Abstract uncertainty principle and geometry of the infinite Grassmann manifold

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Abstract

We study the set $\mathcal{C}$ consisting of pairs of orthogonal projections P, Q acting in a Hilbert space $\mathcal{H}$ such that PQ is a compact operator. These pairs have a rich geometric structure which we describe here. They are parted in three subclasses: $\mathcal{C}_0$ which consists of finite rank projections, $\mathcal{C}_1$ of pairs such that Q lies in the restricted Grassmannian (also called Sato Grassmannian) of the polarization $\mathcal{H} = N(P) \oplus R(P)$, and $\mathcal{C}_\infty$. Belonging to this last subclass one has the pairs

$$P_I f = \chi_I f, \quad Q_J f = \left(\chi_J \hat{f}\right)\check{}, \quad f \in L^2(\mathbb{R}^n),$$

where $I, J \subset \mathbb{R}^n$ are sets of finite Lebesgue measure, χ_I, χ_J denote the corresponding characteristic functions and $\hat{}, \check{}$ denote the Fourier-Plancherel trasnsform $L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ and its inverse. These pairs have been widely studied by several authors in connection with the mathematical formulation of the Uncertainty Principle. We apply known results on the Finsler geometry of the Grassmann manifold of $\mathcal{H}$ to establish that if H is the logarithm of the Fourier transform in $L^2(\mathbb{R}^n)$, then

$$\|[H, P_I]\| \geq \pi/2.$$

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1 Introduction

The study of pairs of subspaces of a Hilbert space $\mathcal{H}$ or, more generally, pairs of orthogonal projections in a C*-algebra started in the early times of spectral theory with Dixmier [13]. Some efforts towards finding more transparent proofs of Dixmier's theorems are due to Davis [11], Pedersen [27], Halmos [19], Raeburn and Sinclair [30], Avron, Seiler and Simon [4], Amrein and Sinha [1], among many others. The excellent survey of Böttcher and Spitkovsky [6] contains a complete description and bibliography. A part of this theory is concerned not only with two projections P, Q in $\mathcal{B}(\mathcal{H})$ (the algebra of bounded linear operators in $\mathcal{H}$) but also with the products PQ and PQP. This paper is an addition to this part of the theory, where PQ or PQ are supposed to be compact. The interest in this type of products is not new. Consider the following example:

Example 1.1. Let $I, J \subset \mathbb{R}^n$ be Lebesgue-measurable sets of finite measure. Let P_I, Q_J be the projections in $L^2(\mathbb{R}^n, dx)$ given by

$$P_I f = \chi_I f \quad \text{and} \quad Q_J f = \left(\chi_J \hat{f}\right)\check{}\,,$$

where χ_L denotes the characteristic function of the set L. Equivalently, denoting by $U_{\mathcal{F}}$ the Fourier transform regarded as a unitary operator acting in $L^2(\mathbb{R}^n, dx)$, then

$$P_I = M_{\chi_I} \text{ and } Q_J = U_{\mathcal{F}}^* P_J U_{\mathcal{F}}.$$

In fact, PQ is Hilbert-Schmidt [14].

An intuitive formulation of Heisenberg's uncertainty principle says that a nonzero function and its Fourier transform cannot be (simultaneously) sharply localized (see [16], page 207). More precisely, it says that if f is essentially zero outside a finite measure set I and its Fourier transform $\hat{f}$ is essentially zero outside J (also of finite measure), then

$$|I||J| \geq 1.$$

See [14], page 906 for a modern presentation.

Probably, the idea of using projections P_I and Q_J to obtain a form of the uncertainty principle is due to Fuchs [17], and it was developed later in a series of papers by Landau, Pollack and Slepian [23], [24], [32]. See the survey by Folland and Sitaram [16].

By a clever modification of the meaning of support of a function, Donoho and Stark [14] significantly extended this mathematical version of the uncertainty principle. Namely, they proved that if $f \in L^2(\mathbb{R}^n)$ with $\|f\|_2 = 1$ and $I, J \subset \mathbb{R}^n$ have finite Lebesgue measure and satisfy that

$$\int_{\mathbb{R}^n - I} |f(t)|^2 dt < \epsilon_I \text{ and } \int_{\mathbb{R}^n - J} |\hat{f}(w)|^2 dw < \epsilon_J$$

then

$$|I||J| \geq (1 - (\epsilon_I + \epsilon_J))^2.$$

Donoho and Stark showed several applications of these ideas to signal processing (and the obstruction to the existence of an instantaneous frequency). Later Smith [33] generalized these results to a locally compact abelian group G where $I \subset G$ and $J \subset \hat{G}$, the dual group of G. The books by Havin and Jöricke [20], Hogan and Lakey [21], and Gröchenig [18] among many others, contain further applications, generalizations and history of the different uncertainty principles.

The proof of the theorem by Donoho and Stark can be divided into the following steps. Define the projections P_I and Q_J as above. If I and J have finite measure then $P_I Q_J$ is a Hilbert-Schmidt operator (thus compact). Next they prove that

$$\|P_I Q_J\| \geq 1 - \epsilon_I - \epsilon_J.$$

The fact that $\|P_I Q_J\| \leq \|P_I Q_J\|_{HS}$ (the Hilbert Schmidt norm) is well known. Finally, the proof that

$$\|P_I Q_J\|_{HS} = \sqrt{|I||J|}$$

is an elementary computation using Fubini's theorem. They observe that any bound c such that

$$\|P_I Q_J\| \leq c < 1$$

is an expression of the uncertainty principle ([14], page 912).

Our main goal in this paper is the study of the geometry of the sets

$$\mathcal{C} = \{(P, Q) : P, Q \text{ are orthogonal projections and } PQ \text{ is compact}\}$$

and, for each projection P,

$$\mathcal{C}(P) = \{Q : PQ \text{ is compact}\}.$$

An application of these geometrical results is a form of the uncertainty principle (see Theorem 8.5 below) Let us describe the contents of the paper.

In Section 2 we state elementary properties of pairs P, Q in $\mathcal{C}$: the spectral description of the entries of Q, written as a 2×2 matrix in terms of P, and the partition of the class $\mathcal{C}$ in three subclasses $\mathcal{C}_0, \mathcal{C}_1$ and $\mathcal{C}_\infty$. In Section 3 we recall the Halmos' decomposition of $\mathcal{H}$ given by a pair of subspaces, and specialize it to the case where the corresponding pair of projections lies in $\mathcal{C}$. In Section 4 we introduce the action of the restricted unitary group induced by P on projections $Q \in \mathcal{C}(P)$. In Section 5 we study the class $\mathcal{C}_1$, on which an index is defined, and prove that the connected components of $\mathcal{C}_1$ are parametrized by this index. In Section 6 we study the class $\mathcal{C}_\infty$, and prove that it is connected. In Section 7 we prove that the sets $\mathcal{C}$ and $\mathcal{C}(P)$ are (non complemented) C^∞-differentiable submanifolds of $\mathcal{B}(\mathcal{H})$. We apply known results [29], [10], [2] on the Finsler geometry of the Grassmann manifold of $\mathcal{H}$ to the special case of pairs P_I, Q_J. For instance we prove that if H is the logarithm of the Fourier transform in $L^2(\mathbb{R}^n)$, and $I \subset \mathbb{R}^n$ is a set of finite Lebesgue measure, then

$$\|[H, P_I]\| = \|[H, Q_I]\| \geq \pi/2.$$

Also it is shown that for any pair of sets $I, J \subset \mathbb{R}^n$ of finite measure, one has

$$N(P_I) + N(Q_J) = L^2(\mathbb{R}^n),$$

where the sum is non-direct (the subspaces have infinite dimensional intersection).

2 Elementary properties

Let $\mathcal{H}$ be a Hilbert space, $\mathcal{B}(\mathcal{H})$ the algebra of bounded linear operators in $\mathcal{H}$, $\mathcal{K}(\mathcal{H})$ the ideal of compact operators and $\mathcal{P}(\mathcal{H})$ the set of selfadjoint (orthogonal) projections. Given $P \in \mathcal{P}(\mathcal{H})$, operators acting in $\mathcal{H}$ can be written as 2×2 matrices. For instance, any projection Q such that PQ is compact is of the form

$$Q = \begin{pmatrix} a & x \\ x^* & b \end{pmatrix}$$

where the fact that Q is a projection is equivalent to the relations

$$\begin{cases} xx^* = a - a^2 \\ x^*x = b - b^2 \\ ax + xb = x \end{cases}, \qquad (1)$$

with $0 \leq a \leq 1_{R(P)}$, $0 \leq b \leq 1_{N(P)}$ and $\|x\| \leq 1/2$. The fact that PQ is compact means that $a \in \mathcal{B}(R(P))$ and $x \in \mathcal{B}(N(P), R(P))$ are compact. Here and throughout $R(T)$ and $N(T)$ denote the range and the nullspace of T.

Let us show another example of pairs of projections with compact product:

Example 2.1. Let $\mathcal{H} = \mathcal{L} \times \mathcal{S}$ and $K : \mathcal{S} \to \mathcal{L}$ a compact operator. Consider the idempotent $E = E_K$ given by the matrix

$$E = \begin{pmatrix} 1_{\mathcal{L}} & K \\ 0 & 0 \end{pmatrix}.$$

Then $P = P_{R(E)} = P_{\mathcal{L}}$ and $Q = P_{N(E)}$ verify that PQ is compact. Indeed, straightforward computations show that $R(E) = \mathcal{L}$ and that

$$P_{N(E)} = (1 - E_K)(1 - E_K - E_K^*)^{-1} = \begin{pmatrix} KK^*(1 + KK^*) - 1 & -K(1 + K^*K)^{-1} \\ -K^*(1 + KK^*)^{-1} & (1 + K^*K)^{-1} \end{pmatrix}.$$

Then

$$PQ = \begin{pmatrix} KK^*(1 + KK^*)^{-1} & -K(1 + K^*K)^{-1} \\ 0 & 0 \end{pmatrix},$$

which is clearly compact. The singular values of PQ are the square roots of the eigenvalues of

$$PQP = \begin{pmatrix} KK^*(1 + KK^*)^{-1} & 0 \\ 0 & 0 \end{pmatrix},$$

i.e. those of $KK^*(1 + KK^*)^{-1}$, which have the same asymptotic behaviour near zero as the singular values of K.

Let us collect several elementary properties of pairs in $\mathcal{C}$. First note that b (in the matrix expression of Q in terms of P) may not be compact. It is positive and $b - b^2$ is compact. This implies that it can be diagonalized, and that its spectrum consists of eigenvalues which can only accumulate (eventually) at 0 or 1, plus 0 and 1 which may not be eigenvalues. All eigenvalues different from 0 or 1 have finite multiplicity.

Moreover, there is a relationship between eigenvalues of a and b, which we state in the following Lemma:

Lemma 2.2. *If $\lambda \neq 0, 1$ is an eigenvalue of b, then $1 - \lambda$ is an eigenvalue of a, and the operator $x|_{N(b - \lambda 1_{N(P)})}$ maps $N(b - \lambda 1_{N(P)})$ isomorphically onto $N(a - (1 - \lambda) 1_{R(P)})$. Thus, in particular, these eigenvalues have the same multiplicity. Moreover,*

$$x P_{N(b - \lambda 1_{N(P)})} = P_{N(a - (1-\lambda) 1_{R(P)})} x.$$

Proof. Let $\xi \in \mathcal{H}$, $\xi \neq 0$, such that $b\xi = \lambda\xi$ (with $\lambda \neq 0, 1$). Then by the third relation in (1) one has

$$x\xi = ax\xi + xb\xi = ax\xi + \lambda x\xi, \text{ i.e. }, ax\xi = (1 - \lambda)x\xi.$$

Also note that

$$N(x) = N(x^*x) = N(b - b^2) = N(b) \oplus N(b - 1_{N(P)}),$$

and thus $x\xi \neq 0$ is an eigenvector for a, with eigenvalue $1 - \lambda$, and the map $x|_{N(b - \lambda 1_{N(P)})}$ is injective from $N(b - \lambda 1_{N(P)})$ to $N(ax - (1 - \lambda) 1_{R(P)})$. Therefore

$$\dim(N(b - \lambda 1_{N(P)})) \leq \dim(N(a - (1 - \lambda) 1_{R(P)}).$$

By a symmetric argument, using x^* (and the relation $bx^* + x^*a = x$), one obtains equality of these dimensions.

Pick now an arbitrary $\xi \in N(P)$, $\xi = \xi_1+\xi_2$, with $\xi_1 \in N(b-\lambda 1_{N(P)})$ and $\xi_2 \perp N(b-\lambda 1_{N(P)})$. Then

$$xP_{N(b-\lambda 1_{N(P)})}\xi = x\xi_1.$$

On the other hand

$$P_{N(a-(1-\lambda)1_{R(P)})}x\xi_1 = x\xi_1,$$

by the fact proven above. Let us see that $P_{N(a-(1-\lambda)1_{R(P)})}x\xi_2 = 0$, which will prove our claim. Since $\xi_2 \perp N(b-\lambda 1_{N(P)})$, $\xi_2 = \sum_{l\geq 2}\eta_l + \eta_0 + \eta_1$, where η_l, $l \geq 2$, are eigenvectors of b corresponding to eigenvalues λ_l different from 0, 1 and λ, $\eta_0 \in N(b)$, $\eta_1 \in N(b-1_{N(P)})$ (where these two latter may be trivial). Note then that $\eta_0, \eta_1 \in N(x)$, and thus

$$x\xi_2 = \sum_{l\geq 2} x\eta_l,$$

where the (non nil) vectors $x\eta_l$ are eigenvectors of a corresponding to eigenvalues $1-\lambda_l$, different from $0, 1$ and $1-\lambda$. Thus $P_{N(a-(1-\lambda)1_{R(P)})}x\xi_2 = 0$. □

Remark 2.3. This result implies that we may write a and b as

$$\begin{array}{l} a = \sum_{n\geq 1}\lambda_n P_n + E_1 \\ b = \sum_{n\geq 1}(1-\lambda_n)P'_n + E'_1 \end{array} \qquad (2)$$

where $1 > \lambda_n > 0$ is a decreasing set, which may be finite or a sequence converging to 0,

$$r(P_n) = r(P'_n) < \infty \ , E_1 = P_{N(a-1_{R(P)})} \ \text{ with } \ r(E_1) < \infty, \ \text{ and } E'_1 = P_{N(b-1_{N(P)})}.$$

Accordingly, the decomposition of the (non selfadjoint) operator x in singular values is

$$x = \sum_{n\geq 1}\alpha_n\Big(\sum_{j=1}^{k_n}\xi_{n,j}\otimes\xi'_{n,j}\Big),$$

where $\alpha_n = \sqrt{\lambda_n - \lambda_n^2}$, and $\{\xi_{n,j} : 1 \leq j \leq k_n\}$ and $\{\xi'_{n,j} : 1 \leq j \leq k_n\}$ are orthonormal systems which span $R(P_n)$ and $R(P'_n)$, respectively.

It is also useful to describe the pairs $(P,Q) \in \mathcal{C}$ by means of the homomorphism onto the Calkin algebra

$$\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H}).$$

Put $p = \pi(P)$, $q = \pi(Q)$ and write the matrix of q in terms of p. Apparently

$$q = \begin{pmatrix} 0 & 0 \\ 0 & q' \end{pmatrix},$$

where q' is a projection (i.e. a selfadjoint idempotent) in $\mathcal{B}(N(P))/\mathcal{K}(N(P))$, the Calkin algebra of $N(P)$. In the Calkin algebra there are three (unitary) equivalence classes of projections: 0, 1 and $e \neq 0, 1$ ($e = \pi(E)$ for any E with $R(E)$ and $N(E)$ infinite dimensional).

Remark 2.4. Fix $P \in \mathcal{P}(\mathcal{H})$. Denote

$$\mathcal{C}(P) = \{Q \in \mathcal{P}(\mathcal{H}) : PQ \text{ is compact}\}.$$

According to the above classification, relative to P there are three classes of projections Q such that PQ is compact:

1. If $q' = 0$:

$$\mathcal{C}_0 = \mathcal{C}_0(P) = \{Q \in \mathcal{P}(\mathcal{H}) : \dim(R(Q)) < \infty\}.$$

   In the spectral picture given above, this means that the sequence $\{\lambda_n\}$ is finite and that $\dim(R(E'_1)) < \infty$. Apparently this class does not depend on the particular P.

2. If $q' = 1$:

$$\mathcal{C}_1(P) = \{Q \in \mathcal{P}(\mathcal{H}) : \pi((1-P)(1-Q)(1-P)) = \pi(1-P)\}.$$

   This means that $\dim(N(b)) < \infty$. We shall describe this class below. It is the restricted Grassmannian induced by the decomposition $\mathcal{H} = N(P) \oplus R(P)$ (in the usual description of the restricted Grassmannian: $N(P)$ plays the main role).

3. If q' is a proper projection in $\mathcal{B}(N(P))/\mathcal{K}(N(P))$:

$$\mathcal{C}_\infty(P) = \{Q \in \mathcal{P}(\mathcal{H}) : \pi((1-P)(1-Q)(1-P)) \neq \pi(1-P), 0\}.$$

   We shall call this the class of *essential* projections relative to P. We shall see that the pairs in Example 1.1 belong to this class.

3 The Halmos decomposition

Given orthogonal projections P and Q , we shall call the *Halmos decomposition* [19] (though it was certainly used before) the following orthogonal decomposition of $\mathcal{H}$:

$$\mathcal{H}_{11} = R(P) \cap R(Q) \ , \quad \mathcal{H}_{00} = N(P) \cap N(Q) \ , \ \mathcal{H}_{10} = R(P) \cap N(Q) \ , \quad \mathcal{H}_{01} = N(P) \cap R(Q)$$

and $\mathcal{H}_0$ the orthogonal complement of the sum of the above. This last subspace is usually called the *generic part* of the pair P, Q. Note also that

$$N(P-Q) = \mathcal{H}_{11} \oplus \mathcal{H}_{00} \ , \quad N(P-Q-1) = \mathcal{H}_{10} \ \text{ and } N(P-Q+1) = \mathcal{H}_{01},$$

so that the generic part depends in fact of the difference $P - Q$.

Halmos proved that there is an isometric isomorphism between $\mathcal{H}_0$ and a product Hilbert space $\mathcal{L} \times \mathcal{L}$ such that in the above decomposition (putting $\mathcal{L} \times \mathcal{L}$ in place of $\mathcal{H}_0$), the projections are

$$P = 1 \oplus 0 \oplus 1 \oplus 0 \oplus \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$Q = 1 \oplus 0 \oplus 0 \oplus 1 \oplus \begin{pmatrix} C^2 & CS \\ CS & S^2 \end{pmatrix},$$

where $C = cos(X)$ and $S = sin(X)$ for some operator $0 < X \leq \pi/2$ in $\mathcal{L}$ with trivial nullspace.

Let us describe the pairs in $\mathcal{C}$ in terms of this decomposition. It should be noted that the operator X and the space $\mathcal{L}$ are uniquely determined up to unitary equivalence.

Proposition 3.1. *The pair (P, Q) belongs to $\mathcal{C}$ if and only if $\mathcal{H}_{11}$ is finite dimensional and $C = cos(X)$ is compact.*

Proof. By direct computation,

$$PQ = 1 \oplus 0 \oplus 0 \oplus 0 \oplus \begin{pmatrix} C^2 & CS \\ 0 & 0 \end{pmatrix}.$$

If C is compact, then C^2 and CS are compact. If additionally $\dim \mathcal{H}_{11} < \infty$, then it is clear that PQ is compact.

Conversely, if PQ is compact, then clearly $\dim \mathcal{H}_{11} < \infty$. If the matrix operator

$$\begin{pmatrix} C^2 & CS \\ 0 & 0 \end{pmatrix}$$

is compact, then its $1,1$ entry is also compact. The square root of a positive compact operator is compact (recall that $C \geq 0$), thus C is compact. □

Remark 3.2. If $(P, Q) \in \mathcal{C}$, then the spectral resolution of X can be easily described. Since $0 < cos(X)$ is compact, it follows that

$$X = \sum_n \gamma_n P_n + \frac{\pi}{2} E,$$

where $0 < \gamma_n < \pi/2$ is an increasing (finite or infinite) sequence which accumulates eventually at $\pi/2$. For all n, $\dim R(P_n) < \infty$, and

$$R(E) \oplus (\oplus_{n \geq 1} R(P_n)) = \mathcal{L}.$$

From the spectral picture of X above, one obtains the following, which states that in their generic part, all pairs in $\mathcal{C}$ are obtained as in Example 2.1

Proposition 3.3. *The pair (P, Q) belongs to $\mathcal{C}$ if and only if the following conditions are satisfied:*

1. *$\mathcal{H}_{11}$ is finite dimensional.*
2. *The subspaces of the generic part $\mathcal{H}_0$, $M = P(\mathcal{H}_0)$ and $N = Q(\mathcal{H}_0)$ satisfy*

$$M \oplus N = \mathcal{H}_0.$$

3. *The idempotent $E = P_{M\|N}$ corresponding with this decomposition has matrix form, in terms of its range M,*

$$E = \begin{pmatrix} 1 & K \\ 0 & 0 \end{pmatrix}$$

for $K : N \to M$ a compact operator.

Proof. The sufficiency of these conditions, in order that $(P, Q) \in \mathcal{C}$, is clear. Also that condition $\dim(\mathcal{H}_{11}) < \infty$ is necessary. Denote by P_0 and Q_0 the reductions of P and Q to their generic part. In Halmos' model $\mathcal{H}_0 = \mathcal{L} \times \mathcal{L}$, it is apparent that $\sin(X)$ is invertible in $\mathcal{L}$. Then (reasoning with matrices in terms of $\mathcal{H}_0 = \mathcal{L} \times \mathcal{L}$)

$$P_0 - Q_0 = \begin{pmatrix} S^2 & CS \\ CS & -S^2 \end{pmatrix} \text{ and thus } (P_0 - Q_0)^2 = \begin{pmatrix} S^2 & 0 \\ 0 & S^2 \end{pmatrix}$$

is invertible. Then $P_0 - Q_0$ is invertible. This means that $M \oplus N = \mathcal{H}_0$ (see [7]). Moreover, by a formula in [7], and after straightforward computations,

$$E = P_{M\|N} = P_0(P_0 - Q_0)^{-1} = \begin{pmatrix} 1 & -CS^{-1} \\ 0 & 0 \end{pmatrix}.$$

Note that $-CS^{-1}$ is compact in $\mathcal{L}$. □

We shall describe the different subclasses of $\mathcal{C}$ in terms of the Halmos decomposition and the spectral resolution of X. The class $\mathcal{C}_0$ is easiest to describe. Recall that $(P, Q) \in \mathcal{C}_0$ if $\dim R(Q) < \infty$. Then apparently

Proposition 3.4. *Let* $(P, Q) \in \mathcal{C}$. *Then* $(P, Q) \in \mathcal{C}_0$ *if and only if the sequence* $\{\gamma_n\}$ *is finite,* $\dim \mathcal{H}_{01} < \infty$ *and* $\dim R(E_1) < \infty$.

4 Unitary actions

We shall use two unitary actions to describe the structure of $\mathcal{C}$. The full unitary group $\mathcal{U}(\mathcal{H})$ acts on pairs in $\mathcal{C}$ by joint inner conjugation:

$$U \cdot (P, Q) = (UPU^*, UQU^*), \quad U \in \mathcal{U}(\mathcal{H}), \quad (P, Q) \in \mathcal{C}.$$

We shall make use of another local unitary action, on pairs in $\mathcal{C}$ with the first coordinate P_0 fixed. Recall the definition of the restricted unitary group, where the restriction is given by the decomposition $\mathcal{H} = R(P_0) \oplus N(P_0)$,

$$\mathcal{U}_{res}(P_0) = \{U \in \mathcal{U}(\mathcal{H}) : [U, P_0] \in \mathcal{K}(\mathcal{H})\}.$$

In matrix form, in terms of the given decomposition, these unitaries are of the form

$$U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$$

where u_{12} and u_{21} are compact operators. Elementary matrix computations, involving the fact that U is unitary, imply that u_{11} and u_{22} are Fredholm operators in $R(P_0)$ and $N(P_0)$, respectively, and that

$$ind(u_{22}) = -ind(u_{11}).$$

The integer $ind(u_{11})$ is usually called the *index* of U. It is known that this index parametrizes the connected components of $\mathcal{U}_{res}(P_0)$: two unitaries $U, W \in \mathcal{U}_{res}(P_0)$ belong to the same connected component if and only if $ind(U) = ind(W)$ (see for instance [28] or [8]).

Let us prove that $\mathcal{U}_{res}(P_0)$ acts by inner conjugation of the classes $\mathcal{C}_x(P_0)$ ($x = 0, 1, \infty$).

Proposition 4.1. *Let* $Q \in \mathcal{C}_x(P_0)$ ($x = 0, 1, \infty$) *and* $U \in \mathcal{U}_{res}(P_0)$. *Then* $UQU^* \in \mathcal{C}_x(P_0)$.

Proof. Straightforward matrix computation:

$$UQU^* = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \begin{pmatrix} a & x \\ x^* & y \end{pmatrix} \begin{pmatrix} u_{11}^* & u_{21}^* \\ u_{12}^* & u_{22}^* \end{pmatrix}.$$

The $1,1$ entry of this product is

$$u_{11}au_{11}^* + u_{12}x^*u_{11}^* + u_{11}xu_{12}^* + u_{12}bu_{12}^*,$$

where a, x and u_{12} are compact, therefore the $1,1$ entry is compact. The $1,2$ entry is

$$u_{11}au_{21}^* + u_{12}x^*u_{21}^* + u_{11}xu_{22}^* + u_{12}bu_{22}^*,$$

which is compact by a similar argument. Then $P_0UQU^* \in \mathcal{K}(\mathcal{H})$. □

We shall mainly use $\mathcal{U}^0_{res}(P_0)$, the connected component of the identity (or zero index component). This component is an exponential group, namely (see [8], [28])

$$\mathcal{U}^0_{res}(P_0) = exp\{iX \in \mathcal{B}(\mathcal{H}) : X^* = X, [X, P_0] \in \mathcal{K}(\mathcal{H})\}.$$

Note that $\mathcal{U}_{res}(P_0)$ is the unitary group of the C*-algebra $\mathcal{A}_{P_0}(\mathcal{H})$ of operators T in $\mathcal{H}$ such that $[T, P_0] \in \mathcal{K}(\mathcal{H})$.

Remark 4.2. Clearly, if $Q, Q' \in \mathcal{C}_0$ (of finite rank) lie in the same connected component, then they have the same rank. Indeed, being homotopic in $\mathcal{C}_0$, in particular they are homotopic in $\mathcal{P}(\mathcal{H})$, and this implies equality of the rank. On the other hand, if Q and Q' have the same (finite) rank, let $\mathcal{J}$ be the (finite dimensional) linear subspace spanned by $R(Q)$ and $R(Q')$. Apparently Q and Q' are reduced by $\mathcal{J}$ (and act trivially on $\mathcal{J}^\perp$), and $Q|_\mathcal{J}$ and $Q'|_\mathcal{J}$ have the same rank. Therefore there exists a unitary operator $U_0 = e^{iX_0}$ in $\mathcal{J}$ (X_0 selfadjoint in $\mathcal{J}$) such that $e^{iX_0}Q|_\mathcal{J}e^{-iX_0} = Q'|_\mathcal{J}$. Then $X = X_0 \oplus 0$ is a finite rank operator acting in $\mathcal{J} \oplus \mathcal{J}^\perp$ which verifies

$$e^{iX}Qe^{-iX} = Q'.$$

Note that $e^{iX} \in exp(\mathcal{K}(\mathcal{H})) \subset \mathcal{U}^0_{res}(P_0)$, and therefore Q and Q' are homotopic in $\mathcal{C}_0$ (for instance, consider the path $Q(t) = e^{itX}Qe^{-itX}$ in $\mathcal{C}_0$).

5 The restricted Grassmannian

Let us recall some elementary facts concerning the restricted Grassmannian of a decomposition of $\mathcal{H}$. Given a projection P_0 and the decomposition $\mathcal{H} = N(P_0) \oplus R(P_0)$, the main role will be played by $\mathcal{N}_0 = N(P_0)$. Denote by E_0 the orthogonal projection onto $\mathcal{N}_0$.

Definition 5.1. [31]*:*

A projection Q belongs to the restricted Grassmannian $\mathcal{P}_{res}(\mathcal{N}_0)$ with respect to the decomposition $\mathcal{H} = \mathcal{N}_0 \oplus \mathcal{N}_0^\perp$, or more precisely, with respect to subspace $\mathcal{N}_0$, if and only if

1. $$E_0Q|_{R(Q)} : R(Q) \to \mathcal{N}_0 \in \mathcal{B}(R(Q), \mathcal{N}_0)$$

   is a Fredholm operator in $\mathcal{B}(R(Q), \mathcal{N}_0)$, and

2. $$(1 - E_0)Q|_{R(Q)} : R(Q) \to \mathcal{N}_0^\perp \in \mathcal{B}(R(Q), \mathcal{N}_0^\perp)$$

   is compact.

The index of the first operator characterizes the connected components of $\mathcal{P}_{res}(\mathcal{N}_0)$. The following result is elementary:

Lemma 5.2. *Let $Q \in \mathcal{P}(\mathcal{H})$ with matrix (in terms of $\mathcal{H} = \mathcal{N}_0 \oplus \mathcal{N}_0^{\perp}$)*

$$Q = \begin{pmatrix} a & x \\ x^* & b \end{pmatrix}.$$

Then $Q \in \mathcal{P}_{res}(\mathcal{N}_0)$ if and only if a is Fredholm in $\mathcal{B}(\mathcal{N}_0)$, and b and x are compact.

Proof. The proof is based on the following elementary facts:

- $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ is a Fredholm operator if and only if AA^* is a Fredholm operator in $\mathcal{H}_1$ and $N(A)$ is finite dimensional.
- $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ is compact if and only if $A^*A \in \mathcal{B}(\mathcal{H}_1)$ is compact.

Suppose first that $Q \in \mathcal{P}_{res}(\mathcal{N}_0)$. Then $E_0 Q \in \mathcal{B}(R(Q), \mathcal{N}_0)$ is Fredholm, and thus

$$E_0 Q (E_0 Q)^*|_{\mathcal{N}_0} = E_0 Q E_0|_{\mathcal{N}_0} = a$$

is Fredholm in $\mathcal{N}_0$. Also $(1 - E_0)Q \in \mathcal{B}(R(Q), \mathcal{N}_0^{\perp})$ is compact, and thus

$$(1 - E_0)Q(1 - E_0 - Q)^*|_{\mathcal{N}_0^{\perp}} = (1 - E_0)Q(1 - E_0)|_{\mathcal{N}_0^{\perp}} = b$$

is compact in $\mathcal{N}_0^{\perp}$. The fact that Q is a projection implies the relation $b - b^2 = x^*x$, and thus x is compact.

Conversely, by the last computations, if x and b are compact, then $(1-E_0)Q \in \mathcal{B}(R(Q), \mathcal{N}_0^{\perp})$ is compact. Similarly, $E_0 Q(E_0 Q)^*|_{\mathcal{N}_0} = a$ is Fredholm, thus $E_0 Q$, as an operator in $\mathcal{B}(R(Q), \mathcal{N}_0)$, has closed range (equal to the range of a) with finite codimension. Let us prove that its nullspace is finite dimensional. Let $\xi = \xi_+ + \xi_- = Q\xi$ such that $E_0 \xi = 0$, ($\xi_+ \in \mathcal{N}_0$, $\xi_- \in \mathcal{N}_0^{\perp}$). This implies that

$$\begin{cases} \xi_+ = a\xi_+ + x\xi_- \\ \xi_- = x^*\xi_+ + b\xi_- \end{cases}$$

and $\xi_+ = 0$. The second equation then reduces to $\xi_- = b\xi_-$, i.e. ξ_- lies in the 1-eigenspace of the compact operator b. Thus ξ_- lies in a finite dimensional space. It follows that $N(E_0 Q|_{R(Q)})$ is finite dimensional. □

Corollary 5.3. *Let $P \in \mathcal{P}(\mathcal{H})$ (with $N(P), R(P)$ infinite dimensional). Then $\mathcal{C}_1(P)$ coincides with the restricted Grassmannian of $\mathcal{H}$ induced by the decomposition $\mathcal{H} = N(P) \oplus R(P)$.*

Proof. In the description of the classes $\mathcal{C}_i(P)$ at Remark 2.4 (given in matrix form in terms of the decomposition $\mathcal{H} = R(P) \oplus N(P)$, note the reversed order), a projection Q belongs to $\mathcal{C}_1(P)$ if and only if, in the Calkin algebra, its $2, 2$ entry is the identity and all other entries are nil. By the above Lemma, this means that Q belongs to the restricted Grassmannian of the decomposition $\mathcal{H} = N(P) \oplus R(P)$. □

From now on we shall refer this set of projections as the restricted Grassmannian of $N(P_0)$.

Remark 5.4. The group $\mathcal{U}^0_{res}(P_0)$ acts transitively on the connected components of $\mathcal{C}_1(P_0)$, which are parametrized by the Fredholm index defined in the restricted Grassmannian of $N(P_0)$.

Let us denote by

$$\mathcal{C}_1 = \{(P, Q) \in \mathcal{C} : Q \in \mathcal{C}_1(P)\},$$

the union of $\mathcal{C}_1(P)$ for all $P \in \mathcal{P}_\infty(\mathcal{H})$, where $\mathcal{P}_\infty(\mathcal{H})$ denotes the (connected) space of projections in $\mathcal{H}$ with infinite dimensional range and nullspace.

Theorem 5.5. *The connected components of $\mathcal{C}_1$ are parametrized by the Fredholm index.*

Namely, $(P, Q), (P', Q') \in \mathcal{C}_1$ lie in the same connected component if and only if the index of Q in the restricted Grassmannian of $N(P)$ coincides with the index of Q' in the restricted Grassmannian of $N(P')$.

Proof. There exists a unitary operator $U \in \mathcal{U}(\mathcal{H})$ such that $U^*P'U = P$. Consider the pair $U^* \cdot (P', Q') = (P, U^*Q'U)$. Apparently $(P, U^*Q'U)$ belongs to the restricted Grassmannian of $N(P)$, and it has the same index as (P', Q'). Since $\mathcal{U}(\mathcal{H})$ is connected, this means that one is reduced to the case $P = P'$, where the result is valid due to the above Corollary. $\square$

Remark 5.6. One can prove, in a similar fashion, that the connected components of $\mathcal{C}_0$ are parametrized by the rank of Q.

Note that the class $\mathcal{C}_1$ can be described in terms of the Halmos decomposition:

Proposition 5.7. *Let $(P, Q) \in \mathcal{C}$. Then the following are equivalent:*

1. $(P, Q) \in \mathcal{C}_1$.
2. $\dim \mathcal{H}_{00} < \infty$.
3. $\dim N(b) < \infty$.

In this case, the index of Q in the restricted Grassmannian of $N(P)$ is given by

$$\dim \mathcal{H}_{01} - \dim \mathcal{H}_{10}.$$

Proof. The (five space) Halmos decomposition induces a (four space) decomposition of $\mathcal{H}$ which reduces both P and Q. Namely

$$\mathcal{H} = \mathcal{H}_{00} \oplus \mathcal{H}_{11} \oplus \mathcal{H}' \oplus \mathcal{H}_0,$$

where $\mathcal{H}' = \mathcal{H}_{10} \oplus \mathcal{H}_{01}$. By Lemma (5.2), the part of Q which acts on $N(P)$ must be a Fredholm operator. By the above reduction, this amounts to show that both 0 acting in $\mathcal{H}_{00}$ and S^2 acting in the space $\mathcal{L}$, are Fredholm operators (recall notations from Section 3). The first assertion means that $\dim \mathcal{H}_{00} < \infty$. With respect to the second,

$$S^2 = sin^2(X) = \sum_n sin^2(\gamma_n) P_n + E$$

is always a Fredholm operator (recall that $N(S) = N(X) = \{0\}$), since $0 < sin^2(\gamma_n)$ is a finite set or a sequence increasing to 1. If $Q \in \mathcal{C}_1(P)$, then b is Fredholm in $N(P)$, and thus $N(B)$ is finite dimensional. Conversely, the fact that $Q \in \mathcal{C}(P)$ implies that the spectral decomposition of b is of the form

$$b = \sum_{n \geq 1} (1 - \lambda_n) P'_n + E'_1,$$

with $1 > \lambda_n > 0$ a finite set or a strictly decreasing sequence converging to 0. If $N(b)$ is finite dimensional, then apparently b is a Fredholm operator, and thus Q belongs to the restricted Grassmannian of $N(P)$, i.e. $Q \in \mathcal{C}_1(P)$.

If Q lies in the restricted Grassmannian, it is well known that the index of Q with respect to $N(P)$ is

$$\dim(R(Q) \cap N(P)) - \dim(N(Q) \cap R(P)) = \dim \mathcal{H}_{01} - \dim \mathcal{H}_{10}.$$

□

Note that if the projection Q has finite rank, then the subspace $\mathcal{H}_{00}$ is infinite dimensional (recall that $P \in \mathcal{P}_\infty(\mathcal{H})$).

Example 5.8. There is a classical example, in the framework of the main application of the restricted Grassmannian [31], to the parametrization of solutions of the KdV equation. In our case we may state it in the following manner. Let

$$\mathcal{H} = L^2(\mathbb{T}, dt)$$

where $\mathbb{T}$ is the 1-torus, and consider the decomposition

$$\mathcal{H} = \mathcal{H}_- \oplus \mathcal{H}_+,$$

where $\mathcal{H}_+$ is the Hardy space. If $\varphi : \mathbb{T} \to \mathbb{C}$ is a continuous non-vanishing function, then $\varphi\mathcal{H}_-$ belongs to the restricted Grassmannian of $\mathcal{H}_-$, and therefore the pair $(P_{\mathcal{H}_+}, P_{\varphi\mathcal{H}_-})$ belongs to $\mathcal{C}_1$. The index of the pair coincides with the winding number of the function φ ([28], [31]).

6 Essential projections

Following the notation of the previous section, denote

$$\mathcal{C}_\infty = \{(P, Q) \in \mathcal{C} : Q \in \mathcal{C}_\infty(P)\},$$

the union of $\mathcal{C}_\infty(P)$ for all $P \in \mathcal{P}_\infty(\mathcal{H})$. Let $(P, Q) \in \mathcal{C}_\infty$. Write Q as a matrix in terms of P as before,

$$Q = \begin{pmatrix} a & x \\ x^* & b \end{pmatrix}$$

with $a = \sum_{n \geq 1} \lambda_n P_n + E_1$ and $b = \sum_{n \geq 1} (1 - \lambda_n) P'_n + E'_1$. Define

$$Q_d = \begin{pmatrix} E_1 & 0 \\ 0 & \sum_n P'_n + E'_1 \end{pmatrix}.$$

Apparently Q_d is a projection; it is also clear that $Q_d \in \mathcal{P}_\infty(\mathcal{H})$. Indeed, since $r(E_1) < \infty$, it follows that $\dim N(Q_d) = \infty$. If the sequence $\{\lambda_n\}$ is finite, the facts that they have finite multiplicities and that b is a Fredholm operator, imply that $r(E'_1) = \infty$. If the sequence is infinite, then $r(\sum_n P'_n) = \infty$. In any case, $r(Q_d) = \infty$.

Lemma 6.1. *$B = Q + Q_d - 1$ is invertible in $\mathcal{H}$.*

Proof. Note that, writing $1_{R(P)} = E_1 + N + \sum_{n\geq 1} P_n$, and $1_{N(P)} = E'_1 + N' + \sum_{n\geq 1} P'_n$, where where N and N' denote the orthogonal projections onto the nullspaces of a and b in $R(P)$ and $N(P)$, respectively, one has

$$B = \begin{pmatrix} \sum_{n\geq 1}(\lambda_n - 1)P_n + E_1 - N & x \\ x^* & \sum(1-\lambda_n)P'_n + E'_1 - N' \end{pmatrix}.$$

The diagonal entries of B are invertible in $R(P)$ and $N(P)$. Indeed, they are diagonal operators with non nil eigenvectors that accumulate (eventually) at -1 and 1, respectively. The codiagonal entries of B are compact. It follows that B is of the form invertible plus compact. Thus it is a Fredholm operator, and in particular it has closed range. Therefore, since B is selfadjoint, it suffices to show that it has trivial nullspace. Note that B is a difference of projections, namely

$$B = Q - (1 - Q_d).$$

It is an elementary fact that the nullspace of a difference of projections is

$$N(B) = (N(Q) \cap N(1-Q_d)) \oplus (R(Q) \cap R(1-Q_d)) = (N(Q) \cap R(Q_d)) \oplus (R(Q) \cap N(Q_d)).$$

Let us see that $N(Q) \cap R(Q_d) = \{0\}$. Let $\xi + \eta \in R(P) \oplus N(P) = \mathcal{H}$ in $N(Q) \cap R(Q_d)$. Then

$$E_1\xi = \xi \quad \text{and} \quad \sum_{n\geq 1} P'_n\eta + E'_1\eta = \eta. \tag{3}$$

This implies that $P_n\xi = 0$ for all n, $N\xi = 0$ and $N'\eta = 0$. Also one has

$$\begin{cases} 0 = \sum_{n\geq 1}\lambda_n P_n\xi + E_1\xi + x\eta = \xi + x\eta \\ 0 = x^*\xi + \sum_{n\geq 1}(1-\lambda_n)P'_n\eta + E'\eta. \end{cases} \tag{4}$$

Recall that $R(x) = \oplus_{n\geq 1} R(P_n)$ which is orthogonal to $R(E_1)$. Thus $\xi = 0$ and $x\eta = 0$. Since the nullspace of x is $R(N') \oplus R(E'_1)$, one has that $\eta = N'\eta + E'_1\eta$. Combining this with the second equality in (3), yields $N'\eta = 0$ and $E'\eta = \eta$ (and $P'_n\eta = 0$ for all n). Using these facts in the second equation of (4), one obtains $\eta = 0$.

The fact that $R(Q) \cap N(Q_d) = \{0\}$ is proved in a similar fashion. □

Remark 6.2. Buckholtz [7] proved that a difference of projections $P_1 - P_2$ is invertible if and only if $\|P_1 + P_2 - 1\| < 1$. In our case, this implies that

$$\|Q - Q_d\| < 1.$$

Lemma 6.3. *The unitary part U of B in the polar decomposition $B = U|B|$ belongs to $\mathcal{U}^0_{res}(P)$.*

Proof. As remarked in the above proof, the off-diagonal entries of B (in its matrix in terms of P) are compact. Therefore B is an invertible element in the C*-algebra $\mathcal{A}_P(\mathcal{H})$. It follows that its unitary part is a unitary element of this algebra, namely $\mathcal{U}_{res}(P)$. We need to show that it has index zero. The index is in fact defined in the whole invertible group of $\mathcal{A}_P(\mathcal{H})$, and it coincides with the index of the $1,1$ entry. As it was also pointed out in the proof above, the $1,1$ entry of B is invertible in $R(P)$, and thus it has trivial index. □

Remark 6.4. It is well known (see for instance [10]) that if an invertible operator intertwines two selfadjoint projections, then its unitary part in the polar decomposition also does. In our case,

$$BQ = QQ_d = Q_d B.$$

Therefore $UQ = Q_d U$, or $UQU^* = Q_d$.

Note also the fact that since B is selfadjoint, U is a *symmetry* (i.e., a selfadjoint unitary: $S^* = S^{-1} = S$), so that $UQU = Q_d$.

Lemma 6.5. *Let E, F be two projections in $\mathcal{C}_\infty$ which commute with P. Then they are unitarily equivalent with a unitary operator in $\mathcal{U}^0_{res}(P)$.*

Proof. In terms of P, one has

$$E = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} \text{ and } F = \begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix},$$

where E_1 and F_1 have finite rank in $R(P)$ and E_2 and F_2 have infinite rank and nullity in $N(P)$. By means of a unitary operator of the form

$$\begin{pmatrix} 1_{R(P)} & 0 \\ 0 & W \end{pmatrix},$$

one is reduced to the case $E_2 = F_2$. Clearly this unitary operator belongs to $\mathcal{U}^0_{res}(P)$. In order to prove that E and F are conjugate with a unitary in $\mathcal{U}^0_{res}(P)$, it suffices to show that any of these projections, for instance E, can be conjugated with

$$E_0 = \begin{pmatrix} 0 & 0 \\ 0 & E_2 \end{pmatrix}.$$

Consider the following orthonormal bases:

- $\{e_n : 1 \leq n\}$ an orthonormal basis of $R(E_2)$ (in $N(P)$).
- $\{e'_l : 1 \leq l\}$ an orthonormal basis of $N(P) \ominus R(E_2)$.
- $\{f_k : 1 \leq k\}$ an orthonormal basis of $R(P)$, with $f_1, \ldots, f_N$ spanning $R(E_1)$.

Consider U defined as follows:

- $U(e_n) = f_n$ if $1 \leq n \leq N$, and $U(e_n) = e_{n-N}$ if $n \geq N + 1$.
- $U(e'_l) = e'_{l+N}$.
- $U(f_k) = e'_k$ if $1 \leq k \leq N$, and $U(f_k) = f_k$ if $n \geq N + 1$.

It is straightforward to verify that U is a unitary operator. Note also that U is not the identity only on a finite number of f_k, and thus UP and PU are of the form P plus compact. Therefore $[U, P]$ is compact, i.e. $U \in \mathcal{U}_{res}(P)$. For the same reason, on $R(P)$, U is the identity plus a finite rank operator, and thus U has index zero. Finally, by construction,

$$U(R(E_0)) = R(E_1) \text{ and } U(N(E_0)) = N(E_1).$$

□

From these facts, the main result of this Section follows:

Theorem 6.6.

1. *Let $P_0 \in \mathcal{P}_\infty(\mathcal{H})$, then the action of $\mathcal{U}^0_{res}(P_0)$ is transitive in $\mathcal{C}_\infty(P_0)$. In particular, $\mathcal{C}_\infty(P_0)$ is connected.*

2. *$\mathcal{C}_\infty$ is connected,*

Proof. Let Q and R be elements of $\mathcal{C}_\infty(P_0)$. By the first two lemmas above, Q is $\mathcal{U}^0_{res}(P_0)$-conjugate to Q_d and R is $\mathcal{U}^0_{res}(P_0)$-conjugated to R_d. R_d and Q_d are $\mathcal{U}^0_{res}(P_0)$-conjugate by the third lemma.

To prove the second assertion, suppose that (P, Q) and (P', Q') belong to $\mathcal{C}_\infty$. Since by hypothesis $P, P' \in \mathcal{P}_\infty(\mathcal{H})$, there exists a unitary operator $W = e^{iX}$ $(X^* = X)$ such that $WPW^* = P'$. Apparently, the pairs (P, Q) and (P', WQW^*) are homotopic in $\mathcal{C}_\infty$ (for instance, by means of the curve $(e^{itX}Pe^{-itX}, e^{itX}Qe^{-itX})$). Thus it suffices to show that (P', WQW^*) and (P', Q') are homotopic in $\mathcal{C}_\infty$. This is the first assertion. □

Note that in particular, this implies that if $Q \in \mathcal{C}_\infty(P)$, then also $Q_d \in \mathcal{C}_\infty(P)$. This fact could have been obtained directly from the definition of Q_d.

Remark 6.7. Consider the example at the beginning of Section 1, namely let I, J be measurable subsets of $\mathbb{R}^n$ of finite measure, and put $P_I, Q_J \in \mathcal{P}(L^2(\mathbb{R}^n, dx))$ given by

$$P_I f = \chi_I f \quad \text{and} \quad Q_J f = \left(\chi_J \hat{f}\right)\check{}.$$

Lenard proved [25] that $N(P_I) \cap N(Q_J)$ is infinite dimensional. Therefore the matrix of Q_J in terms of P_I (whose first column and row are compact) has the $2, 2$ entry which is not a Fredholm operator. Clearly it is not compact (which would mean that Q_J has finite rank). Therefore $(P_I, Q_J) \in \mathcal{C}_\infty$.

From the above result it follows that all pairs constructed in this way can be joined by a continuous curve inside $\mathcal{C}_\infty$.

The above Remark, showing that pairs in the example by Lenard belong to $\mathcal{C}_\infty$, can be generalized. Recall the characterizations of $\mathcal{C}_0$ and $\mathcal{C}_1$ in terms of the Halmos decomposition:

Proposition 6.8. *Let $(P, Q) \in \mathcal{C}$. Then $(P, Q) \in \mathcal{C}_\infty$ if and only if $\dim R(Q) = \infty$ and $\dim \mathcal{H}_{00} = \infty$.*

Remark 6.9. Apparently, $(P, Q) \in \mathcal{C}$ if and only if $(Q, P) \in \mathcal{C}$: PQ is compact if and only if QP is compact. There is a however an abuse of notation in this assertion, because we have supposed from the beginning that the first coordinate of the pair must belong to $\mathcal{P}_\infty$. Assume thus that also $Q \in \mathcal{P}_\infty$.

Note that if furthermore $(P, Q) \in \mathcal{C}_1$, then also $(Q, P) \in \mathcal{C}_1$. This follows in a straightforward manner from the definition of the restricted Grassmannian, by taking adjoints. Also it is clear that the index of the reversed pair changes sign.

As a consequence (since the class $\mathcal{C}_0$ is explicitly excluded), it follows that $(P, Q) \in \mathcal{C}_\infty$ implies that $(Q, P) \in \mathcal{C}_\infty$.

7 Regular structure

Let us recall some basic facts on the differential geometry of the set $\mathcal{P}(\mathcal{H})$ (see for instance [10], [29]).

Remark 7.1.

1. The space $\mathcal{P}(\mathcal{H})$ is a homogeneous space under the action of the unitary group $\mathcal{U}(\mathcal{H})$ by inner conjugation. The orbits of the action coincide with the connected components of $\mathcal{P}(\mathcal{H})$, which are: $\mathcal{P}_{n,\infty}(\mathcal{H})$ (projections of nullity n), $\mathcal{P}_{\infty,n}(\mathcal{H})$ (projections of rank n) and $\mathcal{P}_{\infty}(\mathcal{H})$ (projections of infinite rank and nullity). These components are C^{∞}-submanifolds of $\mathcal{B}(\mathcal{H})$.

2. There is a natural linear connection in $\mathcal{B}(\mathcal{H})$. If $\dim \mathcal{H} < \infty$, it is the Levi-Civita connection of the Riemannian metric which consists of considering the Frobenius inner product at every tangent space. It is based on the diagonal - codiagonal decompositon of $\mathcal{B}(\mathcal{H})$. To be more specific, given $P_0 \in \mathcal{P}(\mathcal{H})$, the tangent space of $\mathcal{P}(\mathcal{H})$ at P_0 consists of all selfadjoint codiagonal matrices (in terms of P_0). The linear connection in $\mathcal{P}(\mathcal{H})$ is induced by a reductive structure, where the horizontal elements at P_0 (in the Lie algebra of $\mathcal{U}(\mathcal{H})$: the space of antihermitian elements of $\mathcal{B}(\mathcal{H})$) are the codiagonal antihermitian operators. The geodesics of $\mathcal{P}$ which start at P_0 are curves of the form

$$\delta(t) = e^{itX} P_0 e^{-itX}, \tag{5}$$

   with $X^* = X$ codiagonal with respect to P_0. It was proved in [29] that if $P_0, P_1 \in \mathcal{P}(\mathcal{H})$ satisfy $\|P_0 - P_1\| < 1$, then there exists a unique geodesic (up to reparametrization) joining P_0 and P_1. This condition is not necessary for the existence of a unique geodesic.

3. In [2] a necessary and sufficient condition was found, in order that there exists a unique geodesic joining two projections P and Q. This is the case if and only if

$$R(P) \cap N(Q) = N(P) \cap R(Q) = \{0\}.$$

4. If $\dim \mathcal{H} = \infty$, the Frobenius metric is not available. If one endows each tangent space of $\mathcal{P}(\mathcal{H})$ with the usual norm of $\mathcal{B}(\mathcal{H})$, one obtains a continuous (non regular) Finsler metric. In [29] it was shown that the geodesics (5) remain minimal among their endpoints for all t such that

$$|t| \leq \frac{\pi}{2\|X\|}.$$

5. It is sometimes useful to parametrize projections using symmetries S ($S^* = S$, $S^2 = 1$), via the affine map

$$P \longleftrightarrow S_P = 2P - 1.$$

   Some algebraic computations are simpler with symmetries. For instance, the condition that the exponent X (of the geodesic) is P_0-codiagonal means that X anti-commutes with S_{P_0}. Thus the geodesic (5), in terms of symmetries, can be expressed

$$S_\delta(t) = e^{itX} S_{P_0} e^{-itX} = e^{2itX} S_{P_0} = S_{P_0} e^{-2itX}.$$

Fix $P_0 \in \mathcal{P}_\infty(\mathcal{H})$. We shall see first that $\mathcal{C}(P_0)$ is a differentiable manifold. If $\mathcal{A}$ is a C*-algebra, denote by $\mathcal{A}_h$ the space of selfadjoint elements of $\mathcal{A}$.

Lemma 7.2. *If $Q, Q' \in \mathcal{C}(P_0)$ and $\|Q - Q'\| < 1$, then there exists $U \in \mathcal{U}^0_{res}(P_0)$ such that $UQU^* = Q'$. This unitary operator U can be chosen as an explicit smooth formula in terms of Q and Q'. In particular, Q and Q' lie in the same (class and) connected component of $\mathcal{C}(P_0)$.*

Proof. If $\|Q - Q'\| < 1$, then there exists a unique geodesic joining Q and Q' in $\mathcal{P}(\mathcal{H})$: $Q' = e^{iX}Qe^{-iX}$ for $X^* = X$ Q-codiagonal with $\|X\| < \pi/2$. As remarked in [10], the fact that X is Q-codiagonal implies that X anti-commutes with $2Q - 1$. Then

$$2Q' - 1 = e^{iX}(2Q - 1)e^{-iX} = e^{2iX}(2Q - 1).$$

Thus

$$e^{2iX} = (2Q' - 1)(2Q - 1).$$

Since $\|2iX\| < \pi$, the spectrum of $(2Q' - 1)(2Q - 1)$ is contained in the subset $\{e^{it} : t \in (-\pi, \pi)\}$ of the unit circle, and thus X can be recovered as a continuous (in fact holomorphic) logarithm of $(2Q' - 1)(2Q - 1)$,

$$X = -\frac{i}{2}\log((2Q' - 1)(2Q - 1)).$$

Note that both $(2Q' - 1)(2Q - 1)P_0$ and $P_0(2Q' - 1)(2Q - 1)$ are of the form P_0 plus compact. It follows that $[(2Q' - 1)(2Q - 1), P_0]$ is compact, and thus $(2Q' - 1)(2Q - 1) \in \mathcal{U}_{res}(P_0)$. This implies that the exponent X belongs to $\mathcal{A}_{P_0}$ (recall that the exponential map is a diffeomorphism between exponents $X^* = X$ in $\mathcal{A}_{P_0}$ of norm less than π and unitaries U in $\mathcal{U}_{res}(P_0)$ such that $\|U - 1\| < 2$). □

Remark 7.3. In particular, the above result provides a way to parametrize elements $Q' \in \mathcal{C}(P_0)$ in the vicinity of a given $Q \in \mathcal{C}(P_0)$. Namely, let

$$\mathcal{V}_Q = \{Q' \in \mathcal{C}(P_0) : \|Q' - Q\| < 1\}.$$

For each $Q' \in \mathcal{V}_Q$, there exists a unique $X = X_Q(Q')$, X^*, $\|X\| < \pi/2$, which is Q-codiagonal and belongs to $\mathcal{A}_{P_0}$, such that $e^{iX}Qe^{-iX} = Q'$.

Conversely, to each X as above, there corresponds an element $Q' = e^{iX}Qe^{-iX} \in \mathcal{C}(P_0)$, with $\|Q' - Q\| < 1$. Apparently both maps

$$Q' \mapsto X \quad \text{and} \quad X \mapsto Q'$$

are smooth maps, and each one is the inverse of the other. Thus one has defined a local chart $\mathcal{V}_Q$ for any $Q \in \mathcal{C}(P_0)$, which is modelled in an open ball of $(\mathcal{A}_{P_0})_h$.

Corollary 7.4. *For any $P_0 \in \mathcal{P}_\infty(\mathcal{H})$, the set $\mathcal{C}(P_0)$ is a smooth manifold modeled in $\mathcal{A}_{P_0}(\mathcal{H})_h$.*

Let us prove now that the whole set of pairs $\mathcal{C}$ is also a smooth manifold:

Theorem 7.5. *The set*

$$\mathcal{C} = \{(P, Q) : P \in \mathcal{P}_\infty(\mathcal{H}), PQ \text{ is compact}\}$$

is a smooth differentiable manifold.

Proof. Fix a pair $(P_0, Q_0) \in \mathcal{C}$. We shall exhibit a local chart for $\mathcal{C}$ near this pair. Let $(P, Q) \in \mathcal{C}$ such that $\|P - P_0\| < 1$. Then, as remarked above, there exists $X = X(P)$ (a smooth map in terms of P, with $X(P_0) = 0$), $X^* = X$, $\|X\| < \pi/2$ and X is P_0-codiagonal, such that

$$P = e^{iX} P_0 e^{-iX}.$$

Then the pair $e^{-iX}(P, Q)e^{iX} = (P_0, e^{-iX}Qe^{iX})$ belongs to $\mathcal{C}(P_0)$. Let (P, Q) be close enough to (P_0, Q_0) so that $e^{-iX}Qe^{iX}$ lies in the local chart $\mathcal{V}_{Q_0}$ for $\mathcal{C}(P_0)$ around Q_0 constructed above. Note that if $P \to P_0$, then $e^{iX} \to 1$, so that

$$\|e^{-iX}Qe^{iX} - Q_0\| \leq \|e^{-iX}Qe^{iX} - Q\| + \|Q - Q_0\|$$

is arbitrarily small if (P, Q) is close to (P_0, Q_0). The chart for (P_0, Q_0) is the open set

$$\mathcal{V}_{(P_0,Q_0)} = \{(P, Q) \in \mathcal{C} : \|P - P_0\| < 1 \text{ and } e^{-iX}Qe^{iX} \in \mathcal{V}_{Q_0}\}.$$

If $e^{-iX}Qe^{iX} \in \mathcal{V}_{Q_0}$, then there exists a unique $Y = X_{Q_0}(e^{-iX}Qe^{iX})$ in $\mathcal{A}_{P_0}$, $Y^* = Y$, $\|Y\| < \pi/2$, which is Q_0-codiagonal, such that

$$e^{-iX}Qe^{iX} = e^{iY}Q_0e^{-iY}.$$

Denote $\mathcal{B}_{P_0} = \{X \in \mathcal{B}_h(\mathcal{H}) : \|X\| < \pi/2 \text{ and } X \text{ is } P_0 - \text{codiagonal}\}$ (and accordingly consider $\mathcal{B}_{Q_0}$). Consider the map

$$\Psi = \Psi_{(P_0,Q_0)} : \mathcal{V}_{(P_0,Q_0)} \to \mathcal{B}_{P_0} \times (\mathcal{B}_{Q_0} \cap (\mathcal{A}_{P_0})_h) \subset \mathcal{B}_h(\mathcal{H}) \times (\mathcal{A}_{P_0})_h$$

given by

$$\Psi(P, Q) = (X, Y).$$

The inverse of Ψ is the map

$$\Psi^{-1}(X, Y) = (e^{iX}P_0e^{-iX}, e^{iY}e^{iX}Q_0e^{-iY}e^{-iX}).$$

It is straightforward to verify that these maps are each other inverses. □

Let us return to $\mathcal{C}(P_0)$ for a fixed $P_0 \in \mathcal{C}$, and the fact stated in Remark (7.3). This remark says a bit more about the geometry of $\mathcal{C}(P_0)$ as a submanifold of $\mathcal{P}(\mathcal{H})$. Recall from the facts pointed out at the beginning of this section, that two projections at distance less than one are joined by a unique minimal geodesic.

Corollary 7.6. *Let $Q, Q' \in \mathcal{C}(P_0)$ such that $\|Q - Q'\| < 1$. Then the unique geodesic of $\mathcal{P}(\mathcal{H})$ remains inside $\mathcal{C}(P_0)$.*

Proof. If $Q, Q' \in \mathcal{C}(P_0)$ with $\|Q - Q'\| < 1$, then then the unique (selfadjoint, Q-codiagonal) exponent $X = X_Q(Q')$ with $\|X\| < \pi/2$ such that $e^{iX}Qe^{-iX} = Q'$, belongs to $\mathcal{A}_{P_0}$. □

Recall from Remark 7.1, the fact that if a weaker condition holds, namely

$$R(Q) \cap N(Q') = N(Q) \cap R(Q') = \{0\},$$

then there exists a unique X as above. A natural question is the following. Suppose that this condition for uniqueness holds, but $\|Q - Q'\| = 1$, does the unique geodesic $\gamma(t) = e^{iX}Qe^{-iX}$ belong to $\mathcal{C}(P)$?

8 Pairs P_I, Q_J

In [2] it was characterized when two projections P, Q are joined by a geodesic, and furthermore when this geodesic is unique. Namely, there exists a geodesic of $\mathcal{P}(\mathcal{H})$ joining P and Q if and only if (in the notation above)

$$\dim \mathcal{H}_{10} = \dim \mathcal{H}_{01}.$$

The geodesic is unique if and only if $\mathcal{H}_{10} = \mathcal{H}_{01} = \{0\}$. Also it was noted the the exponent X of the geodesics joining P and Q is both P and Q-codiagonal.

Lenard proved in [25], that for the projections $P_I, Q_J \in \mathcal{P}(L^2(\mathbb{R}^n, dx))$ given by

$$Pf = \chi_I f \quad \text{and} \quad Qf = \left(\chi_J \hat{f}\right)\check{}$$

for I, J be measurable subsets of $\mathbb{R}^n$ of finite measure, one has

$$R(P_I) \cap N(Q_J) = R(Q_J) \cap N(P_I) = \{0\}.$$

Moreover, $\|P_I - Q_J\| = 1$.

Putting these facts together one obtains the following

Theorem 8.1. *Let I, J be measurable subsets of $\mathbb{R}^n$ of finite measure, and P_I, Q_J the above projections. Then there exists a unique selfadjoint operator $X_{I,J}$ satisfying:*

1. $\|X_{I,J}\| = \pi/2$.
2. *$X_{I,J}$ is P_I and Q_J codiagonal. In other words, $X_{I,J}$ maps functions in $L^2(\mathbb{R}^n, dx)$ with support in I to functions with support in $\mathbb{R}^n - I$, and functions such that $\hat{f}$ has support in J to functions such that the Fourier transform has support in $\mathbb{R}^n - J$.*
3. $e^{iX_{I,J}} P_I e^{-iX_{I,J}} = Q_J$.
4. *If $P(t)$, $t \in [0,1]$ is a smooth curve in $\mathcal{P}(\mathcal{H})$ with $P(0) = P_I$ and $P(1) = Q_J$, then*

$$\ell(P) = \int_0^1 \|\dot{P}(t)\| dt \geq \pi/2.$$

Proof. The fact that $\|P_I - Q_J\| = 1$ means that $\|X_{I,J}\| = \pi/2$ (which is the length of the minimal geodesic joining these projections [2]). □

Remark 8.2. It is known [16] that $\lambda_1 = \|P_I Q_J P_I\| = \|P_I Q_J\|^2 < 1$, and moreover $\sqrt{\lambda_1}$ equals the cosine of the angle between the subspaces $R(P_I)$ and $R(Q_J)$.

One can also relate this number λ_1 with the operator $X_{I,J}$. Using Halmos' decomposition (recall that it consists only of $\mathcal{H}_{00}$ and the generic part $\mathcal{H}_0$ in this case),

$$P_I Q_J P_I = 0 \oplus \begin{pmatrix} C^2 & 0 \\ 0 & 0 \end{pmatrix}$$

and thus $\lambda_1 = \|cos(X)\|^2$. We shall see below that the spectrum of X is a strictly increasing sequence of positive eigenvalues $\gamma_n \to \pi/2$, with finite multiplicity. Moreover, since $P_I Q_J P_I$ belongs to $\mathcal{B}_1(\mathcal{H})$, it follows that $C \in \mathcal{B}_2(\mathcal{L})$. Thus

$$\{cos(\gamma_n)\} \in \ell^2.$$

Theorem 8.3. *Let I, I_0, J be measurable subsets of $\mathbb{R}^n$ of finite Lebesgue measure, with $|I \cap I_0| = 0$. Let $P_0 = P_{I_0}$. Then the selfadjoint operator $X_{I,J}$ has closed infinite dimensional range, in particular it is not compact. The commutant $[X_{I,J}, P_0]$ is compact.*

Proof. Easy matrix computations ([2]) show that, in the decomposition $\mathcal{H} = \mathcal{H}_{00} \oplus (\mathcal{L} \times \mathcal{L})$, $X_{I,J}$ is of the form

$$X_{I,J} = 0 \oplus \begin{pmatrix} 0 & -iX \\ iX & 0 \end{pmatrix}.$$

Note that the spectrum of this operator is symmetric with respect to the origin. Indeed, if V equals the symmetry

$$V = 1 \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

then apparently $VX_{I,J}V = -X_{I,J}$. Also note that

$$X_{I,J}^2 = 0 \oplus \begin{pmatrix} X^2 & 0 \\ 0 & X^2 \end{pmatrix}.$$

Therefore the spectrum of $X_{I,J}$ is

$$\sigma(X_{I,J}) = \{0\} \cup \{\gamma_n : n \geq 1\} \cup \{-\gamma_n : n \geq 1\},$$

with 0 of infinite multiplicity, and the multiplicity of γ_n equal to the multiplicity of $-\gamma_n$, and finite. What matters here, is that the set $\{\gamma_n : n \geq 1\}$ is infinite, and is therefore an increasing sequence converging to $\pi/2$. This holds because otherwise, the operator C would have finite rank, and therefore $P_I Q_J P_I$ would be of finite rank, which is not the case (see [25]). Thus $X_{I,J}$ has closed range. of infinite dimension.

Note that P_I and Q_J belong to $\mathcal{C}_\infty(P_0)$. Indeed $P_I P_0 = 0$ and $Q_J P_0 = Q_J P_{I_0}$ is compact.

Reasoning with symmetries, the fact that $P_I, Q_J \in \mathcal{C}(P_0)$ implies that

$$S_{P_I}, S_{Q_J} \in \mathcal{A}_{P_0}.$$

Since $S_{Q_J} = e^{i2X_{I,J}} S_{P_I}$, this implies that

$$e^{i2X_{I,J}} \in \mathcal{A}_{P_0}.$$

By the spectral picture of $X_{I,J}$ it is clear that $X_{I,J}$ can be obtained as an holomorphic function of $e^{i2X_{I,J}}$. Since $\mathcal{A}_{P_0}$ is a C*-algebra, this implies that $X_{I,J} \in \mathcal{A}_{P_0}$. □

Let us relate the operator $X_{I,J}$ with the mathematical version of the uncertainty principle, according to [14] and [16].

Let $A \in \mathcal{B}(\mathcal{H})$ be an operator with closed range, the reduced minimum modulus γ_A of A is the positive number

$$\gamma_A = \min\{\|A\xi\| : \xi \in N(A)^\perp, \|\xi\| = 1\} = \min\{|\lambda| : \lambda \in \sigma(A), \lambda \neq 0\}.$$

Donoho and Stark [14] underline the role of the number $\|Q_J P_I\|$ and consider any constant c such that $\|Q_J P_I\| \leq c$ a manifestation of the (mathematical) uncertainty principle. By the above Remark, we have

Corollary 8.4. *With the current notations,*

$$\|Q_J P_I\| = \cos(\gamma_{X_{I,J}}).$$

Proof. Indeed, in the above description of the spectrum of $X_{I,J}$, the reduced minimum modulus $\gamma_{X_{I,J}}$ of $X_{I,J}$ coincides with γ_1. □

This co-diagonal exponent $X_{I,J}$ (with respect both to P_I and Q_J) has interesting features when $I = J$.

If we pick $I = J$ (with $|I| < \infty$), and denote by $X_I = X_{I,I}$, we have two unitary operators intertwining P_I and Q_I. Namely, the Fourier transform $U_{\mathcal{F}}$ and the exponential e^{iX_I},

$$U_{\mathcal{F}}^* P_I U_{\mathcal{F}} = Q_I = e^{iX_I} P_I e^{-iX_I}.$$

Let $H = H^*$ be the natural logarithm of the Fourier transform, $e^{iH} = U_{\mathcal{F}}$. Namely, writing E_1, E_{-1}, E_i and E_{-i} the eigenprojections of $U_{\mathcal{F}}$,

$$H = -\pi E_{-1} + \frac{\pi}{2} E_i - \frac{\pi}{2} E_{-i}.$$

Note that $\|H\| = \pi$. Thus one obtains a smooth path joining P_I and Q_I:

$$\varphi(t) = e^{-itH} P_I e^{itH}.$$

Indeed, apparently $\varphi(1) = Q_I$.

Theorem 8.5. *For any Lebesgue measurable set $I \subset \mathbb{R}^n$ with $|I| < \infty$, one has*

$$\|[H, P_I]\| = \|[H, Q_I]\| \geq \pi/2.$$

Proof. The geodesic δ_I with exponent X_I is the shortest curve in $\mathcal{P}(\mathcal{H})$ joining P_I and Q_I. Its length is $\pi/2$. Then

$$\pi/2 \leq \ell(\varphi) = \int_0^1 \|\dot\varphi(t)\| dt = \int_0^1 \|e^{itH}[H, P_I]e^{-itH}\| dt = \|[H, P_I]\|.$$

Note that

$$U_{\mathcal{F}}^*[H, P_I]U_{\mathcal{F}} = [H, U_{\mathcal{F}}^* P_I U_{\mathcal{F}}] = [H, Q_I]$$

because $U_{\mathcal{F}}$ and H commute. □

Remark 8.6.

1. We may write H in terms of $U_{\mathcal{F}}$ using the well known formulas

   $$E_{-1} = \frac{1}{4}(1 - U_{\mathcal{F}} + U_{\mathcal{F}}^2 - U_{\mathcal{F}}^3),\ E_i = \frac{1}{4}(1 - iU_{\mathcal{F}} - U_{\mathcal{F}}^2 + iU_{\mathcal{F}}^3),\ E_{-i} = \frac{1}{4}(1 + iU_{\mathcal{F}} - U_{\mathcal{F}}^2 - iU_{\mathcal{F}}^3),$$

   and thus

   $$H = \frac{\pi}{4}\{-1 + (1+i)U_{\mathcal{F}} - U_{\mathcal{F}}^2 + (1+i)U_{\mathcal{F}}^3\}.$$

   Then

   $$[H, P_I] = \frac{\pi}{4}\{(1+i)[U_{\mathcal{F}}, P_I] - [U_{\mathcal{F}}^2, P_I] + (1+i)[U_{\mathcal{F}}^3, P_I]\}.$$

   The inequality in Corollary 8.5 can be written

   $$\|(1+i)[U_{\mathcal{F}}, P_I] - [U_{\mathcal{F}}^2, P_I] + (1+i)[U_{\mathcal{F}}^3, P_I]\| \geq 2.$$

2. In the special case when the set I is (essentially) symmetric with respect to the origin, P_I commutes with $U_{\mathcal{F}}^2$, so that

$$[U_{\mathcal{F}}^2, P_I] = 0 \quad \text{and} \quad [U_{\mathcal{F}}^3, P_I] = [U_{\mathcal{F}}, P_I]U_{\mathcal{F}}^2 = U_{\mathcal{F}}^2[U_{\mathcal{F}}, P_I]$$

one has

$$[H, P_I] = \frac{(1+i)\pi}{4}[U_{\mathcal{F}}, P_I](1 + U_{\mathcal{F}}^2).$$

The operator $U_{\mathcal{F}}^2 f(x) = f(-x)$ is a symmetry, then $\frac{1}{2}(1+U_{\mathcal{F}}^2)$ is the orthogonal projection E_e onto the the subspace of essentially even functions ($f(x) = f(-x)$ *a.e.*). Then one can write

$$[H, P_I] = \frac{(1+i)\pi}{2}[U_{\mathcal{F}}, P_I]E_e = \frac{(1+i)\pi}{2}E_e[U_{\mathcal{F}}, P_I].$$

Corollary 8.7. *Suppose that I is essentially symmetric, with finite measure.*

1.

$$\|E_e[U_{\mathcal{F}}, P_I]\| = \|E_e[U_{\mathcal{F}}, P_I]E_e\| \geq \frac{1}{\sqrt{2}}.$$

2.

$$\|E_eP_I - E_eQ_I\| \geq \frac{1}{\sqrt{2}},$$

where $E_eP_I = P_IE_e$ and $E_eQ_I = Q_IE_e$ are orthogonal projections.

Proof. Recall that E_e and $U_{\mathcal{F}}$ commute. Then

$$E_e[U_{\mathcal{F}}, P_I]E_e = E_e(U_{\mathcal{F}}P_I - P_IU_{\mathcal{F}})E_e = U_{\mathcal{F}}E_e(P_I - U_{\mathcal{F}}^*P_IU_{\mathcal{F}})E_e$$

$$= U_{\mathcal{F}}E_e(P_I - Q_I)E_e.$$

where E_e, as well as $U_{\mathcal{F}}$, and thus also $Q_I = U_{\mathcal{F}}^*P_IU_{\mathcal{F}}$ commute with E_e. □

The ranges of these two orthogonal projections E_eP_I and E_eQ_I consist of the elements of L^2 which are essentially even and vanish (essentially) outside I, and the analogous set for the Fourier transform.

Let us return to the general setting (I not necessarily equal to J). The ranges and nullspaces of P_I and Q_J have several interesting properties. First we need the following lemma:

Lemma 8.8. *Let P, Q be orthogonal projections such that $\|P - Q\| = 1$. Then one and only one of the following conditions hold:*

1. *$N(P) + R(Q) = \mathcal{H}$, with non direct sum (and this is equivalent to $R(P) + N(Q)$ being a direct sum and a closed proper subspace of $\mathcal{H}$).*

2. *$R(P) + N(Q) = \mathcal{H}$, with non direct sum (and this is equivalent to $N(P) + R(Q)$ being a direct sum and a closed proper subspace of $\mathcal{H}$).*

3. *$R(P) + N(Q)$ is non closed (and this is equivalent to $N(P) + R(Q)$ being non closed).*

Proof. By the Krein-Krasnoselskii-Milman formula (see for instance [22])

$$\|P-Q\| = \max\{\|P(1-Q)\|, \|Q(1-P)\|\},$$

we have that one and only one of the following hold:

1. $\|P(1-Q)\| < 1$ and $\|Q(1-P)\| = 1$,
2. $\|P(1-Q)\| = 1$ and $\|Q(1-P)\| < 1$, or
3. $\|P(1-Q)\| = 1$ and $\|Q(1-P)\| = 1$.

This alternative corresponds precisely with the three conditions in the Lemma. It is known [12] that for two orthogonal projections E and F, $\|EF\| < 1$ holds if and only if $R(E) \cap R(F) = \{0\}$ and $R(E) + R(F)$ closed. The sum $\mathcal{M} + \mathcal{N}$ of two subspaces is closed if and only if the sum $\mathcal{M}^\perp + \mathcal{N}^\perp$ is closed (see [12]). Therefore, $\|EF\| < 1$ is also equivalent to $N(E) + N(F) = \mathcal{H}$.

If we apply these facts to $E = P$ and $F = 1 - Q$, we obtain that the first alternative is equivalent to $R(P) \cap N(Q) = \{0\}$ and $R(P) + N(Q)$ closed, or to $N(P) + R(Q) = \mathcal{H}$.

Analogously, the second alternative is equivalent to $R(Q) \cap N(P) = \{0\}$ and $R(Q) + N(P)$ closed, or to $N(Q) + R(P) = \mathcal{H}$.

Note that in the first case, $R(P) + N(Q)$ is proper, otherwise its orthogonal complement would be $N(P) \cap R(Q) = \{0\}$, which together with the fact that $N(P) + R(Q) = \mathcal{H}$ (closed!), would lead us to the second alternative.

Analogously in the second alternative, $N(P) + R(Q)$ is proper.

If neither of these two happen, it is clear that $R(P) + N(Q)$ (nor, equivalently the sum of the orthogonals $N(P) + R(Q)$) is closed. □

We have the following:

Theorem 8.9. *Let $I, J \subset \mathbb{R}^n$ with finite Lebesgue measure. Then*

1. *$R(P_I) + R(Q_J)$ is a closed proper subset of $L^2(\mathbb{R}^n)$, with infinite codimension. The sum is direct ($R(P_I) \cap R(Q_J) = \{0\}$).*
2. *$N(P_I) + N(Q_J) = L^2(\mathbb{R}^n)$, and the sum is not direct ($N(P_I) \cap N(Q_J)$ is infinite dimensional).*
3. *$R(P_I) + N(Q_J)$ and $N(P_I) + R(Q_J)$ are proper dense subspaces of $L^2(\mathbb{R}^n)$, and $R(P_I) \cap N(Q_J) = N(P_I) \cap R(Q_J) = \{0\}$.*

Proof. By the cited result [12], two projections P, Q, satisfy that $R(P) + R(Q)$ is closed and $R(P) \cap R(Q) = \{0\}$ if and only if $\|PQ\| < 1$. It is also known (see above, [16]) that $\|P_I Q_J\| < 1$. The intersection of these spaces is, in our case (using the notation of the Halmos' decomposition)

$$R(P_I) \cap R(Q_J) = \mathcal{H}_{11} = \{0\}.$$

As remarked above, Lenard proved that $\mathcal{H}_{11} = \mathcal{H}_{10} = \mathcal{H}_{01} = \{0\}$, and $\mathcal{H}_{00}$ is infinite dimensional. The orthogonal supplement of this sum is

$$(R(P_I) + R(Q_J))^\perp = N(P_I) \cap N(Q_J) = \mathcal{H}_{00}.$$

Thus the first assertion follows.

In our case $\|P_I - Q_J\| = 1$ ([16], [25]) thus we may apply the above Lemma.

The first condition cannot happen:

$$(N(P_I) + R(Q_J))^\perp = R(P_I) \cap N(Q_J) = \mathcal{H}_{10} = \{0\}.$$

By a similar argument, neither the second condition can happen. Thus $R(P_I) + R(Q_J)$ is non closed, and its orthogonal supplement is trivial. Thus the second and third assertions follow. □

Remark 8.10. It is known (see for instance [15]), that if P, Q are projections with PQ compact and $R(P) \cap R(Q) = \{0\}$, then

$$\|PQ\| < 1.$$

In [9], the second named author and A. Maestripieri studied the set of operators $T \in \mathcal{B}(\mathcal{H})$ which are of the form $T = PQ$. Among other properties, they proved that T may have many factorizations, but there is a minimal factorization (called canonical factorization of T), namely

$$T = P_{\overline{R(T)}} P_{N(T)^\perp},$$

which satisfies that if $T = PQ$, then $R(T) \subset R(P)$ and $N(T)^\perp \subset R(Q)$ (or equivalently $N(Q) \subset N(T)$). Following this notation,

Proposition 8.11. *The factorization $P_I Q_J$ is canonical.*

Proof. Put $T = P_I Q_J$. Using Halmos' decomposition in this particular case ($\mathcal{H} = \mathcal{H}_{00} \oplus (\mathcal{L} \times \mathcal{L})$), apparently

$$P_I Q_J P_I = 0 \oplus \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix},$$

and thus $R(P_I Q_J P_I) = 0 \oplus (R(C) \times 0)$. Recall that $C^2 > 0$, and thus C^2 has dense range. It follows that

$$\overline{R(T)} = \overline{R(P_I Q_J)} = \overline{R(P_I Q_J P_I)} = 0 \oplus (\mathcal{L} \times 0),$$

which is precisely the range of P_I: $\overline{R(T)} = R(P_I)$. Note the following elementary fact:

$$N(PQ) = N(Q) \oplus (R(Q) \cap N(P)).$$

For the factorization $T = P_I Q_J$ it is known ([25]) that $R(Q_J) \cap N(P_I) = 0$. Thus

$$N(T) = N(P_I Q_J) = N(Q_J)$$

and the proof follows. □

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