is infinite dimensional. Note that $N(x_1) \perp N(x_2)$. Indeed, $N(x_1) = N(x_1^*x_1) = N(1 - x_2^*x_2)$ is the eigenspace of $x_2^*x_2$ associated to the eigenvalue 1, whereas $N(x_2) = N(x_2^*x_2)$ is the eigenspace of $x_2^*x_2$ associated to 0. Let

$$\mathcal{H} = N(x_1) \oplus N(x_2) \oplus L, \text{ where } L = (N(x_1) \oplus N(x_2))^\perp.$$

Since \mathbf{x} is isometric, it maps this orthogonal decomposition to the orthogonal decomposition

$$\mathcal{H} \times \mathcal{H} = (\{0\} \times N(x_1)) \oplus (N(x_2) \times \{0\}) \oplus \mathbb{L}, \quad \mathbf{x}(\nu_1 \oplus \nu_2 \oplus \mu) = (0, \nu_1) \oplus (\nu_2, 0) \oplus (x_1\mu, x_2\mu).$$

If $\dim \mathbb{L} < \infty$, then clearly $\dim L < \infty$, and thus $N(x_1), N(x_2)$ cannot have both finite co-dimension. Suppose $M = N(x_1)^\perp$ has infinite dimension (the other case is similar). Put

$$\mathbb{M} = \{(0, \gamma) : \gamma \in M, (0, \gamma) \perp \mathbb{L}\}.$$

Since $\dim \mathbb{L} < \infty$, it is clear that $\dim \mathbb{M} = \infty$. By construction, \mathbb{M} is orthogonal to $\{0\} \times N(x_1)$, $N(x_2) \times \{0\}$ and to \mathbb{L} . Thus $R(\mathbf{x})^\perp$ has infinite dimension in this case.

Suppose otherwise that $\dim \mathbb{L} = \infty$, and consider

$$\mathbf{x}_0 : \mathbf{x}|_L : L \rightarrow \mathbb{L}.$$

If $R(x_1|_L)^\perp$ is infinite dimensional, then the fact that $R(\mathbf{x}_0) \subset R(x_1|_L) \times R(x_2|_L)$, implies that $R(\mathbf{x}_0)^\perp$ is infinite dimensional. Suppose that $\dim R(x_1|_L)^\perp < \infty$. Then $x_1|_L$ has closed range, and trivial nullspace, therefore it has a Moore-Penrose pseudo-inverse $(x_1|_L)^\dagger : L \rightarrow R(x_1|_L)$. Thus, if $\xi \in L$, $\xi = x_1^\dagger \gamma$, for a unique $\gamma \in R(x_1|_L)$, and $x_1\xi = \gamma$. Then

$$R(\mathbf{x}_0) = \{x_1\xi, x_2\xi\} : \xi \in L\} = \{(\gamma, y\gamma) : \gamma \in R(x_1|_L)\},$$

where $y := x_2(x_1|_L)^\dagger$. The orthogonal complement $R(\mathbf{x}_0)^\perp$ contains the set

$$\{(-y^*\gamma, \gamma) : \gamma \in R(x_1|_L)\},$$

which is infinite dimensional. Therefore \mathbf{x} is an isometry with infinite co-rank.

Let us check now that the sphere \mathbb{S} is connected. Consider a fixed unitary operator $U : \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{H}$. Let $\mathbf{x}, \mathbf{y} \in \mathbb{S}$, again, be regarded as isometries from \mathcal{H} to \mathcal{H}^2 . Then the compositions $U\mathbf{x}$ and $U\mathbf{y}$ are isometries in \mathcal{H} , with infinite co-rank. In [4] it is shown that the connected components of the set of isometries in \mathcal{H} is parametrized by the co-rank. Two isometries belong to the same component if and only if they have the same co-rank. Moreover, also in [4], it is shown that these components are the orbits of the left action of the unitary group of \mathcal{H} , by left multiplication. It follows that there exists a selfadjoint operator $Z \in \mathcal{B}(\mathcal{H})$ such that

$$U\mathbf{y} = e^{iZ}U\mathbf{x}.$$

Then $\mathbf{x}(t) = U^* e^{itZ} U \mathbf{x}$ is a continuous curve of isometries from \mathcal{H} to $\mathcal{H} \times \mathcal{H}$ such that $\mathbf{x}(0) = \mathbf{x}$ and $\mathbf{x}(1) = \mathbf{y}$. Then

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \in \mathbb{S}$$

is a continuous curve in \mathbb{S} which connects \mathbf{x} and \mathbf{y} . \square

Let us denote by $\mathcal{U}_2(\mathcal{A})$ the unitary group of $M_2(\mathcal{A})$. Since $M_2(\mathcal{A})$ is the C^* -algebra of adjointable operators of the module \mathcal{A}^2 , $\mathcal{U}_2(\mathcal{A})$ is the group of invertible elements in $M_2(\mathcal{A})$ which preserve the \mathcal{A} -valued inner product of \mathcal{A}^2 . In particular, this implies that $\mathcal{U}_2(\mathcal{A})$ acts on \mathbb{S} , and on $\mathbb{S}_{\mathcal{O}}$.

Proposition 2.3. $\mathbb{S}_{\mathcal{O}}$ is the orbit of \mathbf{e}_1 under the action of $\mathcal{U}_2(\mathcal{A})$.

Proof. Clearly the orbit $\mathcal{U}_2(\mathcal{A}) \cdot \mathbf{e}_1$ is contained in $\mathbb{S}_{\mathcal{O}}$. Pick $\mathbf{x} = \tilde{g}\mathbf{e}_1 \in \mathbb{S}_{\mathcal{O}}$. We must show that there exists a unitary element \tilde{u} such that $\tilde{u}\mathbf{e}_1 = \tilde{g}\mathbf{e}_1$. Denote $\tilde{r} = \tilde{g}^* \tilde{g}$ and by \tilde{p}_0 the selfadjoint projection $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. The fact that $\tilde{g}\mathbf{e}_1 \in \mathbb{S}_{\mathcal{O}}$ means that

$$\langle \tilde{r}\mathbf{e}_1, \mathbf{e}_1 \rangle = 1,$$

or, equivalently, that $\tilde{r}\tilde{p}_0 = \tilde{p}_0$. It follows that, for any $n \geq 0$, $\tilde{r}^n \tilde{p}_0 = \tilde{p}_0$, and thus, if $q(t) = \sum_{i=0}^n \alpha_i t^i$, then $q(\tilde{r})\tilde{p}_0 = \sum_{i=0}^n \alpha_i \tilde{p}_0 = q(1)\tilde{p}_0$. Let $q_n(t)$ be a sequence of polynomials which converge uniformly to \sqrt{t} in the spectrum of \tilde{r} (a compact subset of $(0, +\infty)$). Then $q_n(\tilde{r}) \rightarrow \tilde{r}^{1/2}$. On the other hand, it holds

$$q_n(\tilde{r})\tilde{p}_0 = q_n(1)\tilde{p}_0 \rightarrow \tilde{p}_0.$$

Then, $\tilde{r}^{1/2}\tilde{p}_0 = \tilde{p}_0$. Consider the polar decomposition of the (invertible) element \tilde{g} in $M_2(\mathcal{A})$: $\tilde{g} = \tilde{u}\tilde{r}^{1/2}$. Then \tilde{u} is a unitary element such that $\tilde{u}\tilde{p}_0 = \tilde{u}\tilde{r}\tilde{p}_0 = \tilde{g}\tilde{p}_0$. In particular,

$$\tilde{u}\mathbf{e}_1 = \text{first column of } \tilde{u}\tilde{p} = \text{first column of } \tilde{g}\tilde{p} = \tilde{g}\mathbf{e}_1.$$

\square

Corollary 2.4. The unitary group \mathcal{U} of \mathcal{A} acts on $\mathbb{S}_{\mathcal{O}}$ by right multiplication.

Proof. Pick $\mathbf{x} = \tilde{u} \cdot \mathbf{e}_1 \in \mathbb{S}_{\mathcal{O}}$ ($\tilde{u} \in \mathcal{U}_2(\mathcal{A})$), and $v \in \mathcal{U}$. Then

$$\mathbf{x} \cdot v = \tilde{u} \cdot \mathbf{e}_1 \cdot v = \tilde{u}v \cdot \mathbf{e}_1,$$

where $\tilde{u}v = \tilde{u} \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix} \in \mathcal{U}_2(\mathcal{A})$. \square

Remark 2.5. There is a retraction $\mathcal{A}_{reg}^2 \rightarrow \mathbb{S}$: $\mathbf{x} \mapsto \mathbf{x}\langle \mathbf{x}, \mathbf{x} \rangle^{-1/2}$.

Corollary 2.6. In the case $\mathcal{A} = \mathcal{B}(\mathcal{H})$, $\mathbb{S}_{\mathcal{O}} = \mathbb{S}$ and $\mathcal{O} = \mathcal{A}_{reg}^2$.

3 The projective line over \mathcal{A}

We shall call *rank one submodule* a closed submodule $0 \neq \ell \subset \mathcal{A}^2$ which is orthocomplemented, i.e.,

$$\ell \oplus \ell^\perp = \mathcal{A}^2, \quad \text{where } \ell^\perp = \{\mathbf{y} \in \mathcal{A}^2 : \langle \mathbf{x}, \mathbf{y} \rangle = 0 \text{ for all } \mathbf{x} \in \ell\},$$

and such that there exists $\mathbf{x}_0 \in \ell$ with $[x_0] = \{x_0 a : a \in \mathcal{A}\} = \ell$. As the result below shows, not every element in \mathcal{A}^2 generates a rank one submodule:

Proposition 3.1. *Let $\mathbf{x} \in \mathcal{A}^2$, $\mathbf{x} \neq 0$. Then $\ell = \{\mathbf{x}a : a \in \mathcal{A}\}$ is a rank one submodule if and only if there exists $a \in \mathcal{A}$ such that $\mathbf{x}_0 = \mathbf{x}a$ is also a generator for ℓ , and $\langle \mathbf{x}_0, \mathbf{x}_0 \rangle$ is a projection in \mathcal{A} .*

Proof. Suppose that there is a generator \mathbf{x}_0 of ℓ such that $\langle \mathbf{x}_0, \mathbf{x}_0 \rangle = p_0$ is a projection. Note that $\mathbf{x}_0 = \mathbf{x}_0 p_0$. Indeed, $\mathbf{x}_0(1 - p_0)$ satisfies $\langle \mathbf{x}_0(1 - p_0), \mathbf{x}_0(1 - p_0) \rangle = (1 - p_0)p_0(1 - p_0) = 0$, i.e., $\mathbf{x}_0(1 - p_0) = 0$. Then $p_{\mathbf{x}_0}$ defined by $p_{\mathbf{x}_0}(\mathbf{y}) = \mathbf{x}_0 \langle \mathbf{x}_0, \mathbf{y} \rangle$ is the orthogonal projection onto ℓ . Clearly, it is selfadjoint. It is a projection:

$$p_{\mathbf{x}_0}^2(\mathbf{y}) = \mathbf{x}_0 \langle \mathbf{x}_0, \mathbf{x}_0 \langle \mathbf{x}_0, \mathbf{y} \rangle \rangle = \mathbf{x}_0 p_0 \langle \mathbf{x}_0, \mathbf{y} \rangle = \mathbf{x}_0 \langle \mathbf{x}_0, \mathbf{y} \rangle = p_{\mathbf{x}_0}(\mathbf{y}).$$

Obviously, the range of $p_{\mathbf{x}_0}$ is ℓ .

Conversely, suppose that $\ell = [\mathbf{x}]$ is orthocomplemented. Then there exists a symmetric adjointable projection $p : \mathcal{A}^2 \rightarrow \ell \subset \mathcal{A}^2$. It is of the form $p(\mathbf{y}) = \mathbf{x}\varphi(\mathbf{y})$, where $\varphi : \mathcal{A}^2 \rightarrow \mathcal{A}$ is \mathcal{A} -linear and bounded. Then (see [12], p. 13) φ is of the form $\varphi(\mathbf{y}) = \langle \mathbf{z}, \mathbf{y} \rangle$ for some $\mathbf{z} \in \mathcal{A}^2$. The adjoint of $p(\mathbf{y}) = \mathbf{x}\langle \mathbf{z}, \mathbf{y} \rangle$ is $p^*(\mathbf{y}) = \mathbf{z}\langle \mathbf{x}, \mathbf{y} \rangle$. Therefore, the fact that $p^* = p$, implies that $\mathbf{z} = \mathbf{x}a$ for some $a \in \mathcal{A}$. Thus, $p = \mathbf{x}a\langle \mathbf{x}, \cdot \rangle = \mathbf{x}\langle \mathbf{x}a, \cdot \rangle = \mathbf{x}a^*\langle \mathbf{x}, \cdot \rangle$. Using that $p^2 = p$, we get that

$$p = p^2 = \mathbf{x}a^*\langle \mathbf{x}, \mathbf{x}a\langle \mathbf{x}, \cdot \rangle \rangle = \mathbf{x}a^*\langle \mathbf{x}, \mathbf{x} \rangle a\langle \mathbf{x}, \cdot \rangle = \mathbf{x}b\langle \mathbf{x}b, \cdot \rangle,$$

where $b = (a^*\langle \mathbf{x}, \mathbf{x} \rangle a)^{1/2}$. Denote $\mathbf{x}_0 = \mathbf{x}b$. The above computation shows that \mathbf{x}_0 is also a generator for ℓ . Then $p = p_{\mathbf{x}_0}$. Using again the fact that $p^2 = p$, we obtain

$$\mathbf{x}_0 \langle \mathbf{x}_0, \cdot \rangle = p = p^2 = \mathbf{x}_0 \langle \mathbf{x}_0, \mathbf{x}_0 \rangle \langle \mathbf{x}_0, \cdot \rangle.$$

Evaluating at \mathbf{x}_0 , we get $\mathbf{x}_0 \langle \mathbf{x}_0, \mathbf{x}_0 \rangle = \mathbf{x}_0 \langle \mathbf{x}_0, \mathbf{x}_0 \rangle^2$; applying $\langle \mathbf{x}_0, \cdot \rangle$ to this equality, we obtain that $c = \langle \mathbf{x}_0, \mathbf{x}_0 \rangle$ satisfies $c^2 = c^3$. Since $c \geq 0$, this means that c has two spectral values 0, 1, and thus c is a (non zero) projection. \square

In this paper, we shall be interested in rank one modules $\ell = [\mathbf{x}]$ with $\mathbf{x} \in \mathcal{O}$. Note that for elements in \mathcal{O} , $\langle \mathbf{x}, \mathbf{x} \rangle$ is invertible in \mathcal{A} . Then, the projection p_0 in the above proposition is $1 \in \mathcal{A}$.

Define

$$\mathcal{AP}_1 = \{\ell = [\mathbf{x}] : \mathbf{x} \in \mathcal{O}\}.$$

An equivalence is defined in \mathcal{O} . Namely, $\mathbf{x} \sim \mathbf{x}'$ iff there exists $a \in G$ such that $\mathbf{x}' = \mathbf{x}a$ (or equivalently, $[\mathbf{x}] = [\mathbf{x}']$). Then, it holds that also $\mathcal{AP}_1 = \mathcal{O}/\sim$.

The map

$$\mathcal{O} \rightarrow \mathcal{AP}_1 \quad \mathbf{x} \mapsto [\mathbf{x}]$$

is onto, and the fibers are the equivalence classes in \mathcal{O} .

Consider the map

$$\mathcal{O} \rightarrow \mathbb{S}_{\mathcal{O}}, \quad \mathbf{x} \mapsto \mathbf{x} \langle \mathbf{x}, \mathbf{x} \rangle^{-1/2}.$$

Clearly, it is a well defined and continuous retraction.

Remark 3.2. It should be noticed that the projective spaces of C^* -modules over \mathcal{A} which appear in [2] and [3] are strictly bigger than \mathcal{AP}_1 .

We end this section by defining the *cross ratio* $CR(\ell_1, \ell_2, \ell_3, \ell_4)$ of four submodules $\ell_1, \ell_2, \ell_3, \ell_4$ in \mathcal{AP}_1 , following ideas by M. I. Zelikin [17].

Definition 3.3. We denote by $CR(\ell_1, \ell_2, \ell_3, \ell_4)$ the (possibly empty) set of module homomorphisms $\varphi : \ell_3 \rightarrow \ell_3$ of the form $\varphi = \psi\eta$, where the homomorphisms $\eta : \ell_3 \rightarrow \ell_2$, $\psi : \ell_2 \rightarrow \ell_3$ satisfy $x - \psi(x) \in \ell_4$ and $y - \eta(y) \in \ell_1$, for all $x \in \ell_3$, $y \in \ell_2$.

4 The hyperbolic part of the \mathcal{A} -projective line: the Poincaré disk of \mathcal{A}

In this section we use facts from [1] concerning the hyperbolic geometry of the unit disk \mathcal{D} of \mathcal{A} . With the appropriate metric, \mathcal{D} is a model for the part of \mathcal{AP}_1 which carries a nonpositively curved geometry. The main feature is a signed sesquilinear form θ , and the group which preserves it. More precisely, define

$$\theta : \mathcal{A}^2 \times \mathcal{A}^2 \rightarrow \mathcal{A}, \quad \theta(\mathbf{x}, \mathbf{y}) = x_1^* y_1 - x_2^* y_2.$$

In [1] it was denoted θ_D . Note that $\theta(\mathbf{x}, \mathbf{y}) = \left\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \rho_\theta \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\rangle$, where

$$\rho = \rho_\theta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The main tool in the study of the geometry of the Poincaré disk of \mathcal{A} is the subgroup $\mathcal{U}(\theta)$ of $Gl_2(\mathcal{A})$ which leaves the form θ invariant:

$$\mathcal{U}(\theta) = \{\tilde{g} \in Gl_2(\mathcal{A}) : \theta(\tilde{g}\mathbf{x}, \tilde{g}\mathbf{y}) = \theta(\mathbf{x}, \mathbf{y}) \text{ for all } \mathbf{x}, \mathbf{y} \in \mathcal{A}^2\}.$$

We call $\mathcal{U}(\theta)$ the *unitary group of the form θ* .

Equivalently, $\tilde{g} \in Gl_2(\mathcal{A})$ belongs to $\mathcal{U}(\theta)$ iff $\tilde{g}^* \rho \tilde{g} = \rho$. In [1] it was shown that $\mathcal{U}(\theta)$ is a complemented Banach-Lie subgroup of $Gl_2(\mathcal{A})$.

Let us define the *hyperbolic part* \mathcal{AP}_1^θ of \mathcal{AP}_1 .

Definition 4.1. The set

$$\mathcal{AP}_1^\theta = \{[\mathbf{x}] \in \mathcal{AP}_1 : \text{there exists a generator } \mathbf{x}_0 \text{ of } [\mathbf{x}] \text{ such that } \theta(\mathbf{x}_0, \mathbf{x}_0) \in G^+\}$$

is called the *hyperbolic part* of \mathcal{AP}_1 .

Remark 4.2. Note that

$$\theta(\mathbf{x}a, \mathbf{x}a) = a^* x_1^* x_1 a - a^* x_2^* x_2 a = a^* (x_1^* x_2 - x_2^* x_1) a = a^* \theta(\mathbf{x}, \mathbf{x}) a.$$

Therefore, if $\theta(\mathbf{x}_0, \mathbf{x}_0) \in G^+$ for a generator \mathbf{x}_0 of $[\mathbf{x}]$, then the same is true for any other generator of $[\mathbf{x}]$.

Also, it holds that θ is positive (semidefinite) in $[\mathbf{x}]$. But the converse does not hold. Namely θ might be positive in $[\mathbf{x}]$ without $\theta(\mathbf{x}_0, \mathbf{x}_0)$ being invertible for any generator \mathbf{x}_0 .

Theorem 4.3. *The group $\mathcal{U}(\theta)$ acts transitively in \mathcal{AP}_1^θ . If $\mathbf{x} \in \mathcal{O}$ and $\tilde{g} \in \mathcal{U}(\theta)$, the action is given by*

$$\tilde{g} \cdot [\mathbf{x}] = [\tilde{g}\mathbf{x}].$$

Before we give the proof of this statement, we mention certain facts on \mathcal{AP}_1^θ . Note that if $\mathbf{x} \in \mathcal{O}$ is a generator of $[\mathbf{x}] \in \mathcal{AP}_1^\theta$, then

$$\theta(\tilde{g}\mathbf{x}, \tilde{g}\mathbf{x}) = \theta(\mathbf{x}, \mathbf{x}) \in G^+.$$

Also it is clear that $\tilde{g}\mathbf{x} \in \mathcal{O}$. Thus, $\mathcal{U}(\theta)$ acts in \mathcal{AP}_1^θ . To prove the above theorem, we only need to show that the action is transitive, i.e., that if \mathbf{x} satisfies $\theta(\mathbf{x}, \mathbf{x}) \in G^+$ (i.e. $\mathbf{x} \in \mathcal{O}$), then there exists $\tilde{h} \in \mathcal{U}(\theta)$ such that $\mathbf{x} = \tilde{h}\mathbf{e}_1$.

Definition 4.4. *Let us denote by \mathcal{K}_θ the hyperboloid defined by the form θ :*

$$\mathcal{K}_\theta = \{\mathbf{x} \in \mathcal{A}^2 : \theta(\mathbf{x}, \mathbf{x}) = 1 \text{ and } x_1 \in G\}.$$

In a sense, the space \mathcal{K}_θ is a coordinate space for \mathcal{AP}_1^θ . Note that $\mathcal{K}_\theta \subset \mathcal{A}_{reg}^2$: $\theta(\mathbf{x}, \mathbf{x}) = 1$ means that $x_1^* x_1 = 1 + x_2^* x_2$, so that $\langle \mathbf{x}, \mathbf{x} \rangle = 1 + 2x_2^* x_2 \in G^+$. Note that $\mathcal{K}_\theta \subset \mathcal{O}$.

Proof. (of Theorem 4.3) Note that $\mathbf{e}_1 \in \mathcal{K}_\theta$. Therefore if $\mathbf{x} \in \mathcal{K}_\theta$, there exists $\tilde{g} \in \mathcal{U}(\theta)$ such that $\tilde{g}\mathbf{e}_1 = \mathbf{x}$. Since π_θ is onto, it follows that any $[\mathbf{x}] \in \mathcal{AP}_1^\theta$ has a generator (say) $\mathbf{x} \in \mathcal{K}_\theta$. Then $[\mathbf{x}] = \tilde{g} \cdot [\mathbf{e}_1]$. \square

Remark 4.5. Let us describe the group \mathcal{B}_θ explicitly:

$$\mathcal{B}_\theta = \left\{ \begin{pmatrix} \frac{g + \hat{g}}{2} - \hat{g}x & \frac{g - \hat{g}}{2} - \hat{g}x \\ \frac{g - \hat{g}}{2} + \hat{g}x & \frac{g + \hat{g}}{2} + \hat{g}x \end{pmatrix} : g \in G, x^* = -x \right\}.$$

The group used in [1] is $\mathcal{B} = \left\{ \begin{pmatrix} g & 0 \\ \hat{g}s & \hat{g} \end{pmatrix} : g \in G, s^* = s \right\}$. The group considered here is conjugate of \mathcal{B} :

$$\mathcal{B}_\theta = U^* \mathcal{B} U,$$

where $U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$. We call \mathcal{B}_θ the *Borel subgroup* of $\mathcal{U}(\theta)$. The corresponding Borel subgroup of $\mathcal{U}(\theta_H)$ for the Poincaré halfspace \mathcal{H} is described in [1]. The facts mentioned in Proposition ?? are proved in [1] for the hyperboloid $\mathcal{K}_{\theta_H} = U\mathcal{K}_\theta$, of the form $\theta_H(\mathbf{x}, \mathbf{y}) = \theta(U^*\mathbf{x}, U^*\mathbf{y})$.

Remark 4.6. There are two natural subgroups in \mathcal{B}_θ , which are homomorphic images of the group G of invertible elements of \mathcal{A} and of the additive group \mathcal{A}_{ah} of anti-Hermitian elements of \mathcal{A} , which we denote, respectively, by G_θ and T_θ . Namely,

$$G_\theta = \{\tilde{\gamma}_g := \frac{1}{2} \begin{pmatrix} g + \hat{g} & g - \hat{g} \\ g - \hat{g} & g + \hat{g} \end{pmatrix} : g \in G\} \quad \text{and} \quad T_\theta = \{\tilde{\tau}_x := \begin{pmatrix} 1-x & -x \\ x & 1+x \end{pmatrix} : x^* = -x\}.$$

Elementary computations show that

1. $G_\theta \subset \mathcal{B}_\theta$ and $T_\theta \subset \mathcal{B}_\theta$.
2. The maps $G \ni g \mapsto \tilde{\gamma}_g \in G_\theta$ and $\mathcal{A}_{ah} \ni x \mapsto \tilde{\tau}_x \in T_\theta$ are group homomorphisms (\mathcal{A}_{ah} considered with its additive structure). In particular, G_θ and T_θ are subgroups of \mathcal{B}_θ .

3. G_θ and T_θ generate \mathcal{B}_θ . More precisely, if $\tilde{g} = \begin{pmatrix} \frac{g + \hat{g}}{2} - \hat{g}x & \frac{g - \hat{g}}{2} - \hat{g}x \\ \frac{g - \hat{g}}{2} + \hat{g}x & \frac{g + \hat{g}}{2} + \hat{g}x \end{pmatrix} \in \mathcal{B}_\theta$, then

$$\tilde{g} = \frac{1}{2} \begin{pmatrix} g + \hat{g} & g - \hat{g} \\ g - \hat{g} & g + \hat{g} \end{pmatrix} \begin{pmatrix} 1-x & -x \\ x & 1+x \end{pmatrix}.$$

4. The above factorization is unique. Or, equivalently, $G_\theta \cap T_\theta$ contains only the identity matrix.

Let us describe an alternative model for \mathcal{AP}_1^θ , namely, the unit disk $\mathcal{D} = \{a \in \mathcal{A} : \|a\| < 1\}$ of \mathcal{A} . There is a natural map from \mathcal{K}_θ to \mathcal{D} :

Lemma 4.7. *The map $\mathcal{K}_\theta \rightarrow \mathcal{D}$, $\mathbf{x} \mapsto x_2 x_1^{-1}$ is well defined, onto and C^∞ .*

Proof. Since $\theta(\mathbf{x}, \mathbf{x}) = 1$, $x_2^* x_2 = x_1^* x_1 - 1$. Then

$$0 \leq (x_2 x_1^{-1})^* x_2 x_1^{-1} = (x_1^{-1})^* (x_1^* x_1 - 1) x_1^{-1} = 1 - (x_1 x_1^*)^{-1}.$$

Then

$$\|x_2 x_1^{-1}\|^2 = \|(x_2 x_1^{-1})^* x_2 x_1^{-1}\| = \|1 - (x_1 x_1^*)^{-1}\| = \sup\{1 - \frac{1}{t} : t \in \sigma(x_1 x_1^*)\} < 1,$$

because $\sigma(x_1 x_1^*)$ is a compact set in $(0, +\infty)$ (recall that x_1 is invertible). The map is clearly C^∞ . Pick $a \in \mathcal{D}$. Then $1 - a^* a \in G$. Put $x_1 = (1 - a^* a)^{-1/2}$ and $x_2 = a x_1$. Therefore $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ belongs to \mathcal{K}_θ and is mapped to a . \square

Proposition 4.8. *The map $\mathcal{K}_\theta \rightarrow \mathcal{D}$ induces a C^∞ diffeomorphism*

$$\mathcal{AP}_1^\theta \xrightarrow{\simeq} \mathcal{D}, \quad [\mathbf{x}] \mapsto x_2 x_1^{-1}.$$

Proof. If $\mathbf{x}, \mathbf{y} \in \mathcal{O}$ satisfy $[\mathbf{x}] = [\mathbf{y}]$, then there exists $g \in G$ such that $\mathbf{y} = \mathbf{x}g$, and thus $y_2 y_1^{-1} = x_2 g (x_1 g)^{-1} = x_2 x_1^{-1}$. Its inverse is

$$\mathcal{D} \ni z \mapsto \left[\begin{pmatrix} 1 \\ z \end{pmatrix} \right] \in \mathcal{AP}_1^\theta.$$

\square

The map $\pi_\theta : \mathcal{K}_\theta \rightarrow \mathcal{AP}_1^\theta$, $\pi_\theta(\mathbf{x}) = [\mathbf{x}]$ (or alternatively, $\tilde{\pi}_\theta : \mathcal{K}_\theta \rightarrow \mathcal{D}$, $\tilde{\pi}_\theta(\mathbf{x}) = x_2 x_1^{-1}$) is an analogue of the classical Hopf fibration.

It is useful to describe the action of $\mathcal{U}(\theta)$ on the model \mathcal{D} . By straightforward computations, if $\tilde{g} \in \mathcal{U}(\theta)$ and $z \in \mathcal{D}$, then

$$\tilde{g} \cdot z = (g_{21} + g_{22}z)(g_{11} + g_{12}z)^{-1}. \quad (1)$$

Remark 4.9. In particular, note that if $\tilde{k} \in \mathcal{U}(\theta)$ satisfies $\tilde{k} \cdot 0 = 0$, then $\tilde{k} = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix}$, with $u_1, u_2 \in \mathcal{U}_\mathcal{A}$.

We finish this section by addressing the characterization of the local regular structure of \mathcal{AP}_1^θ . Instead of exhibiting an atlas of local coordinates, we refer to an intrinsic model for \mathcal{AP}_1^θ . In [1] we studied the space \mathcal{Q}_ρ , defined as the space

$$\mathcal{Q}_\rho = \{\epsilon \in M_2(\mathcal{A}) : \epsilon^2 = 1 \text{ and } \rho\epsilon \in G^+\}.$$

The elements $\epsilon \in M_2(\mathcal{A})$ satisfying $\epsilon^2 = 1$, usually called reflections, are in (natural) one to one correspondence with projections $q \in M_2(\mathcal{A})$, $q^2 = q$: $\epsilon \leftrightarrow q = \frac{1}{2}(1 + \epsilon)$. The condition $\epsilon\rho \in G^+$ implies, in particular, that ϵ , and thus q , is θ -selfadjoint. More precisely, the projections q in \mathcal{Q}_ρ , correspond to the submodules $\ell \in \mathcal{AP}_1^\theta$ (see [1]), by means of the one to one mapping

$$\mathcal{Q}_\rho \ni q \longleftrightarrow \ell = R(q) \in \mathcal{AP}_1^\theta.$$

In [15] it was shown that \mathcal{Q}_ρ is a complemented submanifold of $M_2(\mathcal{A})$. A benefit we obtain from this coordinate free identification $\mathcal{AP}_1^\theta \simeq \mathcal{Q}_\rho$, is a description of the tangent spaces of \mathcal{AP}_1^θ as subspaces of θ -selfadjoint elements in $M_2(\mathcal{A})$. We shall specify this in the next section.

5 The tangent spaces of \mathcal{AP}_1^θ

In this short section we characterize the tangent spaces of the projective line \mathcal{AP}_1 , and its hyperbolic part \mathcal{AP}_1^θ . If $\mathbf{x}_0 \in \mathcal{O}$, we identify \mathcal{AP}_1 with $\mathcal{O}/\mathbf{x}_0 \cdot G$, because \mathcal{O} is open in \mathcal{A}^2 (and G is open in \mathcal{A}). Then we have

$$(T\mathcal{AP}_1)_{[\mathbf{x}_0]} \simeq \mathcal{A}^2 / \mathbf{x}_0 \cdot \mathcal{A} = \mathcal{A}^2 / [\mathbf{x}_0].$$

Note that \mathcal{AP}_1^θ is open in \mathcal{AP}_1 .

Therefore, if $[\mathbf{x}_0] \in \mathcal{AP}_1^\theta$,

$$(T\mathcal{AP}_1^\theta)_{[\mathbf{x}_0]} = (T\mathcal{AP}_1)_{[\mathbf{x}_0]} \simeq \mathcal{A}^2 / \mathbf{x}_0 \cdot \mathcal{A} = \mathcal{A}^2 / [\mathbf{x}_0].$$

Given a submodule $\ell \subset \mathcal{A}^2$, denote by ℓ^{\perp_θ} the θ -orthogonal complement of ℓ , if it exists (not every submodule ℓ has a θ -orthogonal complement).

Lemma 5.1. *If $\mathbf{x}_0 \in \mathcal{O}$, then there exists the submodule $[\mathbf{x}_0]^{\perp_\theta}$. It is generated by an element $\mathbf{y}_0 \in \mathcal{A}_{reg}^2$ (not necessarily in \mathcal{O}).*

Proof. Consider

$$\mathbf{y}_0 = \begin{pmatrix} (x_1^*)^{-1}x_2^* \\ 1 \end{pmatrix}.$$

Straightforward computations show that $\theta(\mathbf{x}_0, \mathbf{y}_0) = 0$. It is easy to see that \mathbf{x}_0 and \mathbf{y}_0 generate \mathcal{A}^2 . \square

Lemma 5.2. *If $[\mathbf{x}_0] \in \mathcal{AP}_1^\theta$, then the form θ is negative and non degenerate in $[\mathbf{x}_0]^{\perp_\theta}$: if $\mathbf{y} \in [\mathbf{x}_0]^{\perp_\theta}$, then $-\theta(\mathbf{y}, \mathbf{y}) \in G^+$.*

Proof. Recall that $[\mathbf{x}_0] \in \mathcal{AP}_1^\theta$ means that for any generator (e.g. $\mathbf{x}_0 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$), $x_1 \in G$ and $\theta(\mathbf{x}_0, \mathbf{x}_0) \in G^+$. Put $\mathbf{y}_0 = \begin{pmatrix} (x_1^*)^{-1}x_2^* \\ 1 \end{pmatrix}$ as above. Then, for any $\mathbf{y} = \mathbf{y}_0 a \in [\mathbf{x}_0]^\perp$,

$$\theta(\mathbf{y}, \mathbf{y}) = a^* \theta(\mathbf{y}_0, \mathbf{y}_0) a,$$

and

$$\theta(\mathbf{y}_0, \mathbf{y}_0) = x_2 x_1^{-1} (x_1^*)^{-1} x_2^* - 1 = x_2 x_1^{-1} (x_2 x_1^{-1})^* - 1.$$

From Lemma 4.7 and Proposition 4.8, we get that $x_2 x_1^{-1} \in \mathcal{D}$, i.e., $\|x_2 x_1^{-1}\| < 1$; thus, $1 - x_2 x_1^{-1} (x_2 x_1^{-1})^* \in G^+$. \square

For $[\mathbf{x}_0] \in \mathcal{AP}_1^\theta$, the tangent space $(T\mathcal{AP}_1^\theta)_{[\mathbf{x}_0]} = \mathcal{A}^2/[\mathbf{x}_0]$ is isomorphic to any supplement of $[\mathbf{x}]$ in \mathcal{A}^2 . We choose to identify

$$(T\mathcal{AP}_1^\theta)_{[\mathbf{x}_0]} \simeq [\mathbf{x}_0]^{\perp_\theta}. \quad (2)$$

Remark 5.3. If $\ell = [\mathbf{x}_0] \in \mathcal{AP}_1^\theta$, the identification of $(T\mathcal{AP}_1^\theta)_\ell$ with ℓ^{\perp_θ} depends on the choice of the generator \mathbf{x}_0 . In order to see how the change of generator affects this identification, we must refer to an intrinsic model for \mathcal{AP}_1^θ . We choose the model \mathcal{Q}_ρ described in the first section. Also, to keep matters more simple, consider generators in \mathcal{K}_θ , i.e. \mathbf{x}_0 satisfies $\theta(\mathbf{x}_0, \mathbf{x}_0) = 1$. If $\ell = q_\ell$ (the unique θ -orthogonal projection onto ℓ) then

$$(T\mathcal{AP}_1^\theta)_\ell = (T\mathcal{Q}_\rho)_{q_\ell} = \{X \in M_2(\mathcal{A}) : X \text{ is } \theta\text{-symmetric and } \rho\text{-co-diagonal}\}.$$

If $\mathbf{x}'_0 \in \mathcal{K}_\theta$ is another generator of ℓ , then there exists a unitary $u \in \mathcal{U}_\mathcal{A}$ such that $\mathbf{x}'_0 = \mathbf{x}_0 u$. In the identification between $(T\mathcal{AP}_1^\theta)_\ell$ and \mathcal{A}^2/ℓ done by means of \mathbf{x}_0 , the tangent vector X is identified with $X\mathbf{x}_0 \in \ell^{\perp_\theta}$.

Thus, if we change to \mathbf{x}'_0 , both identifications differ in right multiplication by u .

6 The geometry of the disk

In the previous section we introduced an $\mathcal{U}(\theta)$ -invariant Finsler metric in the hyperbolic part of the \mathcal{A} -projective line \mathcal{AP}_1^θ induced by the quadratic form θ . Also, we noted that there is a natural diffeomorphism $[\mathbf{x}] \mapsto x_2 x_2^{-1}$ between \mathcal{AP}_1^θ and the unit disk \mathcal{D} of \mathcal{A} .

In [1], we introduced a (nonpositively curved) metric in \mathcal{D} , by establishing in turn an identification between \mathcal{D} and a space of positive operators related to the symmetry ρ (related to θ). Let us briefly describe it:

Remark 6.1. (several results taken from [1])

1. \mathcal{D} is embedded in the space $GL_2(\mathcal{A})^+$ of positive elements in $GL_2(\mathcal{A})$ by means of the map

$$\begin{aligned}\Phi_{\mathcal{D}} : \mathcal{D} &\rightarrow GL_2(\mathcal{A})^+, \quad \Phi_{\mathcal{D}}(a) = \begin{pmatrix} 2(1 - a^*a)^{-1} - 1 & -2(1 - a^*a)^{-1}a^* \\ -2a(1 - a^*a)^{-1} & 2a(1 - a^*a)^{-1}a^* + 1 \end{pmatrix} \\ &= -\rho + 2 \begin{pmatrix} (1 - a^*a)^{-1} & 0 \\ 0 & (1 - aa^*)^{-1} \end{pmatrix} \begin{pmatrix} 1 & -a^* \\ -a & aa^* \end{pmatrix}.\end{aligned}$$

For the last equality, we use that $a(1 - a^*a)^{-1} = (1 - aa^*)^{-1}a$. Also note that

$$\Phi_{\mathcal{D}}(a) = \begin{pmatrix} (1 - a^*a)^{-1} & 0 \\ 0 & (1 - aa^*)^{-1} \end{pmatrix} \begin{pmatrix} 1 + a^*a & -2a^* \\ -2a & 1 + aa^* \end{pmatrix}$$

where both matrices commute.

2. Therefore, given two points $z_0, z_1 \in \mathcal{D}$ there exists a unique geodesic joining them. In [1] we computed the velocity of this unique geodesic in the case $z_0 = 0$ and $z_1 = z$ is an arbitrary element of \mathcal{D} . The geodesic is given by

$$\delta(t) = e^{t \begin{pmatrix} 0 & \alpha^* \\ \alpha & 0 \end{pmatrix}} \cdot 0,$$

where

$$\alpha = z \sum_{k=0}^{\infty} \frac{1}{2k+1} (z^*z)^k, \quad (3)$$

satisfies that $\delta(0) = 0$ and $\delta(1) = z$. Note here that the series $\sum_{k=0}^{\infty} \frac{t^{2k+1}}{2k+1}$ corresponds, in the interval $(-1, 1)$, to the function $f(t) = \frac{1}{2} \log(\frac{1+t}{1-t})$. We shall compute below the explicit form of δ in \mathcal{D} .

3. The norm of $(T\mathcal{D})_0$ is the usual norm of \mathcal{A} .
4. Recall from Proposition ?? the hyperboloid \mathcal{K}_{θ} . We have the following commutative diagram:

$$\begin{array}{ccc} & \mathcal{K}_{\theta} & \\ \pi_{\theta} \swarrow & & \searrow \tilde{\pi}_{\theta} \\ \mathcal{AP}_1^{\theta} & \xrightarrow{\simeq} & \mathcal{D}, \end{array} \quad (4)$$

where the map $\mathcal{K}_{\theta} \rightarrow \mathcal{AP}_1^{\theta}$ is the restriction of the quotient map $(\mathcal{O} \rightarrow \mathcal{AP}_1)$. The group $\mathcal{U}(\theta)$ acts on the three spaces, and the maps are equivariant with respect to the action.

Let us compute explicitly the form of the geodesic δ joining 0 and $z \in \mathcal{D}$ at time $t = 1$:

Lemma 6.2. *Given $z \in \mathcal{D}$, the unique geodesic δ of \mathcal{D} with $\delta(0) = 0$ and $\delta(1) = z$ is given by*

$$\delta(t) = \omega \tanh(t|\alpha|), \quad (5)$$

where α is given in (3) above, and ω is the partial isometry in the polar decomposition of α : $\alpha = \omega|\alpha|$, performed in \mathcal{A}^{**} .

Proof. Straightforward computations show that the even and odd powers of $\begin{pmatrix} 0 & \alpha^* \\ \alpha & 0 \end{pmatrix}$ are, respectively

$$\begin{pmatrix} 0 & \alpha^* \\ \alpha & 0 \end{pmatrix}^{2k} = \begin{pmatrix} (\alpha^* \alpha)^k & 0 \\ 0 & (\alpha \alpha^*)^k \end{pmatrix} \text{ and } \begin{pmatrix} 0 & \alpha^* \\ \alpha & 0 \end{pmatrix}^{2k+1} = \begin{pmatrix} 0 & \alpha^* (\alpha \alpha^*)^k \\ \alpha (\alpha^* \alpha)^k & 0 \end{pmatrix}.$$

We are interested in the first column of $e^{t \begin{pmatrix} 0 & \alpha^* \\ \alpha & 0 \end{pmatrix}}$, which is

$$\begin{pmatrix} \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!} (\alpha^* \alpha)^k \\ \alpha \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} (\alpha^* \alpha)^k \end{pmatrix} = \begin{pmatrix} \cosh(t|\alpha|) \\ \omega \sinh(t|\alpha|) \end{pmatrix}.$$

Then

$$\delta(t) = \omega \sinh(t|\alpha|) (\cosh(t|\alpha|))^{-1} = \omega \tanh(t|\alpha|).$$

□

Notice that $\omega \in \mathcal{A}^{**}$ need not belong to \mathcal{A} . However $\delta(t) \in \mathcal{A}$ for all t .

Let us relate the polar decompositions of z and α .

Proposition 6.3. *If $z \in \mathcal{D}$ and α as in (3), then*

$$\alpha = \frac{1}{2} \omega (\log(1 + |z|) - \log(1 - |z|)), \quad \text{and} \quad z = \omega |z|,$$

*i.e., the partial isometry $\omega \in \mathcal{A}^{**}$ is the same for α and z .*

Proof. First note that

$$|\alpha|^2 = \alpha^* \alpha = \sum_{k=0}^{\infty} \frac{1}{2k+1} (z^* z)^k z^* z \sum_{k=0}^{\infty} \frac{1}{2k+1} (z^* z)^k = \left(\sum_{k=0}^{\infty} \frac{1}{2k+1} |z|^{2k+1} \right)^2,$$

i.e., $|\alpha| = \frac{1}{2} (\log(1 + |z|) - \log(1 - |z|))$. Next, put $z = \mu |z|$ the polar decomposition of z . Note that, since $\alpha = z \sum_{k=0}^{\infty} \frac{1}{2k+1} |z|^{2k}$, we have that

$$\alpha = \mu |z| \sum_{k=0}^{\infty} \frac{1}{2k+1} |z|^{2k} = \mu |\alpha|.$$

Thus, in order to prove our claim, it suffices to show that both partial isometries μ, ω have the same initial and final spaces (the result follows, then, by the uniqueness property of the polar decomposition). The partial isometry μ maps $N(z)^\perp = N(|z|)^\perp$ onto $\overline{R(z)}$, whereas ω maps $N(\alpha)^\perp$ onto $\overline{R(\alpha)}$. If $z\xi = 0$, then

$$\alpha \xi = z \sum_{k=0}^{\infty} \frac{1}{2k+1} (z^* z)^k \xi = \sum_{k=0}^{\infty} \frac{1}{2k+1} (z z^*)^k z \xi = 0,$$

i.e., $N(z) \subset N(\alpha)$. Conversely, in the last expression of α , $\alpha = \sum_{k=0}^{\infty} \frac{1}{2k+1} |z^*|^{2k} z$; observe that

$$\sum_{k=0}^{\infty} \frac{1}{2k+1} |z^*|^{2k} = g(|z^*|),$$

where $g(t) = \frac{1}{2t}(\log(1+t) - \log(1-t))$ (which can be extended continuously as $g(0) = 1$), is defined in $\sigma(|z^*|) \subset [0, 1)$, and is nonvanishing there. Therefore, $g(|z^*|)$ is invertible. Thus, $\alpha\xi = 0$ implies $z\xi = 0$. Again, using the function g , we get $\alpha = zg(|z|)$, and thus $R(z) = R(\alpha)$. \square

Corollary 6.4. *The exponential map Exp_0 of \mathcal{D} at 0, and its inverse Log_0 can be written explicitly as follows: if $z = \omega|z| \in \mathcal{D}$*

$$\text{Log}_0 : \mathcal{D} \rightarrow (T\mathcal{D})_0, \quad \text{Log}_0(z) = \frac{1}{2}\omega \log((1+|z|)(1-|z|)^{-1}).$$

If $\alpha = \omega|\alpha| \in \mathcal{A} \simeq (T\mathcal{D})_0$, then

$$\text{Exp}_0 : (T\mathcal{D})_0 \rightarrow \mathcal{D}, \quad \text{Exp}_0(\alpha) = \omega \tanh(|\alpha|).$$

In particular, if $\alpha = \text{Log}_0(z)$ (or, equivalently, $z = \text{Exp}_0(\alpha)$), then z and α have the same partial isometry in the polar decomposition. Also,

$$|\text{Exp}_0(\alpha)| = \tanh(|\alpha|) \quad \text{and} \quad |\text{Log}_0(z)| = \frac{1}{2} \log((1+|z|)(1-|z|)^{-1}).$$

7 Limit points of geodesics

One of our concerns in computing the geodesic δ , and the above results on the polar decomposition, is to establish the following result:

Theorem 7.1. *For $z \in \mathcal{D}$, let δ be the unique geodesic of \mathcal{D} such that $\delta(0) = 0$ and $\delta(1) = z$. Put $z = \omega|z|$ the polar decomposition (i.e., $\omega \in \mathcal{A}^{**}$); then*

$$\text{SOT} - \lim_{t \rightarrow \infty} \delta(t) = \omega \quad \text{and} \quad \text{SOT} - \lim_{t \rightarrow -\infty} \delta(t) = -\omega.$$

Proof. By formula (5), we only need to compute the limit of $\tanh(t|z|)$ when $t \rightarrow \pm\infty$. The spectrum $\sigma(|z|)$ is contained in $[0, 1)$. Clearly, for any $s \in [0, 1)$,

$$\lim_{t \rightarrow \infty} \tanh(ts) = \begin{cases} 1 & \text{if } s \in (0, 1) \\ 0 & \text{if } s = 0. \end{cases}$$

By Lebesgue's bounded convergence theorem, and the Borel functional calculus for bounded selfadjoint operators, we have that

$$\lim_{t \rightarrow \infty} \tanh(t|\alpha|) = \chi_{(0,1)}(|\alpha|) = P_{N(\alpha)^\perp}.$$

Then,

$$\lim_{t \rightarrow \infty} \delta(t) = \omega P_{N(\alpha)^\perp} = \omega.$$

Similarly, using that $\lim_{t \rightarrow -\infty} \tanh(st) = \begin{cases} -1 & \text{if } s \in (0, 1) \\ 0 & \text{if } s = 0 \end{cases}$, we get that

$$\lim_{t \rightarrow -\infty} \delta(t) = -\omega P_{N(\alpha)^\perp} = -\omega.$$

\square

This geometric role of the partial isometry ω in the polar decomposition of $z \in \mathcal{D}$ (or, more generally, of every $z \in \mathcal{A} \setminus \{0\}$) has not been noticed before, to the authors' knowledge.

In order to compute the limit points of arbitrary geodesics, it will be useful to extend the action of $\mathcal{U}(\theta)$ to the strong operator border

$$\partial\mathcal{D} := \{a \in \mathcal{A}^{**} : \|a\| = 1\}$$

of \mathcal{D} , i.e., to define $\tilde{g} \cdot a$ for $a \in \mathcal{A}^{**}$ with $\|a\| = 1$.

Lemma 7.2. *If $\tilde{g} \in \mathcal{U}(\theta)$ and $a \in \partial\mathcal{D}$, then $g_{11} + g_{12}a$ is invertible in \mathcal{A}^{**} .*

Proof. Note that

$$g_{11} + g_{12}a = \langle \mathbf{e}_1, \tilde{g} \begin{pmatrix} 1 \\ a \end{pmatrix} \rangle.$$

Using the polar decomposition $\tilde{g} = \tilde{u}|\tilde{g}|$, $\tilde{u} = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix}$,

$$\langle \mathbf{e}_1, \tilde{g} \begin{pmatrix} 1 \\ a \end{pmatrix} \rangle = u_1^* \langle \mathbf{e}_1, |\tilde{g}| \begin{pmatrix} 1 \\ a \end{pmatrix} \rangle,$$

i.e., we may suppose $\tilde{g} \geq 0$, $\tilde{g} = \begin{pmatrix} (1 + b^*b)^{1/2} & b^* \\ b & (1 + bb^*)^{1/2} \end{pmatrix}$, for $b \in \mathcal{A}$. Then

$$g_{11} + g_{12}a = (1 + b^*b)^{1/2} + b^*a = (1 + b^*b)^{1/2}(1 + (1 + b^*b)^{-1/2}b^*a).$$

It suffices to show that $\|(1 + b^*b)^{-1/2}b^*a\| < 1$. Note that, since $\|a\| = 1$,

$$\|(1 + b^*b)^{-1/2}b^*a\|^2 \leq \|(1 + b^*b)^{-1/2}b^*\|^2 = \|(1 + b^*b)^{-1/2}b^*b(1 + b^*b)^{-1/2}\| = \max\{f(t) : t \in \sigma(b^*b)\},$$

for $f(t) = \frac{t}{(1+t)}$. Clearly, this number is strictly less than 1. \square

Proposition 7.3. *If $a \in \partial\mathcal{D}$, and $\tilde{g} \in \mathcal{U}(\theta)$, then*

$$\tilde{g} \cdot a := (g_{21} + g_{22}a)(g_{11} + g_{12}a)^{-1} \in \partial\mathcal{D},$$

defines a left action of $\mathcal{U}(\theta)$ on $\partial\mathcal{D}$.

Proof. A density argument (or a proof similar as in the previous lemma), shows that if $x \in \mathcal{A}^{**}$ with $\|x\| < 1$, then $\tilde{g} \cdot x$, defined as above, also satisfies $\|\tilde{g} \cdot x\| < 1$, and defines an action on the unit ball of \mathcal{A}^{**} . Let $a \in \partial\mathcal{D}$. Then, by Kaplansky's density theorem, there exists a sequence $a_n \in \mathcal{D}$ such that $a_n \rightarrow a$ in the strong operator topology. We claim that

Lemma 7.4. *$\tilde{g} \cdot a_n \rightarrow \tilde{g} \cdot a$ strongly.*

Proof. Consider the polar decomposition $\tilde{g} = \tilde{u}|\tilde{g}|$. We check first that $|\tilde{g}| \cdot a_n \rightarrow |\tilde{g}| \cdot a$ strongly.

As before, $\tilde{g} = \begin{pmatrix} (1 + b^*b)^{1/2} & b^* \\ b & (1 + bb^*)^{1/2} \end{pmatrix}$. Then

$$|\tilde{g}| \cdot a_n = (b + (1 + bb^*)^{1/2}a_n)((1 + b^*b)^{1/2} + b^*a_n)^{-1}.$$

Clearly, $b + (1 + bb^*)^{1/2}a_n \rightarrow b + (1 + bb^*)^{1/2}a$ and $(1 + b^*b)^{1/2} + b^*a_n \rightarrow (1 + b^*b)^{1/2} + b^*a$ strongly. Moreover,

$$((1 + b^*b)^{1/2} + b^*a_n)^{-1} = (1 + (1 + b^*b)^{-1/2}b^*a_n)^{-1}(1 + b^*b)^{-1/2}.$$

Let us show that the norms of these inverses are uniformly bounded. It suffices to see that $\|(1 + (1 + b^*b)^{-1/2}b^*a_n)^{-1}\|$ are uniformly bounded. Denote $d_n = (1 + b^*b)^{-1/2}b^*a_n$. Then, as seen above,

$$\|d_n\|^2 \leq \max\{f(t) : t \in \sigma(b^*b)\} = r^2 < 1.$$

Thus,

$$\|(1 + (1 + b^*b)^{-1/2}b^*a_n)^{-1}\| \leq \frac{1}{1 - r}.$$

Therefore the inverses $(1 + (1 + b^*b)^{-1/2}b^*a_n)^{-1}$ converges strongly to $(1 + (1 + b^*b)^{-1/2}b^*a)^{-1}$. Then, clearly, $c_n := |\tilde{g}| \cdot a_n$ converge strongly to $c := |\tilde{g}| \cdot a$. Since $\tilde{u} = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix}$, it is clear that

$$\tilde{g} \cdot a_n = \tilde{u} \cdot (|\tilde{g}| \cdot a_n) = u_2 c_n u_1^* \rightarrow u_2 c u_1^* = \tilde{g} \cdot a.$$

□

Let us proceed with the proof of Proposition 7.3. Since $\|a_n\| < 1$, we know that $\|\tilde{g} \cdot a_n\| < 1$. From Lemma 7.4, it follows that $\|\tilde{g} \cdot a\| \leq 1$. Suppose that $\|\tilde{g} \cdot a\| < 1$. Using again the fact that $\mathcal{U}(\theta)$ acts on \mathcal{D} , this would imply that

$$\tilde{g}^{-1} \cdot (\tilde{g} \cdot a) = a \in \mathcal{D},$$

a contradiction. Thus, $\|\tilde{g} \cdot a\| = 1$.

The fact that this rule defines, indeed, a left action, follows from similar density arguments. □

Using this result, we can compute the limit points of arbitrary geodesics. Since the action of $\mathcal{U}(\theta)$ is transitive, given $z_1, z_2 \in \mathcal{D}$, there exists $\tilde{g} \in \mathcal{U}(\theta)$ such that $\tilde{g} \cdot z_1 = 0$.

Corollary 7.5. *Let $z_0, z_1 \in \mathcal{D}$ and $\tilde{g} \in \mathcal{U}(\theta)$ such that $\tilde{g} \cdot z_0 = 0$. Let δ be the unique geodesic of \mathcal{D} such that $\delta(0) = z_0$ and $\delta(1) = z_1$. Denote by $\dot{\delta}_0$ the initial velocity of δ . Then*

$$\text{SOT} - \lim_{t \rightarrow +\infty} \delta(t) = \tilde{g} \cdot \omega_0 \text{ and } \text{SOT} - \lim_{t \rightarrow -\infty} \delta(t) = \tilde{g} \cdot (-\omega_0),$$

where $\omega_0 \in \mathcal{A}^{**}$ is the partial isometry in the polar decomposition of $\dot{\delta}_0$: $\dot{\delta}_0 = \omega_0 |\dot{\delta}_0|$.

Remark 7.6. In order to identify these limit points in \mathcal{D} , following the notation of the above Corollary, note that if $\tilde{g} = |\tilde{g}^*| \tilde{v} = \begin{pmatrix} (1 + b^*b)^{1/2} & b^* \\ b & (1 + bb^*)^{1/2} \end{pmatrix} \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix}$ (the reversed polar decomposition), then $\tilde{v} \cdot \omega_0 = v_2 \omega_0 v_1^*$ is a partial isometry. Therefore, the limit points of geodesics are elements in $\partial \mathcal{D}$ of the form

$$(b + (1 + bb^*)^{1/2}\omega)((1 + b^*b)^{1/2} + b^*\omega)^{-1} \text{ and } (b - (1 + bb^*)^{1/2}\omega)((1 + b^*b)^{1/2} - b^*\omega)^{-1},$$

where $b \in \mathcal{A}$ is arbitrary and $\omega \in \mathcal{A}^{**}$ is a partial isometry.

Note that not every partial isometry in \mathcal{A}^{**} occurs in the polar decomposition of an element in \mathcal{D} . For instance, if $\mathcal{A} = C([0, 1])$ (continuous functions in the unit interval), the polar decomposition of $f \in \mathcal{A}$ is $f = w|f|$, where $w \in L^\infty(0, 1)$ is given by $w(t) = \begin{cases} f(t)/|f(t)| & \text{if } f(t) \neq 0 \\ 0 & \text{if } f(t) = 0 \end{cases}$.

An arbitrary partial isometry in $L^\infty(0, 1)$ is a measurable function whose values are zero or complex numbers of modulus 1. The set of zeros of such a function is an arbitrary measurable set, whereas the set of zeros of partial isometries which occur in the polar decomposition of a continuous function, are closed subsets of $[0, 1]$.

Another way to study the limit points of geodesics, is by using the Borel subgroup $\mathcal{B}_\theta \subset \mathcal{U}(\theta)$ instead. Indeed, since the action of this group is transitive in \mathcal{D} , any limit point of a geodesic is either of the form $\tilde{g} \cdot v$ or $\tilde{g} \cdot (-v)$, for $\tilde{g} \in \mathcal{B}_\theta$. Consider the following example:

Example 7.7. Suppose that \mathcal{A} is a von Neumann algebra, and let $p \neq 0$ be a projection in \mathcal{A} .

For $\tilde{g} = \begin{pmatrix} \frac{g + \hat{g}}{2} - \hat{g}x & \frac{g - \hat{g}}{2} - \hat{g}x \\ \frac{g - \hat{g}}{2} + \hat{g}x & \frac{g + \hat{g}}{2} + \hat{g}x \end{pmatrix}$, let us compute $\tilde{g} \cdot p$. After straightforward computations,

$$\tilde{g} \cdot p = (g(1 + p) + \hat{g}(-1 + p + 2x(1 + p))) (g(1 + p) + \hat{g}(1 - p - 2x(1 + p)))^{-1}.$$

Note that $1 + p$ is invertible and that $(1 - p)(1 + p)^{-1} = 1 - p$. Then

$$\begin{aligned} \tilde{g} \cdot p &= (1 + \hat{g}(p - 1 + 2x(1 + p)) (1 + p)^{-1} g^{-1}) (1 + \hat{g}(1 - p - 2x(1 + p)) (1 + p)^{-1} g^{-1})^{-1} = \\ &= (1 + \hat{g}(p - 1 + 2x)g^{-1}) (1 + \hat{g}(1 - p - 2x)g^{-1})^{-1}. \end{aligned}$$

Denote $\alpha = \hat{g}(p - 1)g^{-1}$ and $\beta = 2\hat{g}xg^{-1}$. Observe that α is a non-invertible selfadjoint element, $\alpha \leq 0$ and its range is proper and closed; β is an arbitrary anti-selfadjoint element. Then

$$\tilde{g} \cdot p = (1 + \alpha + \beta) (1 - (\alpha + \beta))^{-1}.$$

Note that $1 - (\alpha + \beta)$ is invertible because $\operatorname{Re}(1 - (\alpha + \beta)) = 1 - \alpha \geq 1$. If one picks $p = 1$, then $\alpha = 0$ and

$$\tilde{g} \cdot 1 = (1 + \beta)(1 - \beta)^{-1},$$

which is a unitary operator such that -1 does not belong to its spectrum. In particular, this shows that the action of \mathcal{B}_θ ceases to be transitive in $\partial\mathcal{D}$.

Our next result shows a necessary condition for an element $a \in \mathcal{A}^{**}$ with $\|a\| = 1$ to be a limit point of a geodesic of \mathcal{D}

Proposition 7.8. *If $a \in \mathcal{A}^{**}$ is the limit point at $+\infty$ of a geodesic in \mathcal{D} , then $1 - a^*a = hqh$, where $h \in G^+$ and $q \in \mathcal{A}^{**}$ is a projection. In particular, not every element of norm 1 in \mathcal{A}^{**} is the limit point of a geodesic: such elements satisfy that the defect element $1 - a^*a$ has closed range.*

Proof. If a is the limit point of a geodesic if and only if there exists $\tilde{g} \in \mathcal{U}(\theta)$ and a partial isometry $\omega \in \mathcal{A}^{**}$ such that $a = \tilde{g} \cdot a$. The definition of the action implies that this equality can be read as an usual matrix equality

$$\tilde{g} \begin{pmatrix} 1 \\ \omega \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix},$$

with $b_1 \in G$ and $b_2 b_1^{-1} = a$. Using the form θ , and the fact that $\tilde{g} \in \mathcal{U}(\theta)$,

$$\theta\left(\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\right) = \theta\left(\begin{pmatrix} 1 \\ \omega \end{pmatrix}, \begin{pmatrix} 1 \\ \omega \end{pmatrix}\right),$$

i.e.,

$$b_1^* b_1 - b_2^* b_2 = 1 - \omega^* \omega = q',$$

which is a projection in \mathcal{A}^{**} . Let $b_1 = u|b_1|$ be the polar decomposition, with u unitary. Then $au = b_2 b_1^{-1} u = b_2 |b_1|^{-1}$. Thus,

$$q' = b_1^* b_1 - b_2^* b_2 = |b_1|(1 - |b_1|^{-1} b_2^* b_2 |b_1|^{-1})|b_1| = |b_1|(1 - u^* a^* a u)|b_1|,$$

i.e.,

$$1 - a^* a = u|b_1|^{-1} q' |b_1|^{-1} u^* = h q h,$$

for $q = u q' u^*$ a projection in \mathcal{A}^{**} and $h = u|b_1|^{-1} u^* \in G^+$. \square

Remark 7.9. We believe that the characterization of the partial isometries which appear in the polar decompositions of all limit points $a \in \mathcal{A}^{**}$ is an interesting open problem.

8 The invariant metric in \mathcal{AP}_1^θ

The characterization of the tangent spaces done in (2), Section 5, enables us to define a Finsler metric, that is, a continuous distribution of norms on the tangent bundle of \mathcal{AP}_1^θ .

Given $\ell \in \mathcal{AP}_1^\theta$ and $V \in (T\mathcal{AP}_1^\theta)_\ell$, fix a generator $\mathbf{x}_0 \in \mathcal{K}_\theta$ for ℓ , i.e., $[\mathbf{x}_0] = \ell$ and $\theta(\mathbf{x}_0, \mathbf{x}_0) = 1$. Recall, from Remark 5.3, that \mathbf{x}_0 is determined up to a unitary element of \mathcal{A} : if \mathbf{x}'_0 is another such generator, then there exists $u \in \mathcal{U}_\mathcal{A}$ such that $\mathbf{x}'_0 = \mathbf{x}_0 u$. Having fixed a generator for ℓ , as we saw in Section 5, $(T\mathcal{AP}_1^\theta)_\ell$ identifies with ℓ^{\perp_θ} , and to the tangent vector V corresponds an element $\mathbf{v} \in \ell^{\perp_\theta}$. We define

$$|V|_\ell := \|\theta(\mathbf{v}, \mathbf{v})\|^{1/2}. \quad (6)$$

Note that $|V|_\ell$ does not depend on the choice of the generator. If we choose $\mathbf{x}'_0 = \mathbf{x}_0 u$ instead, the tangent vector V is represented by $\mathbf{v}' = \mathbf{v} u \in \ell^{\perp_\theta}$, and therefore

$$\|\theta(\mathbf{v}', \mathbf{v}')\| = \|\theta(\mathbf{v} u, \mathbf{v} u)\| = \|u^* \theta(\mathbf{v}, \mathbf{v}) u\| = \|\theta(\mathbf{v}, \mathbf{v})\|.$$

Next, recall from Lemma 5.2, that θ is negative definite (non degenerate), and therefore the expression (6) above defines a proper norm in $(T\mathcal{AP}_1^\theta)_\ell$.

Remark 8.1. If $V \in (T\mathcal{AP}_1^\theta)_{[\mathbf{x}_0]}$, by the identification in (2), Section 5, V is represented by some $\mathbf{v} \in [\mathbf{x}_0]^{\perp_\theta}$; since $\mathbf{y}_0 = \begin{pmatrix} (x_1^*)^{-1} x_2 \\ 1 \end{pmatrix}$, for $\mathbf{x}_0 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, is a generator of $[\mathbf{x}_0]^{\perp_\theta}$, there exists $a \in \mathcal{A}$ such that $\mathbf{v} = \mathbf{y}_0 a$. Then

$$|V|_{[\mathbf{x}_0]} = \|\theta(\mathbf{y}_0 a, \mathbf{y}_0 a)\|^{1/2} = \|a^* (1 - x_2 x_1^{-1})^* a\|^{1/2} = \|(1 - |(x_2 x_1^{-1})^*|^2)^{1/2} a\|.$$

Note also that the norm of $\mathbf{v} = \mathbf{y}_0 a$ in \mathcal{A}^2 is

$$\|\mathbf{y}_0 a\| = \|\langle \mathbf{y}_0 a, \mathbf{y}_0 a \rangle\|^{1/2} = \|(1 + |(x_2 x_1^{-1})^*|^2)^{1/2} a\|.$$

Proposition 8.2. *For any $\ell \in \mathcal{AP}_1^\theta$, the norm $|\cdot|_\ell$ of $(T\mathcal{AP}_1^\theta)_\ell$ is complete.*

Proof. With the current notations, if $V = \mathbf{y}_0 a \in (T\mathcal{AP}_1^\theta)_{[\mathbf{x}_0]}$,

$$\begin{aligned} |V|_{[\mathbf{x}_0]} &= \|(1 - |(x_2 x_1^{-1})^*|^2)^{1/2} a\| = \|(1 - |(x_2 x_1^{-1})^*|^2)^{1/2} (1 + |(x_2 x_1^{-1})^*|^2)^{-1/2} (1 + |(x_2 x_1^{-1})^*|^2)^{1/2} a\| \\ &\leq \|\{(1 - |(x_2 x_1^{-1})^*|^2)(1 + |(x_2 x_1^{-1})^*|^2)^{-1}\}^{1/2}\| \|\mathbf{y}_0 a\|. \end{aligned}$$

Similarly

$$\|\mathbf{y}_0 a\| \leq \|\{(1 + |(x_2 x_1^{-1})^*|^2)(1 - |(x_2 x_1^{-1})^*|^2)^{-1}\}^{1/2}\| |V|_{[\mathbf{x}_0]}.$$

It follows that on $(T\mathcal{AP}_1^\theta)_\ell \simeq [\mathbf{y}_0]$, the metric $|\cdot|_\ell$ and the norm of \mathcal{A}^2 are equivalent. Since $[\mathbf{y}_0]$ is closed in \mathcal{A}^2 , it is complete, and the proof follows. \square

The distribution $\mathcal{AP}_1^\theta \ni \ell \mapsto |\cdot|_\ell$ is clearly continuous. Thus, \mathcal{AP}_1^θ is endowed with a Finsler metric.

The following result is tautological, but of the utmost importance for our discussion:

Theorem 8.3. *The Finsler metric defined in (6) is invariant under the action of $\mathcal{U}(\theta)$.*

Proof. Pick $\ell = [\mathbf{x}_0] \in \mathcal{AP}_1^\theta$, $V \in (T\mathcal{AP}_1^\theta)_\ell$ (as before, V is identified to $\sim \mathbf{y}_0 a$) and $\tilde{g} \in \mathcal{U}(\theta)$. The action of $\mathcal{U}(\theta)$ on \mathcal{AP}_1^θ induces an action on the tangent spaces. As quotients: \tilde{g} maps $\mathcal{A}^2/[\mathbf{x}_0]$ onto $\mathcal{A}^2/[\tilde{g}\mathbf{x}_0]$, because \tilde{g} , being \mathcal{A} -linear, maps $[\mathbf{x}_0]$ onto $[\tilde{g}\mathbf{x}_0]$. But as θ -orthogonal submodules as well: since \tilde{g} preserves θ ,

$$\tilde{g}([\mathbf{y}_0]^{\perp_\theta}) = [\tilde{g}\mathbf{y}_0]^{\perp_\theta}.$$

Then

$$|\tilde{g}V|_{[\tilde{g}\mathbf{x}_0]} = \|\theta(\tilde{g}(\mathbf{y}_0 a), \tilde{g}(\mathbf{y}_0 a))\|^{1/2} = \|\theta(\mathbf{y}_0 a, \mathbf{y}_0 a)\|^{1/2} = |V|_{[\mathbf{x}_0]}.$$

\square

8.1 Invariant metric in \mathcal{D}

We need to compute the differential of the map $\mathcal{AP}_1^\theta \rightarrow \mathcal{D}$ at $[\mathbf{e}_1]$. To do so, we use the above diagram (4). Recall that $(T\mathcal{AP}_1^\theta)_{[\mathbf{x}_0]} = \mathcal{A}^2/\{\mathbf{x}_0 a : a \in \mathcal{A}\}$; in particular

$$(T\mathcal{AP}_1^\theta)_{[\mathbf{e}_1]} = \mathcal{A}^2/\mathcal{A} \times \{0\}.$$

Thus, any tangent element $V \in (T\mathcal{AP}_1^\theta)_{[\mathbf{e}_1]}$ has a unique representative $V = \left[\begin{pmatrix} 0 \\ x \end{pmatrix} \right]$, for $x \in \mathcal{A}$.

Lemma 8.4. *The differential of the map $\mathcal{AP}_1^\theta \rightarrow \mathcal{D}$, $[\mathbf{x}] \mapsto x_2 x_1^{-1}$, at $[\mathbf{e}_1]$ is the map*

$$\left[\begin{pmatrix} 0 \\ x \end{pmatrix} \right] \mapsto x, \quad x \in \mathcal{A}.$$

Proof. Fix $x \in \mathcal{A}$. We use the commutative diagram (4). Let $\mathbf{x}(t) \in \mathcal{K}_\theta$ be a smooth curve such that $\mathbf{x}(0) = \mathbf{e}_1$, and the derivative of $[\mathbf{x}(t)]$ (in \mathcal{AP}_1^θ) at $t = 0$ is $\left[\begin{pmatrix} 0 \\ x \end{pmatrix} \right]$. Then $x_1(0) = 1, x_2(0) = 0, \dot{x}_2(0) = x$. If we map $\mathbf{x}(t)$ onto \mathcal{D} , and differentiate at $t = 0$ we get:

$$\frac{d}{dt} x_2(t) x_1^{-1}(t) \big|_{t=0} = \dot{x}_2(0) x_1^{-1}(0) - x_2(0) x_1^{-1}(0) \dot{x}_1(0) x_1^{-1}(0) = x,$$

which proves our claim. \square

Theorem 8.5. *The identification $[\mathbf{x}] \longleftrightarrow x_2 x_1^{-1}$ between \mathcal{AP}_1^θ and \mathcal{D} , is isometric.*

Proof. The group $\mathcal{U}(\theta)$ acts isometrically on both \mathcal{AP}_1^θ and \mathcal{D} . The action is isometric in both spaces. Therefore, it suffices to show that the differential at $[\mathbf{e}_1]$ is an isometry. By Lemma 8.4, this map is

$$\left[\begin{pmatrix} 0 \\ x \end{pmatrix} \right] \mapsto x.$$

The norm of $\left[\begin{pmatrix} 0 \\ x \end{pmatrix} \right]$ is computed via the identification $(T\mathcal{AP}_1^\theta)_{[\mathbf{e}_1]} \simeq [\mathbf{e}_1]^\perp = [\mathbf{e}_2]$, which sends $\left[\begin{pmatrix} 0 \\ x \end{pmatrix} \right]$ to $\mathbf{e}_2 \mathbf{x}$. The norm (in \mathcal{A}^2) of this element is

$$\|x^* \theta(\mathbf{e}_2, \mathbf{e}_2) x\|^{1/2} = \|x^* x\|^{1/2} = \|x\|.$$

On the other hand, the norm of x as a tangent element of \mathcal{D} at 0 is the usual norm $\|x\|$ (in \mathcal{A}). \square

8.2 The action of the Borel subgroup \mathcal{B}_θ

As we mentioned in Proposition ?? and Remark ??, the group \mathcal{B}_θ , which we call the Borel group of the form θ , is a subgroup of $\mathcal{U}(\theta)$ which acts transitively in \mathcal{D} . Since it is a subgroup of $\mathcal{U}(\theta)$, the action is also isometric (recall also that the action is free in the hyperboloid \mathcal{K}_θ). Therefore it acts transitively and isometrically in \mathcal{AP}_1^θ .

Given $z \in \mathcal{D}$, let us find an element $\tilde{g} \in \mathcal{B}_\theta$ such that $\tilde{g} \cdot 0 = z$. First, we describe the action of $\mathcal{U}(\theta)$ on \mathcal{D} , which factors through the hyperboloid \mathcal{K}_θ :

Remark 8.6. Given $\tilde{g} \in \mathcal{U}(\theta)$ and $z \in \mathcal{D}$, we lift z to \mathcal{K}_θ by means of the global section

$$\mathcal{D} \ni z \mapsto \begin{pmatrix} (1 - z^* z)^{-1/2} \\ z(1 - z^* z)^{-1/2} \end{pmatrix}.$$

Next, we multiply $\tilde{g} \begin{pmatrix} (1 - z^* z)^{-1/2} \\ z(1 - z^* z)^{-1/2} \end{pmatrix}$, and finally we compose with the fibration

$$\mathcal{K}_\theta \ni \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto z_2 z_1^{-1} \in \mathcal{D}.$$

Definition 8.7. If $z \in \mathcal{D}$, we define

$$\begin{aligned} \tilde{g}_z &= \begin{pmatrix} (1 - z^* z)^{-1/2} & (1 + z^*)^{-1} z^* (z + 1) (1 - z^* z)^{-1/2} \\ z(1 - z^* z)^{-1/2} & (1 + z^*)^{-1} (z + 1) (1 - z^* z)^{-1/2} \end{pmatrix} \\ &= \begin{pmatrix} (1 + z^*)^{-1} & 0 \\ 0 & (1 + z^*)^{-1} \end{pmatrix} \begin{pmatrix} 1 + z^* & z^* (z + 1) \\ (1 + z^*) z & z + 1 \end{pmatrix} \begin{pmatrix} (1 - z^* z)^{-1/2} & 0 \\ 0 & (1 - z^* z)^{-1/2} \end{pmatrix}. \end{aligned}$$

Observe that the diagonal matrices on the right and left-hand sides do not belong to $\mathcal{U}(\theta)$, so that this is not a proper factorization of \tilde{g}_z .

Lemma 8.8. If $z \in \mathcal{D}$, then $\tilde{g}_z \in \mathcal{B}_\theta$ and satisfies $\tilde{g}_z \cdot 0 = z$.

Proof. The origin $0 \in \mathcal{D}$ lifts to $\mathbf{e}_1 \in \mathcal{K}_\theta$. Recall from Remark 4.5 the form of the elements in \mathcal{B}_θ :

$$\mathcal{B}_\theta = \left\{ \begin{pmatrix} \frac{g+\hat{g}}{2} - \hat{g}x & \frac{g-\hat{g}}{2} - \hat{g}x \\ \frac{g-\hat{g}}{2} + \hat{g}x & \frac{g+\hat{g}}{2} + \hat{g}x \end{pmatrix} : g \in G, x^* = -x \right\}.$$

Thus, we are looking for $g \in G$ and $x \in \mathcal{A}$ with $x^* = -x$ such that

$$\frac{1}{2}(g + \hat{g}) - \hat{g}x = (1 - z^*z)^{-1/2} \quad \text{and} \quad \frac{1}{2}(g - \hat{g}) + \hat{g}x = z(1 - z^*z)^{-1/2}.$$

That is, $g = (1 + z)(1 - z^*z)^{-1/2}$, thus $\hat{g} = (1 + z^*)^{-1}(1 - z^*z)^{1/2}$ and

$$\hat{g}x = \frac{1}{2}(1 + z^*)^{-1}\{z - z^*\}(1 - z^*z)^{-1/2}.$$

Then,

$$x = \frac{1}{2}g^*(1 + z^*)^{-1}\{z - z^*\}(1 - z^*z)^{-1/2} = \frac{1}{2}(1 - z^*z)^{-1/2}(z - z^*)(1 - z^*z)^{-1/2},$$

which is clearly anti-Hermitian, and thus $\tilde{g}_z \in \mathcal{B}_\theta$. \square

Remark 8.9. Denote by the d the distance defined by the the Finsler metric introduced in \mathcal{D} . In [1] (Section 8) it was shown that

$$d(0, z) = \frac{1}{2} \log\left(\frac{1 + \|z\|}{1 - \|z\|}\right).$$

Using the fact that the action of \mathcal{B}_θ on \mathcal{D} is isometric, and the above construction of \tilde{g}_z , for arbitrary $z_1, z_2 \in \mathcal{D}$, we can compute $d(z_1, z_2)$ as follows:

$$d(z_1, z_2) = d(0, \tilde{g}_{z_1}^{-1} \cdot z_2) = \frac{1}{2} \log\left(\frac{1 + \|\tilde{g}_{z_1}^{-1} \cdot z_2\|}{1 - \|\tilde{g}_{z_1}^{-1} \cdot z_2\|}\right). \quad (7)$$

In order to compute \tilde{g}_z^{-1} , recall that elements \tilde{g} in $\mathcal{U}(\theta)$ are characterized by the relation $\rho \tilde{g}^* \rho = \rho^{-1}$. Then

$$\tilde{g}_z^{-1} = \begin{pmatrix} (1 - z^*z)^{-1/2} & 0 \\ 0 & (1 - z^*z)^{-1/2} \end{pmatrix} \begin{pmatrix} 1 + z & -z^*(1 + z) \\ -(1 + z^*)z & 1 + z^* \end{pmatrix} \begin{pmatrix} (1 + z)^{-1} & 0 \\ 0 & (1 + z)^{-1} \end{pmatrix}.$$

Then, after straightforward computations,

$$\tilde{g}_{z_1}^{-1} \cdot z_2 = (1 - z_1^*z_1)^{-1/2}(1 + z_1^*)(1 + z_1)^{-1}(z_2 - z_1)(1 - z_1^*z_2)^{-1}(1 - z_1^*z_1)^{1/2}.$$

9 Operator cross ratio in the hyperbolic part of the projective line

Here we state our main result, relating the metric of \mathcal{AP}_1^θ introduced in Section 8, with the so called operator cross ratio, as defined in the Grassmann manifold of a Hilbert space by M.I.

Zelikin [17]. We shall apply these ideas to the rank one submodules in \mathcal{AP}_1^θ . To this effect, the isometry between \mathcal{AP}_1^θ and the disk \mathcal{D} will be important.

Consider $\ell = \left[\begin{pmatrix} 1 \\ z \end{pmatrix} \right] \in \mathcal{AP}_1^\theta$, for $z \in \mathcal{D}$. Let $z = \omega|z|$ be the polar decomposition. Let δ be the geodesic of \mathcal{AP}_1^θ such that $\delta(0) = \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \right]$ and $\delta(1) = \left[\begin{pmatrix} 1 \\ z \end{pmatrix} \right]$. Equivalently, regarded in \mathcal{D} : $\delta(0) = 0$ and $\delta(1) = z$. As seen in Section 6,

$$\text{SOT} - \lim_{t \rightarrow +\infty} \delta(t) = \omega \text{ and } \text{SOT} - \lim_{t \rightarrow -\infty} \delta(t) = -\omega.$$

Four points are determined: $-\omega, 0, z, \omega$, or better, four submodules

$$\ell_{-\infty} := \left[\begin{pmatrix} 1 \\ -\omega \end{pmatrix} \right], \ell_0 := \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \right], \ell = \left[\begin{pmatrix} 1 \\ z \end{pmatrix} \right], \ell_{+\infty} := \left[\begin{pmatrix} 1 \\ \omega \end{pmatrix} \right],$$

where the limit lines lie in $\partial\mathcal{AP}_1^\theta$.

In Section 3 we defined the operator cross ratio of four elements in \mathcal{AP}_1 , as a (possibly empty) set of module endomorphisms, following ideas in [17]. Here we compute the operator cross ratio $CR(\ell_{-\infty}, \ell_0, \ell, \ell_{+\infty})$, proving that it is non empty, and that there exists a natural ℓ -endomorphism to choose from this set.

Recall that elements of $CR(\ell_{-\infty}, \ell_0, \ell, \ell_{+\infty})$ are (module) endomorphisms of ℓ , defined as the composition of *the projection from ℓ to ℓ_0 parallel to $\ell_{-\infty}$, followed by the projection from ℓ_0 to ℓ parallel to $\ell_{+\infty}$* .

In coordinates, by choosing generators in the respective submodules

$$\begin{pmatrix} 1 \\ z \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix} \lambda, \begin{pmatrix} 1 - \lambda \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -\omega \end{pmatrix} \mu.$$

Then $1 - \lambda = \mu$ and $z = -\omega\mu$. Then $\omega|z| = -\omega\mu$. If z is invertible (and then ω is unitary) this implies $\mu = -|z|$, otherwise this is just one possible solution. Non uniqueness of solutions of these equations reflect the geometric fact that the modules ℓ_0 and ℓ_∞ may not be in direct sum. Explicitly, all solutions of these equations are of the form

$$\lambda = 1 + |z| - \Omega, \quad \mu = -|z| + \Omega,$$

where $\Omega \in \mathcal{A}^{**}$ is such that $\omega\Omega = 0$. In particular, $|z|\Omega = |z|\omega^*\omega\Omega = 0$. We choose the solution with $\Omega = 0$. Note that $\lambda = 1 + |z|$, and therefore the first projection in the above composition is given by

$$\begin{pmatrix} 1 \\ z \end{pmatrix} \mapsto \begin{pmatrix} 1 + |z| \\ 0 \end{pmatrix}.$$

Next

$$\begin{pmatrix} 1 + |z| \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ z \end{pmatrix} \gamma, \begin{pmatrix} 1 + |z| - \gamma \\ -z\gamma \end{pmatrix} = \begin{pmatrix} 1 \\ \omega \end{pmatrix} \epsilon.$$

So that $1 + |z| - \gamma = \epsilon$ and $-z\gamma = \omega\epsilon$, and then (the unique solution if ω is unitary, or a possible solution that we choose, otherwise) $1 + |z| - \gamma = -|z|\omega$, i.e.,

$$\gamma = (1 - |z|)^{-1}.$$

Other solutions of the above equation are of the form

$$\gamma = (1 - |z|)^{-1} + (1 - |z|)^{-1}\Omega',$$

where $\Omega' \in \mathcal{A}^{**}$ is such that $\omega\Omega' = 0$. In general, the possible endomorphisms $\ell \rightarrow \ell$ are given (in these coordinates) by

$$\begin{pmatrix} 1 \\ z \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ z \end{pmatrix} (1 + |z| + \Omega)(1 - |z|)^{-1}(1 + \Omega') = (1 + |z|)(1 - |z|)^{-1} + \Omega' + \Omega(1 - |z|)^{-1} + \Omega\Omega',$$

where we use that $\omega\Omega = \omega\Omega' = 0$, and thus $(1 \pm |z|)^{\pm 1}\Omega = \Omega$ (and the same for Ω').

As noted, if z is invertible, there is a unique solution with $\Omega = \Omega' = 0$. Our choice of cross ratio, picking $\Omega = \Omega' = 0$ in any case, is justified below.

Remark 9.1. The following fact is known (see for instance [11]). Let $a_n \in \mathcal{A}$ with $\|a_n\| \leq 1$. If $a_n \rightarrow a$ strongly, then $|a_n| \rightarrow |a|$ strongly.

Remark 9.2. Dixmier and Marechal [9] proved that the set on invertible elements of a von Neumann algebra is strong operator dense in the algebra. The argument in [9] proceeds as follows. Let $a = u|a|$ be the polar decomposition of a ($u \in \mathcal{A}^{**}$). First, the algebra \mathcal{A}^{**} is factored in its finite and properly infinite parts. In the finite part u can be chosen unitary. In the properly infinite part, one readily sees that it suffices to consider the cases in which u is an isometry or a co-isometry. In the case that u is an isometry, Dixmier and Maréchal prove that u is the strong limit of unitaries u_n . If u is a co-isometry, they show that there exist invertible elements g_n which converge strongly to u , with norms $\|g_n\| = 1$ (this is clear in the proof, though it is not stated in their result). Summarizing, if \mathcal{A} is a von Neumann algebra, and $a \in \mathcal{A}$, there exist $g_n \in G$ with $\|g_n\| \leq \|a\|$ such that

$$\text{SOT} - \lim_{n \rightarrow \infty} g_n = a.$$

Using this fact, it is clear that, if \mathcal{A} is a von Neumann algebra, and $z \in \mathcal{D}$, then there exists and $z_n \in G$ with $\|z_n\| \leq \|z\|$, such that $z_n \rightarrow z$ strongly.

Proposition 9.3. Let $z_n, z \in \mathcal{D}$ such that $z_n \rightarrow z$ strongly and $\|z_n\| \leq \|z\|$. Then

$$(1 + |z_n|)(1 - |z_n|)^{-1} \rightarrow (1 + |z|)(1 - |z|)^{-1} \text{ strongly.}$$

Proof. First note that $|z_n| \rightarrow |z|$ strongly. Then $1 \pm |z_n|$ converges strongly to $1 \pm |z|$. Clearly $\|1 + |z_n|\| \leq 2$. Let us check that also $\|(1 - |z_n|)^{-1}\|$ are uniformly bounded:

$$\|(1 - |z_n|)^{-1}\| = \left\| \sum_{k=0}^{\infty} |z_n|^k \right\| \leq \sum_{k=0}^{\infty} \|z_n^k\| \leq \frac{1}{1 - \|z\|} < \infty.$$

Therefore, $(1 - |z_n|)^{-1} \rightarrow (1 - |z|)^{-1}$ strongly, and since the product is strongly continuous on norm bounded sets, the proof follows. \square

Definition 9.4. Let $z \in \mathcal{D}$. We define $cr(0, z) \in CR(\ell_{-\infty}, \ell_0, \ell_z, \ell_{\infty})$, for $\ell_z = \left[\begin{pmatrix} 1 \\ z \end{pmatrix} \right]$, as the endomorphism

$$cr(0, z) : \ell_z \rightarrow \ell_z, \quad cr(0, z) \left(\begin{pmatrix} 1 \\ z \end{pmatrix} a \right) = \begin{pmatrix} 1 \\ z \end{pmatrix} (1 + |z|)(1 - |z|)^{-1} a,$$

for $a \in \mathcal{A}$.

We use the action of $\mathcal{U}(\theta)$ to extend this definition to any pair $z_0 \neq z_1 \in \mathcal{D}$.

Definition 9.5. Let $z_0, z_1 \in \mathcal{D}$, $z_0 \neq z_1$. Pick $\tilde{g} \in \mathcal{U}(\theta)$ such that $\tilde{g} \cdot 0 = z_0$ and denote $z = \tilde{g}^{-1} \cdot z_1$. We define

$$cr(z_0, z_1) = \tilde{g} \, cr(0, z) \tilde{g}^{-1}.$$

Before checking that the definition does not depend on the choice of \tilde{g} , we remark the following. Let δ be the unique geodesic of \mathcal{D} such that $\delta(0) = z_0$ and $\delta(1) = z_1$, and let

$$z_{-\infty} = \text{SOT} - \lim_{t \rightarrow -\infty} \delta(t) \quad \text{and} \quad z_{+\infty} = \text{SOT} - \lim_{t \rightarrow +\infty} \delta(t).$$

Then $cr(z_0, z_1) \in CR(\ell_{z_{-\infty}}, \ell_{z_0}, \ell_{z_1}, \ell_{z_{+\infty}})$, because \tilde{g} is a module homomorphism which maps ℓ_z onto ℓ_{z_1} . Indeed, if $\mathbf{x} = \begin{pmatrix} 1 \\ z \end{pmatrix} a \in \ell_z$, then clearly

$$\tilde{g}\mathbf{x} = \begin{pmatrix} 1 \\ \tilde{g} \cdot z \end{pmatrix} (g_{11} + g_{12}z)a = \begin{pmatrix} 1 \\ z_1 \end{pmatrix} (g_{11} + g_{12}z)a \in \ell_{z_1}.$$

Let us check that $cr(z_1, z_2)$ is well defined, i.e., that it does not depend on the choice of \tilde{g} .

To prove this, recall from Remark 4.9 that if $\tilde{k} \in \mathcal{U}(\theta)$ satisfies $\tilde{k} \cdot 0 = 0$, then $\tilde{k} = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix}$, with $u_1, u_2 \in \mathcal{U}_{\mathcal{A}}$.

Proposition 9.6. With the above notations, the endomorphism

$$cr(z_0, z_1) \in CR(\ell_{z_{-\infty}}, \ell_{z_0}, \ell_{z_1}, \ell_{z_{+\infty}})$$

does not depend on the choice of \tilde{g} . Namely, if $\tilde{h} \in \mathcal{U}(\theta)$ satisfies $\tilde{h} \cdot 0 = z_0$, and $z' = \tilde{h}^{-1} \cdot z_1$, then

$$\tilde{h} \, cr(0, z') \tilde{h}^{-1} = \tilde{g} \, cr(0, z) \tilde{g}^{-1}.$$

Proof. Since $\tilde{h} \cdot 0 = \tilde{g} \cdot 0$, it follows that $(\tilde{g}^{-1}\tilde{h}) \cdot 0 = 0$, and therefore $\tilde{h} = \tilde{g} \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix}$, for $u_1, u_2 \in \mathcal{U}_{\mathcal{A}}$. Then

$$z' = \tilde{h}^{-1} \cdot z_1 = \begin{pmatrix} u_1^* & 0 \\ 0 & u_2^* \end{pmatrix} \cdot (\tilde{g}^{-1} \cdot z_1) = \begin{pmatrix} u_1^* & 0 \\ 0 & u_2^* \end{pmatrix} \cdot z = u_2^* z u_1,$$

and

$$\tilde{h} \, cr(0, z') \tilde{h}^{-1} = \tilde{g} \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} cr(0, u_2^* z u_1) \begin{pmatrix} u_1^* & 0 \\ 0 & u_2^* \end{pmatrix} \tilde{g}^{-1}.$$

Thus, we must show that $\begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} cr(0, u_2^* z u_1) \begin{pmatrix} u_1^* & 0 \\ 0 & u_2^* \end{pmatrix} = cr(0, z)$. Let us see how the left-hand side endomorphism transforms the element $\begin{pmatrix} 1 \\ z \end{pmatrix} a \in \ell_z$. First, it is sent to

$$\begin{pmatrix} u_1^* & 0 \\ 0 & u_2^* \end{pmatrix} \begin{pmatrix} 1 \\ z \end{pmatrix} a = \begin{pmatrix} 1 \\ u_2^* z u_1 \end{pmatrix} u_1^* a.$$

The map $cr(0, z')$ maps this element to

$$\begin{pmatrix} 1 \\ u_2^* z u_1 \end{pmatrix} (1 + |u_2^* z u_1|)(1 - |u_2^* z u_1|)^{-1} u_1^* a.$$

Note that $|u_2^* z u_1| = ((u_2^* z u_1)^* u_2^* z u_1)^{1/2} = (u_1^* z^* z u_1)^{1/2} = u_1^* |z| u_1$. Therefore, the above element equals

$$\begin{pmatrix} u_1^* \\ u_2^* z \end{pmatrix} (1 + |z|)(1 - |z|)^{-1} a.$$

Finally, multiplying on the left by the matrix $\begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix}$ yields

$$\begin{pmatrix} 1 \\ z \end{pmatrix} (1 + |z|)(1 - |z|)^{-1} a = cr(0, z) \begin{pmatrix} 1 \\ z \end{pmatrix} a.$$

□

Proposition 9.7. *Let \mathcal{A} be a von Neumann algebra.*

1. *If $z \in \mathcal{D} \cap G_{\mathcal{A}}$, the set $CR(\ell_{-\infty}, \ell_0, \ell_z, \ell_{\infty})$ consists of a single element $cr(0, z)$.*
2. *If $z \in \mathcal{D}$ is non invertible, there exist $z_n \in \mathcal{D}$ which are invertible such that $z_n \rightarrow z$ strongly and $\|z_n\| \leq \|z\|$. Then $cr(0, z_n), cr(0, z)$ are endomorphisms of different submodules. In order to compare them, we regard them as \mathcal{A} -module morphisms of \mathcal{A}^2 , embedding each module in \mathcal{A}^2 using the θ -orthogonal projections $p_{\ell_{z_n}}, p_{\ell_z}$ onto the submodules ℓ_{z_n}, ℓ_z , respectively. For $z' \in \mathcal{D}$,*

$$p_{\ell_{z'}}(\mathbf{x}) = (1 - |z'|^2)^{-1/2} \theta \left(\begin{pmatrix} 1 \\ z' \end{pmatrix}, \mathbf{x} \right) \begin{pmatrix} 1 \\ z' \end{pmatrix} (1 - |z'|^2)^{-1/2}.$$

Then

$$cr(0, z_n) p_{\ell_{z_n}}(\mathbf{x}) \rightarrow cr(0, z) p_{\ell_z}(\mathbf{x})$$

strongly in \mathcal{A}^2 .

Proof. By Remark 9.3, we know that $(1 + |z_n|)(1 - |z_n|)^{-1} \rightarrow (1 + |z|)(1 - |z|)^{-1}$ strongly. By a similar argument, it also holds that $(1 - |z_n|^2)^{-1/2} \rightarrow (1 - |z|^2)^{-1/2}$ strongly. Also these operators are uniformly bounded. Therefore, using that the product is strongly continuous in bounded sets, our claim follows. □

Remark 9.8. As a corollary we get that, even if the set $CR(\ell_1, \ell_2, \ell_3, \ell_4)$ may be empty for general $\ell_1, \ell_2, \ell_3, \ell_4$, the particular set $CR(\ell_{-\infty}, \ell_0, \ell_z, \ell_{\infty})$ is not, and $cr(0, z)$ is a distinguished element of this set.

As a first approximation of the deep relationship between the cross ratio and the metric in \mathcal{AP}_1^{θ} , we can state the following result. It can be regarded of the scalar version of our main result, here we state the equality between the norm of the endomorphism $cr(0, z)$ and the distance between 0 and z in the Poincaré disk \mathcal{D} .

Theorem 9.9. *Let $z \in \mathcal{D}$, then*

$$\frac{1}{2} \|cr(0, z)\|_{\mathcal{B}(\ell_z)} = d(0, z),$$

where $\|\cdot\|_{\mathcal{B}(\ell_z)}$ denotes the norm of operators acting in $\ell_z \subset \mathcal{A}^2$.

Proof. Choose for ℓ_z the unital basis $\mathbf{e}_z = \begin{pmatrix} (1 - z^*z)^{-1/2} \\ z(1 - z^*z)^{-1/2} \end{pmatrix}$. Then for any $\mathbf{x} = \mathbf{e}_z a \in \ell_z$,

$$cr(0, z)\mathbf{x} = \mathbf{e}_z \log((1 + |z|)(1 - |z|)^{-1})a,$$

and thus

$$\begin{aligned} \langle cr(0, z)\mathbf{x}, cr(0, z)\mathbf{x} \rangle &= a^* \log((1 + |z|)(1 - |z|)^{-1}) \langle \mathbf{e}_z, \mathbf{e}_z \rangle \log((1 + |z|)(1 - |z|)^{-1})a \\ &= a^* (\log((1 + |z|)(1 - |z|)^{-1}))^2 a. \end{aligned}$$

Since $(\log((1 + |z|)(1 - |z|)^{-1}))^2 \leq \|\log((1 + |z|)(1 - |z|)^{-1})\|^2$, it follows that

$$a^* (\log((1 + |z|)(1 - |z|)^{-1}))^2 a \leq a^* a \|\log((1 + |z|)(1 - |z|)^{-1})\|^2,$$

and therefore

$$\begin{aligned} \|\langle cr(0, z)\mathbf{x}, cr(0, z)\mathbf{x} \rangle\|^{1/2} &\leq \|\log((1 + |z|)(1 - |z|)^{-1})\| \|a^* a\|^{1/2} \\ &= \|\log((1 + |z|)(1 - |z|)^{-1})\| \|\mathbf{x}\|. \end{aligned}$$

This implies that $\|cr(0, z)\|_{\mathcal{B}(\ell_z)} \leq \|\log((1 + |z|)(1 - |z|)^{-1})\|$. Note that

$$cr(0, z)\mathbf{e}_z = \mathbf{e}_z \log((1 + |z|)(1 - |z|)^{-1}),$$

so that

$$\begin{aligned} \|cr(0, z)\mathbf{e}_z\|^2 &= \|\langle \mathbf{e}_z \log((1 + |z|)(1 - |z|)^{-1}), \mathbf{e}_z \log((1 + |z|)(1 - |z|)^{-1}) \rangle\| \\ &= \|\log((1 + |z|)(1 - |z|)^{-1})\|^2, \end{aligned}$$

i.e.,

$$\|cr(0, z)\|_{\mathcal{B}(\ell_z)} = \|\log((1 + |z|)(1 - |z|)^{-1})\| = 2d(0, z).$$

□

10 The coefficient bundle

Consider the slight variant of the commutative diagram in (4)

$$\begin{array}{ccc} & \mathcal{K}_\theta & \\ \pi_\theta \swarrow & & \searrow \tilde{\pi}_\theta \\ \mathcal{Q}_\rho & \xrightarrow{\simeq} & \mathcal{D}, \end{array}$$

Recall (from the end of Section 4), that \mathcal{Q}_ρ denotes the space of θ -orthogonal rank one projections (considered here the coordinate free version of \mathcal{AP}_1^θ). Let us introduce the *canonical bundle*

$$\xi \rightarrow \mathcal{Q}_\rho,$$

whose fiber over $q \in \mathcal{Q}_\rho$ is the module $R(q)$. This is a fiber bundle of right \mathcal{A} -modules, which has a canonical connection. Elements of ξ are pairs (q, \mathbf{x}) , with $q \in \mathcal{Q}_\rho$ and $q(\mathbf{x}) = \mathbf{x}$.

Let $q \in \mathcal{Q}_\rho$ and $\varphi : R(q) \rightarrow R(q)$ a right module endomorphism. Pick a normalized generator $\mathbf{x} \in R(q)$: $\theta(\mathbf{x}, \mathbf{x}) = 1$ (i.e., an element of $R(q)$ in \mathcal{K}_θ). Then the endomorphism φ is determined by the value $\varphi(\mathbf{x}) = \mathbf{x}a$. That is, for any element $\mathbf{y} = \mathbf{x}\lambda \in R(q)$, $\varphi(\mathbf{y}) = \mathbf{x}a\lambda$. In other words, if we regard \mathbf{x} as a basis for $R(q)$, φ can be expressed as $\lambda \mapsto a\lambda$. We call $a \in \mathcal{A}$ the matrix of φ in the basis \mathbf{x} . If \mathbf{x}' is another basis of $R(q)$ in \mathcal{K}_θ , then there exists a unitary $u \in \mathcal{U}_\mathcal{A}$ such that $\mathbf{x}' = \mathbf{x}u$. If b is the matrix of φ in the basis \mathbf{x}' (i.e., $\varphi(\mathbf{x}') = \mathbf{x}'b$), then

$$\mathbf{x}ub = \mathbf{x}'b = \varphi(\mathbf{x}') = \varphi(\mathbf{x}u) = \varphi(\mathbf{x})u = \mathbf{x}'au.$$

Then $ub = au$, which means that the matrix of φ in the basis \mathbf{x}' is $b = u^*au$.

This shows that we can regard the set $\text{End}(R(q))$ of endomorphisms of $R(q)$, as the set of pairs (\mathbf{x}, a) , where $\mathbf{x} \in \mathcal{K}_\theta$ with $q(\mathbf{x}) = \mathbf{x}$, and $a \in \mathcal{A}$, with the identification

$$(\mathbf{x}, a) \sim (\mathbf{x}u, u^*au), \quad u \in \mathcal{U}_\mathcal{A}.$$

The map $\Gamma \rightarrow \mathcal{Q}_\rho$ defined by $(q, \varphi) \mapsto q$ for $q \in \mathcal{Q}_\rho$ and $\varphi \in \text{End}(R(q))$, is a fiber bundle which we call the *coefficient bundle*; alternatively $(\mathbf{x}, a) \sim (\mathbf{x}u, u^*au) \mapsto [\mathbf{x}]$. Each fiber of Γ is a C^* -algebra, which is isomorphic to \mathcal{A} . The canonical connection of the bundle ξ induces a connection in Γ , by the rule

$$(D_X \varphi)(\mathbf{y}) = (D_X \varphi)\mathbf{y} + \varphi(D_X \mathbf{y}).$$

Here, φ is a local cross section of Γ and \mathbf{y} is a local cross section of ξ . We remark that the connections of ξ and Γ are compatible with the action of $\mathcal{U}(\theta)$.

We define the *basic 1-form*. Given $\mathbf{x} \in \mathcal{K}_\theta$ and $X \in (T\mathcal{Q}_\rho)_q$, with $q = \mathbf{x}\theta(\mathbf{x}, \cdot)$, put

$$\kappa_\mathbf{x}(X) = X\mathbf{x}.$$

Note that X is a matrix in $M_2(\mathcal{A})$, θ -symmetric and q -codiagonal: $X\mathbf{x} \in \mathbf{x}^\perp = N(q)$.

Given $X, Y \in (T\mathcal{Q}_\rho)_q$, we define the product

$$\langle X, Y \rangle_\mathbf{x} = -\theta(\kappa_\mathbf{x}(X), \kappa_\mathbf{x}(Y)) = -\theta(X\mathbf{x}, Y\mathbf{x}).$$

If the generator \mathbf{x} is changed for $\mathbf{x}' = \mathbf{x}u$, we have

$$\langle X, Y \rangle_{\mathbf{x}'} = -\theta(X(\mathbf{x}u), Y(\mathbf{x}u)) = -u^*\theta(X\mathbf{x}, Y\mathbf{x})u = u^*\langle X, Y \rangle_\mathbf{x}u.$$

This means that, given $q = [\mathbf{x}] = [\mathbf{x}']$, the product $\langle X, Y \rangle_q$ is well defined as an element of the fiber Γ_q .

This product is therefore a Hilbertian product in $T\mathcal{Q}_\rho$, with values in Γ . To this effect, note that $T\mathcal{Q}_\rho$ is a right module over the bundle Γ of coefficients. Indeed, if we fix $\mathbf{x} \in \mathcal{K}_\theta$ with $q = \mathbf{x}\theta(\mathbf{x}, \cdot)$, the map $X \mapsto \kappa_\mathbf{x}(X) = X\mathbf{x}$ from $(T\mathcal{Q}_\rho)_q$ to $N(q)$ is one to one. If we change \mathbf{x} with $\mathbf{x}u$, $\kappa_\mathbf{x}(X)$ changes to $\kappa_{\mathbf{x}u}(X) = \kappa_\mathbf{x}(X)u$. If $X \in (T\mathcal{Q}_\rho)_q$ and $\varphi \in \Gamma_q$, we define $X\varphi$ as $\kappa_\mathbf{x}(X\varphi) = X\mathbf{x}a$, where φ is represented by the class of (\mathbf{x}, a) . With this definition we have

$$\langle X, Y\varphi \rangle_q = \langle X, Y \rangle_q \varphi.$$

10.1 The cross ratio, the logarithm and the exponential

We have just defined a Hilbertian Γ -valued structure in $\mathcal{Q}_\rho \simeq \mathcal{AP}_1^\theta$, or, equivalently, in \mathcal{D} . In particular, the product

$$\langle \text{Log}_0(z), \text{Log}_0(z) \rangle_0$$

takes values in the set of endomorphisms of $\ell_0 = \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \right]$, where Log_0 is defined in Corollary 6.4. It is a positive module endomorphism (given by multiplying the generator \mathbf{e}_1 by a positive element of \mathcal{A}). Thus, it has a unique positive square root $\langle \text{Log}_0(z), \text{Log}_0(z) \rangle_0^{1/2}$, which we call the θ -modulus, and denote it $\text{mod}_0(\text{Log}_0(z))$ of $\text{Log}_0(z)$. Explicitly, in the generator \mathbf{e}_1 , $\text{mod}_0(\text{Log}_0(z))$ consists in multiplying the generator by $\log((1+|z|)(1-|z|)^{-1})$.

On the other hand, we saw that, for $z \in \mathcal{D}$, the endomorphism of ℓ_z denoted by $cr(0, z)$, is given by the same coefficient $\log((1+|z|)(1-|z|)^{-1})$, which multiplies the generator $\begin{pmatrix} 1 \\ z \end{pmatrix}$ of ℓ_z .

We shall translate the endomorphism $cr(0, z)$ from ℓ_z to ℓ_0 by means of the parallel transport of \mathcal{AP}_1^θ , along the geodesic δ , with $\delta(0) = \ell_0$ and $\delta(1) = \ell_z$ (i.e., the same former δ , which under the identification $\mathcal{D} \simeq \mathcal{AP}_1^\theta$ joins $\delta(0) = 0$ and $\delta(1) = z$ in \mathcal{D} : $\delta(t) = \omega \tanh(t|\alpha|)$).

The parallel transport of elements of \mathcal{D} (or \mathcal{AP}_1^θ) along the geodesic $\delta(t) = e^{t \begin{pmatrix} 0 & \alpha^* \\ \alpha & 0 \end{pmatrix}} \cdot 0$, where α is, as in Remark 6.1.2

$$\alpha = z \sum_{k=0}^{\infty} \frac{1}{2k+1} (z^* z)^k,$$

is given by the left action of the invertible matrix

$e^{t \begin{pmatrix} 0 & \alpha^* \\ \alpha & 0 \end{pmatrix}} : \ell_0 \rightarrow \ell_{\delta(t)}$. The endomorphism $cr(0, z)$ of ℓ_z is transported to ℓ_0 as

$$cr(0, z)_0 := e^{-\begin{pmatrix} 0 & \alpha^* \\ \alpha & 0 \end{pmatrix}} cr(0, z) e^{\begin{pmatrix} 0 & \alpha^* \\ \alpha & 0 \end{pmatrix}} : \ell_0 \rightarrow \ell_0.$$

Our main result (for the origin) is the following:

Theorem 10.1. *With the current notation, if $z \in \mathcal{D}$ (or $\ell_z \in \mathcal{AP}_1^\theta$),*

$$e^{\text{mod}_0(\text{Log}_0(z))} = cr(0, z)_0 \quad \text{or, equivalently,} \quad \text{mod}_0(\text{Log}_0(z)) = \log(cr(0, z)_0), \quad (8)$$

where the exponential in the first equality is the usual exponential of \mathcal{A} , \log in the second equality is the usual logarithm of G , and each endomorphism of ℓ_0 is identified with its coefficient in the basis $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Proof. Let us prove the first equality. Since we are comparing endomorphisms of ℓ_0 , it suffices to show that they carry the generator \mathbf{e}_1 to the same element in \mathcal{A}^2 . Note that

$$cr(0, z)_0(\mathbf{e}_1) = e^{-\begin{pmatrix} 0 & \alpha^* \\ \alpha & 0 \end{pmatrix}} cr(0, z) e^{\begin{pmatrix} 0 & \alpha^* \\ \alpha & 0 \end{pmatrix}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= e^{-\begin{pmatrix} 0 & \alpha^* \\ \alpha & 0 \end{pmatrix}}_{cr(0, z)} \begin{pmatrix} \cosh(|\alpha|) \\ \omega \sinh(|\alpha|) \end{pmatrix};$$

using that $\omega \tanh(|\alpha|) = \delta(1) = z$ (Lemma 6.2), this yields

$$e^{-\begin{pmatrix} 0 & \alpha^* \\ \alpha & 0 \end{pmatrix}}_{cr(0, z)} \begin{pmatrix} 1 \\ z \end{pmatrix} \cosh(|\alpha|) = e^{-\begin{pmatrix} 0 & \alpha^* \\ \alpha & 0 \end{pmatrix}} \begin{pmatrix} 1 \\ z \end{pmatrix} (1 + |z|)(1 - |z|)^{-1} \cosh(|\alpha|).$$

By the same computation that showed that $e^{-\begin{pmatrix} 0 & \alpha^* \\ \alpha & 0 \end{pmatrix}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ z \end{pmatrix} \cosh(|\alpha|)$ (see the proof of Lemma 6.2), we have that

$$e^{-\begin{pmatrix} 0 & \alpha^* \\ \alpha & 0 \end{pmatrix}} \begin{pmatrix} 1 \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cosh(|\alpha|)^{-1},$$

i.e.,

$$cr(0, z)_0(\mathbf{e}_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1 + |z|)(1 - |z|)^{-1}.$$

On the other hand, the endomorphism $mod_0(\text{Log}_0(z))$ sends \mathbf{e}_1 to $\mathbf{e}_1 \log((1 + |z|)(1 - |z|)^{-1})$, and thus

$$e^{mod_0(\text{Log}_0(z))}(\mathbf{e}_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1 + |z|)(1 - |z|)^{-1}.$$

□

As in Definition 9.5, let $z_0 \neq z_1 \in \mathcal{D}$ ($\ell_{z_0} \neq \ell_{z_1} \in \mathcal{AP}_1^\theta$). Pick $\tilde{g} \in \mathcal{U}(\theta)$ such that $\tilde{g} \cdot 0 = z_0$, and denote by $z = \tilde{g}^{-1} \cdot z_1$ as before. Let δ be the geodesic such that $\delta(0) = 0$ and $\delta(1) = z_1$. Then $\delta_{z_0, z_1} = \tilde{g} \cdot \delta$ is the geodesic which joins z_0 and z_1 at $t = 0$ and $t = 1$, respectively. Recall that $cr(z_0, z_1) = \tilde{g} cr(z_0, z_1) \tilde{g}^{-1}$. Likewise, we put

$$\text{Log}_{z_0}(z_1) := \tilde{g} \text{Log}_0(z) \tilde{g}^{-1}, \quad \text{and} \quad mod_{z_0}(z_1) = \langle \text{Log}_{z_0}(z_1), \text{Log}_{z_0}(z_1) \rangle_{z_0}^{1/2},$$

where $\langle \varphi, \psi \rangle_{z_0} = \tilde{g} \langle \tilde{g} \varphi \tilde{g}^{-1}, \tilde{g} \psi \tilde{g}^{-1} \rangle_0 \tilde{g}^{-1}$, and Log_{z_0} is the inverse of the exponential $exp_{z_0} : (T\mathcal{D})_{z_0} \rightarrow \mathcal{D}$. It is not difficult to verify that these definitions do not depend on the choice of \tilde{g} .

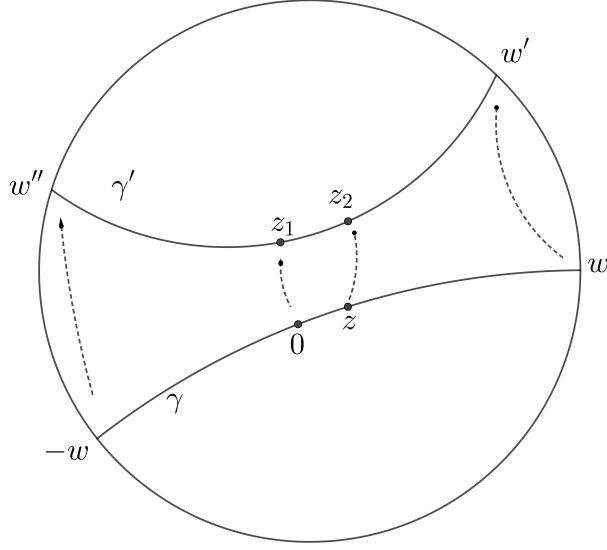
Finally, let us denote by $cr(z_0, z_1)_{z_0}$ the parallel transport of $cr(z_0, z_1)$ from ℓ_{z_1} to ℓ_{z_0} along the geodesic δ_{z_0, z_1} (obtained by conjugation as in the case of the origin, by the value at $t = 1$ of the one parameter group in $\mathcal{U}(\theta)$ which determines δ_{z_0, z_1}). The $\mathcal{U}(\theta)$ -covariance of the data involved enables one to prove the following:

Corollary 10.2. *With the current notations,*

$$mod_{z_0}(\text{Log}_{z_0}(z_1)) = \log(cr(z_0, z_1)_{z_0}).$$

In particular, $\|\text{Log}_{z_0}(z_1)\|_{z_0} = \|\log(cr(z_0, z_1)_{z_0})\|$.

Figure 2



11 An example

Suppose that the algebra \mathcal{A} has a trace \mathbf{tr} onto a central subalgebra, that is, there exists a C^* -subalgebra $\mathcal{B} \subset Z(\mathcal{A})$ of the center of \mathcal{A} and a conditional expectation $\mathbf{tr} : \mathcal{A} \rightarrow \mathcal{B}$ satisfying $\mathbf{tr}(xy) = \mathbf{tr}(yx)$ for all $x, y \in \mathcal{A}$. This happens, for instance, if \mathcal{A} is a finite von Neumann algebra.

A relevant case of this situation is the following. Consider a complex vector bundle $E \rightarrow B$ with compact base space B , endowed with a Riemannian metric $\langle e, e' \rangle_b$, $b \in B$, $e, e' \in E_b$ (the fiber of E over b). Consider the fiber bundle $\text{End}(E) \rightarrow B$ of endomorphisms of the vector bundle E , and let \mathcal{A} be the algebra $\Gamma(\text{End}(E))$ of the continuous global cross sections of $\text{End}(E)$. Since each E_b is a (finite dimensional) Hilbert space, $\text{End}(E_b)$ is a C^* -algebra. The space $\Gamma(\text{End}(E))$ of cross sections has therefore the norm $\|\varphi\| = \sup_{b \in B} \|\varphi_b\|$, where $\varphi_b : E_b \rightarrow E_b$ and $\|\varphi_b\|$ is the usual norm of linear operators. With this norm, \mathcal{A} is a C^* -algebra. The center $Z(\mathcal{A})$ of this algebra is the space of scalar sections λ in $\text{End}(E)$ (homotetic in each fiber). The central trace is given by $\mathbf{tr} : \mathcal{A} \rightarrow Z(\mathcal{A})$, $\mathbf{tr}(\sigma)_b = \text{Tr}(\sigma_b)$, $b \in B$, with Tr the usual trace of E_b . More specifically, B could be a compact manifold, and E the complexification of its tangent bundle, with an Hermitian metric. This case is interesting due to the following observation: in our previous work [1], we noticed the equivalence, as homogeneous spaces, of the disk \mathcal{D} and the Poincaré halfspace \mathcal{H} of the algebra \mathcal{A} . This homogeneous space can be thought as the tangent bundle TG^+ of the space G^+ of positive and invertible elements of \mathcal{A} , as explained in [1]. In this context, an element of \mathcal{H} is a pair (X, a) with $a \in G^+$ and $X \in (TG^+)_a$. The element $a \in G^+$ represents a Riemannian metric in B , and a possible vector X (a selfadjoint element of \mathcal{A}) could be the Ricci curvature of the metric a . In this manner, the geometry of \mathcal{H} is linked to the deformation of the pairs (*Riemannian metric*, *Ricci curvature*), viewed as elements of TG^+ .

Back to the general case (of this example):

$$\mathbf{tr} \mathcal{A} \rightarrow \mathcal{B} \subset Z(\mathcal{A}),$$

we can define a Hilbertian \mathcal{B} -valued inner product, by means of

$$\langle X, Y \rangle_{\mathbf{tr}, q} = -\mathbf{tr}(\theta(X\mathbf{x}, Y\mathbf{y})).$$

Indeed, since \mathbf{tr} is tracial, the value of $-\mathbf{tr}(\theta(X\mathbf{x}, Y\mathbf{y}))$ is independent of the choice of $\mathbf{x} \in \mathcal{K}_\theta$ satisfying $q = \mathbf{x}\theta(\mathbf{x}, \cdot)$. On the other hand, $cr(z_0, z_1)$ is an element of Γ_{z_0} , which has matrix a in a unital base $\mathbf{x} \in R(q)$, as explained before. We put $cr(z_0, z_1)_{\mathbf{tr}}$, for $\mathbf{tr}(a)$. Clearly, $cr(z_0, z_1)_{\mathbf{tr}}$ does not depend on the basis \mathbf{x} . With these notations, the formula in Corollary 10.2, can be written

$$\langle \text{Log}_{z_0} z_1, \text{Log}_{z_0} z_1 \rangle_{\mathbf{tr}}^{1/2} = \log cr(z_0, z_1)_{\mathbf{tr}}, \quad (9)$$

which is an identity involving elements in \mathcal{B} .

More specifically, if \mathcal{A} is commutative, we can choose \mathbf{tr} the identity $\mathcal{A} = \mathcal{B}$, and we have

$$|\text{Log}_{z_0} z_1| = \log cr(z_0, z_1), \quad (10)$$

as elements in \mathcal{A} .

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