A GRÜSS TYPE OPERATOR INEQUALITY

T. BOTTAZZI[1] and C. CONDE[1,2]

ABSTRACT. In 2001, P. Renaud obtained a Grüss type operator inequality involving the usual trace functional. In this article, we give a refinement of such result and we answer positively Renaud's open problem.

1. INTRODUCTION

In 1935, Grüss [6] obtained the following inequality: if f, g are integrable real functions on $[a, b]$ and there exist real constant $\alpha, \beta, \gamma, \delta$ such that $\alpha \leq f(x) \leq \beta, \gamma \leq g(x) \leq \delta$ for all $x \in [a, b]$ then

$$\left| \frac{1}{b-a} \int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2} \int_a^b f(x)dx \int_a^b g(x)dx \right| \leq \frac{1}{4}(\beta - \alpha)(\delta - \gamma),$$

and the inequality is sharp, in the sense that the constant $\frac{1}{4}$ cannot be replaced by a smaller one. This inequality has been investigated, applied and generalized by many mathematicians, for example: Banić, Bourin, Matharu, Moslehian, Ilišević, Renaud and Varošanec, among others, in different areas of mathematics. We recommend see [8] and references within.

In this work $\mathcal{H}$ denotes a (complex, separable) Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Let $(\mathbb{B}(\mathcal{H}), \| \cdot \|)$ be the C^*-algebra of all bounded linear operators acting on $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ with the uniform norm. We denote by Id the identity operator, and for any $A \in \mathbb{B}(\mathcal{H})$ we consider A^* its adjoint and $|A| = (A^*A)^{\frac{1}{2}}$ the absolute value of A. For $A \in \mathbb{B}(\mathcal{H})$ we use $R(A), N(A)$, respectively, to denote the range and kernel of A.

By $\mathbb{B}(\mathcal{H})^+$ we denote the cone of positive operators of $\mathbb{B}(\mathcal{H})$, i.e. $\mathbb{B}(\mathcal{H})^+ := \{T \in \mathbb{B}(\mathcal{H}) : \langle Th, h \rangle \geq 0 \; \forall h \in \mathcal{H}\}$. In the case when $\dim \mathcal{H} = n$, we identify $\mathbb{B}(\mathcal{H})$ with the full matrix algebra $\mathcal{M}_n$ of all $n \times n$ matrices with entries in the complex field $\mathbb{C}$. For each $T \in \mathbb{B}(\mathcal{H})$, we denote its spectrum by $\sigma(T)$, that is, $\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda Id \text{is not invertible}\}$ and a complex number $\lambda \in \mathbb{C}$ is said to be in the approximate

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point spectrum of the operator T, and we denote by $\sigma_{ap}(T)$, if there is a sequence $\{x_n\}$ of unit vectors satisfying $(T-\lambda)x_n \to 0$.

For each operator T we consider

$$r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\} \qquad \text{spectral radius of } T,$$

$$W(T) = \{\langle Th, h\rangle : \|h\| = 1\} \qquad \text{numerical range of } T$$

and

$$w(T) = \sup\{|\lambda| : \lambda \in W(T)\} \qquad \text{numerical radius of } T.$$

Recall that for all $T \in \mathbb{B}(\mathcal{H})$, $r(T) \leq w(T) \leq \|T\| \leq 2w(T)$, $\sigma(T) \subseteq \overline{W(T)}$ and by the Toeplitz-Hausdorff's Theorem $W(T)$ is convex.

Renaud [11] gave a bounded linear operator analogue of Grüss inequality by replacing integrable functions by operators and the integration by a trace function as follows: let $A, T \in \mathbb{B}(\mathcal{H})$, suppose that $W(A)$ and $W(T)$ are contained in disks of radii R_A and R_T, respectively. Then for any positive trace class operator P with $tr(P) = 1$ holds

$$|tr(PAT) - tr(PA)tr(PT)| \leq 4R_A R_T, \tag{1.1}$$

and if A and T are normal (i.e. $T^*T = TT^*$), the constant 4 can be replaced by 1. We can easily see that if $A = \alpha Id$ or $T = \beta Id$ with $\alpha, \beta \in \mathbb{C}$ then the left hand side is equal to zero. In the same article, Renaud proposed the following open problem: to characterise $k(A,T)$, where

$$|tr(PAT) - tr(PA)tr(PT)| \leq k(A,T) R_A R_T, \tag{1.2}$$

with $1 \leq k(A,T) \leq 4$. In particular, whether it depends on A and T separately, i.e. whether we can write $k(A,T) = h(A)h(T)$, where $h(A), h(B)$ are suitably defined constants.

In this paper we give a positive answer to the open problem proposed by Renaud and we obtain an explicit formula for $k(A,T) = h(A)h(T)$. Also, we generalize the inequality (1.1) for normal to transloid operators.

2. Preliminaries

Let us begin with the notation and the necessary definitions.

The set of compact operators in $\mathcal{H}$ is denoted by $B_0(\mathcal{H})$. If $T \in B_0(\mathcal{H})$ we denote by $\{s_n(T)\}$ the sequence of singular values of T, i.e., the eigenvalues of $|T|$ (decreasingly ordered).

The notion of unitary invariant norms can be defined also for operators on Hilbert spaces- a norm $|||.|||$ that satisfies the invariance property $|||UXV||| = |||X|||$, for pair of unitary operators U, V. Recall that each unitarily invariant norm is defined on a natural subclass $\mathcal{J}_{|||.|||}$ of $B_0(\mathcal{H})$ called the norm ideal associated with the norm $|||.|||$. There is a one-to-one correspondence between symmetric gauge functions defined on sequences of real numbers and unitarily invariant norms defined on norm ideals of operators. More precisely, if $|||.|||$ is a unitarily invariant norm, then there is a unique symmetric gauge function g, such that $|||T||| = g(\{s_n(T)\})$ for any $T \in \mathcal{J}_{|||.|||}$. If $\dim R(T) = 1$, then $|||T||| = s_1(T)g(e_1) = g(e_1)\|T\|$. By convention, we assume that $g(e_1) = 1$. If $x, y \in \mathcal{H}$, then we denote $x \otimes y$ the rank 1 operator defined on $\mathcal{H}$ by $(x \otimes y)(z) = \langle z, y\rangle x$ then $\|x \otimes y\| = \|x\|\|y\| = |||x \otimes y|||$.

The most known examples of unitary invariant norms are the Schatten p-norms For $1 \leq p < \infty$, let

$$\|T\|_p^p = \sum_n s_n(T)^p = tr\,|T|^p\,,$$

and

$$B_p(\mathcal{H}) = \{T \in \mathcal{H} : \|T\|_p < \infty\},$$

called the $p-Schatten\,class$ of $\mathbb{B}(\mathcal{H})$. This is the subset of compact operators with singular values in l_p. The positive operators with trace 1 are called density operator (or states) and we denote this set by $\mathcal{S}(\mathcal{H})$. The ideal $B_2(\mathcal{H})$ is called the Hilbert-Schmidt class and it is a Hilbert space with the inner product $\langle S, T\rangle_2 = tr(ST^*)$. On the theory of norm ideals and their associated unitarily invariant norms, a reference for this subject is [5].

An operator $A \in \mathbb{B}(\mathcal{H})$ is called normaloid if $r(A) = \|A\| = \omega(A)$. If $A - \mu Id$ is normaloid for all $\mu \in \mathbb{C}$, then A is called transloid.

Finally, for $A, T \in \mathbb{B}(\mathcal{H})$ and $P \in \mathcal{S}(\mathcal{H})$ we introduce the following notation

$$V_P(A, T) = tr(PAT) - tr(PA)tr(PT).$$

In the particular case $T = A^*$ we get the variance of A respect to P. More precisely, Audenaert in [1] consider the following notion, given $A, P \in \mathcal{M}_n, P \geq 0, tr(P) = 1$ the variance of A respect to the matrix P

$$V_P(A) = tr(|A|^2 P) - |tr(AP)|^2 = V_P(A, A^*).$$

Note that $V_P(A - \lambda Id) = V_P(A)$. Furthermore, he showed that if $A \in \mathcal{M}_n$ then

$$\max\{tr(|A|^2 P) - |tr(AP)|^2 : P \in \mathcal{M}_n^+, tr(P) = 1\} = dist(A, \mathbb{C}Id)^2, \tag{2.1}$$

and the maximization over P on the left hand side can be restricted to density matrices of rank 1.

3. Distance formulas and Renaud's inequality

Let A and T linear bounded operators acting on $\mathcal{H}$, the vector-function $A-\lambda T$ is known as the pencil generated by A and T. Evidently there is at least one complex number λ_0 such that

$$\|A-\lambda_0 T\| = \inf_{\lambda\in\mathbb{C}} \|A-\lambda T\|.$$

The number λ_0 is unique if $0 \notin \sigma_{ap}(T)$ (or equivalently if $\inf\{\|Tx\| : \|x\|=1\} > 0$). Different authors, following [12], called to this unique number as center of mass of A respect to T and we denote it by $c(A,T)$ and when $T = Id$ we write $c(A)$. Following Paul, for $A, T \in \mathbb{B}(\mathcal{H})$ such that $0 \notin \sigma_{ap}(T)$ we consider

$$M_T(A) = \sup_{\|x\|=1} \left[\|Ax\|^2 - \frac{|\langle Ax, Tx\rangle|^2}{\langle Tx, Tx\rangle} \right]^{1/2} = \sup_{\|x\|=1} \left\| Ax - \frac{\langle Ax, Tx\rangle}{\langle Tx, Tx\rangle} Tx \right\|, \tag{3.1}$$

in [9], he proved that $M_T(A) = dist(A, \mathbb{C}T)$. The unique minimizer is characterized by the following conditions: there exists a sequence of unit vectors $\{x_n\}$ such that

$$\|(A-\lambda_0 T)x_n\| \to \|A-\lambda_0 T\| \qquad \text{and} \qquad \langle (A-\lambda_0 T)x_n, x_n\rangle \to 0.$$

In [4], Gevorgyan proved that

$$c(A,T) = \lim_{n\to\infty} \frac{\langle Ay_n, Ty_n\rangle}{\langle Ty_n, Ty_n\rangle}, \tag{3.2}$$

where $\{y_n\}$ is a sequence of unit vectors which approximate the supremum in (3.1). In the particular case that $T = Id$ and A is a Hermitian operator then it is easy to see that

$$\min_{\lambda\in\mathbb{C}} \|A-\lambda Id\| = \frac{\lambda_{max}(A) - \lambda_{min}(A)}{2}, \tag{3.3}$$

where $\lambda_{max}(A)$ (resp. $\lambda_{max}(A)$) denotes the maximum (resp. minimum) eigenvalue of A. Observe that the minimum is

$$c(A) = \frac{\lambda_{max}(A) + \lambda_{min}(A)}{2}.$$

We recall other formulas that express the distance from A to the one-dimensional subspace $\mathbb{C}T$. Then

$$dist(A, \mathbb{C}T) = \sup\{|\langle Ax, y\rangle| : \|x\| = \|y\| = 1,\ \langle Tx, y\rangle = 0\}, \tag{3.4}$$

if $A, T \in \mathbb{B}(\mathcal{H})$ and $0 \notin \sigma_{ap}(T)$. In the particular case, where $T = Id$ we get

$$\begin{aligned} dist(A,\mathbb{C}Id) &= \frac{1}{2}\sup\{\|AX-XA\| : X\in\mathbb{B}(\mathcal{H}), \|X\|=1\} \\ &= \sup\{\|(Id-Q)AQ\| : Q \text{ is a rank one projection}\} \\ &= \sup\{\|(Id-Q)AQ\| : Q \text{ is a projection}\}. \end{aligned} \tag{3.5}$$

In the following statement we present a new proof of the relation between the variance of A respect to P and the distance from A to the unidimensional subspace $\mathbb{C}Id$.

Proposition 3.1. *Let $A\in\mathbb{B}(\mathcal{H})$ and $P\in\mathcal{S}(\mathcal{H})$ then*

$$\begin{aligned} tr(|A|^2P)-|tr(AP)|^2 &= \|AP^{1/2}\|_2^2-|\langle AP^{1/2},P^{1/2}\rangle_2|^2 \\ &= \|AP^{1/2}-\langle AP^{1/2},P^{1/2}\rangle_2 P^{1/2}\|_2^2 \\ &= \min_{\lambda\in\mathbb{C}}\|AP^{1/2}-\lambda P^{1/2}\|_2^2 \le \min_{\lambda\in\mathbb{C}}\|A-\lambda Id\|. \end{aligned}$$

Proof. These inequalities are simple consequences from the following general statement for any Hilbert space $\mathcal{H}$: let $x,y\in\mathcal{H}$ with $y\neq 0$ then

$$\inf_{\lambda\in\mathbb{C}}\|x-\lambda y\|^2=\frac{\|x\|^2\|y\|^2-|\langle x,y\rangle|^2}{\|y\|^2}.$$

□

The following statement is an extension of the Audenaert's formula to infinite dimension.

Remark 3.2 (Audenaert's formula for infinite dimensional spaces). We exhibit that the equality (2.1) holds in the infinite dimensional context, that is for $A\in\mathbb{B}(\mathcal{H})$

$$\sup\{[tr(|A|^2P)-|tr(AP)|^2]^{1/2} : P\in\mathcal{S}(\mathcal{H})\} = dist(A,\mathbb{C}Id). \tag{3.6}$$

First, we obtain this equality from a Prasanna's result in [10]. Indeed, note that

$$\begin{aligned} dist(A,\mathbb{C}Id)^2 &= \sup_{\|x\|=1}\|Ax\|^2-|\langle Ax,x\rangle|^2 \\ &\le \sup\{tr(|A|^2P)-|tr(AP)|^2 : P\in\mathcal{S}(\mathcal{H})\} \\ &\le dist(A,\mathbb{C}Id)^2. \end{aligned}$$

On the other hand, another way to prove (3.6) is to reduce the problem to finite dimension and use the classical Audenaert's formula. Now we give the idea of this proof.

For the sake of clarity, we denote

$$m := \min_{\lambda\in\mathbb{C}} \|A-\lambda Id\|$$

and

$$M := \sup\{[tr(|A|^2P) - |tr(AP)|^2]^{1/2} : P \in \mathcal{S}(\mathcal{H})\}.$$

By Proposition 3.1 we have that $M \leq m$. Suppose by contradiction that $M < m$ then there exists $\epsilon > 0$ such that

$$M < \|A-\lambda Id\| - \epsilon, \tag{3.7}$$

for any $\lambda \in \mathbb{C}$. By the equality (3.2), we have that $c(A) \in \overline{W(A)}$ and then $|c(A)| \leq w(A)$. As any closed ball in the complex plane is a compact set, we can find $\lambda_1, ..., \lambda_m \in \mathcal{H}$ such that

$$B(0,\omega(A)) \subseteq \cup_{j=1}^{m}\{\lambda\in\mathbb{C} : |\lambda-\lambda_j| < \frac{\epsilon}{2}\}.$$

Now, we choose unit vectors $h_1, ..., h_m \in \mathcal{H}$ with the following property: $\|(A - \lambda_j Id)h_j\| > \|A-\lambda_j Id\| - \frac{\epsilon}{2}$. Let $\mathcal{H}' = \operatorname{span}\{h_1, ..., h_m, Ah_1, ..., Ah_m\}$ and $n = \dim \mathcal{H}'$. Applying (2.1) to the compressions of A and Id respectively, we get

$$dist(A', \mathbb{C}Id_n) = \max\{[tr(|A'|^2P') - |tr(A'P')|^2]^{1/2} : P' \in \mathcal{M}_n^+, tr(P') = 1\} = M'. \tag{3.8}$$

One easily verifies that if $\lambda \in B(0,\omega(A))$ there exists $j \in \{1, ..., m\}$ such that

$$\begin{aligned} \|A'-\lambda Id_n\| &> \|A'-\lambda_j Id_n\| - \frac{\epsilon}{2} \geq \|(A'-\lambda_j Id_n)h_j\| - \frac{\epsilon}{2} \\ &= \|(A-\lambda_j Id)h_j\| - \frac{\epsilon}{2} > \|A-\lambda_j Id\| - \epsilon. \end{aligned} \tag{3.9}$$

Thus, combining (3.7) and (3.9) we get

$$\min_{\lambda\in\mathbb{C}} \|A'-\lambda Id_n\| > M \geq M', \tag{3.10}$$

and we have here a contradiction with (3.8), therefore $m = M$.

The following two results give upper bounds for $V_P(A,T)$.

Lemma 3.3. *Let $A, T \in \mathbb{B}(\mathcal{H})$ and $P \in \mathcal{S}(\mathcal{H})$. Then, for any $\lambda, \mu \in \mathbb{C}$ holds*

$$|V_P(A,T) \leq \|A-\lambda Id\|\,\|T-\mu Id\| - |tr\,(P(A-\lambda Id))\,tr\,(P(T-\mu Id))|\,.$$

Proof. Define the following semi-inner product for $X, Y \in \mathbb{B}(\mathcal{H})$ and $P \in \mathcal{S}(\mathcal{H})$:

$$(X,Y)_{2,P} = \left\langle P^{1/2}X, P^{1/2}Y\right\rangle_2.$$

Following the proof given by Dragomir in [[3],Theorem 2], holds for any $E \in \mathbb{B}(\mathcal{H})$ such that $(E,E)_{2,P}=1$

$$\begin{aligned} |(X,Y)_{2,P}-(X,E)_{2,P}(E,Y)_{2,P}| &\le (X,X)_{2,P}^{1/2}(Y,Y)_{2,P}^{1/2}-|(X,E)_{2,P}(E,Y)_{2,P}| \\ &= (X,X)_{2,P}^{1/2}(Y,Y)_{2,P}^{1/2}-G_E(X,Y). \end{aligned}$$

Since $(Id,Id)_{2,P}=1$, then

$$|V_P(A,T)|=|V_P(A-\lambda Id,T-\mu Id)|$$

$$\begin{aligned} &= |(A-\lambda Id,(T-\mu Id)^*)_{2,P}-(A-\lambda Id,Id)_{2,P}(Id,(T-\mu Id)^*)_{2,P}| \\ &\le (A-\lambda Id,A-\lambda Id)_{2,P}^{1/2}(T^*-\bar{\mu}Id,T^*-\bar{\mu}Id)_{2,P}^{1/2}-G_{Id}(A-\lambda Id,T^*-\bar{\mu}Id) \\ &= tr\left(P|(A-\lambda Id)^*|^2\right)^{1/2}tr\left(P|T-\mu Id|^2\right)^{1/2}-G_{Id}(A-\lambda Id,T^*-\bar{\mu}Id) \\ &\le \left\||(A-\lambda Id)^*|^2\right\|^{1/2}\left\||T-\mu Id|^2\right\|^{1/2}-|tr\left(P(A-\lambda Id)\right)tr\left(P(T-\mu Id)\right)| \\ &= \|A-\lambda Id\|\,\|T-\mu Id\|-|tr\left(P(A-\lambda Id)\right)tr\left(P(T-\mu Id)\right)|. \end{aligned}$$

□

Proposition 3.4. *Let $A,T\in\mathbb{B}(\mathcal{H})$ and $P\in\mathcal{S}(\mathcal{H})$. Then,*

$$|V_P(A,T)|\le\sup_{\widetilde{P}\in\mathcal{S}(\mathcal{H})}|tr(\widetilde{P}AT)-tr(\widetilde{P}A)tr(\widetilde{P}T)|\le dist(A,\mathbb{C}Id)dist(T,\mathbb{C}Id). \quad (3.11)$$

Proof. By Lemma 3.3,

$$|V_P(A,T)|\le\|A-\lambda Id\|\,\|T-\mu Id\|-|tr\left(P(A-\lambda Id)\right)tr\left(P(T-\mu Id)\right)|,$$

for $A,T\in\mathbb{B}(\mathcal{H})$, $P\in\mathcal{S}(\mathcal{H})$ and any $\lambda,\mu\in\mathbb{C}$. Therefore,

$$|V_P(A,T)|\le\sup_{\widetilde{P}\in\mathcal{S}(\mathcal{H})}|tr(\widetilde{P}AT)-tr(\widetilde{P}A)tr(\widetilde{P}T)|\le dist(A,\mathbb{C}Id)dist(T,\mathbb{C}Id).$$

□

Remark 3.5. If we define $V_P:\mathbb{B}(\mathcal{H})\times\mathbb{B}(\mathcal{H})\to\mathbb{C}$, $V_P(A,T):=tr(PAT)-tr(PA)tr(PT)$. Then V_P is a bilinear function and by (3.11) a continuous mapping with $\|V_P\|\le 1$.

Now, we give a new proof and a refinement of (1.1).

Proposition 3.6. *Let $A, T \in \mathbb{B}(\mathcal{H})$ and we suppose that $W(A), W(T)$ are contained in closed disk $D(\lambda_0, R_A), D(\mu_0, R_T)$ respectively. Then for any $P \in \mathcal{S}(\mathcal{H})$*

$$\begin{aligned} |tr(PAT) - tr(PA)tr(PT)| &\leq \sup_{\widetilde{P} \in \mathcal{S}(\mathcal{H})} |tr(\widetilde{P}AT) - tr(\widetilde{P}A)tr(\widetilde{P}T)| \\ &\leq dist(A, \mathbb{C}Id)dist(T, \mathbb{C}Id) \\ &\leq \|A - \lambda_0 Id\| \|T - \mu_0 Id\| \\ &\leq 4w(A - \lambda_0 Id)w(T - \mu_0 Id) \\ &\leq 4R_A R_T. \end{aligned} \tag{3.12}$$

In particular, if A and T are normal operators, we have

$$\begin{aligned} |tr(PAT) - tr(PA)tr(PT)| &\leq \sup_{\widetilde{P} \in \mathcal{S}(\mathcal{H})} |tr(\widetilde{P}AT) - tr(\widetilde{P}A)tr(\widetilde{P}T)| \\ &\leq dist(A, \mathbb{C}Id)dist(T, \mathbb{C}Id) = r_A r_T, \end{aligned} \tag{3.13}$$

where r_S denotes the radius of the unique smallest disc containing $\sigma(S)$ for any $S \in \mathbb{B}(\mathcal{H})$.

Proof. The inequalities are consequence of (3.11). In the last inequality we use the fact that $W(A - \lambda_0 Id) \subset D(0, R_A)$ and $W(T - \mu_0 Id) \subset D(0, R_T)$ respectively.

On the other hand, Björck and Thomée [2] have shown that for a normal operator A

$$dist(A, \mathbb{C}Id) = \sup_{\|x\|=1} (\|Ax\|^2 - |\langle Ax, x\rangle|^2)^{1/2} = r_A, \tag{3.14}$$

and this completes the proof. $\square$

Remark 3.7. From (3.13), if we let A be a positive invertible operator, $T = A^{-1}$ and $P = x \otimes x$ with $x \in \mathcal{H}$ with $\|x\| = 1$, then

$$\begin{aligned} |tr(PAT) - tr(PA)tr(PT)| &= |1 - \langle Ax, x\rangle \langle A^{-1}x, x\rangle| \\ &\leq dist(A, \mathbb{C}Id)dist(A^{-1}, \mathbb{C}Id) = r_A r_{A^{-1}}, \end{aligned}$$

i.e. we obtain the Kantorovich inequality for an operator A acting on an infinite dimensional Hilbert space $\mathcal{H}$ with $0 < m \leq A \leq M$.

In 1972, Istratescu ([7]) generalized the equality (3.14) to the transloid class operators, then we have the following statement:

Proposition 3.8. *Let $A, T \in \mathbb{B}(\mathcal{H})$ with A and T transloid operators then*

$$\begin{aligned} |tr(PAT) - tr(PA)tr(PT)| &\leq \sup_{\widetilde{P} \in \mathcal{S}(\mathcal{H})} |tr(\widetilde{P}AT) - tr(\widetilde{P}A)tr(\widetilde{P}T)| \\ &\leq dist(A, \mathbb{C}Id)dist(T, \mathbb{C}Id) = r_A r_T. \end{aligned} \tag{3.15}$$

Proof. It follows from the same arguments in the proof of inequality (3.13). □

The previous proposition generalizes Renaud's result for normal operators, since the classes of transloid and normal operators are related by the inclusion as follows

$$\text{normal} \subseteq \text{quasinormal} \subseteq \text{subnormal} \subseteq \text{hyponormal} \subseteq \text{transloid},$$

where at least the first inclusion is proper.

In the following statement we obtain a parametric refinement of (1.1).

Theorem 3.9. *Let $A, T \in \mathbb{B}(\mathcal{H})$ with $A, T \notin \mathbb{C}Id$ and suppose that $W(A), W(T)$ are contained in the closed disk $D(\lambda_0, R_A)$ and $D(\mu_0, R_T)$ respectively. Thus for any $P \in \mathcal{S}(\mathcal{H})$ we get*

$$\begin{aligned} |tr(PAT) - tr(PA)tr(PT)| &\leq \sup_{\widetilde{P} \in \mathcal{S}(\mathcal{H})} |tr(\widetilde{P}AT) - tr(\widetilde{P}A)tr(\widetilde{P}T)| \\ &\leq dist(A, \mathbb{C}Id)dist(T, \mathbb{C}Id) \\ &\leq h_\lambda(A)h_\mu(T)\omega(A - \lambda_0 Id)\omega(T - \mu_0 Id) \\ &\leq h_\lambda(A)h_\mu(T)R_A R_T, \end{aligned} \tag{3.16}$$

where

$$h_\lambda(A) = 2(1-\lambda) + \lambda \frac{\|A - c(A)Id\|}{w(A - \lambda_0 Id)}, \quad h_\mu(T) = 2(1-\mu) + \mu \tfrac{\|T - c(T)Id\|}{w(T - \mu_0 Id)}$$

and $1 \leq h_\lambda(A)h_\mu(T) \leq 4$, for any $\lambda, \mu \in [0, 1]$.

Proof. Let $\lambda \in [0, 1]$. Then,

$$\begin{aligned} \|A - c(A)Id\| &\leq \lambda\|A - c(A)Id\| + (1-\lambda)\|A - \lambda_0 Id\| \\ &\leq \lambda\|A - c(A)Id\| + 2(1-\lambda)w(A - \lambda_0 Id) \\ &= w(A - \lambda_0 Id)\left(2(1-\lambda) + \lambda\frac{\|A - c(A)Id\|}{w(A - \lambda_0 Id)}\right) \\ &= w(A - \lambda_0 Id)h_\lambda(A), \end{aligned}$$

where $1 \leq h_\lambda(A) \leq 2$ since $\|A - c(A)Id\| \leq \|A - \lambda_0 Id\| \leq 2w(A - \lambda_0 Id)$. This inequality completes the proof. □

Note that the previous result gives a positive answer to Renaud's open question (1.2).

Corollary 3.10. *Under the same notation as in Theorem 3.9, if $A-\lambda_0 Id$ and $T-\mu_0 Id$ are normaloid operators then, for any $\lambda, \mu \in [0,1]$*

$$\begin{aligned}
|tr(PAT) - tr(PA)tr(PT)| &\leq \sup_{\widetilde{P}\in\mathcal{S}(\mathcal{H})} |tr(\widetilde{P}AT) - tr(\widetilde{P}A)tr(\widetilde{\mathrm{P}}T)| \\
&\leq dist(A, \mathbb{C}Id)dist(T, \mathbb{C}Id) \\
&\leq (2-\lambda)(2-\mu)\omega(A-\lambda_0 Id)\omega(T-\mu_0 Id) \\
&\leq (2-\lambda)(2-\mu)R_A R_T.
\end{aligned}$$

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[1] Instituto Argentino de Matemática "Alberto P. Calderón", Saavedra 15, 3 piso, (C1083ACA) Buenos Aires, Argentina.

[2] Instituto de Ciencias, Universidad Nacional de General Sarmiento, J. M. Gutierrez 1150, (B1613GSX) Los Polvorines, Argentina.

E-mail address: tpbottaz@ungs.edu.ar

E-mail address: cconde@ungs.edu.ar