

UNITARY SUBGROUPS AND ORBITS OF COMPACT SELF-ADJOINT OPERATORS

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ABSTRACT. Let \mathcal{H} be a separable Hilbert space, and $\mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$ the anti-Hermitian bounded diagonals in some fixed orthonormal basis and $\mathcal{K}(\mathcal{H})$ the compact operators. We study the group of unitary operators

$$\mathcal{U}_{k,d} = \{u \in \mathcal{U}(\mathcal{H}) : \exists D \in \mathcal{D}(\mathcal{B}(\mathcal{H}))^{ah} \text{ such that } u - e^D \in \mathcal{K}(\mathcal{H})\}$$

in order to obtain a concrete description of short curves in unitary Fredholm orbits $\mathcal{O}_b = \{e^K b e^{-K} : K \in \mathcal{K}(\mathcal{H})^{ah}\}$ of a compact self-adjoint operator b with spectral multiplicity one. We consider the rectifiable distance on \mathcal{O}_b defined as the infimum of curve lengths measured with the Finsler metric defined by means of the quotient space $\mathcal{K}(\mathcal{H})^{ah}/\mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})$. Then for every $c \in \mathcal{O}_b$ and $x \in T(\mathcal{O}_b)_c$ there exist a minimal lifting $Z_0 \in \mathcal{B}(\mathcal{H})^{ah}$ (in the quotient norm, not necessarily compact) such that $\gamma(t) = e^{tZ_0} c e^{-tZ_0}$ is a short curve on \mathcal{O}_b in a certain interval.

1. INTRODUCTION

Let $\mathcal{B}(\mathcal{H})$ be the algebra of bounded operators on a separable Hilbert space \mathcal{H} , $\mathcal{K}(\mathcal{H})$ and $\mathcal{U}(\mathcal{H})$ the compact and unitary operators respectively. If an orthonormal basis is fixed we can consider matricial representations of each $A \in \mathcal{B}(\mathcal{H})$ and diagonal operators which we denote with $\mathcal{D}(\mathcal{B}(\mathcal{H}))$.

Consider the following subset of the unitary group $\mathcal{U}(\mathcal{H})$ of $\mathcal{B}(\mathcal{H})$:

$$(1.1) \quad \mathcal{U}_{k,d} = \{u \in \mathcal{U}(\mathcal{H}) : \exists D \in \mathcal{D}(\mathcal{B}(\mathcal{H}))^{ah} \text{ such that } u - e^D \in \mathcal{K}(\mathcal{H})\}.$$

In the present work we prove that $\mathcal{U}_{k,d}$ is a subgroup of $\mathcal{U}(\mathcal{H})$. Moreover, $\mathcal{U}_{k,d}$ is closed, arc-connected and shares the topology of $\mathcal{U}(\mathcal{H})$ given by the operator norm. Therefore $\mathcal{U}_{k,d}$ is a Lie subgroup in the sense of [9] and [10].

We did not find any reference to the subgroup $\mathcal{U}_{k,d}$ mentioned in the literature and so we included here a detailed study of it. In Theorem 3.18 we prove that $\mathcal{U}_{k,d}$ is a Lie subgroup of $\mathcal{U}(\mathcal{H})$ according to the definition mentioned before. The Lie algebra of $\mathcal{U}_{k,d}$ turns to be $\mathcal{K}(\mathcal{H})^{ah} + \mathcal{D}(\mathcal{B}(\mathcal{H}))^{ah}$

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which is not complemented in $\mathcal{B}(\mathcal{H})^{ah}$ and therefore a stronger notion of Lie subgroup cannot be used (see Proposition 3.16).

This subgroup admits a generalization to $\mathcal{U}_{\mathcal{J},\mathcal{A}}$ for certain ideals \mathcal{J} and subalgebras \mathcal{A} of $\mathcal{B}(\mathcal{H})^h$ (see 3.19).

Our particular interest in $\mathcal{U}_{k,d}$ relies on the geometric study of the orbits

$$\mathcal{O}_b^{\mathcal{V}} = \{ubu^* : u \in \mathcal{V}\}$$

where b a self-adjoint operator. If the spectrum of b is finite \mathcal{O}_b is a complemented submanifold of $b + \mathcal{K}(\mathcal{H})$ (see [1]). If we consider a compact diagonal self-adjoint operator b with spectral multiplicity one then the orbit \mathcal{O}_b can have a smooth structure (see Lemma 1 in [6]).

The subgroup $\mathcal{U}_{k,d}$ has the following properties.

- If $\mathcal{U}_k = \{u \in \mathcal{U}(\mathcal{H}) : \exists D \in \mathcal{D}(\mathcal{K}(\mathcal{H})^{ah}) \text{ such that } u - 1 \in \mathcal{K}(\mathcal{H})\}$, the following orbits coincide

$$\begin{aligned} \mathcal{O}_b &= \mathcal{O}_b^{\mathcal{U}_k} = \{ubu^* : u \in \mathcal{U}_k\} \\ &= \mathcal{O}_b^{\mathcal{U}_{k,d}} = \{ubu^* : u \in \mathcal{U}_{k,d}\} \end{aligned}$$

- The natural Finsler metric defined in $T(\mathcal{O}_b^{\mathcal{U}_{k,d}})_1$ and $T(\mathcal{O}_b^{\mathcal{U}_k})_1$ by means of the quotient norm coincides if b is a compact self-adjoint diagonal operator and we consider the identifications of the tangent spaces with the quotients

$$\begin{aligned} T(\mathcal{O}_b^{\mathcal{U}_k})_1 &\cong (T\mathcal{O}_b)_c \cong (T\mathcal{U}_k)_1 / (T\mathcal{I}_b)_1 = \mathcal{K}(\mathcal{H})^{ah} / \mathcal{D}(\mathcal{K}(\mathcal{H})^{ah}) \\ T(\mathcal{O}_b^{\mathcal{U}_{k,d}})_1 &\cong (T\mathcal{U}_{k,d})_1 / (T\mathcal{I}_b)_1 \cong \left(\mathcal{K}(\mathcal{H})^{ah} + \mathcal{D}(\mathcal{B}(\mathcal{H}))^{ah} \right) / \mathcal{D}(\mathcal{B}(\mathcal{H}))^{ah} \\ &\cong \mathcal{K}(\mathcal{H})^{ah} / \mathcal{D}(\mathcal{B}(\mathcal{H}))^{ah} \end{aligned}$$

(see Remark 4.5 for details).

These properties allow the construction of minimum length curves of \mathcal{O}_b considering the rectifiable distance defined in the Preliminaries (see (2.6)).

Next we describe minimal vectors of the tangent space and their relation with the short curves in these homogeneous spaces. We say that a self-adjoint operator $Z \in \mathcal{B}(\mathcal{H})$ is *minimal* for a subalgebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ if

$$(1.2) \quad \|Z\| = \inf_{D \in \mathcal{A}} \|Z + D\|,$$

for $\|\cdot\|$ the usual operator norm in $\mathcal{B}(\mathcal{H})$. Given a fixed Z we say that $D_0 \in \mathcal{A}$ is *minimal* for Z if $\|Z + D_0\| = \inf_{D \in \mathcal{A}} \|Z + D\|$, that is, if $Z + D_0$ is minimal for \mathcal{A} . These minimal operators Z allow the concrete description of short curves $\gamma(t) = e^{itZ} A e^{-itZ}$ in the unitary orbit \mathcal{O}_A of a some fixed self-adjoint operator $A \in \mathcal{B}(\mathcal{H})^h$, when considered with a certain natural Finsler metric (see (2.4), [8], [1] and [6] for details and different examples).

If we fix an orthonormal basis in \mathcal{H} we can consider matricial representations and diagonal operators in $\mathcal{B}(\mathcal{H})$. In [6] we studied the orbit \mathcal{O}_A of a diagonal compact self-adjoint operator $b \in \mathcal{B}(\mathcal{H})$ under the action of the Fredholm unitary subgroup $\mathcal{U}_k = \{e^K : K \in \mathcal{K}(\mathcal{H})^{ah}\}$ where $\mathcal{K}(\mathcal{H})^{ah}$ denotes the compact anti-Hermitian operators. We used a particular element $Z_r \in \mathcal{K}(\mathcal{H})^{ah}$ with the property that there does not exist a compact diagonal D_0 such that the quotient norm $\|Z_r + D_0\| = \inf_{D \in \mathcal{D}(\mathcal{K}(\mathcal{H}))} \|Z_r + D\|$ is attained. This example posted an interesting geometric question, since the existence of such minimal compact diagonal D_0 would allow the explicit description of a short path with initial velocity $[Z_r, b]$ (see [8, 5]).

Using that $\lim (Z_r)_{jj}$ converges to a non-zero constant when $j \rightarrow \infty$ we showed in [6] that the curve parametrized by

$$\beta(t) = e^{tZ_r} b e^{-tZ_r}$$

with $|t| \leq \frac{\pi}{2\|Z_r\|}$, is still a geodesic even though Z_r is not a minimal operator. Moreover, β can be approximated uniformly by minimal length curves of finite matrices β_n (with minimal initial velocity vectors) satisfying $\beta_n(0) = \beta(0) = b$ and $\beta'_n(0) = \beta'(0)$.

Nevertheless, in the same paper, we showed examples of compact operators Z_o whose unique minimal diagonals had several limits. In these cases the techniques used with Z_r were not enough to prove either that $\gamma(t) = e^{tZ_o} b e^{-tZ_o}$ was a short curve nor that γ could be approximated by curves of matrices.

In the present work we describe short curves that include those cases. In order to do so we consider the unitary subgroup $\mathcal{U}_{k,d}$. The action of this group on a diagonal self-adjoint operator b produces the same orbit as \mathcal{U}_k but permits a concrete description of geodesics using minimal operators of its Lie algebra $\mathcal{K}(\mathcal{H})^{ah} + \mathcal{D}(\mathcal{B}(\mathcal{H}))^{ah}$ (see 4.2 and 4.6).

2. PRELIMINARIES

Let $(\mathcal{H}, \langle, \rangle)$ be a separable Hilbert space. As usual, $\mathcal{B}(\mathcal{H})$, $\mathcal{U}(\mathcal{H})$ and $\mathcal{K}(\mathcal{H})$ denote the sets of bounded, unitary and compact operators on \mathcal{H} . We denote with $\|\cdot\|$ the usual operator norm in $\mathcal{B}(\mathcal{H})$. It should be clear from the context the use of the same notation $\|\cdot\|$ to refer to the operator norm or the norm on the Hilbert space $\|h\| = \langle h, h \rangle^{1/2}$ for $h \in \mathcal{H}$.

Given $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$, we use the superscript ah (respectively h) to note the subset of anti-Hermitian (respectively Hermitian) elements of \mathcal{A} .

Consider the Fredholm subgroup of $\mathcal{U}(\mathcal{H})$ defined as

$$\mathcal{U}_k = \{u \in \mathcal{U}(\mathcal{H}) : u - I \in \mathcal{K}(\mathcal{H})\} = \{u \in \mathcal{U}(\mathcal{H}) : \exists K \in \mathcal{K}(\mathcal{H})^{ah}, u = e^K\}$$

(see [1] and Proposition 3.1).

$\mathcal{U}(\mathcal{H})$ is a Lie-Banach group and its Lie algebra $T_1(\mathcal{U}(\mathcal{H})) = \mathcal{B}(\mathcal{H})^{ah}$. We consider the usual analytical exponential map $\exp : \mathcal{B}(\mathcal{H})^{ah} \rightarrow \mathcal{U}(\mathcal{H})$, given for any $X \in \mathcal{B}(\mathcal{H})^{ah}$ by $\exp(X) = \sum_{n=0}^{\infty} \frac{1}{n!} X^n = e^X$. Then $\mathcal{B}(\mathcal{H})^{ah}$ can be made into a contractive Lie algebra (i.e., $\|[X, Y]\|_c \leq \|X\|_c \|Y\|_c$ for all $X, Y \in \mathcal{B}(\mathcal{H})^{ah}$) defining $\|\cdot\|_c := 2\|\cdot\|$. Then, by Proposition 1.29 in [4]

$$\|\log(e^X e^Y)\|_c \leq -\log(2 - e^{\|X\|_c + \|Y\|_c})$$

if $\|X\|_c + \|Y\|_c < \log 2$. Consequently, it can be proved that if

$$(2.1) \quad \|X\| + \|Y\| < \frac{\log 2}{2}$$

the Baker-Campbell-Hausdorff (B-C-H) series expansion converges absolutely for all $X, Y \in \mathcal{B}(\mathcal{H})^{ah}$. This B-C-H series can be defined as

$$\log(e^X e^Y) = \sum_{n=1}^{\infty} c_n(T),$$

where each c_n is a polynomial map of $\mathcal{B}(\mathcal{H})^{ah} \times \mathcal{B}(\mathcal{H})^{ah}$ into $\mathcal{B}(\mathcal{H})^{ah}$ of degree n , $\forall n \in \mathbb{N}$. For instance, the first terms are:

$$\begin{cases} c_1(X, Y) = X + Y, \\ c_2(X, Y) = \frac{1}{2}[X, Y], \\ c_3(X, Y) = \frac{1}{12}[X, [X, Y]] + \frac{1}{12}[Y, [Y, X]]. \end{cases}$$

Also, each c_n is a sum of commutators for all $n > 1$. Therefore, the formula of the series can be rewritten as follows

$$(2.2) \quad \log(e^X e^Y) = X + Y + \sum_{n=2}^{\infty} c_n(T).$$

To see the complete general expression or other properties of the B-C-H series for Lie algebras see [4] or [12].

Definition 2.1. *Given $X \in \mathcal{B}(\mathcal{H})^{ah}$ we will say that X is sufficiently close to 0 if $\|X\| < \frac{\log 2}{4}$.*

Using the previous definition the B-C-H series (2.2) converges for every $X, Y \in \mathcal{B}(\mathcal{H})^{ah}$ sufficiently close to 0, since this condition implies (2.1).

We define the unitary Fredholm orbit of a fixed self-adjoint $A \in \mathcal{B}(\mathcal{H})$ as

$$(2.3) \quad \mathcal{O}_A = \{uAu^* : u \in \mathcal{U}_k(\mathcal{H})\} \subset A + \mathcal{K}(\mathcal{H}).$$

Considering the action $\pi_b : \mathcal{U}_k \rightarrow \mathcal{O}_A$, $\pi_b(u) = L_u \cdot b = ubu^*$ then \mathcal{O}_A becomes a homogeneous space in some cases. If A has finite spectrum then \mathcal{O}_A is a submanifold of $A + \mathcal{K}(\mathcal{H})$ (see Theorem 4.4 in [1]) and if A is

a compact operator with spectral multiplicity one then \mathcal{O}_A has a smooth structure (see Lemma 1 in [6]).

Denote by $[\cdot, \cdot]$ the commutator operator in $\mathcal{B}(\mathcal{H})$, that is, for any $T, S \in \mathcal{B}(\mathcal{H})$, $[T, S] = TS - ST$.

For each $b \in \mathcal{O}_A$, the isotropy group \mathcal{I}_b is

$$\begin{aligned}\mathcal{I}_b &= \{u \in \mathcal{U}_k : ubu^* = b\} = \{e^K \in \mathcal{U}_k : K \in \mathcal{K}(\mathcal{H})^{ah}, [K, b] = 0\} \\ &= \{b\}' \cap \mathcal{K}(\mathcal{H})^{ah},\end{aligned}$$

where $\{b\}'$ is the set of all operators in $\mathcal{B}(\mathcal{H})$ that commute with b (i.e., $[T, b] = 0$).

For each $b \in \mathcal{O}_A$, its tangent space is $(T\mathcal{O}_A)_b = \{Yb - bY : Y \in \mathcal{K}(\mathcal{H})^{ah}\}$ and can be identified as follows

$$(T\mathcal{O}_A)_b \cong (T\mathcal{U}_k)_1 / (T\mathcal{I}_b)_1 \cong \mathcal{K}(\mathcal{H})^{ah} / (\{b\}' \cap \mathcal{K}(\mathcal{H})^{ah}).$$

In this context we consider the following Finsler metric defined for $x \in (T\mathcal{O}_A)_b$ as

$$\begin{aligned}(2.4) \quad \|x\|_b &= \inf \{ \|Y\| : Y \in \mathcal{K}(\mathcal{H})^{ah} \text{ such that } [Y, b] = x \} \\ &= \inf_{C \in (\{b\}' \cap \mathcal{K}(\mathcal{H})^{ah})} \|Y_0 + C\|.\end{aligned}$$

where $Y_0 + C$ is any element of the class $[Y_0] = \{Y \in \mathcal{K}(\mathcal{H})^{ah} : [Y, b] = x\}$. Note that this norm is invariant under the action of \mathcal{U}_k .

An element $Z \in \mathcal{B}(\mathcal{H})^{ah}$ such that $[Z, b] = x$ and $\|Z\| = \|x\|_b$ is called a minimal lifting for x . This operator Z may not be compact and/or unique (see [5]). Consider piecewise smooth curves $\beta : [a, b] \rightarrow \mathcal{O}_A$. We define

$$(2.5) \quad L(\beta) = \int_a^b \|\beta'(t)\|_{\beta(t)} dt, \text{ and}$$

$$(2.6) \quad \text{dist}(c_1, c_2) = \inf \{L(\beta) : \beta \text{ is smooth, } \beta(a) = c_1, \beta(b) = c_2\}$$

as the rectifiable length of β and distance between two points $c_1, c_2 \in \mathcal{O}_A$, respectively.

If \mathcal{A} is any C^* -algebra of $\mathcal{B}(\mathcal{H})^h$ and $\{e_k\}_{k=1}^\infty$ is a fixed orthonormal basis of \mathcal{H} , we denote with $\mathcal{D}(\mathcal{A})$ the set of diagonal operators with respect to this basis, that is

$$\mathcal{D}(\mathcal{A}) = \{T \in \mathcal{A} : \langle Te_i, e_j \rangle = 0, \text{ for all } i \neq j\}.$$

Given an operator $Z \in \mathcal{A}$, if there exists an operator $D_1 \in \mathcal{D}(\mathcal{A})$ such that

$$\|Z + D_1\| \leq \|Z + D\|$$

for all $D \in \mathcal{D}(\mathcal{A})$, we say that D_1 is a best approximant of Z in $\mathcal{D}(\mathcal{A})$. The operator $Z + D_1$ satisfies

$$\|Z + D_1\| = \text{dist}(Z, \mathcal{D}(\mathcal{A})),$$

and $Z + D_1$ is a minimal operator in the class $[Z]$ of the quotient space $\mathcal{A}/\mathcal{D}(\mathcal{A})$, or similarly we say that D_1 is minimal for Z .

These minimal operators play an important role in the concrete description of minimal length curves on \mathcal{O}_A (see [8] and [1]).

If Z is anti-Hermitian it holds that

$$\text{dist}(Z, \mathcal{D}(\mathcal{A})) = \text{dist}(Z, \mathcal{D}(\mathcal{A}^{ah})),$$

since $\|Im(X)\| \leq \|X\|$ for every $X \in \mathcal{A}$.

Let $T \in \mathcal{B}(\mathcal{H})$ and consider for the fixed basis of \mathcal{H} the coefficients $T_{ij} = \langle Te_i, e_j \rangle$ for each $i, j \in \mathbb{N}$. This define an infinite matrix $(T_{ij})_{i,j \in \mathbb{N}}$ such that their j th-column and i th-row of T are the vectors in ℓ^2 given by $c_j(T) = (T_{1j}, T_{2j}, \dots)$ and $f_j(T) = (T_{i1}, T_{i2}, \dots)$, respectively.

We use $\sigma(T)$ and $R(T)$ to denote the spectrum and range of $T \in \mathcal{B}(\mathcal{H})^h$, respectively.

We define $\Phi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{D}(\mathcal{B}(\mathcal{H}))$, $\Phi(X) = \text{Diag}(X)$, as the map that builds a diagonal operator with the same diagonal as X (i.e., $\Phi(X)_{ii} = \text{Diag}(X)_{ii} = X_{ii}$ and 0 elsewhere). For a given bounded sequence $\{d_n\}_{n \in \mathbb{N}} \subset \mathbb{C}$ we denote with $\text{Diag}(\{d_n\}_{n \in \mathbb{N}})$ the diagonal (infinite) matrix with $\{d_n\}_{n \in \mathbb{N}}$ in its diagonal and 0 elsewhere.

3. THE UNITARY SUBGROUP $\mathcal{U}_{k,d}$

Recall the unitary Fredholm group

$$\mathcal{U}_k = \{u \in \mathcal{U}(\mathcal{H}) : u - 1 \in \mathcal{K}(\mathcal{H})\}$$

(see [1] and [6]) and define the following subsets of $\mathcal{U}(\mathcal{H})$:

$$\begin{aligned} \mathcal{U}_{k,d} &= \{u \in \mathcal{U}(\mathcal{H}) : \exists D \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah}) \text{ such that } u - e^D \in \mathcal{K}(\mathcal{H})\}, \\ \mathcal{U}_d &= \{u \in \mathcal{U}(\mathcal{H}) : \exists D \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah}) \text{ such that } u = e^D\} \\ (3.1) \quad &= \mathcal{U}(\mathcal{H}) \cap \mathcal{D}(\mathcal{B}(\mathcal{H})) \\ \mathcal{U}_{k+d} &= \{u \in \mathcal{U}(\mathcal{H}) : \exists K \in \mathcal{K}(\mathcal{H})^{ah} \text{ and } D \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah}) \\ &\quad \text{such that } u = e^{K+D}\}. \end{aligned}$$

Also denote with

$$\mathcal{O}_b^{\mathcal{F}} = \{ubu^* : u \in \mathcal{F}\},$$

where \mathcal{F} is any of the unitary sets defined in (3.1). The main purpose of this section is the study of these unitary sets and its relations.

The following Proposition has been proved in [6] using arguments of Lemma 2.1 in [1].

Proposition 3.1. $\mathcal{U}_k = \{e^K : K \in \mathcal{K}(\mathcal{H})^{ah}, \|K\| \leq \pi\}$.

Remark 3.2. Let $S_0 \in \mathcal{K}(\mathcal{H})^{ah}$ and $D_0 \in \mathcal{D}(\mathcal{B}(\mathcal{H}))^{ah}$. Then, the exponential series

$\sum_{n=0}^{\infty} \frac{1}{n!} (S_0 + D_0)^n$ converges absolutely and

(3.2)

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{1}{n!} (S_0 + D_0)^n &= \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(S_0^n + \binom{n}{1} S_0^{n-1} D_0 + \cdots + \binom{n}{n-1} S_0 D_0^{n-1} + D_0^n \right) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} S_0 \left(S_0^{n-1} + \binom{n}{1} S_0^{n-2} D_0 + \cdots + \binom{n}{n-1} D_0^{n-1} \right) + \frac{1}{n!} D_0^n \\ &= S_0 \underbrace{\sum_{n=1}^{\infty} \frac{1}{n!} \left(S_0^{n-1} + \binom{n}{1} S_0^{n-2} D_0 + \cdots + \binom{n}{n-1} D_0^{n-1} \right)}_{\parallel \Psi(S_0, D_0)} + \sum_{n=0}^{\infty} \frac{1}{n!} D_0^n \\ &= S_0 \Psi(S_0, D_0) + e^{D_0}. \end{aligned}$$

with $S_0 \Psi(S_0, D_0) \in \mathcal{K}(\mathcal{H})$.

Proposition 3.3. $\mathcal{U}_{k,d}$ is a unitary subgroup of $\mathcal{U}(\mathcal{H})$ and it equals

$$\mathcal{U}_k \mathcal{U}_d = \{u \in \mathcal{U}(\mathcal{H}) : \exists K \in \mathcal{K}(\mathcal{H})^{ah}, \|K\| \leq \pi, \text{ and } D \in \mathcal{D}(\mathcal{B}(\mathcal{H}))^{ah} \text{ such that } u = e^K e^D\}.$$

Moreover

$$\mathcal{U}_{k,d} = \mathcal{U}_k \mathcal{U}_d = \mathcal{U}_d \mathcal{U}_k.$$

Proof. Let $u \in \mathcal{U}_{k,d}$, then there exists $D \in \mathcal{D}(\mathcal{B}(\mathcal{H}))^{ah}$ such that $u - e^D \in \mathcal{K}(\mathcal{H})$. Then

$$\begin{aligned} u e^{-D} - 1 &\in \mathcal{K}(\mathcal{H}) \Rightarrow \exists K \in \mathcal{K}(\mathcal{H})^{ah}, \|K\| \leq \pi \text{ such that } u e^{-D} = e^K \\ &\Rightarrow u = e^K e^D, \end{aligned}$$

and therefore $u \in \mathcal{U}_k \mathcal{U}_d$.

Conversely, if there exists $K' \in \mathcal{K}(\mathcal{H})^{ah}$, and $D' \in \mathcal{D}(\mathcal{B}(\mathcal{H}))^{ah}$ such that $u = e^{K'} e^{D'} \in \mathcal{U}(\mathcal{H})$, then

$$u e^{-D'} = e^{K'} \in \mathcal{U}_k \Rightarrow u e^{-D'} - 1 \in \mathcal{K}(\mathcal{H}) \Rightarrow (u e^{-D'} - 1) e^{D'} = u - e^{D'} \in \mathcal{K}(\mathcal{H}).$$

These calculations prove that $\mathcal{U}_{k,d} = \mathcal{U}_k \mathcal{U}_d$.

Similar computations (with left multiplication of e^{-D}) lead to the equality $\mathcal{U}_{k,d} = \mathcal{U}_d \mathcal{U}_k$.

Now we will prove that $\mathcal{U}_{k,d}$ is a group. Let $u, v \in \mathcal{U}_{k,d}$:

- There exists $D \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$ such that $u - e^D \in \mathcal{K}(\mathcal{H})$. Then $u^* - e^{-D} \in \mathcal{K}(\mathcal{H}) \Rightarrow u^* \in \mathcal{U}_{k,d}$.
- Using that $\mathcal{U}_{k,d} = \mathcal{U}_k \mathcal{U}_d$ we can write $u = e^{K_1} e^{D_1}$ and $v = e^{K_2} e^{D_2}$, with $K_1, K_2 \in \mathcal{K}(\mathcal{H})^{ah}$ and $D_1, D_2 \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$. Then using Remark 3.2

$$uv = e^{D_1+D_2} + K_1 \Psi(K_1, D_1) + e^{D_2} K_2 \Phi(K_2, D_2) + K_1 \Psi(K_1, D_1) K_2 \Phi(K_2, D_2).$$

Therefore $uv - e^{D_1+D_2} \in \mathcal{K}(\mathcal{H})$ which implies that $uv \in \mathcal{U}_{k,d}$.

Then, $\mathcal{U}_{k,d}$ is a unitary subgroup of $\mathcal{U}(\mathcal{H})$. \square

Proposition 3.4. *Let $\mathcal{U}_{k,d}$, \mathcal{U}_d and \mathcal{U}_{k+d} be as defined in (3.1), then the following statements hold:*

- (1) $\mathcal{U}_d \subsetneq \mathcal{U}_{k,d}$.
- (2) $\mathcal{U}_k \subsetneq \mathcal{U}_{k,d}$.
- (3) $\mathcal{U}_{k+d} \subseteq \mathcal{U}_{k,d}$.
- (4) If $u \in \mathcal{U}_{k,d}$ then $u = K' + D'$, with $K' \in \mathcal{K}(\mathcal{H})$ and $D' \in \mathcal{D}(\mathcal{U}(\mathcal{H}))$.
- (5) For every $K \in \mathcal{K}(\mathcal{H})^{ah}$ and $D \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$ there exists $K' \in \mathcal{K}(\mathcal{H})^{ah}$ such that $e^K e^D = e^D e^{K'}$.
- (6) $\mathcal{U}_k \subsetneq \mathcal{U}_{k+d}$.
- (7) $\mathcal{U}_k = \{u \in \mathcal{U}(\mathcal{H}) : \exists D \in \mathcal{D}(\mathcal{K}(\mathcal{H})^{ah}) \text{ such that } u - e^D \in \mathcal{K}(\mathcal{H})\}$.

Proof. (1) It is apparent.

(2) $u \in \mathcal{U}_k \Leftrightarrow u - 1 \in \mathcal{K}(\mathcal{H}) \Leftrightarrow u - e^0 \in \mathcal{K}(\mathcal{H})$.

(3) Let e^{K+D} with $K \in \mathcal{K}(\mathcal{H})^{ah}$ and $D \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$. Then

$$\begin{aligned} e^{K+D} &= 1 + (K + D) + \frac{1}{2!}(K + D)^2 + \dots = e^D + K \Psi(K, D) \Rightarrow \\ &\Rightarrow e^{K+D} - e^D \in \mathcal{K}(\mathcal{H}) \Rightarrow e^{K+D} \in \mathcal{U}_{k,d}. \end{aligned}$$

(4) If $u \in \mathcal{U}_{k,d} \Rightarrow u - e^D \in \mathcal{K}(\mathcal{H})$ with $D \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah}) \Rightarrow \exists K' \in \mathcal{K}(\mathcal{H}) / u = K' + e^D$.

(5) If $K \in \mathcal{K}(\mathcal{H})^{ah}$ and $D \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$ then Proposition 3.1 implies that $e^K - 1 \in \mathcal{K}(\mathcal{H})$ which gives $(e^K - 1)e^D = e^K e^D - e^D \in \mathcal{K}(\mathcal{H})$ and $e^{-D} e^K e^D - 1 \in \mathcal{K}(\mathcal{H})$. Then, $e^{-D} e^K e^D \in \mathcal{U}_k$ and there exists $K' \in \mathcal{K}(\mathcal{H})^{ah}$ such that $e^{-D} e^K e^D = e^{K'}$. The result follows easily.

(6) It is apparent.

- (7) If $u \in \mathcal{U}_k$ then $u - 1 = u - e^0 \in \mathcal{K}(\mathcal{H})$ with $0 \in \mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})$ and then $u \in \{u \in \mathcal{U}(\mathcal{H}) : \exists D \in \mathcal{D}(\mathcal{K}(\mathcal{H})^{ah}) \text{ such that } u - e^D \in \mathcal{K}(\mathcal{H})\}$. Conversely, let $u \in \mathcal{U}(\mathcal{H})$ and $D \in \mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})$ such that $u - e^D \in \mathcal{K}(\mathcal{H})$. Then

$$u - 1 = u - e^D + e^D - 1 \in \mathcal{K}(\mathcal{H}),$$

since $e^D \in \mathcal{U}_k$, which completes the proof. \square

Proposition 3.5. *Let $K_1, K_2 \in \mathcal{K}(\mathcal{H})^{ah}$, $D_1, D_2 \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$, then the following statements are equivalent:*

- a) $e^{K_1}e^{D_1} = e^{K_2}e^{D_2}$
- b) *There exists $d \in \mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})$ such that*

$$e^{K_2} = e^{K_1}e^{-d} \quad \text{and} \quad e^{D_2} = e^d e^{D_1} = e^{d+D_1}.$$

Proof. b) \implies a) is apparent after computing $e^{K_2}e^{D_2}$.

Let us consider a) \implies b).

If $e^{K_1}e^{D_1} = e^{K_2}e^{D_2}$ then $e^{D_1-D_2} = e^{-K_1}e^{K_2}$. Since $e^{-K_1}, e^{K_2} \in \mathcal{U}_k$ which is a group, then there exists $K_{1,2} \in \mathcal{K}(\mathcal{H})^{ah}$ such that $\|K_{1,2}\| \leq \pi$ and $e^{-K_1}e^{K_2} = e^{K_{1,2}}$ (see Proposition 3.1). Moreover, there exists a diagonal $D_{1,2} \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$ with $\|D_{1,2}\| \leq \pi$ such that $e^{D_1-D_2} = e^{D_{1,2}}$. Therefore

$$e^{K_{1,2}} = e^{-K_1}e^{K_2} = e^{D_1-D_2} = e^{D_{1,2}}$$

with $K_{1,2} \in \mathcal{K}(\mathcal{H})^{ah}$ and $D_{1,2} \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$. Using Theorem 3.1 in [7] we can conclude that $|K_{1,2}| = |D_{1,2}|$ which implies that $K_{1,2}$ and $D_{1,2}$ are both diagonal and compact operators. If we chose $-d = D_{1,2} \in \mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})$, then

$$e^{K_2} = e^{K_1}e^{D_1-D_2} = e^{K_1}e^{D_{1,2}} = e^{K_1}e^{-d}$$

and

$$e^{D_2} = e^{D_2-D_1}e^{D_1} = e^{-D_{1,2}}e^{D_1} = e^d e^{D_1}$$

which proves the proposition. \square

Proposition 3.6. *Let $u \in \mathcal{U}(\mathcal{H})$. Then the following statements are equivalent*

- a) $u \in \mathcal{U}_{k,d}$
- b) $u - \text{Diag}(u) \in \mathcal{K}(\mathcal{H})$ and $|u_{j,j}| \xrightarrow{j \rightarrow \infty} 1$.

Proof. a) \implies b) If $u \in \mathcal{U}_{k,d}$ then there exists $D \in \mathcal{D}(\mathcal{B}(\mathcal{H}))^{ah}$ such that $u - e^D \in \mathcal{K}(\mathcal{H})$. Then $\text{Diag}(u - e^D)_{j,j} = u_{j,j} - e_{j,j}^D \xrightarrow{j \rightarrow \infty} 0$ and therefore

$\text{Diag}(u) - e^D \in \mathcal{K}(\mathcal{H})$ and $|u_{j,j}| \xrightarrow{j \rightarrow \infty} 1$. Then since

$$u - e^D = u - \text{Diag}(u) + \underbrace{\text{Diag}(u) - e^D}_{\in \mathcal{K}(\mathcal{H})} \in \mathcal{K}(\mathcal{H}),$$

we obtain that $u - \text{Diag}(u) \in \mathcal{K}(\mathcal{H})$.

b) \Leftarrow a) If $u_{j,j} \xrightarrow{j \rightarrow \infty} 1$ then there exists $D \in \mathcal{D}(\mathcal{B}(\mathcal{H}))^{ah}$ such that $(u_{j,j} - e_{j,j}^D) \xrightarrow{j \rightarrow \infty} 0$ (for example take $e_{j,j}^D = \frac{1}{|u_{j,j}|} u_{j,j}$ when j is sufficiently large). Then $\text{Diag}(u) - e^D \in \mathcal{K}(\mathcal{H})$. Using the hypothesis that $\text{Diag}(u) - e^D \in \mathcal{K}(\mathcal{H})$, then

$$\underbrace{u - \text{Diag}(u)}_{\in \mathcal{K}(\mathcal{H})} + \underbrace{\text{Diag}(u) - e^D}_{\in \mathcal{K}(\mathcal{H})} = u - e^D \in \mathcal{K}(\mathcal{H})$$

and therefore $u \in \mathcal{U}_{k,d}$. \square

Remark 3.7. The previous proposition allow us to prove easily that $\mathcal{U}_{k,d} \subsetneq \mathcal{U}(\mathcal{H})$ since the block diagonal defined symmetry $u = \begin{pmatrix} s & 0 & 0 & \cdots \\ 0 & s & 0 & \cdots \\ 0 & 0 & s & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$ with $s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ clearly does not satisfy conditions b) of Proposition 3.6, but $u \in \mathcal{U}(\mathcal{H})$.

The following proposition is a consequence of results present in [7].

Proposition 3.8. Let $K, K' \in \mathcal{K}(\mathcal{H})^{ah}$ satisfying $\|K\|, \|K'\| \leq \pi$ and $D \in \mathcal{D}(\mathcal{B}(\mathcal{H}))^{ah}$ be such that $e^D e^K = e^{K'} e^D$. Then $\|K\| = \|K'\|$ and

- a) if $\|K\| = \|K'\| = \pi$, then
 - (1) $|K| = e^{-D} |K'| e^D$,
 - (2) $v \in \mathcal{H}$ is an eigenvector of K with corresponding eigenvalue $\lambda \in i\mathbb{R}$, $|\lambda| < \pi \iff e^D v$ is an eigenvector of K' with corresponding eigenvalue $\lambda \in i\mathbb{R}$, $|\lambda| < \pi$,
 - (3) if E_X is the spectral measure of the operator X , then

$$K - e^{-D} K' e^D = 2\pi i (E_K(\mathbb{R} + i\pi) - E_{e^{-D} K' e^D}(\mathbb{R} + i\pi))$$

- b) and moreover, if $\|K\| = \|K'\| < \pi$ then $K = e^{-D} K' e^D$.

Proof. Observe first that since $e^D e^K = e^{K'} e^D$ then

$$(3.3) \quad e^K = e^{-D} e^{K'} e^D = e^{e^{-D} K' e^D}$$

and therefore $|K| = |e^{-D} K' e^D| = e^{-D} |K'| e^D$ (see Theorem 3.1 i) in [7]) which implies $\|K\| = \|K'\|$.

- a) (1) This is a direct consequence of (3.3), the fact that $\sigma(K)$ and $\sigma(e^{-D} K' e^D)$ are contained in $\mathcal{S} = \{z \in \mathbb{C} : -\pi \leq \text{Im}(z) \leq \pi\}$ and Theorem 3.1 i) of [7].

- (2) Consider $\lambda \in \sigma(K) \subset i\mathbb{R}$, $|\lambda| < \pi$ and $v \in \mathcal{H}$ such that $Kv = \lambda v$. Then $e^K v = e^\lambda v$ and the equation (3.3) imply that e^λ is an eigenvalue of $e^{e^{-D}K'e^D}$ with eigenvector v . Then $\lambda \in \sigma(e^{-D}K'e^D)$ (because $|\lambda| < 1$) with eigenvector v . Therefore $\lambda \in \sigma(K')$ with eigenvector $e^D v$. The other implication follows similarly.
- (3) This statement follows from $\sigma(K), \sigma(e^{-D}K'e^D) \subset \mathcal{S}$, Remark 2.4 and Theorem 4.1 in [7].
- b) If the strict inequality $\|K\| = \|K'\| < \pi$ holds then (3.3) and Corollary 4.2 iii) in [7] imply directly that $K = e^{-D}K'e^D$.

□

Corollary 3.9. *Let $K, K' \in \mathcal{K}(\mathcal{H})^{ah}$, $\|K\|, \|K'\| \leq \pi$ and $D \in \mathcal{D}(\mathcal{B}(\mathcal{H}))^{ah}$. Then*

- a) *if $\|K\| = \|K'\| = \pi$, the following equivalence holds*

$$e^D e^K = e^{K'} e^D \iff K - e^{-D}K'e^D = 2\pi i (E_K(\mathbb{R} + i\pi) - E_{e^{-D}K'e^D}(\mathbb{R} + i\pi))$$

- b) *and if $\|K\|, \|K'\| < \pi$, the following equivalence holds*

$$e^D e^K = e^{K'} e^D \iff K = e^{-D}K'e^D$$

Proof. a) If $e^D e^K = e^{K'} e^D$ then

$$K - e^{-D}K'e^D = 2\pi i (E_K(\mathbb{R} + i\pi) - E_{e^{-D}K'e^D}(\mathbb{R} + i\pi))$$

follows from a) (3) of the previous Proposition 3.8.

The converse is proved using that $K - 2\pi i E_K(\mathbb{R} + i\pi) = e^{-D}K'e^D - E_{e^{-D}K'e^D}(\mathbb{R} + i\pi)$ implies that

$$e^{K - 2\pi i E_K(\mathbb{R} + i\pi)} = e^{e^{-D}K'e^D - 2\pi i E_{e^{-D}K'e^D}(\mathbb{R} + i\pi)}$$

and since K commutes with E_K and $e^{-D}K'e^D$ with $E_{e^{-D}K'e^D}$, follows that

$$e^K e^{-2\pi i E_K(\mathbb{R} + i\pi)} = e^{e^{-D}K'e^D} e^{-2\pi i E_{e^{-D}K'e^D}(\mathbb{R} + i\pi)}.$$

Since $e^{-2\pi i E_K(\mathbb{R} + i\pi)} = e^{-2\pi i E_{e^{-D}K'e^D}(\mathbb{R} + i\pi)} = 1$ then

$$e^K = e^{e^{-D}K'e^D} = e^{-D}e^{K'}e^D$$

which ends the proof.

- b) It is apparent using the previous Proposition 3.8 (b) and the fact that $e^{e^{-D}K'e^D} = e^{-D}e^{K'}e^D$.

□

In [11] Thompson proved for any $X, Y \in M_n(\mathbb{C})^{ah}$ that there exist unitaries U, V such that

$$(3.4) \quad e^X e^Y = e^{U^* X U + V^* Y V}.$$

Subsequently, in [3] Antezana et. al. proved a generalization of (3.4) for compact operators: given $K_1, K_2 \in \mathcal{K}(\mathcal{H})^{ah}$, there exist unitaries U_n, V_n , for $n \in \mathbb{N}$, such that

$$e^{K_1} e^{K_2} = \lim_{n \rightarrow \infty} e^{U_n^* K_1 U_n + V_n^* K_2 V_n}$$

where the convergence is considered in the usual operator norm. The following proposition adds a new simple case where this equality holds.

Proposition 3.10. *Let $K \in \mathcal{K}(\mathcal{H})^{ah}$ and $D \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$ and suppose that there exists $\lambda \in i\mathbb{R}$ such that $\lim_{n \rightarrow \infty} D_{nn} = \lambda$. Then, there exist unitaries U_n, V_n , for $n \in \mathbb{N}$, such that*

$$e^K e^D = \lim_{n \rightarrow \infty} e^{U_n^* K U_n + V_n^* D V_n}.$$

Proof. Observe that $D - \lambda I \in \mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})$. Then, using Theorem 3.1 and Remark 3.3 in [3] there exist unitaries U_n, V_n , for $n \in \mathbb{N}$, such that

$$\begin{aligned} e^K e^D e^{-\lambda I} &= e^K e^{D - \lambda I} = \lim_{n \rightarrow \infty} e^{U_n^* K U_n + V_n^* (D - \lambda I) V_n} \\ &= \lim_{n \rightarrow \infty} e^{U_n^* K U_n + V_n^* D V_n} e^{-\lambda I}. \end{aligned}$$

Therefore, $e^K e^D = \lim_{n \rightarrow \infty} e^{U_n^* K U_n + V_n^* D V_n}$. \square

Proposition 3.11. *Let $K_1, K_2 \in \mathcal{K}(\mathcal{H})^{ah}$ and $D_1, D_2 \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$ such that $K_1 + D_1$ and $K_2 + D_2$ are sufficiently close to 0 (see Definition 2.1). Then, there exists $K \in \mathcal{K}(\mathcal{H})^{ah}$ such that*

$$e^{K_1 + D_1} e^{K_2 + D_2} = e^{K + D_1 + D_2}.$$

Proof. Let $K_1, K_2 \in \mathcal{K}(\mathcal{H})^{ah}$ and $D_1, D_2 \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$ such that $K_1 + D_1$ and $K_2 + D_2$ are sufficiently close to 0. Using the B-C-H formula (2.2), then $X = \log(e^{K_1 + D_1} e^{K_2 + D_2}) = K_1 + D_1 + K_2 + D_2 + \sum_{n \geq 2} c_n(K_1 + D_1, K_2 + D_2)$.

Also, observe that $c_n(K_1 + D_1, K_2 + D_2) \in \mathcal{K}(\mathcal{H})^{ah}$ for every n , since $\mathcal{K}(\mathcal{H})$ is a two-sided closed ideal and $[D_1, D_2] = 0$. Therefore, $X = K + D$, with $K \in \mathcal{K}(\mathcal{H})^{ah}$ and $D \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$ and

$$(3.5) \quad e^{K_1 + D_1} e^{K_2 + D_2} = e^{K + D} \in \mathcal{U}_{k+d}.$$

In particular $D = D_1 + D_2$, since each c_n is a sum of commutators and $\text{Diag}([A, B]) = 0$ for every $A, B \in \mathcal{B}(\mathcal{H})^{ah}$. \square

Corollary 3.12. *Let $K_1, K_2 \in \mathcal{K}(\mathcal{H})^{ah}$ and $D_1, D_2 \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$.*

- (1) If $K_1 + D_1$ and $K_2 + D_2$ are sufficiently close to 0 (see Definition 2.1), then

$$e^{K_1+D_1}e^{K_2+D_2} = e^{\tilde{K}+D_1+D_2}, \text{ with}$$

$$\tilde{K} = K_1 + K_2 + \sum_{n \geq 2} c_n(K_1 + D_1, K_2 + D_2) \in \mathcal{K}(\mathcal{H})^{ah}$$

$$\text{and } \text{Diag}(\tilde{K}) = \text{Diag}(K_1 + K_2).$$

- (2) If K_1 and D_1 are sufficiently close to 0, there exist $K', K'' \in \mathcal{K}(\mathcal{H})^{ah}$ such that

$$(3.6) \quad e^{K_1}e^{D_1} = e^{D_1}e^{K'} = e^{K''+D_1}.$$

Proof. These equalities are due to item (3) of the Proposition 3.4, Proposition 3.11 and some calculations from its proof. \square

Theorem 3.13. $\mathcal{U}_{k,d}$ is arc-connected and closed in $\mathcal{U}(\mathcal{H})$.

Proof. Every $u = e^K e^D \in \mathcal{U}_{k,d}$ (with $K \in \mathcal{K}(\mathcal{H})^{ah}$ and $D \in \mathcal{D}(\mathcal{B}(\mathcal{H}))^{ah}$) is connected to 1 by the curve $\gamma(t) = e^{tK} e^{tD}$, for $t \in [0, 1]$.

Consider now the closedness of $\mathcal{U}_{k,d}$. Let $\{u_n\}_{n \in \mathbb{N}} \subset \mathcal{U}_{k,d}$, $u_n = e^{K_n} e^{D_n}$ for $n \in \mathbb{N}$, $K_n \in \mathcal{K}(\mathcal{H})^{ah}$ and $D_n \in \mathcal{D}(\mathcal{B}(\mathcal{H}))^{ah}$ be a sequence such that $\lim_{n \rightarrow \infty} u_n = u_0$ in the usual operator norm in $B(H)$. We will prove that $u_0 \in \mathcal{U}_{k,d}$.

Since $u_0 - u_n = u_0 - e^{D_n} + e^{D_n} - u_n$ tends to 0 as $n \rightarrow \infty$ and $e^{D_n} - u_n \in \mathcal{K}(\mathcal{H})$, for all $n \in \mathbb{N}$, then $\text{dist}(\{u_0 - e^{D_n}\}_{n \in \mathbb{N}}, \mathcal{K}(\mathcal{H})) = 0$.

Observe that

$$(3.7) \quad \begin{aligned} & \text{dist} \left(\{ \text{Diag}(u_0) - e^{D_n} \}_{n \in \mathbb{N}}, \mathcal{K}(\mathcal{H}) \right) = \\ & = \text{dist} \left(\{ \text{Diag}(u_0) - \text{Diag}(u_n) + \text{Diag}(u_n) - e^{D_n} \}_{n \in \mathbb{N}}, \mathcal{K}(\mathcal{H}) \right) \\ & \leq \inf_{K \in \mathcal{K}(\mathcal{H})} \| \text{Diag}(u_0) - \text{Diag}(u_n) \| + \| \text{Diag}(u_n) - e^{D_n} - K \| \end{aligned}$$

for any $n \in \mathbb{N}$.

Note that $u_n - e^{D_n} \in \mathcal{K}(\mathcal{H})$ which implies that $\text{Diag}(u_n - e^{D_n}) = \text{Diag}(u_n) - e^{D_n} \in \mathcal{D}(\mathcal{K}(\mathcal{H}))$. Since $u_n \rightarrow u_0$ then $\text{Diag}(u_n) \rightarrow \text{Diag}(u_0)$. Then the first summand in the last inequality (3.7) can be chosen to be arbitrarily small for big n and the infimum of the second term is zero because $\text{Diag}(u_n) - e^{D_n} \in \mathcal{K}(\mathcal{H})$. Then

$$(3.8) \quad \begin{aligned} & \text{dist} \left(\{ \text{Diag}(u_0) - e^{D_n} \}_{n \in \mathbb{N}}, \mathcal{K}(\mathcal{H}) \right) = 0 \\ & = \text{dist} \left(\{ \text{Diag}(u_0) - e^{D_n} \}_{n \in \mathbb{N}}, \mathcal{D}(\mathcal{K}(\mathcal{H})) \right). \end{aligned}$$

Moreover, using that $u_n - e^{D_n} \in \mathcal{K}(\mathcal{H})$, for every $K \in \mathcal{K}(\mathcal{H})$ holds that

$$\begin{aligned}
 \text{dist}(u_0 - \text{Diag}(u_0), \mathcal{K}(\mathcal{H})) &= \inf_{K \in \mathcal{K}(\mathcal{H})} \|u_0 - \text{Diag}(u_0) - K\| \\
 (3.9) \quad &\leq \|u_0 - \text{Diag}(u_0) - u_n + e^{D_n} - K\| \\
 &\leq \|u_0 - u_n\| + \|e^{D_n} - \text{Diag}(u_0) - K\|
 \end{aligned}$$

Here both summands of the last term can be chosen to be arbitrarily small. It is enough to take n appropriately, since $u_n \rightarrow u_0$ and the distance from $\{\text{Diag}(u_0) - e^{D_n}\}_{n \in \mathbb{N}}$ to $\mathcal{K}(\mathcal{H})$ is null as seen above in (3.8). Then $\text{dist}(u_0 - \text{Diag}(u_0), \mathcal{K}(\mathcal{H})) = 0$ and therefore

$$(3.10) \quad u_0 - \text{Diag}(u_0) \in \mathcal{K}(\mathcal{H}).$$

If there exists $\delta > 0$ such that for a subsequence $\{e^{D_{n_k}}\}_{k \in \mathbb{N}}$, for $k \in \mathbb{N}$, holds that $|(\text{Diag}(u_0) - e^{D_{n_k}})_{j,j}| \geq \delta$ for infinite $j \in \mathbb{N}$ which contradicts (3.8). Therefore, given $\delta > 0$, only finite $n \in \mathbb{N}$ satisfy that $|(\text{Diag}(u_0) - e^{D_n})_{j,j}| \geq \delta$ for infinite $j \in \mathbb{N}$. Then, if $k \in \mathbb{N}$ and we choose $\delta = \frac{1}{k}$, there exists $n_k \in \mathbb{N}$ such that if $n \geq n_k$ then $|(\text{Diag}(u_0) - e^{D_n})_{j,j}| \geq \frac{1}{k}$ only for finite $j \in \mathbb{N}$. Observe that the subsequence n_k could be chosen to be strictly increasing. For each $k \in \mathbb{N}$, we will define a sub-index $j_k \in \mathbb{N}$ such that $|(\text{Diag}(u_0) - e^{D_{n_k}})_{j,j}| < \frac{1}{k}$ for all $j \geq j_k$, $j \in \mathbb{N}$. Moreover j_k can be chosen to be strictly increasing in k and $j_1 > 1$. Therefore, for each $k \in \mathbb{N}$, there exists $n_k, j_k \in \mathbb{N}$ such that

$$(3.11) \quad |(\text{Diag}(u_0) - e^{D_{n_k}})_{j,j}| < \frac{1}{k}, \text{ for all } j \geq j_k$$

Then define the following unitary diagonal matrix e^D in terms of its j, j entries (and zero elsewhere) whose construction is based in the $e^{D_{n_k}}$, and the corresponding j_k mentioned above:

$$(3.12) \quad (e^D)_{j,j} = \begin{cases} 1, & \text{if } 1 \leq j < j_1 \\ (e^{D_{n_1}})_{j,j}, & \text{if } j_1 \leq j < j_2 \\ (e^{D_{n_2}})_{j,j}, & \text{if } j_2 \leq j < j_3 \\ \dots & \dots \\ (e^{D_{n_k}})_{j,j}, & \text{if } j_k \leq j < j_{k+1} \\ \dots & \dots \end{cases}.$$

D can be chosen as the anti-Hermitian diagonal matrix formed with the corresponding parts of 0, D_{n_1} , D_{n_2} , \dots , D_{n_k} , \dots .

If we define $j_0 = 1$ and take any $j \in \mathbb{N}$, then equation (3.11) and definition (3.12) imply that

$$|(\text{Diag}(u_0) - e^D)_{j,j}| = |(\text{Diag}(u_0) - e^{D_{n_k}})_{j,j}| < \frac{1}{k}, \quad \text{provided } j_k \leq j < j_{k+1}$$

Then $(\text{Diag}(u_0) - e^D)_{j,j} \rightarrow 0$ as $j \rightarrow \infty$, and therefore $\text{Diag}(u_0) - e^D \in \mathcal{K}(\mathcal{H})$ since it is a diagonal matrix.

Using that also $u_0 - \text{Diag}(u_0) \in \mathcal{K}(\mathcal{H})$ (see (3.10)) we conclude that $(u_0 - \text{Diag}(u_0)) + (\text{Diag}(u_0) - e^D) = u_0 - e^D \in \mathcal{K}(\mathcal{H})$, which implies that $u_0 \in \mathcal{U}_{k,d}$ and therefore $\mathcal{U}_{k,d}$ is closed. \square

Lemma 3.14. *There exists $\varepsilon_0 > 0$ such that if $u \in \mathcal{U}_{k,d}$ and $\|u - 1\| < \varepsilon_0$ then $u \in \mathcal{U}_{k+d}$.*

Moreover, there exist $K \in \mathcal{K}(\mathcal{H})^{ah}$ and $D \in \mathcal{D}(\mathcal{B}(\mathcal{H}))^{ah}$ such that

- $u = e^{K+D}$ with $K, D \in \exp^{-1}(B(1, 3\varepsilon_0))$,
- K, D are sufficiently close to 0 and
- $K + D \in \exp^{-1}(B(1, \varepsilon_0)) \cap \mathcal{B}(\mathcal{H})^{ah}$.

Proof. Let us fix $\delta_0 > 0$ such that fulfills two conditions. One of them is that if $V \in B(0, \delta_0) \cap \mathcal{B}(\mathcal{H})^{ah}$ then V is sufficiently close to 0 as in Definition 2.1. The other one is that $\exp : B(0, \delta_0) \cap \mathcal{B}(\mathcal{H})^{ah} \rightarrow \exp(B(0, \delta_0)) \cap \mathcal{U}(\mathcal{H})$ is a diffeomorphism considering the usual operator norm. The last requirement can be fulfilled after applying the inverse map theorem for Banach spaces.

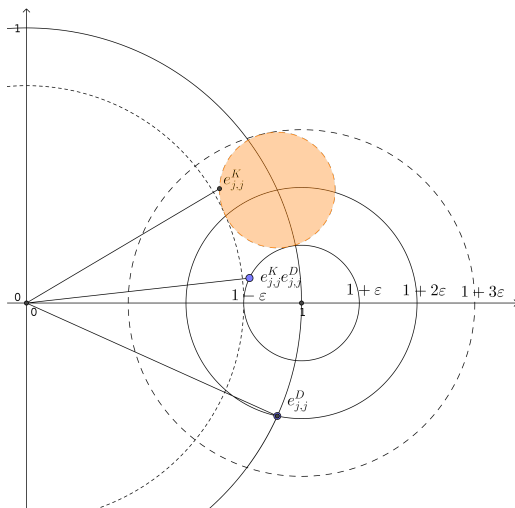
Then define $\varepsilon_0 = \varepsilon > 0$ such that

$$(3.13) \quad B(1, \varepsilon) \subset B(1, 3\varepsilon) \subset \exp(B(0, \delta_0)).$$

If we take $u \in \mathcal{U}_{k,d} \cap B(1, \varepsilon)$, then there exists $K_1 \in \mathcal{K}(\mathcal{H})^{ah}$ and $D_1 \in \mathcal{D}(\mathcal{B}(\mathcal{H}))^{ah}$ such that $u = e^{K_1} e^{D_1}$. Observe that the j, j entries of the diagonal of $u = e^{K_1} e^{D_1}$ are $e_{j,j}^{K_1} e_{j,j}^{D_1}$.

Then $\|u - 1\| < \varepsilon$ implies that $|e_{j,j}^{K_1} e_{j,j}^{D_1} - 1| < \varepsilon$ for all $j \in \mathbb{N}$. Suppose that $|e_{j,j}^{D_1} - 1| \geq 2\varepsilon$ for infinite $j \in \mathbb{N}$. Then, using that $|e_{j,j}^{D_1}| = 1$ we obtain that $|e_{j,j}^{-D_1} - 1| = |e_{j,j}^{D_1} (e_{j,j}^{-D_1} - 1)| = |1 - e_{j,j}^{D_1}| \geq 2\varepsilon$, and that $|e_{j,j}^{K_1} - e_{j,j}^{-D_1}| = |(e_{j,j}^{K_1} - e_{j,j}^{-D_1}) e_{j,j}^{D_1}| = |e_{j,j}^{K_1} e_{j,j}^{D_1} - 1| < \varepsilon$ for infinite $j \in \mathbb{N}$. Therefore, there must exist infinite $j \in \mathbb{N}$ such that

$$(3.14) \quad |e_{j,j}^{K_1} - e_{j,j}^{-D_1}| < \varepsilon \text{ and } |e_{j,j}^{-D_1} - 1| \geq \varepsilon$$



Therefore

$$\begin{aligned} |e_{j,j}^{K_1} - 1| &= |e_{j,j}^{K_1} - e_{j,j}^{-D_1} + e_{j,j}^{-D_1} - 1| \geq ||e_{j,j}^{K_1} - e_{j,j}^{-D_1}| - |e_{j,j}^{-D_1} - 1|| \\ &= |e_{j,j}^{-D_1} - 1| - |e_{j,j}^{K_1} - e_{j,j}^{-D_1}| \geq 2\varepsilon - \varepsilon = \varepsilon. \end{aligned}$$

for infinite $j \in \mathbb{N}$, where we used (3.14) in the last equality and inequality. This is a contradiction because $e^{K_1} \in \mathcal{U}_k$ and then $e^{K_1} - 1 \in \mathcal{K}(\mathcal{H})$ which implies that the diagonal of e^{K_1} tends to 1. Then $|e_{j,j}^{D_1} - 1| \geq 2\varepsilon$ only for finite $j \in \mathbb{N}$. Choosing appropriately a compact anti-Hermitian diagonal d we can construct $D_2 = D_1 + d \in \mathcal{D}(\mathcal{B}(\mathcal{H}))^{ah}$ and $K_2 \in \mathcal{K}(\mathcal{H})^{ah}$ such that $u = e^{K_1}e^{D_1} = e^{K_2}e^{D_2}$ and $|e_{j,j}^{D_2} - 1| \leq 2\varepsilon$ for all $j \in \mathbb{N}$ (see Proposition 3.5). Then, $\|e^{D_2} - 1\| < 2\varepsilon$ and $\|e^{D_2} - 1\| = \|e^{-D_2}(e^{D_2} - 1)\| = \|1 - e^{-D_2}\| = \|e^{-D_2} - 1\| < 2\varepsilon$. Moreover, since e^{D_2} is unitary,

$$\begin{aligned} (3.15) \quad \|e^{K_2} - 1\| &\leq \|e^{K_2} - e^{-D_2}\| + \|e^{-D_2} - 1\| \\ &= \|(e^{K_2} - e^{-D_2})e^{D_2}\| + \|e^{-D_2} - 1\| \\ &= \|e^{K_2}e^{D_2} - 1\| + \|e^{-D_2} - 1\| \\ &= \|u - 1\| + \|e^{-D_2} - 1\| < \varepsilon + 2\varepsilon = 3\varepsilon \end{aligned}$$

We obtained that $\|e^{D_2} - 1\| < 2\varepsilon$ and $\|e^{K_2} - 1\| < 3\varepsilon$ which implies that e^{D_2} and $e^{K_2} \in \exp(B(0, \delta_0))$ (see the definition of ε in (3.13)). Therefore, using that $\exp : B(0, \delta_0) \cap \mathcal{B}(\mathcal{H})^{ah} \rightarrow \exp(B(0, \delta_0)) \cap \mathcal{U}(\mathcal{H})$ is a diffeomorphism, there exist unique D and K in $\exp^{-1}(B(0, 3\varepsilon)) \cap \mathcal{B}(\mathcal{H})^{ah} \subset B(0, \delta_0) \cap \mathcal{B}(\mathcal{H})^{ah}$ such that $e^D = e^{D_2}$ and $e^K = e^{K_2}$. Standard calculations can show that under these conditions D must be diagonal and K compact. Hence $D \in \exp^{-1}(B(0, 3\varepsilon)) \cap \mathcal{D}(\mathcal{B}(\mathcal{H}))^{ah}$ and $K \in \exp^{-1}(B(0, 3\varepsilon)) \cap \mathcal{K}(\mathcal{H})^{ah}$. Moreover, since $D, K \in B(0, \delta_0)$ they are sufficiently close to 0. Then using (3.6)

$$u = e^K e^D = e^{K+D} \in \mathcal{U}_{k+d}$$

with $K \in \mathcal{K}(\mathcal{H})^{ah}$ and $D \in \mathcal{D}(\mathcal{B}(\mathcal{H}))^{ah}$, $K, D \in \exp^{-1}(B(0, 3\varepsilon))$ and K, D sufficiently close to 0 as required in a) and b).

Since $e^{K+D} = u \in \mathcal{U}_{k,d}$, and K and D are sufficiently close to 0, then $\|K+D\| < \pi$. Hence, since $\exp : \exp^{-1}(B(1, \varepsilon)) \rightarrow B(1, \varepsilon)$ is a diffeomorphism there exists $V \in \exp^{-1}(B(1, \varepsilon))$ such that $e^V = u = e^{K+D}$. Then Corollary 4.2 in [7] implies that $V = K + D$ and therefore $K + D \in \exp^{-1}(B(1, \varepsilon))$ as required in c). \square

Proposition 3.15. *There exists $\mathcal{V} \subset \mathcal{B}(\mathcal{H})^{ah}$ an open neighborhood of 0 such that*

$$\exp\left(\mathcal{V} \cap \left(\mathcal{K}(\mathcal{H})^{ah} + \mathcal{D}(\mathcal{B}(\mathcal{H}))^{ah}\right)\right) = \exp(\mathcal{V}) \cap \mathcal{U}_{k,d}$$

Proof. Take $\mathcal{V} = \exp^{-1}(B(1, \varepsilon_0)) \cap \mathcal{B}(\mathcal{H})^{ah}$, where ε_0 is the one from Lemma 3.14. Then, as seen in that lemma, for every $u \in \mathcal{U}_{k,d}$, and $u = e^{K_1} e^{D_1} \in B(1, \varepsilon_0)$ with $K_1 \in \mathcal{K}(\mathcal{H})^{ah}$ and $D_1 \in \mathcal{D}(\mathcal{B}(\mathcal{H}))^{ah}$, there exist $K \in \mathcal{K}(\mathcal{H})^{ah} \cap \exp^{-1}(B(1, 3\varepsilon_0))$ and $D \in \mathcal{D}(\mathcal{B}(\mathcal{H}))^{ah} \cap \exp^{-1}(B(1, 3\varepsilon_0))$ such that

$$(3.16) \quad u = e^{K+D}, \quad \text{with } (K+D) \in \exp^{-1}(B(1, \varepsilon_0)) \cap \mathcal{B}(\mathcal{H})^{ah}.$$

Suppose first that $V \in \mathcal{V}$ and $e^V \in \exp(\mathcal{V}) \cap \mathcal{U}_{k,d}$. Then, as commented in (3.16), there exists $K \in \mathcal{K}(\mathcal{H})^{ah}$ and $D \in \mathcal{D}(\mathcal{B}(\mathcal{H}))^{ah}$ such that $e^V = e^{K+D}$. Since the exponential is a diffeomorphism restricted to the neighborhood \mathcal{V} then $V = K + D$. Therefore

$$\exp(\mathcal{V}) \cap \mathcal{U}_{k,d} \subset \exp\left(\mathcal{V} \cap \left(\mathcal{K}(\mathcal{H})^{ah} + \mathcal{D}(\mathcal{B}(\mathcal{H}))^{ah}\right)\right).$$

Now suppose that $V \in \mathcal{V}$ and

$$e^V = e^{K+D} \in \exp\left(\mathcal{V} \cap \left(\mathcal{K}(\mathcal{H})^{ah} + \mathcal{D}(\mathcal{B}(\mathcal{H}))^{ah}\right)\right)$$

with $K \in \mathcal{K}(\mathcal{H})^{ah}$ and $D \in \mathcal{D}(\mathcal{B}(\mathcal{H}))^{ah}$. Then clearly $e^V \in \exp(\mathcal{V})$ and using (3) from Properties 3.4 we obtain that also $e^V = e^{K+D} \in \mathcal{U}_{k,d}$ holds. This proves that $\exp\left(\mathcal{V} \cap \left(\mathcal{K}(\mathcal{H})^{ah} + \mathcal{D}(\mathcal{B}(\mathcal{H}))^{ah}\right)\right) \subset \exp(\mathcal{V}) \cap \mathcal{U}_{k,d}$ which concludes the proof. \square

Proposition 3.16. $\{X \in \mathcal{B}(\mathcal{H})^{ah} : e^{tX} \in \mathcal{U}_{k,d}, \forall t \in \mathbb{R}\} = \mathcal{K}(\mathcal{H})^{ah} + \mathcal{D}(\mathcal{B}(\mathcal{H}))^{ah}.$

Proof. The property (3) of Proposition 3.4 directly implies that $e^{t(K+D)} = e^{tK+tD} \in \mathcal{U}_{k,d}$ for all $t \in \mathbb{R}$ and therefore $\mathcal{K}(\mathcal{H})^{ah} + \mathcal{D}(\mathcal{B}(\mathcal{H}))^{ah} \subset L(\mathcal{U}_{k,d})$.

Suppose now that $X \neq 0$ (0 is a trivial case) and let $X \in L(\mathcal{U}_{k,d})$, then $e^{tX} \in \mathcal{U}_{k,d}, \forall t \in \mathbb{R}$. In particular $e^{tX} \in \mathcal{U}_{k,d}$ holds for small $|t|$, for example for $t_0 = \frac{\delta_0}{2\|X\|} < \frac{\delta_0}{\|X\|}$ where $\delta_0 > 0$ is the constant used in the proof of Lemma 3.14. Then, $\|t_0 X\| = |t_0| \|X\| < \delta_0$ and $u = e^{t_0 X} \in \mathcal{U}_{k,d}$. Therefore using Lemma 3.14, there exists $K \in \mathcal{K}(\mathcal{H})^{ah}$ and $D \in \mathcal{D}(\mathcal{B}(\mathcal{H}))^{ah}$ such that

$$e^{t_0 X} = e^{K+D}.$$

The constant δ_0 of the proof of Lemma 3.14 is chosen such that $\exp : B(0, \delta_0) \cap \mathcal{B}(\mathcal{H})^{ah} \rightarrow \exp(B(0, \delta_0)) \cap \mathcal{U}(\mathcal{H})$ is a diffeomorphism. Then $t_0 X = K + D$ and therefore $X = 1/t_0(K + D) = 1/t_0 K + 1/t_0 D \in \mathcal{K}(\mathcal{H})^{ah} + \mathcal{D}(\mathcal{B}(\mathcal{H}))^{ah}$ as required. \square

Remark 3.17. Following V.2.3 [9] and [10] (page 428) we call H a Lie subgroup of G if H is a closed subgroup of a Banach–Lie group G which is itself a Lie group relative to the induced topology.

Therefore the previous results allow us to state the following.

Theorem 3.18. $\mathcal{U}_{k,d}$ is a Lie subgroup of $\mathcal{U}(\mathcal{H})$ and its Lie algebra is $L(\mathcal{U}_{k,d}) = \mathcal{K}(\mathcal{H})^{ah} + \mathcal{D}(\mathcal{B}(\mathcal{H}))^{ah}$.

Proof. According to the definition of Lie subgroup mentioned in the previous Remark 3.17, Theorem 3.13 and Proposition 3.15 imply that $\mathcal{U}_{k,d}$ is a Lie subgroup of $\mathcal{U}(\mathcal{H})$.

The equality $L(\mathcal{U}_{k,d}) = \mathcal{K}(\mathcal{H})^{ah} + \mathcal{D}(\mathcal{B}(\mathcal{H}))^{ah}$ follows from Corollary V.2.2 in [9] and Proposition 3.16. \square

Although there exist stronger notions of Lie subgroups those cannot be used for $\mathcal{U}_{k,d}$ since its Lie algebra $\mathcal{K}(\mathcal{H})^{ah} + \mathcal{D}(\mathcal{B}(\mathcal{H}))^{ah}$ is not complemented in $\mathcal{B}(\mathcal{H})^{ah}$ (the Lie algebra of $\mathcal{U}(\mathcal{H})$).

Remark 3.19. Generalization of the $\mathcal{U}_{k,d}$ group.

The proofs of some of the basic properties we use in the study of $\mathcal{U}_{k,d}$ require that the exponential $\exp : \mathcal{K}(\mathcal{H})^{ah} \rightarrow \mathcal{U}_k$ must be surjective. This is the reason why the following generalization involves ideals \mathcal{J} with this property.

If $\mathcal{J} \subset \mathcal{B}(\mathcal{H})$ is either of the two-sided closed ideals of p -Schatten operators (for $p \in [0, \infty)$) or $\mathcal{K}(\mathcal{H})$, and \mathcal{A} is any C^* subalgebra of $\mathcal{B}(\mathcal{H})$, then the following unitary sets of $\mathcal{U}(\mathcal{H})$ can be defined, by analogy with (3.1):

$$\begin{aligned} \mathcal{U}_{\mathcal{J}} &= \{u \in \mathcal{U}(\mathcal{H}) : u - 1 \in \mathcal{J}\}, \\ \mathcal{U}_{\mathcal{J},\mathcal{A}} &= \{u \in \mathcal{U}(\mathcal{H}) : \exists A \in \mathcal{A}^{ah} \text{ such that } u - e^A \in \mathcal{J}\}, \\ \mathcal{U}_{\mathcal{A}} &= \{u \in \mathcal{U}(\mathcal{H}) : \exists A \in \mathcal{A}^{ah} \text{ such that } u = e^A\}, \\ \mathcal{U}_{\mathcal{J}+\mathcal{A}} &= \{u \in \mathcal{U}(\mathcal{H}) : \exists J \in \mathcal{J}^{ah} \text{ and } A \in \mathcal{A}^{ah} \text{ such that } u = e^{J+A}\}. \end{aligned}$$

The groups $\mathcal{U}_{\mathcal{J}}$, where \mathcal{J} is any p -Schatten ideal of $\mathcal{B}(\mathcal{H})$, were studied in [2].

It can be proved that the previous unitary sets satisfy the following properties:

- (1) $\mathcal{U}_{\mathcal{J}} = \{e^J \in \mathcal{U}(\mathcal{H}) : J \in \mathcal{J}^{ah}, \|J\| \leq \pi\}$.
- (2) $\mathcal{U}_{\mathcal{J},\mathcal{A}}$ is a group, equals

$$\begin{aligned} \mathcal{U}_{\mathcal{J},\mathcal{A}} &= \{u \in \mathcal{U}(\mathcal{H}) : \exists J \in \mathcal{J}^{ah}, \|J\| \leq \pi, \text{ and} \\ &\quad A \in \mathcal{A}^{ah} \text{ such that } u = e^J e^A\}. \end{aligned}$$

$$\text{and } \mathcal{U}_{\mathcal{J},\mathcal{A}} = \mathcal{U}_{\mathcal{J}} \mathcal{U}_{\mathcal{A}} = \mathcal{U}_{\mathcal{A}} \mathcal{U}_{\mathcal{J}}.$$

- (3) $\mathcal{U}_{\mathcal{A}} \subsetneq \mathcal{U}_{\mathcal{J},\mathcal{A}}$.
- (4) $\mathcal{U}_{\mathcal{J}} \subsetneq \mathcal{U}_{\mathcal{J},\mathcal{A}}$.

- (5) $\mathcal{U}_{\mathcal{J}+\mathcal{A}} \subseteq \mathcal{U}_{\mathcal{J},\mathcal{A}}$.
- (6) If $u \in \mathcal{U}_{\mathcal{J},\mathcal{A}}$ then $u = J' + A'$, with $J' \in \mathcal{J}$ and $A' \in \mathcal{U}_{\mathcal{A}}$.
- (7) For every $J \in \mathcal{J}^{ah}$ and $A \in \mathcal{A}^{ah}$ there exists $J' \in \mathcal{K}(\mathcal{H})^{ah}$ such that $e^J e^A = e^A e^{J'}$.
- (8) $\mathcal{U}_{\mathcal{J}} \subsetneq \mathcal{U}_{\mathcal{J}+\mathcal{A}}$
- (9) If $\mathcal{A}^{ah} \cap \mathcal{J} \neq \emptyset$, then

$$\mathcal{U}_{\mathcal{J}} = \{u \in \mathcal{U}(\mathcal{H}) : \exists A \in \mathcal{A}^{ah} \cap \mathcal{J} \text{ such that } u - e^A \in \mathcal{J}\}.$$

- (10) For every $J \in \mathcal{J}^{ah}$ and $A \in \mathcal{A}^{ah}$ sufficiently close to 0, there exist $J'' \in \mathcal{J}^{ah}$ and $A' \in \mathcal{A}^{ah}$ such that

$$e^J e^A = e^{J''+A'}.$$

- (11) For every $J_1, J_2 \in \mathcal{J}^{ah}$ and $A_1, A_2 \in \mathcal{A}^{ah}$ the following statements are equivalent:

- $e^{J_1} e^{A_1} = e^{J_2} e^{A_2}$.
- There exists $a \in \mathcal{A}^{ah} \cap \mathcal{J}$ such that $e^{J_2} = e^{J_1} e^{-a}$ and $e^{A_2} = e^a e^{A_1}$.

Property (1) has been proved for p -Schatten ideals in [2] (Remark 3.1) and for $\mathcal{K}(\mathcal{H})$ see Proposition 3.1. Properties (2)-(9) and (11) may be proved in much the same way as Propositions 3.3, 3.4 and 3.5. Property (10) involves the B - C - H series expansion $\log(e^J e^A)$ for the Lie algebra $\mathcal{B}(\mathcal{H})^{ah}$ (see the Preliminaries).

4. MINIMAL LENGTH CURVES IN THE ORBIT OF A COMPACT SELF-ADJOINT OPERATOR

Consider the unitary Fredholm orbit of a compact operator

$$b = \text{Diag}(\{\lambda_i\}_{i \in \mathbb{N}}) \in \mathcal{D}(\mathcal{K}(\mathcal{H})^h)$$

with $\lambda_i \neq \lambda_j$ for each $i \neq j$, and the orbit

$$\mathcal{O}_b = \{ubu^* : u \in \mathcal{U}_k\}.$$

The isotropy subgroup of $c = e^{K_0} b e^{-K_0} \in \mathcal{O}_b$, with $K_0 \in \mathcal{K}(\mathcal{H})^{ah}$ for the action $L_u \cdot c = u c u^*$, with $u \in \mathcal{U}_k$, is $\mathcal{I}_c = \{e^{K_0} e^d e^{-K_0} : d \in \mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})\} = \{e^{e^{K_0} d e^{-K_0}} : d \in \mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})\}$. $(T\mathcal{O}_b)_c$ can be identified with the quotient space $\mathcal{K}(\mathcal{H})^{ah} / \mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})$ for every $c \in \mathcal{O}_b$. The projection to the quotient $\mathcal{K}(\mathcal{H})^{ah} / \mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})$ defines a Finsler metric as

$$\|x\|_{e^{K_0} e^d e^{-K_0}} = \|[Y]\| = \inf_{D \in \mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})} \|Y + e^{K_0} D e^{-K_0}\|$$

for each class $[Y] = \{Y + e^{K_0} D e^{-K_0} : D \in \mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})\}$ and $x = Yc - cY \in (T\mathcal{O}_b)_c$. This metric is invariant under the action of L_{e^K} for $e^K \in \mathcal{U}_k$ (see [6]).

This invariance implies that the curve $\gamma : [a, b] \rightarrow \mathcal{O}_b$, such that $\gamma(0) = b$ and $\dot{\gamma}(0) = x$ has the same length than $\beta(t) = L_{e^K} \cdot \gamma : [a, b] \rightarrow \mathcal{O}_b$ and satisfies $\beta(0) = L_{e^K} \cdot b$, $\dot{\beta}(0) = L_{e^K} \cdot \dot{\gamma}(0) = L_{e^K} \cdot x$, for $e^K \in \mathcal{U}_k$.

Proposition 4.1. *Let $b = \text{Diag}(\{\lambda_i\}_{i \in \mathbb{N}}) \in \mathcal{D}(\mathcal{K}(\mathcal{H}))$ with $\lambda_i \neq \lambda_j$ for each $i \neq j$ and $Z_0 = S_0 + D_0$, with $S_0 \in \mathcal{K}(\mathcal{H})^{ah}$ and $D_0 \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$. Then $e^{Z_0} b e^{-Z_0} \in \mathcal{O}_b$.*

Proof. Observe that D_0 is not necessarily compact. Using Remark 3.2 the exponential $e^{Z_0} = e^{S_0 + D_0}$ can be rewritten as

$$e^{Z_0} = e^{D_0} + S_0 \Psi(S_0, D_0).$$

Then, $(e^{D_0} + S_0 \Psi(S_0, D_0)) e^{-D_0}$ is unitary and $S_0 \Psi(S_0, D_0) e^{-D_0} = e^{D_0 - D_0} - 1 + S_0 \Psi(S_0, D_0) e^{-D_0} \in \mathcal{K}(\mathcal{H})$ since $S_0 \in \mathcal{K}(\mathcal{H})$. Moreover

$$(e^{D_0} + S_0 \Psi(S_0, D_0)) e^{-D_0} - 1 \in \mathcal{K}(\mathcal{H}),$$

which implies that $e^{S_0 + D_0} e^{-D_0} - 1 \in \mathcal{K}(\mathcal{H})$. Therefore, by Proposition 3 in [6] there exists $K \in \mathcal{K}(\mathcal{H})^{ah}$ such that

$$e^{S_0 + D_0} e^{-D_0} = e^K, \text{ and therefore } e^{S_0 + D_0} = e^K e^{D_0}.$$

Then

$$e^{Z_0} b e^{-Z_0} = e^{S_0 + D_0} b e^{-S_0 - D_0} = e^K e^{D_0} b e^{-D_0} e^{-K} = e^K b e^{-K} \in \mathcal{O}_b.$$

□

Theorem 4.2. *Let $Z_0 = S_0 + D_0$ such that $S_0 \in \mathcal{K}(\mathcal{H})^{ah}$, $D_0 \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$ and $b = \text{Diag}(\{\lambda_i\}_{i \in \mathbb{N}}) \in \mathcal{D}(\mathcal{K}(\mathcal{H}))$ with $\lambda_i \neq \lambda_j$ for each $i \neq j$, and $\gamma(t) = e^{tZ_0} b e^{-tZ_0}$, $\forall t \in \mathbb{R}$. Then,*

- a) $\gamma(t) \in \mathcal{O}_b$, $\forall t \in \mathbb{R}$, and
- b) if Z_0 is minimal for $\mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$ (see (1.2) and the Preliminaries) then $\gamma : \left[-\frac{\pi}{2\|Z_0\|}, \frac{\pi}{2\|Z_0\|}\right] \rightarrow \mathcal{O}_b$ is a minimal length curve on \mathcal{O}_b considering the distance (2.6).

Proof. The assertion of item a) follows directly from Proposition 4.1. Note that the a) holds even though Z_0 may not be compact.

In order to prove b) consider $\mathcal{P}_b = \{ubu^* : u \in \mathcal{U}(\mathcal{H})\}$, then by Theorem II in [8], since Z_0 is minimal, the curve γ has minimal length over all the smooth curves in \mathcal{P}_b that join $\gamma(0) = b$ and $\gamma(t)$, with $|t| \leq \frac{\pi}{2\|Z_0\|}$. Since clearly $\mathcal{O}_b \subseteq \mathcal{P}_b$, then for each $t_0 \in \left[-\frac{\pi}{2\|Z_0\|}, \frac{\pi}{2\|Z_0\|}\right]$ follows that γ is minimal in \mathcal{O}_b , that is

$$L(\gamma) = \text{dist}(b, \gamma(t_0)),$$

where $\text{dist}(b, \gamma(t_0))$ is the rectifiable distance between b and $\gamma(t_0)$ defined in (2.6) of the Preliminaries. \square

Remark 4.3. Recall that for every S_0 there always exists a minimal $Z_0 \in \mathcal{B}(\mathcal{H})^{ah}$ as that mentioned in Theorem 4.2 although it may not be compact (see [8], [5]).

Proposition 4.4. If $b = \text{Diag}(\{\lambda_i\}_{i \in \mathbb{N}}) \in \mathcal{D}(\mathcal{K}(\mathcal{H}))$ with $\lambda_i \neq \lambda_j$ for each $i \neq j$, then $\mathcal{O}_b = \mathcal{O}_b^{\mathcal{U}_{k,d}}$ and $\mathcal{O}_b^{\mathcal{U}_{k+d}} \subseteq \mathcal{O}_b$.

Proof. Since b is diagonal and $e^K e^D b e^{-D} e^{-K} = e^K b e^{-K}$ follows that $\mathcal{O}_b = \mathcal{O}_b^{\mathcal{U}_{k,d}}$. The inclusion $\mathcal{O}_b^{\mathcal{U}_{k+d}} \subseteq \mathcal{O}_b$ is trivial because b is diagonal and $\mathcal{U}_{k,d} \supset \mathcal{U}_{k+d}$ (see (3) in Proposition 3.4). \square

Remark 4.5. Under the same assumptions of Proposition 4.4, if $c \in \mathcal{O}_b$ the following identifications can be made

$$(T\mathcal{O}_b)_c \cong (T\mathcal{U}_k)_1 / (T\mathcal{I}_b)_1 = \mathcal{K}(\mathcal{H})^{ah} / \mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})$$

and

$$\begin{aligned} (T\mathcal{U}_{k,d})_1 / (T\mathcal{I}_b)_1 &\cong \left(\mathcal{K}(\mathcal{H})^{ah} + \mathcal{D}(\mathcal{B}(\mathcal{H}))^{ah} \right) / \mathcal{D}(\mathcal{B}(\mathcal{H}))^{ah} \\ &\cong \mathcal{K}(\mathcal{H})^{ah} / \mathcal{D}(\mathcal{B}(\mathcal{H}))^{ah}. \end{aligned}$$

Moreover the norm on each quotient coincides on every class since for $K \in \mathcal{K}(\mathcal{H})^{ah}$ holds $\|[K]\| = \inf_{d \in \mathcal{D}(\mathcal{K}(\mathcal{H})^{ah})} \|K + d\| = \inf_{D \in \mathcal{D}(\mathcal{B}(\mathcal{H}))^{ah}} \|K + D\|$ (see for example Proposition 5 in [5]). Therefore the Finsler metrics defined by the subgroups \mathcal{U}_k and $\mathcal{U}_{k,d}$ coincide on \mathcal{O}_b .

Let $c = L_{e^{K_0}} \cdot b = e^{K_0} b e^{-K_0} \in \mathcal{O}_b$ (for $K_0 \in \mathcal{K}(\mathcal{H})^{ah}$) and $x \in T(\mathcal{O}_b)_c$. Then there always exists a vector $z_c = L_{e^{K_0}} \cdot Z_0$, with $z_c c - z_c c = x$ minimal for $\{F \in \mathcal{B}(\mathcal{H})^{ah} : Fc - cF = 0\} = \{F \in \mathcal{B}(\mathcal{H})^{ah} : F = e^{K_0} D e^{-K_0}, \text{ for } D \in \mathcal{D}(\mathcal{B}(\mathcal{H}))^{ah}\}$ such that Z_0 is minimal for $\mathcal{D}(\mathcal{B}(\mathcal{H}))^{ah}$ as in Theorem 4.2 b) and Remark 4.3. That is,

$$\begin{aligned} \|x\|_c &= \|[z_c]\|_c = \inf_{F \in \mathcal{B}(\mathcal{H})^{ah} \cap \{c\}'} \|z_c + F\| \\ &= \inf_{D \in \mathcal{D}(\mathcal{B}(\mathcal{H}))^{ah}} \|e^{K_0} Z_0 e^{-K_0} + e^{K_0} D e^{-K_0}\| \\ &= \inf_{D \in \mathcal{D}(\mathcal{B}(\mathcal{H}))^{ah}} \|Z_0 + D\| = \|[Z_0]\| \end{aligned}$$

This equality and the left invariance of the action L_{e^K} imply that the curve

$$\beta(t) = e^{tz_c} c e^{-tz_c}$$

for $t \in [-\frac{\pi}{2\|z_c\|}, \frac{\pi}{2\|z_c\|}]$, satisfies $\beta(0) = c$, $\dot{\beta}(0) = x = z_c c - c z_c$ and $L(\beta|_{[a,b]}) = L(\gamma|_{[a,b]})$ for the curve γ mentioned in Theorem 4.2 and every $[a, b] \subset$

$[-\frac{\pi}{2\|z_c\|}, \frac{\pi}{2\|z_c\|}]$. The previous comments and results allow us to prove the following.

Corollary 4.6. *Let $b = \text{Diag}(\{\lambda_i\}_{i \in \mathbb{N}}) \in \mathcal{D}(\mathcal{K}(\mathcal{H}))$ with $\lambda_i \neq \lambda_j$ for each $i \neq j$, $c = e^{K_0} b e^{-K_0} \in \mathcal{O}_b$, with $K_0 \in \mathcal{K}(\mathcal{H})^{ah}$, and $x \in T(\mathcal{O}_b)_c$. Then there exists $Z_0 \in \mathcal{B}(\mathcal{H})^{ah}$ minimal for $\mathcal{D}(\mathcal{B}(\mathcal{H}))^{ah}$ such that $\beta(t) = e^{tz_c} c e^{-tz_c} \in \mathcal{O}_b$ for all $t \in \mathbb{R}$, $z_c = e^{K_0} Z_0 e^{-K_0}$ and $x = L_{e^{K_0}} \cdot (Z_0 b - b Z_0)$. Moreover, $\beta : [-\frac{\pi}{2\|z_c\|}, \frac{\pi}{2\|z_c\|}] \rightarrow \mathcal{O}_b$ is a minimal length curve in \mathcal{O}_b such that $\beta(0) = c$, $\dot{\beta}(0) = x$ considering the distance (2.6).*

Proof. Given $x \in T(\mathcal{O}_b)_c$ we can choose $Z_0 \in \mathcal{K}(\mathcal{H})^{ah}$, such that $Z_0 b - b Z_0 = L_{e^{-K_0}} \cdot x \in T(\mathcal{O}_b)_b$ and that satisfies $\|Z_0\| = \inf_{D \in \mathcal{D}(\mathcal{B}(\mathcal{H}))^{ah}} \|Z_0 + D\|$ as in Theorem 4.2 and Remark 4.3. Z_0 is minimal for $\mathcal{D}(\mathcal{B}(\mathcal{H}))^{ah}$ and therefore Theorem 4.2 b) applies and $\gamma(t) = e^{tZ_0} b e^{-tZ_0}$ is a short curve for $t \in [-\frac{\pi}{2\|Z_0\|}, \frac{\pi}{2\|Z_0\|}]$. Direct calculations show that $x = L_{e^{K_0}} \cdot (Z_0 b - b Z_0) = (L_{e^{K_0}} \cdot Z_0)(L_{e^{K_0}} \cdot b) - (L_{e^{K_0}} \cdot b)(L_{e^{K_0}} \cdot Z_0)$. If $z_c = L_{e^{K_0}} \cdot Z_0 = e^{K_0} Z_0 e^{-K_0}$ it is apparent that if $\beta(t) = e^{tz_c} c e^{-tz_c}$, for $t \in \mathbb{R}$ and $c = L_{e^{K_0}} \cdot b$, then $\beta(0) = c$ and $\dot{\beta}(0) = z_c c - c z_c = L_{e^{K_0}} \cdot (Z_0 b - b Z_0) = x$

Similar considerations as those in Proposition 4.1 using that

$$\beta(t) = e^{K_0} e^{tZ_0} e^{-K_0} e^{K_0} b e^{K_0} e^{-K_0} e^{-tZ_0} e^{-K_0} = L_{e^{K_0}} (e^{tZ_0} b e^{-tZ_0}) = L_{e^{K_0}} (\gamma(t))$$

imply that $\beta(t) \in \mathcal{O}_b$ for all $t \in \mathbb{R}$.

Standard arguments of homogeneous spaces (invariance of the Finsler metric) imply that β is a curve of minimal length when is defined in the interval $[-\frac{\pi}{2\|z_c\|}, \frac{\pi}{2\|z_c\|}] = [-\frac{\pi}{2\|Z_0\|}, \frac{\pi}{2\|Z_0\|}]$ as γ is. \square

Remark 4.7. *Theorem 4.2, Remark 4.5 and Corollary 4.6 allow us to describe short curves β in \mathcal{O}_b with initial condition $\beta(0) = c$ even for velocity vectors $x \in T(\mathcal{O}_b)_c$ that do not have a minimal compact lifting Z_0 . Thus \mathcal{U}_k is an example of a group whose action on \mathcal{O}_b has short curves that need not to be described necessarily with minimal vectors F that belong to $\{F \in \mathcal{B}(\mathcal{H})^{ah} : Fc - cF = 0\}$. Nevertheless there exists another group $\mathcal{U}_{k,d}$ acting on \mathcal{O}_b such that its b -orbit coincides with that of \mathcal{U}_k , defines the same Finsler metric on it and where every short curve can be described by means of a minimal lifting.*

The previous geometric properties allow the following results relating the quotient norm $\|[K]\| = \inf_{D \in \mathcal{D}(\mathcal{B}(\mathcal{H}))^{ah}} \|K + D\|$ of two anti-Hermitian compact operators.

Proposition 4.8. *Let $K_1, K_2 \in \mathcal{K}(\mathcal{H})^{ah}$ and $D_1, D_2 \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$ such that $e^{tK_1}e^{D_1} = e^{tK_2}e^{D_2}$ for all $t \in [0, 1]$. Then,*

$$\|[K_1]\| = \|[K_2]\|.$$

Proof. Let $b = \text{Diag}(\{\lambda_i\}_{i \in \mathbb{N}}) \in \mathcal{D}(\mathcal{K}(\mathcal{H})^h)$ with $\lambda_i \neq \lambda_j$ for each $i \neq j$. The equality $e^{tK_1}e^{D_1} = e^{tK_2}e^{D_2}$ implies that

$$e^{tK_1}be^{-tK_1} = e^{tK_2}be^{-tK_2},$$

for all $t \in [0, 1]$. If we consider $\alpha, \beta : [0, 1] \rightarrow \mathcal{O}_b$, defined by

$$\beta(t) = e^{tK_1}be^{-tK_1} \text{ and } \alpha(t) = e^{tK_2}be^{-tK_2},$$

then

$$\begin{aligned} L(\beta) = L(\alpha) &\Rightarrow \int_0^1 \|\beta'(t)\|_{\beta(t)} dt = L(\alpha) = \int_0^1 \|\alpha'(t)\|_{\alpha(t)} dt \\ &\Rightarrow \|[K_1]\| = \|[K_2]\|. \end{aligned}$$

This concludes the proof. \square

Proposition 4.9. *Let $K \in \mathcal{K}(\mathcal{H})^{ah}$ and $D \in \mathcal{D}(\mathcal{B}(\mathcal{H})^{ah})$ such that $K + D$ is minimal and $\|K + D\| < \frac{\pi}{2}$. Then, if $K' \in \mathcal{K}(\mathcal{H})^{ah}$ is such that $e^{K+D} = e^{K'}e^D$ the inequality*

$$\|[K]\| \leq \|[K']\|$$

holds.

Proof. Let $b = \text{Diag}(\{\lambda_i\}_{i \in \mathbb{N}}) \in \mathcal{D}(\mathcal{B}(\mathcal{H})^h)$ with $\lambda_i \neq \lambda_j$ for each $i \neq j$, and consider $\alpha, \beta : [0, 1] \rightarrow \mathcal{O}_b$, defined by

$$\beta(t) = e^{t(K+D)}be^{-t(K+D)} \text{ and } \alpha(t) = e^{tK'}be^{-tK'}.$$

Observe that since $e^{K+D} = e^{K'}e^D$ then $\beta(0) = \alpha(0)$ and $\beta(1) = \alpha(1)$.

But β has minimal length between all rectifiable unitary curves that join b with $\beta(1) = e^{K+D}be^{-(K+D)} = e^{K'}be^{-K'} = \alpha(1)$ (see Corollary 4.6 and [8]).

Therefore

$$\begin{aligned} L(\beta) &\leq L(\alpha) \\ &\Rightarrow \int_0^1 \|\beta'(t)\|_{\beta(t)} dt = \|Kb - bK\|_b = \|[K]\| \\ &\leq \int_0^1 \|\alpha'(t)\|_{\alpha(t)} dt = \|K'b - bK'\|_b = \|[K']\|. \end{aligned}$$

\square

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